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EXTREMES OF PRECIPITATION IN PERIODS OF CONSECUTIVE MONTHS

BY

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Abstract:

The greatest and least amounts of precipitation which can occur during specified periods are of great importance to hydrologists and others. It is desirable to establish formulae valid for large areas by which the extremes may be estimated from the annual mean precipitation. The subject has been investigated by Glasspoole, and the present paper is designed to introduce a simple procedure for calculating the results, with examples from Irish meteorological stations. The associated theory is discussed, including the question of persistence of wet and dry spells: Irish data are shown to be inconsistent with the idea of persistence affecting monthly totals.

Introduction:

A very important practical problem is determining the extremes of precipitation which can occur in different areas. Drainage engineers must know just how much they have to cope with. Water supply designers must know both extremes - what supplies they can rely on in the driest years, and how much surplus will be available for storage during the wet ones. It is especially desirable to discover formulae which yield the necessary data for regions where actual measurements are few - in effect formulae to provide interpolation between places where adequately long precipitation records are available.

The evaluation for the long established stations is laborious, but not difficult - what Conrad and Pollak [1] classify as a primitive climatological characteristic. One merely takes the monthly precipitation for each month, set out in tables or on long strips. Ideally the total should be computed for every possible set of consecutive months, every pair, every group of three, of four, and so on. In practice it is easy to select the generally wet and dry periods and concentrate on them, eventually selecting the wettest pair, the driest pair, etc. To put the results into a non-dimensional form and to assist comparison with figures for other places, they are then expressed as percentages of the annual mean precipitation for the station.

If the results are plotted against time the curves (e.g. diagram 4) suggest an asymptotic approach to the normal, as can be expected from our use of the word normal - the average of the basic data for some long period.

The next step is to express the curves as mathematical functions. One method would be to find enveloping curves such that none of the plotted points fell outside it. This would certainly give a result but not one applicable to everyday problems. It would express the extreme extremes instead of the extremes likely to be experienced in a cosmically short time. Further there does not seem to be any rigorous method of evaluating such an envelope, a curve for which all the departures of the individual points must be of the same sign. Rejecting this possibility we can fit curves to

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the points by the "least squares" procedure. Curves so obtained must clearly be interpreted carefully. They represent likely extremes of wetness and dryness and not the real limits. There must also be some dependence on the length of the period analysed. Thus engineers must allow a further margin for these factors in addition to allowances for evaporation, percolation, etc.

Glasspoole [2] in 1930 published the results of his calculations and gave the equations of the curves valid for Great Britain and Ireland as

$$R_W = 9.8M + 17M^{\frac{1}{2}} - 0.18M^2$$
$$R_D = 0.55 - 5.5M^{\frac{1}{2}} + 4.88M + 0.162M^2$$

where R_W and R_D are respectively the greatest and least precipitation during M consecutive months, expressed as percentages of the annual "normal" precipitation. It appears that Glasspoole evaluated R_W and R_D separately. This is an unnecessary elaboration. The curves seem to be a symmetrical pair and theory suggests that such symmetry is to be expected.

Theory:

The theory is basically that of random deviations from the long-period average, with minor correcting terms. If the monthly precipitation totals show a Gaussian distribution, all scattered about the same mean $\bar{F}$ with standard deviation σ, the total for M consecutive months should range between $M\bar{F} \pm n\sigma M^{\frac{1}{2}}$. The $\sigma M^{\frac{1}{2}}$ represents the standard deviation for the M-months total and the factor n is necessary as we are dealing with extremes. $n \approx 3$ in many similar meteorological problems of extremes, e.g. Valentia Annual precipitation = 1418 mm. with s.d. 164 mm. and the largest departures in 87 years were +379 and -338mm.

A term in M^2 is also to be expected. According to Brooks and Carruthers [3] Glasspoole included it to allow for the effects of persistence but I find it arises from the existence of a regular annual variation, so that $\bar{F}$ varies from month to month. The point is easily understood by considering data for Valentia Observatory. Table I gives the average precipitation there for each month of the year and Table II the resultant totals for sequences of "average" months.

TABLE I. Average Precipitation at Valentia Observatory, 1866 - 1953

Month	Average Precipitation (mm.)	Percentage of Annual
January	159	11.2
February	122	8.6
March	104	7.3
April	89	6.3
May	82	5.8
June	81	5.7
July	100	7.0
August	116	8.2
September	113	8.0
October	146	10.3
November	142	10.0
December	164	11.6
Year	1418	-

TABLE II Extremes of Precipitation for periods of consecutive months in normal years at Valentia Observatory, 1866 - 1953

Number of Months	Maximum (% of annual)	Minimum (% of annual)
1	11.6	5.7
2	22.8	11.5
3	32.8	17.8
4	43.1	24.8
5	51.7	32.1
6	59.7	40.3
7	67.9	48.3
8	75.2	56.9
9	82.2	67.2
10	88.5	77.2
11	94.3	88.4
12	100.0	100.0

The figures in Table II fit very well to the expressions

$$8.33M \pm (3.5M - .29M^2)$$

as is shown by Diagram 1.

