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# MODULARITY IN SUPERSYMMETRIC GAUGE THEORY

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# Declaration

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Parts of this thesis are based on the publications [1–4] and the preprint [5], which contains work in collaboration with Johannes Aspman, Georgios Korpas, Jan Manschot, Zhi-Cong Ong and Meng-Chwan Tan.

Elias Furrer

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# Publications

This thesis is based on the following publications [1–5], as well as unpublished work:

- [1] Elliptic Loci of  $SU(3)$  Vacua  
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with J. Aspman and J. Manschot.
- [2] Cutting and gluing with running couplings in  $\mathcal{N} = 2$  QCD  
*Phys. Rev. D* **105** (Jan, 2022) 025021, [[hep-th/2107.04600](#)]  
with J. Aspman and J. Manschot.
- [3] Four flavours, triality and bimodular forms  
*Phys. Rev. D* **105** (Jan, 2022) 025017, [[hep-th/2110.11969](#)]  
with J. Aspman and J. Manschot.
- [4] The  $u$ -plane integral, mock modularity and enumerative geometry  
*Letters in Mathematical Physics* **112** (2022) 30, [[hep-th/2109.04302](#)]  
with J. Aspman, G. Korpas, Z.-C. Ong and M.-C. Tan.
- [5] Topological twists of massive SQCD, Part I  
[\[hep-th/2206.08943\]](#)  
with J. Aspman and J. Manschot.

# Summary

Supersymmetry provides a rich ground for qualitative and quantitative analyses in both quantum field theory and gravity. Many supersymmetric theories exhibit dualities such as the strong-weak coupling duality, which manifests itself through the equivalence of the dynamics of a theory for distinct values of its coupling constants. The mathematical formulation of dualities in many cases involves modular and automorphic forms, whose number theoretic properties are often as intriguing as the dualities themselves. It is a natural problem to determine a domain for the coupling constant parametrising inequivalent theories, and finding the relations between the couplings giving rise to the same dynamics.

In this thesis, we obtain several results on this question, particularly regarding the duality and modularity of Coulomb branches for four-dimensional  $\mathcal{N} = 2$  supersymmetric quantum field theories. For pure  $\mathcal{N} = 2$  super Yang–Mills theory (SYM) with gauge group  $SU(2)$ , the Coulomb branch (CB) can be identified with the modular fundamental domain for the duality group of the theory. We extend this result for higher rank gauge groups and for the inclusion of matter.

There are several applications and further directions of study that our results enable. First, our exact results on fundamental domains allow to determine BPS spectra for a larger class of  $\mathcal{N} = 2$  supersymmetric theories. Furthermore, our precise statements of duality allow to study the strongly coupled regions in great detail. Finally, they provide simple and exact formulas for the computation of correlation functions of topological theories. We discuss these results in more detail in the following.

To begin with, we study the asymptotically free  $\mathcal{N} = 2$   $SU(2)$  theories with fundamental matter. The duality groups were only known for  $N_f = 2$  and 3 massless flavours, whereas the generic mass case including the peculiar role of massless  $N_f = 1$  have remained elusive. We find a completely general description for arbitrary masses  $m_i$  and number  $N_f$  of flavours: The space of inequivalent couplings is a fundamental domain  $\mathcal{F}(m_i)$ , endowed with a collection of branch cuts and branch points. For special choices of the masses, the branch points annihilate and  $\mathcal{F}(m_i)$  is a modular fundamental domain. The CB parameter  $u$  is the root of a sextic polynomial, and for certain mass configurations can be determined explicitly as a function of the effective coupling. The description incorporates all possible mass limits, such as decoupling of hypermultiplets and merging of local as well as non-local singularities, giving an analytic handle on the superconformal Argyres–Douglas theories.

The superconformal  $N_f = 4$  theory with gauge group  $SU(2)$  distinguishes itself from the asymptotically free theories through a nontrivial dependence of the theory on an ultraviolet coupling  $\tau_{UV}$ , and is a building block for four-dimensional  $\mathcal{N} = 2$  superconformal field theories. We prove that for special mass configurations, the order parameter  $u$  transforms as a modular form for

the infrared coupling  $\tau$  as well as the ultraviolet coupling  $\tau_{UV}$ . A simultaneous transformation on  $\tau$  and  $\tau_{UV}$  permutes the moduli spaces according to the triality group of the flavour symmetry  $SO(8)$ , and we show that the characteristic CB functions organise into vector-valued *bimodular forms*. These results allow to derive the S-duality transformations of the topologically twisted four flavour theory.

In a different direction, we study the duality of pure  $\mathcal{N} = 2$  SYM with gauge group  $SU(3)$ , where the two-dimensional space of vacua parametrises an intricate family of genus two Seiberg-Witten curves. We prove that two natural complex one-dimensional loci of the CB are parametrised each by elliptic (i.e. genus one) curves, such that these *elliptic loci* allow a modular parametrisation. The locus where mutually local dyons become massless provides a natural generalisation of the pure  $SU(2)$  fundamental domain, while the locus containing the superconformal Argyres-Douglas points is a fundamental domain for a Fricke group.

We furthermore consider topological twists of four-dimensional  $\mathcal{N} = 2$  supersymmetric quantum chromodynamics with gauge group  $SU(2)$  and  $N_f \leq 3$  fundamental hypermultiplets. The twists are labelled by a choice of background fluxes for the flavour group, which provides an infinite family of topological partition functions. We demonstrate that in the presence of such fluxes the theories can be formulated for arbitrary gauge bundles on a compact four-manifold. Moreover, we consider arbitrary masses for the hypermultiplets, which introduce new intricacies for the evaluation of the low-energy path integral on the Coulomb branch. We develop techniques for the evaluation of these path integrals.

When the topological version of pure  $\mathcal{N} = 2$   $SU(2)$  SYM is formulated on a simply connected four-manifold  $X$ , it is well-known that a physical correlation function computes the celebrated Donaldson invariants of  $X$ , which are smooth four-manifold invariants. Based on earlier work by Mariño and Moore, we generalise recent results by Korpas and Manschot to the case where  $X$  is non-simply connected, and show that correlation functions on such manifolds can be evaluated using mock modular forms. For a product ruled surface  $X = \Sigma_g \times \mathbb{CP}^1$ , by shrinking the genus  $g$  Riemann surface  $\Sigma_g$ , this allows to make a connection between mock modular forms and the topological A-model with target space the moduli space of flat connections on  $\Sigma_g$ .

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# 1 Introduction

Theoretical physics has been enormously successful in predicting the behaviour of nature. In the last century, two main fundamental theories have emerged. On one hand, general relativity describes the fundamental interaction between objects with mass or energy. On the other hand, quantum field theory combines special relativity and quantum mechanics into a framework that describes subatomic particles, which underlie the electromagnetic, strong and weak interactions. The combination of both theories is necessary in order to understand aspects of the natural world, such as the nature of black holes and the formation of the universe. Since there are many conceptual and technical obstacles to the problem of combining both theories, it is of utmost interest to study the intrinsic properties of candidate unified theories in great detail.