Since the extremes being investigated represent extreme departures from such varying normals it is reasonable to expect the expressions for R_W and R_D to include a term in M^2 arising from the existence of the regular annual variation. One can predict also that the coefficient of M^2 will be largest at stations having the best-marked annual variations.

Persistence

It is well known among meteorologists that wetness and dryness are among the characteristics which show persistence (See e.g. Ref.4). That is, the statistical probability of tomorrow being dry is greater if to-day and previous days were dry than if they were wet. As an example, the general probability of a particular day yielding under .04 inches of water at Fassaroe (Co. Wicklow) is .62, but if a dry spell of 15 consecutive such days has been established the chance of the 16th day being dry is .84 [5]. It is a nice paradox that the probability of a spell's continuance seems to approach unity for a very long spell, whereas common sense insists that there must be a limit beyond which the probability drops suddenly to zero..

Do persistent spells occur often enough and last long enough to influence the monthly totals? Or is there persistence of monthly as well as daily wetness and dryness? Some evidence, such as that collected by Beer, Drummond and Fürth [6], suggests a positive answer, although not statistically significant. Some simple analyses are sufficient to show that at Irish stations the month-to-month fluctuations satisfy rigorous tests for randomness. For example the numbers of spells at Valentia were counted, a spell being defined as a sequence of months for each of which the departure from the long-period average was in the same direction. The numbers are presented in Table III and Diagram 2, together with the theoretical results for a random distribution. The theoretical figures used were derived from Gold's treatment [7]. If the chance of a wet month is "p" and of a dry is "q", the chance of a spell of n or more consecutive wet months is $p^n q$.

TABLE III Sequences of Wet and Dry months at Valentia Observatory, 1866 - 1953

Number in sequence		1	2	3	4	5	6	7	≥ 8	Total	X^2	Prob. of randomness
All wet	Actual	145	66	32	15	8	3	1	0	498	1.9	.97
	Expected	139	66	31	15	7	3	2	1	499		
All dry	Actual	128	62	38	20	12	6	2	0	554	3.6	.8
	Expected	125	66	35	18	11	5	3	1.5	554		

The similarity of the actual and expected figures is much too close to give support to ideas of persistence, and X^2 tests confirm this.

A simpler test for persistence is to select the wettest and the driest months and to average the data for a few months before and after them. This was done for Valentia, using the five wettest and driest Januaries, Februaries, etc. See Table IV, and Diag. 3.

TABLE IV Average Precipitation, as percentage of normal, in months before and after the wettest and driest. Valentia Observatory, 1866 - 1953

No. of month	N-6	N-5	N-4	N-3	N-2	N-1	N	N+1	N+2	N+3	N+4	N+5	N+6
N very wet	106	112	115	94	95	95	191	100	100	99	92	100	113
N very dry	99	106	91	105	96	108	29	94	101	106	104	98	97

Again we find no real evidence supporting persistence.

It must be pointed out that the methods used here both eliminated the effects of the regular annual variation, which will have a small influence in that the absolutely wettest months must tend to occur in the rainy season.

Curve-fitting

The expressions for R_W and R_D should therefore include three terms, one determined by the normal, one by random departures from the normal, and one arising from the regular annual variation. A fourth can be expected, due to the skewness of the frequency distribution of the monthly data. Since precipitation is a variable with a fixed lower limit (zero) positive deviations can be larger than the negative. In Ireland this effect is well marked in monthly totals, but small in totals for longer periods. The effect of skewness should be to make the coefficient of $M^{\frac{1}{2}}$ larger for R_W than R_D and to make it decrease with increasing M. It is found, however, that an alternative solution is obtained by including a constant term to allow for this asymmetry.

(1) Then $\begin{matrix} R_W \\ R_D \end{matrix} = a + bM \pm (cM^{\frac{1}{2}} - dM^2)$ and we have to ascertain the values of a, b, c and d which give the best fit to the observed values of R_W and R_D. There are various short methods of fitting a polynomial expression to a given curve, but they depend on the powers being integral, and do not work when there is a term such as $cM^{\frac{1}{2}}$. The straightforward least-squares method is easy if only two coefficients have to be found. i.e. if $y = a + bx$ is to be fitted to a number of pairs of values of x and y, then the required a and b are those which make $\Sigma\,(y-a-bx)^2$, the sum of the squares of the deviations from the curve, a minimum. It is easily shown that the required values are

$$(2)\qquad \frac{a}{\sum_1^n y_i \sum_1^n x_i^2 - \sum_1^n x_i \sum_1^n x_i y_i} = \frac{b}{n\sum_1^n x_i y_i - \sum_1^n x_i \sum_1^n y_i} = \frac{1}{n\sum_1^n x_i^2 - (\sum_1^n x_i)^2}$$

Cases with more than two coefficients can be treated similarly but the arithmetic is tedious, it being necessary to work to many more significant figures than are required for the final result.

In the present case the complication can be avoided by assuming the same a, b, c, d to be applicable to R_W and R_D. Then equations (1) can be rearranged as

$$(3)\qquad \begin{cases} R_W + R_D = 2(a + bM) \\ R_W - R_D = 2(cM^{\frac{1}{2}} - dM^2) \end{cases}$$

Applying the result (2) to these expressions the required solutions are obtained as

$$(4)\qquad \frac{a}{\sum_1^n (R_W + R_D) \sum_1^n M^2 - \sum_1^n M \sum_1^n M(R_W + R_D)} = \frac{b}{n\sum_1^n M(R_W + R_D) - \sum_1^n M \sum_1^n (R_W + R_D)}$$
$$= \frac{1}{2[n\sum_1^n M^2 - (\sum_1^n M)^2]}$$

$$\frac{c}{\sum_1^n M^{\frac{1}{2}}(R_W - R_D) \sum_1^n M^4 - \sum_1^n M^{\frac{5}{2}} \sum_1^n M^2(R_W - R_D)} = \frac{d}{\sum_1^n M^{\frac{5}{2}} \sum_1^n M^{\frac{1}{2}}(R_W - R_D) - \sum_1^n M \sum_1^n M^2(R_W - R_D)}$$
$$= \frac{1}{2[\sum_1^n M \sum_1^n M^4 - (\sum_1^n M^{\frac{5}{2}})^2]}$$

In the present case, if attention is confined to periods not longer than a year, n = 12 and the summations needed are $\Sigma M = 78$, $\Sigma M^2 = 650$, $\Sigma M^{5/2} = 1968.35$ and $\Sigma M^4 = 60710$. Substituting these in (4) and simplifying we obtain

$$(5)\quad \begin{cases} a = 0.18939\sum_1^{12}(R_W + R_D) - 0.022727\sum_1^{12} M(R_W + R_D) \\ b = 0.0034965\sum_1^{12} M(R_W + R_D) - 0.022727\sum_1^{12}(R_W + R_D) \\ c = 0.035257\sum_1^{12} M^{\frac{1}{2}}(R_W - R_D) - 0.0011431\sum_1^{12} M^2(R_W - R_D) \\ d = 0.0011431\sum_1^{12} M^{\frac{1}{2}}(R_W - R_D) - 0.00004530\sum_1^{12} M^2(R_W - R_D) \end{cases}$$

Substituting the Valentia data as given in the "actual" columns of Table V, the values of the coefficients are found to be

$$\begin{aligned} a &= 6.9 \\ b &= 7.7 \\ c &= 14.3 \\ d &= 0.14 \end{aligned}$$

The synthesis from these values is included in the Table in columns A, and is plotted in Diagram 4 which shows also the actual data.

The agreement is seen to be quite good. To test how far my simplified technique approximates to the results obtained by other methods, Table V incorporates under B the results of separate calculations for R_W and R_D using the same form of expression, and under C similar calculations without the constant term "a". The last line of the table gives the sums of the squares of the deviations (Observed minus Expected). It is seen that although the B columns fit better than the A, the difference is very small, and A is much better than C.

TABLE V Actual and calculated extremes of precipitation in periods of consecutive months at Valentia Observatory 1866 - 1953

WETTEST

Number of months	Date	Actual (0)		Calculated (E)		
		Amount (mm.)	% of annual normal	A $6.9 + 7.7M + 14.3M^{\frac{1}{2}} - .14M^2$	B $-1.0 + 5.3M + 23.0M^{\frac{1}{2}} - .08M^2$	C $5.2M + 23.7M^{\frac{1}{2}} - .10M^2$
1	1934	364	26	29	27	29
2	1929-30	573	40	42	44	44
3	1929-30	823	58	54	54	53
4	1929-30	981	69	64	65	67
5	1882-83	1041	73	74	75	76
6	1929-30	1151	83	83	84	87
7	1929-30	1285	91	92	93	94
8	1882-83	1426	101	100	101	102
9	1882-83	1525	108	108	109	110
10	1946-47	1614	114	114	117	117
11	1929-30	1732	122	122	121	123
12	1929-30	1886	133	129	132	130
$\sum(0-E)^2$				74	70	102

TABLE V (Contd.)

DRIEST

Number of months	Date	Actual (0) Amount (mm.)	% of annual normal	Calculated (E) A $6.9 + 7.7M - 14.3M^{\frac{1}{2}} + .14M^2$	B $8.1 + 7.2M - 14.2M^{\frac{1}{2}} + .17M^2$	C $3.6M - 3.14M^{\frac{1}{2}} + 25M^2$
1	1938	3	0.1	0.4	1.1	0.8
2	1896	39	3	3	3	4
3	1921	112	8	7	7	8
4	1909	160	11	11	11	12
5	1940	246	17	17	17	17
6	1940	326	23	23	23	23
7	1940	411	29	30	29	29
8	1952	486	34	37	36	36
9	1893	621	44	45	44	43
10	1952	756	53	53	52	51
11	1893	886	62	61	61	59
12	1952-53	972	69	70	70	68
$\sum(0-E)^2$				14	9	21

Other Irish Stations

An important test of the usefulness of an expression such as that discussed above is ascertaining whether it applies to data for different stations, or to data for one station at different periods. The coefficients a, b, c, d, have therefore been calculated for several Irish stations and the results are set out in Table VI.