One area of such theories which is poorly understood is the problem of strong coupling. The couplings of a theory measure the strength of the interactions of its constituents. When a coupling is known to be small, we can use the powerful method of perturbation theory to study the interactions. In many cases, when the interactions are turned off the theory becomes rather simple and can be solved exactly. Perturbation theory then allows to express the effect of a small coupling as a small correction to the exact solution. This only provides a good solution if the coupling is small compared to the quantities involved.

When the coupling on the other hand is large, the correction to the exact solution is large compared to the original solution, and thus does not provide a good approximation. Therefore, perturbation theory typically becomes obsolete when studying strong interactions. For most relevant physical theories such as the Standard Model, studying the physics of strong coupling in full generality has remained an important open problem.

One way to gain access to the properties of strong coupling is through supersymmetry. Supersymmetry is a natural feature of many unified theories. It allows us to study theories in a wider range of parameters (such as the couplings), as well as to calculate interactions exactly rather than approximately. The broad motivation for this thesis is thus to study exact methods to access the physics of strong coupling and to demonstrate that these analyses allow to compute supersymmetric quantities exactly and in full detail.

This thesis considers the relation between supersymmetry, dualities, modular forms and topological field theory. The remainder of this first section is devoted to giving an introduction to some aspects of these theories, which are necessary to understand the following sections. We hopefully provide enough references in order for the reader to fill in gaps, and also refer to the [Appendix A](#) for more references.

## 1.1 Supersymmetry

Supersymmetry is a symmetry between two basic classes of particles in quantum field theory: bosons, which have integer-valued spin, and fermions, which have half-integer spin (see [6–8] for excellent introductions to supersymmetry). They have very different thermodynamic properties. While bosons follow the Bose-Einstein statistics, fermions obey the Pauli exclusion principle and follow the Fermi-Dirac statistics. In a supersymmetric theory, each particle from one class has a *superpartner* from the other class, whose spin differs by a half-integer number. Moreover, the supersymmetric partners share the same mass and internal quantum numbers other than spin.

Supersymmetry is motivated by solutions to several problems in quantum field theory which are difficult to study in non-supersymmetric theories. Due to the constraints it gives on the dynamics, supersymmetric theories are often simpler to analyse. In many cases, quantities such as path integrals, correlation functions etc. can be determined exactly rather than only perturbatively, which gives an analytic handle on the physics. For instance, many supersymmetric theories contain interesting *dualities*, which manifests itself through the equivalence of the dynamics of a theory for distinct values of its coupling constants.

One instance of a duality is the strong-weak coupling duality, relating the dynamics for large and small values of the coupling constants. This allows in many cases to study strongly coupled phases of quantum field theory, which are not accessible through traditional perturbative calculations. A particularly interesting example of such a theory is quantum chromodynamics (QCD), as it is the theory of the strong interaction between quarks and gluons and thus an important part of the Standard Model of particle physics. QCD exhibits many interesting features, such as asymptotic freedom and colour confinement. Asymptotic freedom causes interactions between fields to become weaker as the energy scale increases. In particular, quarks interact weakly at high energies, and therefore can be studied through perturbation theory. At low energies however, the interactions become strong, which leads to the *confinement* of quarks and gluons. Colour confinement prohibits quarks and gluons to be isolated from each other. The supersymmetric version of QCD is asymptotically free as well, however the emergent strong-weak coupling duality allows to study the perturbative as well as non-perturbative dynamics at both weak and strong coupling in great detail.

Supersymmetry can be incorporated into a theory by extending the familiar Poincaré algebra by the supersymmetry algebra. The supersymmetry algebra is generated by spinors  $Q_\alpha^I$  and  $(Q_\alpha^I)^\dagger$  called *supercharges*. Here,  $\alpha$  are spinor indices, while  $I = 1, \dots, \mathcal{N}$  are the supersymmetry indices. The integer  $\mathcal{N}$  counts the number of independent supersymmetries of the algebra. Representations of the supersymmetry algebra are called *supermultiplets*, or *multiplets* for short. They consist of a collection of fields which are superpartners.

In four dimensions, the possible amounts of supersymmetry for a quantum field theory are  $\mathcal{N} = 0, 1, 2, 3$  and  $4$ . For  $\mathcal{N} = 1$  supersymmetry, the most commonly used supermultiplets are the  $\mathcal{N} = 1$  vector multiplet and the chiral multiplet. In the vector multiplet, the highest component is the gauge field, while that of a chiral multiplet is a spinor. In  $\mathcal{N} = 2$ , common multiplets are the  $\mathcal{N} = 2$  vector multiplet, containing the gauge field, and the hypermultiplet, containing a spinor as the highest component. In  $\mathcal{N} = 4$ , the vector multiplet contains again all the superpartners of the gauge field.

The space of theories with  $\mathcal{N}$  supersymmetries depends in an interesting way on  $\mathcal{N}$ :

- $\mathcal{N} = 0$  theories do not contain any supercharges. They are constrained only by physical principles, and a classification is currently out of reach.
- $\mathcal{N} = 1$  theories form an enormously big class of theories, and are believed to not be exactly solvable. However, certain subsectors are dictated by supersymmetry, such as the chiral superpotential. This allows to obtain many exact results on the low-energy dynamics [9–12].
- $\mathcal{N} = 2$  supersymmetry is a rich class of theories capable of describing many interesting phenomena, while allowing full solutions of many aspects of the theories. For instance, the low-energy effective action is governed by the so-called *prepotential*. The  $\mathcal{N} = 2$  supersymmetry forces the prepotential to be a holomorphic function of the moduli, and the perturbative quantum corrections can be found exactly. This is analogous to the power of holomorphicity of conformal field theories in two dimensions. This thesis will be almost exclusively about aspects of  $\mathcal{N} = 2$  supersymmetry.
- The number  $\mathcal{N} = 3$  is interesting in its own right, for the following reason. If we consider a Lagrangian  $\mathcal{N} = 3$  theory, it might not by itself be simultaneously charge conjugate, parity and time reversal (CPT) invariant. CPT is a possible symmetry of quantum field theories, and generally holds for all physical phenomena. When a Lagrangian  $\mathcal{N} = 3$  theory is not CPT invariant, we can take instead the direct sum with its CPT conjugate. If the theory is assumed to contain particles of helicities bounded by 1, for  $\mathcal{N} = 3$  and  $\mathcal{N} = 4$  the particle spectra coincide precisely.