TABLE VI. Coefficients of $\frac{R_W}{R_D} = a + bM \pm (cM^{\frac{1}{2}} - dM^2)$

Station	Period	Average Annual Precipitation (mm.)	a	b	c	d
Valentia Observatory	1866-1915	1428	3.2	8.0	12.5	0.10
	1916-1953	1404	7.3	7.7	14.2	0.13
	1866-1953	1418	6.9	7.7	14.3	0.14
Markree Co. Sligo	1833-1863	(1140)	4.3	8.5	12.4	0.09
	1916-1953	1166	5.1	7.8	12.1	0.08
Birr Co. Offaly	1866-1915	842	6.7	7.7	12.2	0.09
	1916-1953	876	2.6	8.7	11.3	0.01
	1866-1953	860	3.8	8.6	12.1	0.01
Phoenix Park Dublin	1837-1880	707	10.5	8.2	16.6	0.11
	1881-1915	701	6.1	7.7	13.4	0.04
	1916-1953	748	2.7	9.1	12.3	-0.05
	1837-1953	ø	10.2	8.2	16.1	0.02
Fassaroe Co. Wicklow	1857-1914	1034	8.4	8.2	16.8	0.05

ø Coefficients calculated from the extreme R_W and R_D of the sub-periods.

Coefficient a seems to be largest at the eastern stations, but the distribution is erratic. A comparison of the values with the skewness of the annual precipitation showed no connection.

Coefficient b should be $\frac{100\%}{12 \text{ months}}$ = 8.3, and in most cases the agreement is quite good. The largest anomaly is at Phoenix Park for the period 1916-53, which was exceptional with all the coefficients.

Coefficient c, dependent on the deviations from normal, is the most interesting. The theoretical $c = n\sigma$ where σ is the standard deviation of monthly precipitation, or $n\frac{\sigma A}{\sqrt{12}}$ where σA is the standard deviation of annual precipitation. The value of n should be about 3 for short periods and not much greater for long ones, judging by other meteorological results. Table VII sets out the relevant figures for the data used in this paper.

TABLE VII

Station	Period	Standard Deviation of annual precipitation (% of average)	c	n
Valentia Observatory	1866-1915 1916-1953 1866-1953	10.9 12.4 11.5	12.5 14.2 14.3	4.0 4.0 4.3
Markree Co. Sligo	1833-1863 1916-1953	12.3 13.1	12.4 12.1	3.5 3.2
Birr Co. Offaly	1866-1915 1916-1953 1866-1953	10.5 12.8 11.2	12.2 11.3 12.1	4.0 3.0 3.7
Phoenix Park Dublin	1837-1880 1881-1915 1916-1953	14.8 14.9 15.0	16.6 13.4 12.3	3.9 3.1 2.8
Fassaroe Co. Wicklow	1857-1914	15.0	16.8	3.9

The values of n for the periods ending 1915 are remarkably consistent except for Phoenix Park, which is also exceptional in having the only value less than 3. It is clear that the years since 1915 have had less extreme conditions than those before, and that n=4.0 is likely to be a good approximation at both eastern and western stations, when sufficiently long records are examined.

Coefficient d is largest at Valentia, consistent with that station having the simplest annual fluctuation of precipitation. Elsewhere there is a Summer maximum as well as a Winter one, and in some places August is the wettest month of the year. It seems likely that the M^2 term can be neglected at eastern and midland stations, but must be allowed for in western regions.

Extension to periods of more than 12 months

The analysis can obviously be extended to periods of more than 12 months, but it is more difficult to fit a formula to the figures.

Glasspoole published values of R_W and R_D for M up to 40, but without any derived formulae. The difficulty is that for $M > 12$ the term depending on the regular annual variation must be a periodic one, which complicates the calculations unduly. The alternative is to omit the term in M^2 but the resultant fit is not very satisfactory. As an example the Valentia data are given for up to 24 months, in Table VIII and Diagram 5.

TABLE VIII. Actual and calculated extremes of precipitation in periods of consecutive months at Valentia Observatory, 1866-1953 as percentages of the normal annual total

No. of months	WETTEST		DRIEST	
	Actual	$9.5 + 12.5M^{\frac{1}{2}} + 7.1M$	Actual	$4.0 - 13.7M^{\frac{1}{2}} + 8.9M$
1	26	29	0.1	-1
2	40	41	3	2
3	58	52	8	7
4	69	63	11	12
5	73	73	17	18
6	83	83	23	24
7	91	92	29	30
8	101	102	34	36
9	108	111	44	43
10	114	120	53	50
11	122	129	62	57
12	133	138	69	63
13	145	147	74	70
14	156	156	78	77
15	167	164	84	84
16	174	173	91	92
17	184	181	95	99
18	194	190	106	106
19	198	199	114	113
20	204	207	119	121
21	214	216	124	128
22	225	224	131	135
23	239	233	144	143
24	248	241	157	151

The deviation of the actual figures from the curve of best fit is most noticeable at the top of the curve, and it is clearly useless for extrapolation.