If we drop the requirement that the theory is Lagrangian, then it can be shown that genuine  $\mathcal{N} = 3$  theories are necessarily isolated superconformal field theories (SCFT) [13]. However, it was shown recently that purely  $\mathcal{N} = 3$  theories exist with an infinite family of them realised as the worldvolume of stacks of D3-branes probing so-called S-folds [14–17].

- $\mathcal{N} = 4$  theories finally are highly constrained. The most prominent example is maximally supersymmetric  $\mathcal{N} = 4$  Yang–Mills theory (SYM),

which is featured in the first example of the conjectured AdS/CFT holography [18]. It is believed to be self-dual [19], that is, invariant under the exchange of electric and magnetic charges. The physics of an  $\mathcal{N} = 4$  theory is generally simpler than that of an  $\mathcal{N} = 2$  theory. For instance, the low energy dynamics obtained from the Seiberg-Witten (SW) solution is rather trivial.

A particular feature of supersymmetry is that it can be broken to lower supersymmetry. For instance, maximally supersymmetric  $\mathcal{N} = 4$  SYM in four dimensions can be broken to the so-called  $\mathcal{N} = 2^*$  theory, which is an  $\mathcal{N} = 2$  supersymmetry-preserving mass deformation of  $\mathcal{N} = 4$ . This gives an analytic handle on structures of  $\mathcal{N} = 4$  supersymmetric theories that are difficult to understand without breaking the supersymmetry. For instance, any  $\mathcal{N} = 2$  theory, such as  $\mathcal{N} = 4$  SYM, contains a *Coulomb branch* of inequivalent vacua. The mass terms of the  $\mathcal{N} = 2^*$  deformation break the conformal symmetry of  $\mathcal{N} = 4$ , and give rise to a richer structure of the Coulomb branch. This allows for instance to study correlation functions in much greater detail than without any deformation.

Supersymmetry is an integral part of string theory, which aims to provide a unified description of gravity and quantum field theory. Indeed, a major motivation to study  $\mathcal{N} = 2$  theories comes from string theory: Many  $\mathcal{N} = 2$  field theories in four dimensions can be obtained from geometric engineering in Type IIA string theory on Calabi-Yau manifolds [20–23]. While all the theories studied in this thesis have a natural origin in string theory, our results mostly concern ordinary quantum field theory, i.e. without gravity. We will discuss relations to string theory sparsely throughout the text, but will expand on those in the conclusions in section 7.

## 1.2 Duality

In maximally supersymmetric  $\mathcal{N} = 4$  SYM, one obtains a physically equivalent theory if the gauge coupling constant  $g_{\text{YM}}$  is replaced by its inverse  $2\pi/g_{\text{YM}}$ . To describe this duality, it is convenient to combine the theta angle  $\theta$  and the gauge coupling  $g_{\text{YM}}$  into the complexified gauge coupling,

$$\tau = \frac{\theta}{2\pi} + \frac{2\pi i}{g_{\text{YM}}^2}. \quad (1.1)$$

It takes values in the upper half-plane

$$\mathbb{H} = \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}, \quad (1.2)$$

since the imaginary part of  $\tau$  is positive. When the theta angle is included, the *Montonen–Olive* duality acts by  $\tau \mapsto -1/\tau$  [19, 24]. Furthermore, the periodicity of the theta angle  $\theta \mapsto \theta + 2\pi$  is translated to  $\tau \mapsto \tau + 1$ . These

two transformations  $\tau \mapsto -1/\tau$  and  $\tau \mapsto \tau + 1$  can be considered as linear fractional transformations

$$\tau \longmapsto \gamma \cdot \tau := \frac{a\tau + b}{c\tau + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}), \quad (1.3)$$

where  $\mathrm{SL}(2, \mathbb{Z})$  is the group of unit determinant  $2 \times 2$  matrices with integer coefficients. In particular, for  $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ , we have that  $S \cdot \tau = -1/\tau$  and  $T \cdot \tau = \tau + 1$ . The elements  $S$  and  $T$  *generate*  $\mathrm{SL}(2, \mathbb{Z})$ , i.e. any element of  $\mathrm{SL}(2, \mathbb{Z})$  can be written as a word in  $S$  and  $T$ . Put differently, the  $\mathcal{N} = 4$  theory contains an  $\mathrm{SL}(2, \mathbb{Z})$  symmetry acting on both  $g_{\mathrm{YM}}$  and  $\theta$  nontrivially. If the theta angle is set to zero, the  $S$ -duality exchanges weak coupling ( $g_{\mathrm{YM}} \ll 1$ ) with strong coupling ( $g_{\mathrm{YM}} \gg 1$ ), which is why Montonen–Olive duality is an example of a strong-weak coupling duality (see [25] for a review). This demonstrates the power of dualities: Strong-weak dualities allow to gain insight into the strongly coupled phases, which are not accessible through traditional perturbative calculations.

The strong-weak coupling duality of  $\mathcal{N} = 4$  SYM is of course not the only example of a duality in a physical theory. In fact, dualities are ubiquitous in quantum field theories, string theory and gravity. The most basic one is perhaps the electric-magnetic duality of electrodynamics, which is characterised by an equivalence of the theory under an exchange of electric and magnetic fields. One of the best-understood examples is the gauge/gravity duality, which relates a strongly coupled quantum field theory to a weakly coupled gravitational theory, and vice versa [18]. Another example is mirror symmetry, which gives a relation between the topological A-model and B-model [26–29]. Finally, there is a web of string dualities in string theory, M-theory and F-theory, such as the T-duality, S-duality and U-duality. Dualities also have far-reaching consequences and relations to pure mathematics, such as the study of enumerative invariants or the geometric Langlands program [24, 30–32].

We can schematically formalise a class of dualities as follows. Let us assume that a theory  $\mathcal{T}$  has a space  $\mathcal{M}$  of couplings  $\tau \in \mathcal{M}$ . These couplings can be considered as external parameters of  $\mathcal{T}$ , on which observables such as the partition function, correlation functions etc. depend explicitly. We can denote this dependence by  $\mathcal{T}[\tau]$ , which indicates the theory with a coupling  $\tau$ . The symbol  $\mathcal{T}$  can denote a theory as e.g. a collection of correlation functions, or a sector of a specific theory such as the infrared.<sup>1</sup> Let us make this idea more precise in two examples.

One example carried throughout this introduction is  $\mathcal{N} = 4$  SYM, for instance with gauge group  $\mathrm{SU}(N)$ . This theory enjoys superconformal invariance, and has a complex one-dimensional  $\mathcal{N} = 4$  supersymmetry-preserving conformal manifold. A conformal manifold is the space of couplings of a conformal field theory (CFT) to exactly marginal operators, and thus can be understood

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<sup>1</sup>We want to stress that the notions introduced below should be understood as schematic, rather than precise.

as the space that parametrises a CFT. For  $\mathcal{N} = 4$  SYM, this conformal manifold may be parametrised by the complexified gauge coupling  $\tau$  (1.1). Thus an observable  $\mathcal{O}$  in  $\mathcal{N} = 4$  for fixed rank  $N$  depends only on  $\tau$ , and as such is a function  $\mathcal{O}(\tau)$ . Then by  $\mathcal{T}_{\mathcal{N}=4}[\tau]$  we mean the collection of observables  $\mathcal{O}(\tau)$ .

The main focus of this thesis is on  $\mathcal{N} = 2$  supersymmetry, which is not superconformal unless it is coupled to matter in specific representations. As will be discussed in detail below,  $\mathcal{N} = 2$  gauge theories have a moduli space of inequivalent vacua, a component of which is the Coulomb branch. The Coulomb branch can be parametrised by order parameters  $u_i$ , which are functions of the low-energy effective coupling  $\tau$ . As an example, in this context, by  $\mathcal{T}_{\mathcal{N}=2}[\tau]$  we could mean such a function  $u_i(\tau)$ .

By comparing  $\mathcal{T}[\tau_1]$  and  $\mathcal{T}[\tau_2]$ , we can use the “map”  $\mathcal{T}$  to test whether two couplings  $\tau_1$  and  $\tau_2$  generate two different theories or two equivalent theories. For instance, in  $\mathcal{N} = 4$  SYM “equivalent” means that correlation functions take the same values, and thus give rise to equivalent dynamics of the theory. This idea naturally leads to the

**Definition 1** (Fundamental domain). *For a theory  $\mathcal{T}$  with space of couplings  $\mathcal{M}$ , a fundamental domain  $\mathcal{F}[\mathcal{T}]$  is a subset of  $\mathcal{M}$  with the property that*

- *for any coupling  $\tau \notin \mathcal{F}[\mathcal{T}]$ , there exists a  $\tau' \in \mathcal{M}$  such that  $\mathcal{T}[\tau] = \mathcal{T}[\tau']$ , and*
- *for any two couplings  $\tau, \tau' \in \mathcal{F}[\mathcal{T}]$  with  $\tau \neq \tau'$  we have that  $\mathcal{T}[\tau] \neq \mathcal{T}[\tau']$ .*

Let us now assume that a group  $G \ni g$  acts on the coupling  $\tau$ , which we denote by  $g \cdot \tau$ . If

$$\mathcal{T}[g \cdot \tau] = \mathcal{T}[\tau] \tag{1.4}$$

for all  $g \in G$  and for all  $\tau \in \mathcal{M}$ , then we say that  $G$  is the duality group of the theory  $\mathcal{T}$  for the action on the coupling  $\tau \in \mathcal{M}$ . When the theory has such a duality group  $G$ , there is different way to formulate a fundamental domain, as two couplings in  $\mathcal{M}$  can be compared as follows:

**Definition 2** (Fundamental domain). *Let  $\mathcal{T}$  be a theory with space  $\mathcal{M}$  of couplings and duality group  $G$  acting on  $\mathcal{M}$ . Then  $\tilde{\mathcal{F}}[\mathcal{T}]$  is a fundamental domain, if*

- *for any  $\tau \notin \tilde{\mathcal{F}}[\mathcal{T}]$ , there exists a  $g \in G$  such that  $g \cdot \tau \in \tilde{\mathcal{F}}[\mathcal{T}]$ , and*
- *for any two distinct  $\tau, \tau' \in \tilde{\mathcal{F}}[\mathcal{T}]$ , there is no  $g \in G$  such that  $\tau' = g \cdot \tau$ .*

Fundamental domains are in general not unique, and there is no canonical choice. For families of theories that are related by deformations, such as supersymmetric quantum chromodynamics (SQCD), fixing a frame for the choice of fundamental domains becomes important, since also the fundamental domains must be related by certain deformations.



An important class of dualities are those which have a nontrivial action on the gauge group. For instance, Montonen–Olive duality of  $\mathcal{N} = 4$  SYM in general is a symmetry which under a generalised  $S$ -duality  $\tau \mapsto -\frac{1}{n_G \tau}$  replaces the gauge group  $G$  with its so-called Langlands dual group  ${}^L G$ , where  $n_G = 1, 2, 3$  is an integer that depends on  $G$ . For instance, for  $G = \text{SU}(N)$ , the Langlands dual is  ${}^L \text{SU}(N) = \text{SU}(N)/\mathbb{Z}_N$ , while  ${}^L \text{U}(N) = \text{U}(N)$  is Langlands self-dual. When the gauge group  $G$  is not Langlands self-dual, the  $S$ -transformation is interesting because it explains the strong coupling dynamics in terms of a weakly coupled description of a different theory with gauge group  ${}^L G$ . Dualities affecting the gauge groups are not restricted to  $\mathcal{N} = 4$ , they are also found in  $\mathcal{N} = 2$  supersymmetric theories [21, 33, 34]. In our schematic setup introduced above, we will only consider the cases where the duality keeps the gauge group fixed.

In this thesis, we will mainly focus on the case where  $\mathcal{M}$  is a subset of the upper half-plane  $\mathbb{H}$ , (1.2). On  $\mathbb{H}$  there is a natural group action of  $\text{SL}(2, \mathbb{Z})$  or any subgroup thereof, as defined in (1.3). For such subgroups, the definition 2 for a fundamental domain is widely used in the literature of modular groups and automorphic forms. From the generators of a group, there is an algorithm that draws a certain choice of fundamental domain (see Appendix A.3).

We find many examples where  $g \cdot \tau$  is not a linear fractional transformation for  $G \ni g$  an honest group, and the action needs to be properly generalised. Then definition 1 can be used instead of definition 2. This distinction will be made clear in this thesis.

Let us briefly return to the above example of  $\mathcal{N} = 4$  SYM, whose duality group is  $\text{SL}(2, \mathbb{Z})$  acting on the coupling  $\tau \in \mathcal{M} = \mathbb{H}$  through the transformation (1.3).<sup>2</sup> The duality can be encoded in the equation

$$\mathcal{T}_{\mathcal{N}=4}[\gamma \cdot \tau] = \mathcal{T}_{\mathcal{N}=4}[\tau], \quad (1.5)$$

which is a formulation of the Montonen–Olive duality. This allows to extract the physical consequences of the duality based on  $\text{SL}(2, \mathbb{Z})$  spectral theory [35]. For the duality group  $\text{SL}(2, \mathbb{Z})$ , there is a standard choice of fundamental domain  $\mathcal{F}$ , which is given in Fig. 1.