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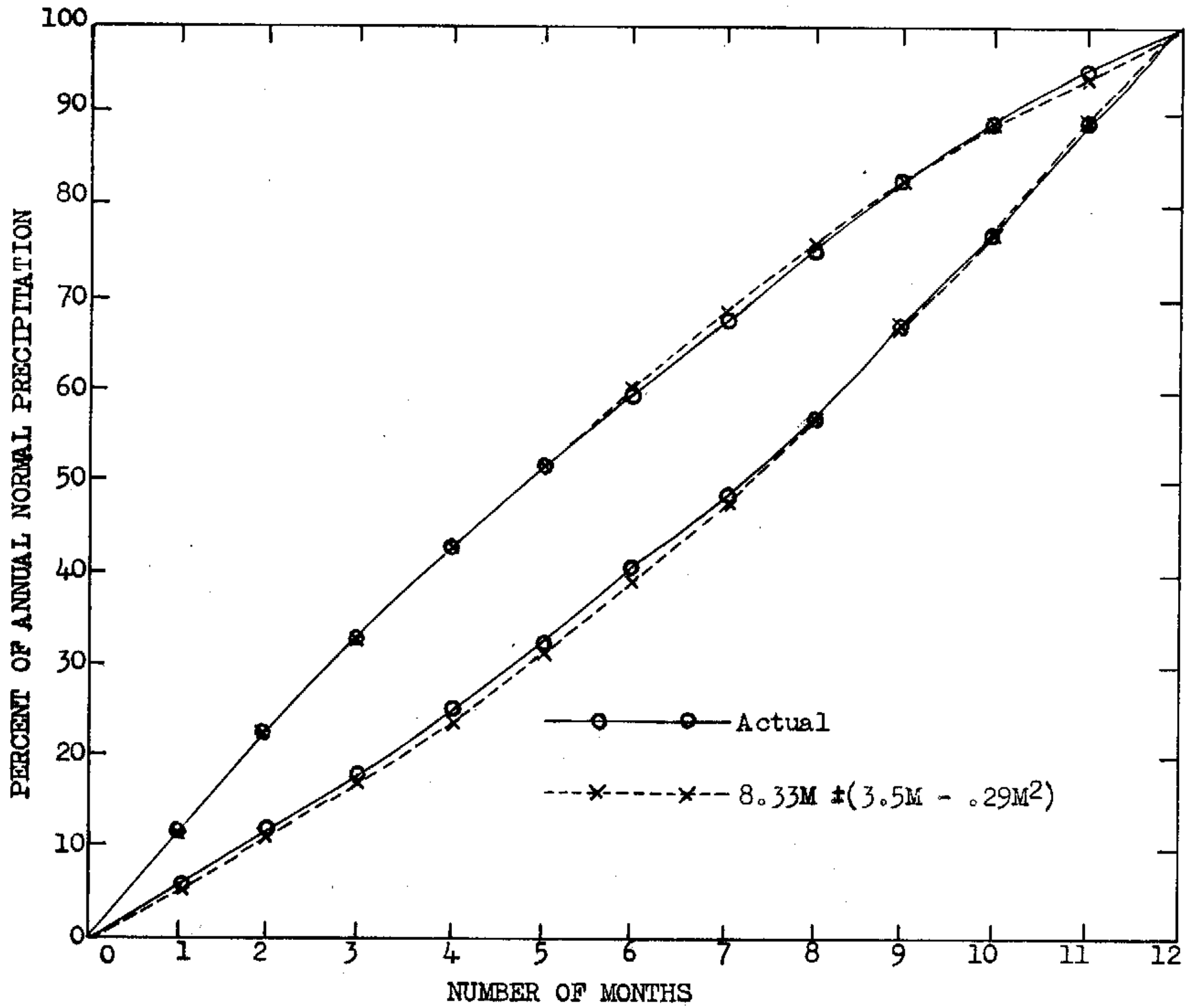

Fig. 1. Extremes of Precipitation in periods of consecutive months during "normal" years. Valentia 1866-1953.