From definition 2, it can be understood as follows. For some  $\tau = x + iy \in \mathbb{H}$ ,  $x \in \mathbb{R}$  is a real number. As such, there exists a unique  $n \in \mathbb{Z}$  such that  $\tau + n \in [-\frac{1}{2}, \frac{1}{2}]$  lies in the unit interval between  $\pm \frac{1}{2}$ . Using the action (1.3), we can write  $\tau + n = T^n \tau$ , and the  $\text{SL}(2, \mathbb{Z})$  element  $T^n$  maps  $\tau$  inside this strip. This implies that a fundamental domain  $\mathcal{F}$  for  $\text{SL}(2, \mathbb{Z})$  must be contained in the strip  $-\frac{1}{2} \leq x < \frac{1}{2}$ . On the other hand, if  $|\tau| < 1$ , then  $|-1/\tau| > 1$ . This can be seen by writing a complex number as an absolute value times a phase. Consequently,  $\mathcal{F}$  must be contained in the region defined by  $|\tau| \geq 1$ . The

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<sup>2</sup>We ignore several important aspects (such as modular weights, multiplier systems, 't Hooft fluxes, spin structures, non-holomorphicity, modular anomaly, line operators etc.) in order to give a pedagogical example.

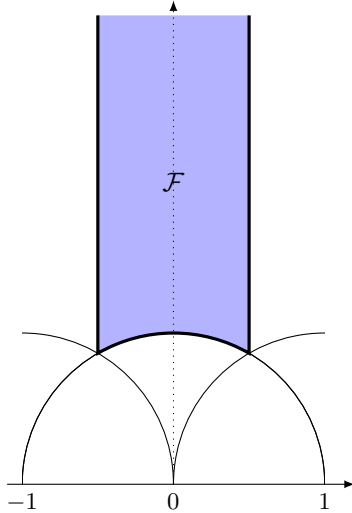


Figure 1: The key-hole fundamental domain  $\mathcal{F}$  of  $\mathrm{SL}(2, \mathbb{Z})$ . The vertical sides are identified, as well as the two halves of the boundary arc on the unit circle.

intersection of both constraints from the  $T$  and  $S$  transformations precisely gives the fundamental region in Fig. 1.

This explanation follows the construction of a fundamental domain as given in Appendix A.3: First, we write down a list of generators for the group. The modular group  $\mathrm{SL}(2, \mathbb{Z})$  is generated by  $T$  and  $S$ . The generator fixing  $\tau = i\infty$  is  $T$ , and determines a vertical strip of width 1, say from  $-\frac{1}{2}$  to  $+\frac{1}{2}$ . The  $S$ -transformation then reflects the argument on the half-circle  $\mathbb{S}^1 \cap \mathbb{H}$  around  $\tau = 0$  with radius 1. The half-circle defines an “interior” and “exterior”, depending on whether a number is contained in the disk bounded by the unit circle or not. A choice of fundamental domain is then given by the intersection of all exterior regions with the strip defined by the element fixing  $\tau = i\infty$ . For  $\mathrm{SL}(2, \mathbb{Z})$ , this gives precisely Figure 1.

### 1.3 Modular forms

A powerful tool to study dualities are modular forms. Modular or automorphic forms are functions that satisfy a functional equation such as (1.4) or generalisations thereof. A part of their importance comes from the fact that if certain growth conditions are included in the definition, they live in finite-dimensional vector spaces. This gives powerful constraints on their behaviour, while still being rich enough to encode nontrivial information. In this section, we present basic aspects of modular forms and modular functions. For a pedagogical introduction, we refer to the references in Appendix A.

Modular forms for  $\mathrm{SL}(2, \mathbb{Z})$  are defined as follows:

**Definition 3** (Modular form). *Let  $k$  be an integer. A function  $f : \mathbb{H} \rightarrow \mathbb{C}$  is a modular form of weight  $k$ , if*

- $f$  is holomorphic in  $\mathbb{H}$ ,
- $f(\gamma \cdot \tau) = (c\tau + d)^k f(\tau)$  for all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$  and all  $\tau \in \mathbb{H}$ ,
- $f$  is holomorphic at  $\infty$ .

The space of modular forms for a fixed weight  $k$  is a vector space over  $\mathbb{H}$ , and the third condition makes it finite-dimensional. For  $\gamma = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ , the functional equation becomes  $f(\tau + 1) = f(\tau)$ . This shows that  $f$  admits a Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n, \quad (1.6)$$

where  $q = e^{2\pi i\tau}$ . The Fourier coefficients  $a_n \in \mathbb{C}$  encode the same information as the function  $f$  itself, and in many interesting cases are in fact integers. The expansion (1.6) does not contain terms with nonzero  $a_n$  for negative  $n$ , since for  $\tau \rightarrow i\infty$  we have  $q \rightarrow 0$ , which would give a pole for  $f$  as  $\tau \rightarrow i\infty$ .

When  $\gamma = -I$  in Definition 3, then  $f = (-1)^k f$  implies that the only odd weight modular form for  $SL(2, \mathbb{Z})$  is the zero function. For even  $k$ , a class of examples are the Eisenstein series. They can be defined as the  $q$ -series

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \quad (1.7)$$

with  $\sigma_k(n) = \sum_{d|n} d^k$  the divisor sum and  $B_k$  the Bernoulli numbers. For  $E_4$  and  $E_6$ , the Fourier series reads

$$\begin{aligned} E_4(\tau) &= 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \mathcal{O}(q^5), \\ E_6(\tau) &= 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 + \mathcal{O}(q^5). \end{aligned} \quad (1.8)$$

It can be shown that for  $k \geq 4$  even,  $E_k$  is a modular form of weight  $k$  for  $SL(2, \mathbb{Z})$ . For  $k = 2$ ,  $E_2$  is a so-called quasi-modular form, which means that the  $SL(2, \mathbb{Z})$  transformation includes a shift in addition to the weight factor (see Appendix A.1).

If we denote the finite-dimensional vector space of modular forms of weight  $k$  for  $\Gamma = SL(2, \mathbb{Z})$  by  $M_k(\Gamma)$ , the direct sum

$$M_*(\Gamma) = \bigoplus_k M_k(\Gamma) \quad (1.9)$$

can be shown to be an algebra over  $\mathbb{C}$ : A modular form can be multiplied by a complex number, the sum of two modular forms of the same weight is a modular form, and the product of two modular forms of weight  $k$  and  $k'$  is a modular form of weight  $k + k'$ . This gives a simple way to construct new modular forms. One can show that for  $\Gamma = SL(2, \mathbb{Z})$ ,  $M_*(\Gamma)$  is freely generated by  $E_4$  and  $E_6$ . This implies in particular that the Eisenstein series of weight 8 and larger can always be written as sums of products of  $E_4$  and  $E_6$ . For instance, it is easy to confirm that  $E_4^2 = E_8$ ,  $E_4 E_6 = E_{10}$ , etc. A proof of

these identities is given by the insight that the vector spaces  $M_k(\Gamma)$  are finite-dimensional, and as such it is enough to compare a finite number of Fourier coefficients on both sides of the equation.

These relations furthermore predict an infinite number of functional identities among the Fourier coefficients of  $E_k$ , which are given by the divisor sums  $\sigma_k(n)$ . In this way, many problems in number theory can be cast into problems involving identities of modular forms. For instance, the divisor sum enjoys a multiplicative property: whenever  $a$  and  $b$  are coprime, then  $\sigma_k(ab) = \sigma_k(a)\sigma_k(b)$ .

A natural question suggested by (1.9) is whether the ring structure also admits a division. In particular, if two functions  $f$  and  $f'$  have weight  $k$ , then  $f/f'$  has weight 0 under  $\mathrm{SL}(2, \mathbb{Z})$ . However, since for a nonzero holomorphic function  $f : \mathbb{H} \rightarrow \mathbb{C}$ , the inverse  $1/f$  is generally *meromorphic* (i.e. it contains poles), the definition 3 needs to be adapted.

**Definition 4** (Modular function). *A function  $f : \mathbb{H} \rightarrow \mathbb{C}$  is called a modular function, if*

- $f$  is meromorphic in  $\mathbb{H}$ ,
- $f(\gamma \cdot \tau) = f(\tau)$  for all  $\gamma \in \mathrm{SL}(2, \mathbb{Z})$  and all  $\tau \in \mathbb{H}$ ,
- $f$  is meromorphic at  $\infty$ , i.e. the Fourier series is of the form  $f(\tau) = \sum_{n=-m}^{\infty} a_n q^n$  for some  $m \in \mathbb{Z}$ .

From the generators  $E_4$  and  $E_6$  of  $M_*(\Gamma)$ , the simplest modular function that can be constructed is a quotient of two modular forms of weight 12, which is the least common multiple of 4 and 6. One such quotient is the modular  $j$ -invariant,

$$j = \frac{12^3 E_4^3}{E_4^3 - E_6^2}, \quad (1.10)$$

which is a modular function for  $\mathrm{SL}(2, \mathbb{Z})$ ,

$$j(\gamma \cdot \tau) = j(\tau), \quad \gamma \in \mathrm{SL}(2, \mathbb{Z}). \quad (1.11)$$

Its Fourier coefficients

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \mathcal{O}(q^3) \quad (1.12)$$

enjoy a remarkable relation to group theory, a phenomenon called *monstrous moonshine* [36]. It can be proven that the Fourier coefficients are the dimensions of the graded part of an infinite-dimensional algebra representation of the monster group  $\mathbb{M}$ , the largest sporadic simple group of order  $\sim 10^{54}$ . The Monster vertex algebra can be constructed as a conformal field theory describing 24 free bosons compactified on the torus induced by the 24-dimensional self-dual Leech lattice, followed by a  $\mathbb{Z}_2$  orbifold [37].

The  $j$ -invariant has many other important properties. First of all, it generates modular functions for  $\mathrm{SL}(2, \mathbb{Z})$ . In particular, every modular function

for  $\mathrm{SL}(2, \mathbb{Z})$  is a rational function in  $j$ , and conversely, every rational function in  $j$  is a modular function. This furthermore enhances (1.9) for weight 0 functions to an algebraic field: If  $g$  is a nonzero modular function, then  $f/g$  is a modular function. For  $\mathrm{SL}(2, \mathbb{Z})$ , it can be shown that this *function field* can be generated by a single transcendental function,  $j$  being an example. Up to Möbius transformations and normalisations, such a function is unique and traditionally called *Hauptmodul*.

Another property of the  $j$ -function is that it knows about the fundamental domain  $\mathcal{F}$  of its invariance group  $\mathrm{SL}(2, \mathbb{Z})$ : Its fundamental domain obtained from Def. 1 coincides with the one from Def. 2. In other words,  $j : \mathcal{F} \rightarrow \mathbb{C}$  is an isomorphism. This illustrates that in the presence of an invariance group, for a good choice of test function the definitions 1 and 2 can be equivalent.

For both modular forms and modular functions there is a variety of generalisations that have been studied in great detail. The most immediate generalisation is to subgroups of  $\mathrm{SL}(2, \mathbb{Z})$ , in particular the *congruence* subgroups. The most common examples are the groups denoted by  $\Gamma(n)$ ,  $\Gamma_0(n)$  and  $\Gamma^0(n)$ , with  $n \in \mathbb{N}$ . They are defined through congruences of entries of the  $2 \times 2$  matrices, for instance

$$\Gamma^0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \mid b \equiv 0 \pmod{n} \right\}. \quad (1.13)$$

The group  $\Gamma_0(n)$  is similarly defined as the set of such matrices with  $c \equiv 0 \pmod{n}$ , while an element  $\gamma$  of  $\Gamma(n)$  satisfies  $\gamma \equiv \mathbb{1} \pmod{n}$  for all components. The importance of those subgroups comes from the fact that certain elementary functions transform precisely under elements of those groups, rather than under all of  $\mathrm{SL}(2, \mathbb{Z})$ . The simplest class of modular forms for congruence subgroups of  $\mathrm{SL}(2, \mathbb{Z})$  are arguably the Jacobi theta functions, which are defined as

$$\begin{aligned} \vartheta_2(\tau) &= \sum_{r \in \mathbb{Z} + \frac{1}{2}} q^{r^2/2}, \\ \vartheta_3(\tau) &= \sum_{n \in \mathbb{Z}} q^{n^2/2}, \\ \vartheta_4(\tau) &= \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2/2}. \end{aligned} \quad (1.14)$$

They transform into each other under the action of  $\mathrm{SL}(2, \mathbb{Z})$ , however they are invariant themselves under proper subgroups. See Appendix A.1 for the transformations and further properties.

## 1.4 Elliptic curves

Modular forms are intimately related to the theory of elliptic curves. An elliptic curve is a plane algebraic curve, which can be defined as the solution space  $\{(x, y)\}$  of a cubic equation

$$E : y^2 = 4x^3 - g_2x - g_3, \quad (1.15)$$

where  $g_2$  and  $g_3$  are called Weierstraß invariants. The curve is required to be non-singular, which means that it should not have cusps, self-intersections or isolated points. Algebraically, this holds if and only if the discriminant

$$\Delta = g_2^3 - 27g_3^2 \tag{1.16}$$

is nonzero. The formulation of elliptic curves over the complex numbers corresponds to an embedding of the torus into the complex projective plane. This follows from the functional equation  $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$  for the Weierstraß elliptic function  $\wp$ . The Weierstraß function is doubly periodic, i.e. periodic with respect to a lattice  $\Lambda$ , and as such can be considered to be defined on a torus  $T = \mathbb{C}/\Lambda$ . The torus can be embedded in the complex projective plane using the map  $z \mapsto [1 : \wp(z) : \frac{1}{2}\wp'(z)]$ . This map is an isomorphism of Riemann surfaces from the torus to the cubic curve (1.15), and thus topologically an elliptic curve is a torus. The lattice  $\Lambda$  generating the torus can be understood as the linear span of a pair of nonzero complex numbers  $\alpha_1$  and  $\alpha_2$ . This pair defines a smallest cell, which is called the fundamental domain<sup>3</sup> of the torus. Geometrically, the torus is then obtained by identifying the opposite edges of the fundamental domain.