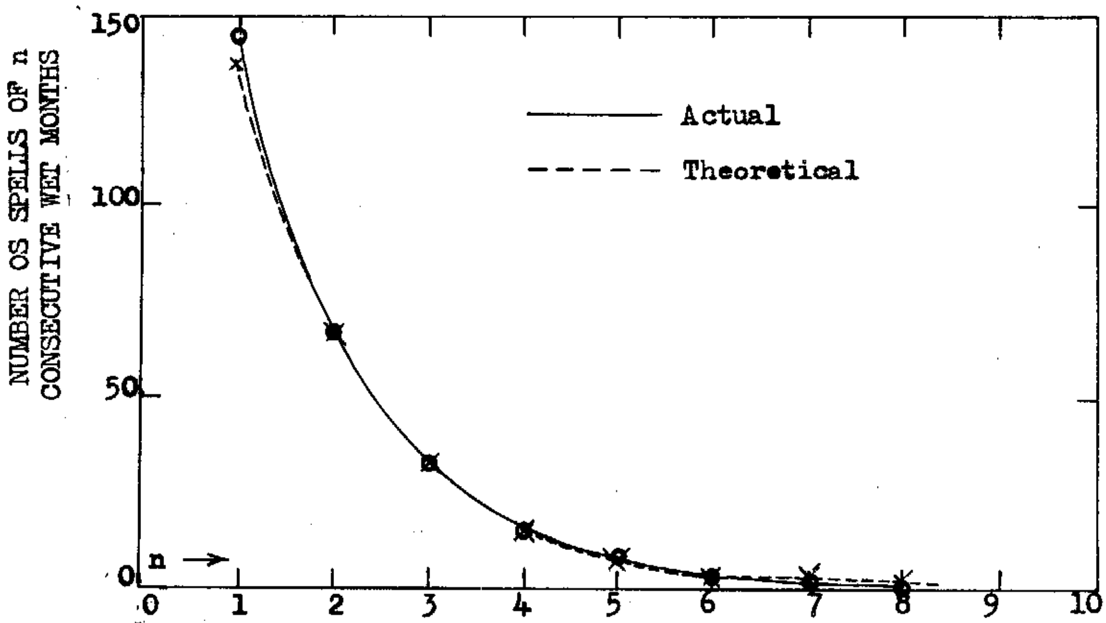

Fig. 2a. Sequences of wet months at Valentia Observatory, 1866-1953.

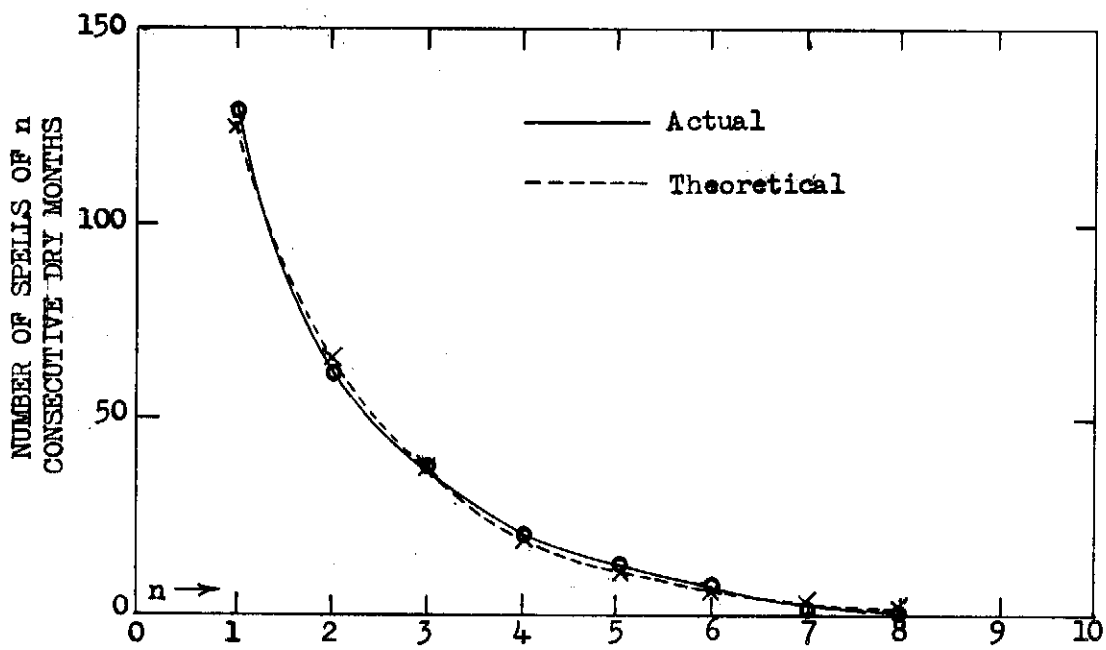

Fig. 2b. Sequences of dry months at Valentia Observatory, 1866-1953.