The shape of the domain and thus of the torus is described by the ratio  $\tau = \alpha_2/\alpha_1$ , and is the *complex structure* or the *modular parameter* of the torus. It is clear that a different choice  $\beta_1, \beta_2$  of complex numbers can describe the same torus. Indeed, this is true if

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \tag{1.17}$$

where  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ . Since the ratio  $\tau = \alpha_2/\alpha_1$  furthermore cancels overall signs of the vector  $(\alpha_1, \alpha_2)^\top$ , we can ignore overall signs also of  $\gamma$  and identify  $\gamma \sim -\gamma$ . In this way, we find the modular group of the torus as the isometry group  $\text{PSL}(2, \mathbb{Z})$  acting on the modular parameter  $\tau$ .

From the curve (1.15) we can compute the  $\mathcal{J}$ -function

$$\mathcal{J} = 12^3 \frac{g_2^3}{\Delta}. \tag{1.18}$$

This quantity is an invariant for isomorphism classes of elliptic curves, meaning that two elliptic curves have the same  $\mathcal{J}$ -invariant if and only if they are isomorphic.

The types of elliptic curves relevant for this thesis are the ones that are parametrised by a complex variable. A particular class of 1-parameter families of elliptic curves are the *elliptic surfaces*. These are obtained by promoting the Weierstraß invariants to functions on a base space  $\mathbb{CP}^1$ . Then the elliptic fibration

$$E \longrightarrow \mathcal{S} \longrightarrow \mathbb{CP}^1 \tag{1.19}$$

---

<sup>3</sup>Not to be confused with definition 1 or 2.

is called an elliptic surface, where  $\mathbb{CP}^1$  can be understood as a compactification of the complex plane  $\mathbb{C}$  by adding a point  $\infty$ . When  $g_2$  and  $g_3$  are polynomials on  $\mathbb{CP}^1$ , then  $\mathcal{J} : \mathbb{CP}^1 \rightarrow \mathbb{C}$  is a rational function. Let  $u$  denote a coordinate on  $\mathbb{CP}^1$ . Then (1.18) is a rational function of  $u$ , and by the above arguments can be identified with the modular  $j$ -invariant, which is a function of the complex structure of the curve,

$$\mathcal{J}(u) = j(\tau). \tag{1.20}$$

This gives interesting relations between the coordinate  $u$  and the modular parameter  $\tau$ , which are of central importance for this thesis. See [38–44] for more details on elliptic surfaces.

## 1.5 Seiberg-Witten theory

A beautiful synthesis of all the above introduced concepts is the full non-perturbative solution for the low-energy dynamics of four-dimensional  $\mathcal{N} = 2$  supersymmetric Yang–Mills theory with gauge group  $SU(2)$ , due to Seiberg and Witten [45, 46]. See [8, 47–54] for reviews.

Seiberg-Witten duality can be regarded as a lift of the electric-magnetic duality to  $\mathcal{N} = 2$  supersymmetric quantum field theory. It however cannot be a duality analogous to Montonen–Olive duality in  $\mathcal{N} = 4$ , for several reasons. The maximally supersymmetric  $\mathcal{N} = 4$  SYM theory is superconformal, and as such the gauge coupling is fixed. The pure  $\mathcal{N} = 2$  supersymmetric Yang–Mills theory has the rather different feature that it is *asymptotically free*, such as the non-supersymmetric quantum chromodynamics for example is. Asymptotic freedom is the property of a theory that causes interactions between fields to become weaker as the energy scale increases. Correspondingly, the couplings should be considered as a function of the energy scale  $\Lambda$ . If duality exchanged the coupling  $g_{\text{YM}}$  with its inverse  $1/g_{\text{YM}}$  measured at a certain scale  $\Lambda$ , this would in general not be true at another scale  $\Lambda'$ . Thus rather than  $\mathcal{N} = 2$  SYM theory itself, we can study its Wilsonian low-energy effective action. Seiberg-Witten duality is then an exact duality that acts on the *effective* coupling, rather than of the microscopic  $SU(2)$  theory.

The Seiberg-Witten geometry underlies the Coulomb branch of  $\mathcal{N} = 2$  gauge theory. The  $\mathcal{N} = 2$  vector multiplet consists of a gauge field  $A$ , a complex scalar field  $\phi$ , and two Weyl fermions  $\lambda$  and  $\psi$ . They are all in the adjoint representation of the gauge group  $G$ , which we fix to  $G = SU(2)$ . The Coulomb branch is the phase of the theory where  $SU(2)$  is broken to  $U(1)$  by a vacuum expectation value (vev) of the vector multiplet scalar  $\phi$ . The potential of the theory is  $V(\phi) = \frac{1}{g^2} \text{Tr}[\phi, \phi^\dagger]^2$ , and we are interested in the moduli space of flat directions. These are found by setting

$$\phi = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \tag{1.21}$$

with  $a$  a complex parameter. However, note that the Weyl group of  $SU(2)$  acts

on  $a$  by  $a \mapsto -a$ . The gauge invariant order parameter is the Coulomb branch expectation value of the theory in  $\mathbb{R}^4$ ,

$$u = \frac{1}{16\pi} \langle \text{Tr}(\phi^2) \rangle_{\mathbb{R}^4}, \quad (1.22)$$

which parametrises the moduli space of inequivalent vacua. The renormalisation group flow generates a quantum scale  $\Lambda$ , at which the gauge coupling becomes strong.

The effective action then describes the physics of the remaining massless U(1) supersymmetric multiplets in terms of a holomorphic function, the *prepotential*  $F$ . By differentiating the prepotential with respect to the coordinate  $a$ , one finds the magnetic dual period  $a_D = \frac{\partial F}{\partial a}$ . A further differentiation results in the low-energy effective gauge coupling

$$\tau = \frac{\partial^2 F}{\partial a^2}, \quad (1.23)$$

which becomes a combination of the effective coupling constant and the effective theta angle, similar to (1.1). The spectrum of the theory contains so-called *dyonic* states or *dyons*, which are states carrying electric as well as magnetic charges. The central charge of such a dyonic state with electric and magnetic charges  $\gamma = (n_m, n_e)$  is given by

$$Z = \gamma \cdot \pi = n_m a_D + n_e a, \quad (1.24)$$

where  $\pi = (a_D, a)^\top$  is the period vector.