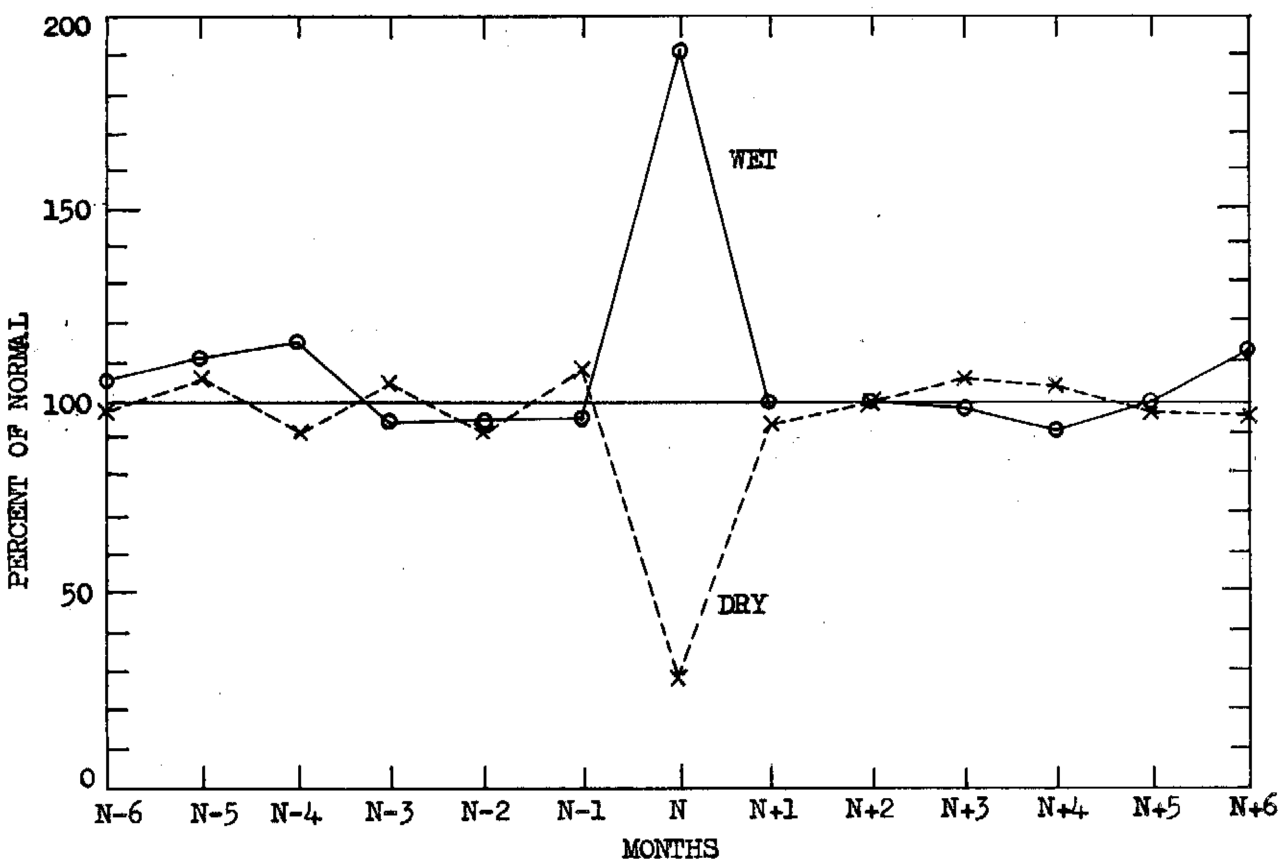

Fig. 3. Precipitation variations before and after individual wet and dry months. Valentia 1866-1953.

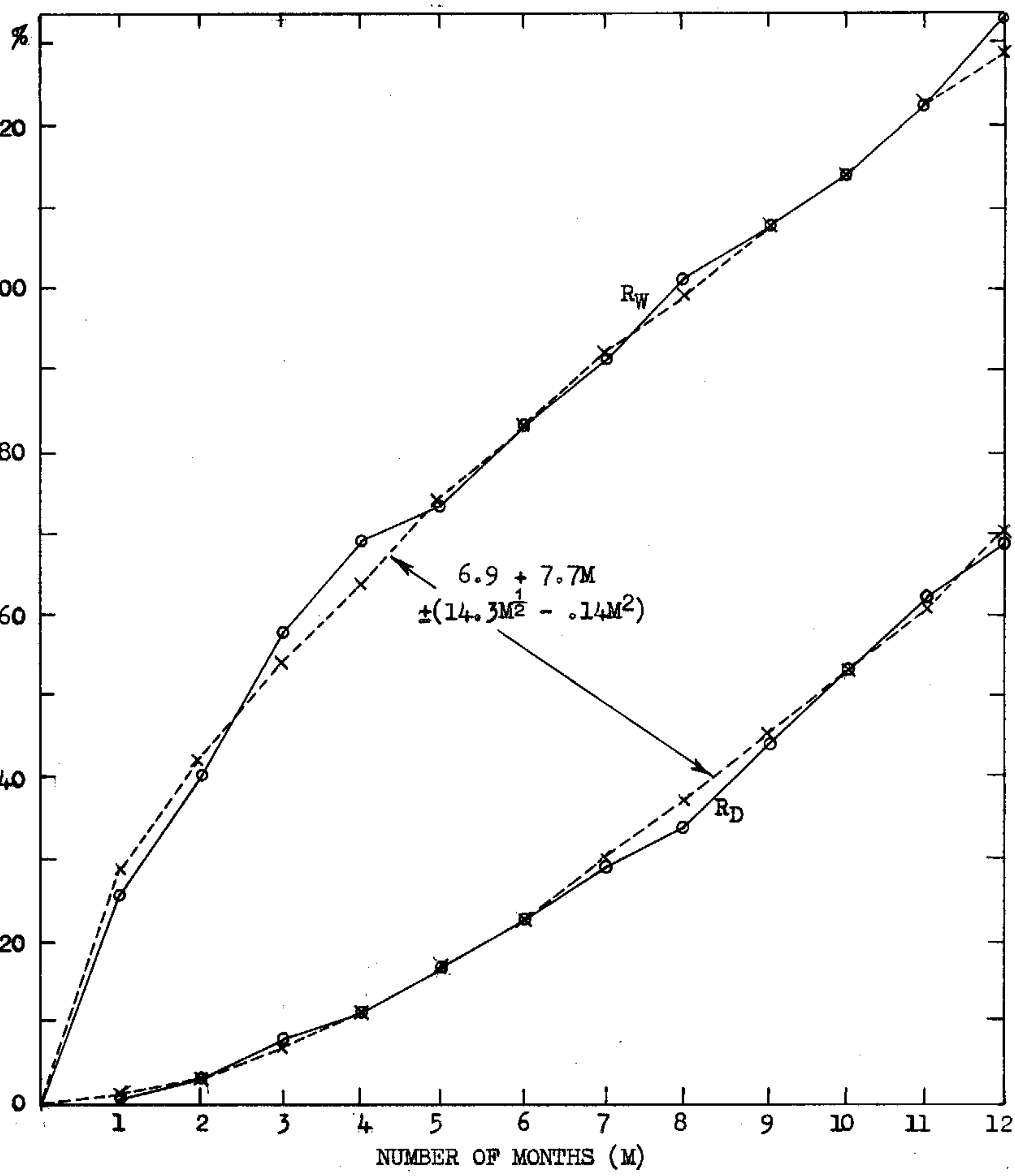

Fig. 4. Extremes of Precipitation totals for M consecutive months as percentages of the annual mean. Valentia 1866-1953.

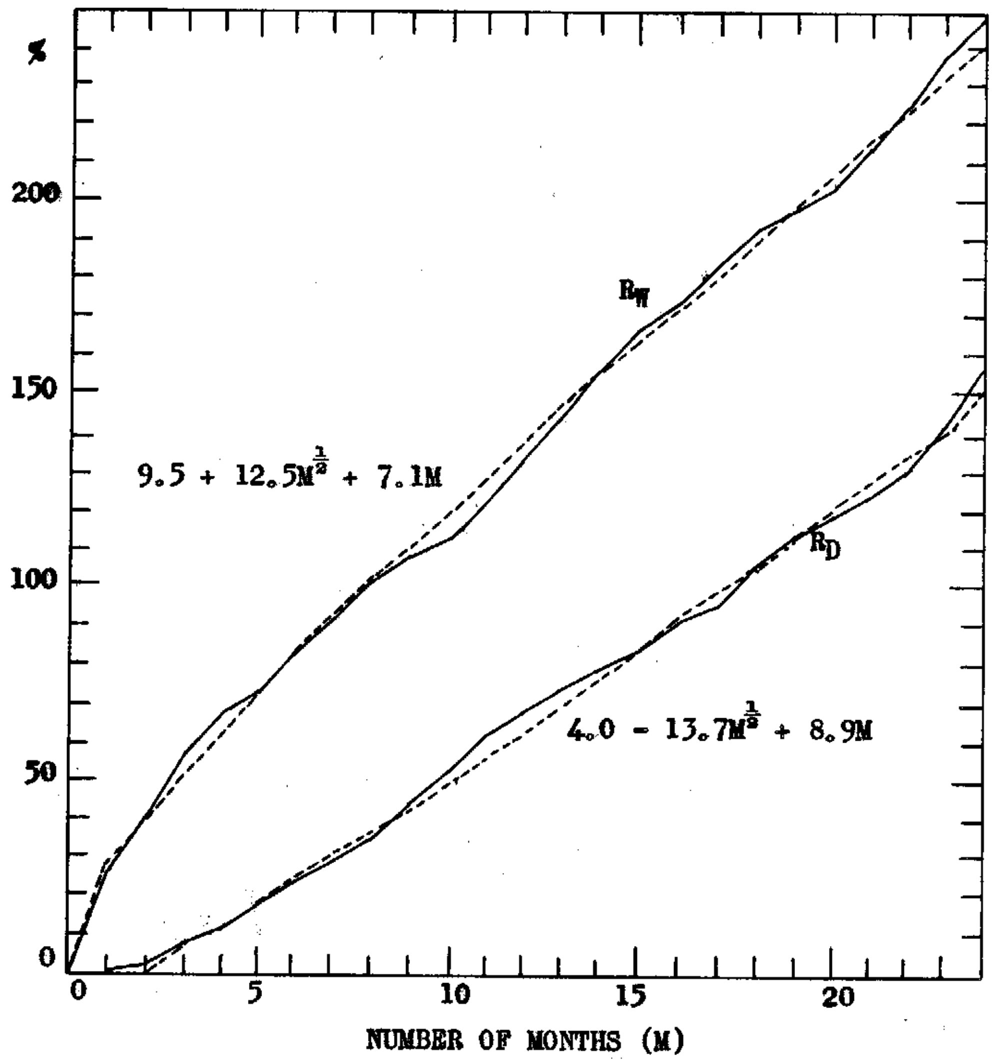

Fig. 5. Extremes of Precipitation totals for M consecutive months as percentages of the annual mean. Valentia 1866-1953.