The Seiberg-Witten (SW) solution provides a family of elliptic curves parametrised by the order parameter  $u$ , whose complex structure corresponds to the running coupling  $\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$ . For the pure SU(2) theory, it is given by

$$y^2 = x^3 - ux^2 + \frac{1}{4}\Lambda^4 x. \quad (1.25)$$

The curve provides exact results for the vevs of the scalar  $a$  and its dual  $a_D$  as period integrals. We have explicitly,

$$a = \int_A \lambda, \quad a_D = \int_B \lambda, \quad (1.26)$$

where  $\lambda$  is a meromorphic 1-form with  $\frac{d\lambda}{du} = \frac{\sqrt{2}}{4\pi} \frac{dx}{y}$ , and  $A$  and  $B$  are one-cycles which form a symplectic basis of the SW curve. The  $A$ -cycle corresponds to a straight line from 0 to 1, while the  $B$ -cycle corresponds to a straight line from 0 to  $\tau$ .

As  $a$  and  $a_D$  are given by period integrals of a meromorphic 1-form over the elliptic curve, they satisfy a system of solutions to a set of Picard-Fuchs equations. This allows to express the periods in terms of hypergeometric functions [55]<sup>4</sup>,

$$\begin{aligned} a_D(u) &= \frac{i}{2}(u-1) {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 2; \frac{1-u}{2}\right), \\ a(u) &= \sqrt{\frac{(u+1)}{2}} {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}, 1; \frac{2}{1+u}\right). \end{aligned} \quad (1.27)$$

<sup>4</sup>We momentarily set  $\Lambda = 1$ .



At the strong coupling points, the periods  $\pi(u) = (a_D(u), a(u))$  become  $\pi(1) = (0, \frac{2}{\pi})$  and  $\pi(-1) = (-\frac{4i}{\pi}, -\frac{2i}{\pi})$ . According to the central charge formula (1.24), these values confirm that for  $u = 1$ , the monopole  $\gamma = (1, 0)$  becomes massless, while for  $u = -1$  the dyon  $\gamma = (-1, 2)$  becomes massless. The limits  $\lim_{u \rightarrow \pm 1} a(u)$  depend on the direction from which  $\pm 1$  are approached, which is due to the branch cut in the hypergeometric function [55].

The curve (1.25) is not in Weierstraß form (1.15) yet, however by rescaling  $x \rightarrow x + \frac{u}{3}$  and  $y \rightarrow \frac{y}{2}$ , we find

$$y^2 = 4x^3 - \left(\frac{4}{3}u^2 - \Lambda^2\right)x - \left(\frac{8}{27}u^3 + \frac{1}{3}u\Lambda^4\right), \quad (1.28)$$

from which we can read off the Weierstraß invariants

$$g_2 = \frac{4}{3}u^2 - \Lambda^4, \quad g_3 = \frac{8}{27}u^3 + \frac{1}{3}u\Lambda^4. \quad (1.29)$$

Using (1.18), we can then compute the  $\mathcal{J}$ -invariant

$$\mathcal{J}(\mathbf{u}) = 64 \frac{(4\mathbf{u}^2 - 3)^3}{\mathbf{u}^2 - 1}, \quad (1.30)$$

which is a rational function of the dimensionless quantity  $\mathbf{u} = \frac{u}{\Lambda^2}$ , as a consequence of the SW curve (1.25) being a rational elliptic surface.

## 1.6 Fundamental domains

By identifying the  $\mathcal{J}$ -invariant of the SW curve with the modular  $j$ -invariant as in (1.20), the order parameter  $u$  is related to the coupling  $\tau$  through

$$\mathcal{J}(\mathbf{u}(\tau)) = j(\tau). \quad (1.31)$$

Naively,  $\mathbf{u}(\tau)$  should be invariant under  $\text{SL}(2, \mathbb{Z})$ , since  $j(\tau)$  is (see (1.11)). Below, we will show explicitly that this is not true. Acting with  $\text{SL}(2, \mathbb{Z})$  on (1.31) leaves the rhs invariant, and so must be the lhs. Since  $\mathcal{J}(\mathbf{u})$  is a rational function, this implies that for any  $\gamma \in \text{SL}(2, \mathbb{Z})$  there exists a function  $h_\gamma$  such that  $\mathbf{u}(\gamma \cdot \tau) = h_\gamma(\mathbf{u}(\tau))$ , which is subject to the constraint that  $\mathcal{J}(\mathbf{u}) = \mathcal{J}(h_\gamma(\mathbf{u}))$ . In order to see this, let us solve the relation (1.31) for  $\mathbf{u}$ . Using (1.30), we can write (1.31) as a sextic polynomial equation

$$(\mathbf{u}^2 - 1)j - 64(4\mathbf{u}^2 - 3)^3 = 0, \quad (1.32)$$

where we suppress the  $\tau$  dependence of  $\mathbf{u}$  and  $j$ . Since the sextic equation contains only even powers of  $\mathbf{u}$ , it is a cubic equation in  $\mathbf{u}^2$  and can thus be solved exactly. One particular solution is

$$\mathbf{u} = -\frac{1}{8\sqrt{3}} \sqrt[2]{\frac{\sqrt[3]{9j}}{\sqrt[3]{(\sqrt{3}\sqrt{12^3 - j} - 72)j}} + \sqrt[3]{3}\sqrt[3]{(\sqrt{3}\sqrt{12^3 - j} - 72)j} + 12^3}, \quad (1.33)$$

such that  $u$  contains square and cube roots of modular functions. The other five solutions take a similar form. Generally, roots of modular functions are not modular, since roots induce branch points at zeros of the modular functions, thus spoiling the holomorphicity. However, here this is not the case. From the  $q$ -expansion of the  $j$ -invariant (1.12), we can easily compute the Fourier series

$$u(\tau) = -\frac{1}{8}(q^{-1/4} + 20q^{1/4} - 62q^{3/4} + 216q^{5/4} + \mathcal{O}(q^{7/4})). \quad (1.34)$$

It appears that up to a simple normalisation, all Fourier coefficients are integers. Considering (1.33) with convoluted roots which generally induce rational coefficients with growing denominators, this might be rather surprising. In fact, the  $q$ -series (1.34) is known in the mathematics literature as the McKay-Thompson series of class 4C for the Monster group [36, 56–58]. As such, it is an example of a *replicable* function (see Appendix A.4 for a definition). It can be checked to arbitrary order that the  $q$ -expansion (1.34) agrees with that of

$$u(\tau) = -\frac{1}{2} \frac{\vartheta_2(\tau)^4 + \vartheta_3(\tau)^4}{\vartheta_2(\tau)^2 \vartheta_3(\tau)^2}, \quad (1.35)$$

where the Jacobi theta functions are defined in (1.14). Using the formulas given in Appendix A.1, one may check that this function is not invariant under the generators  $T$  and  $S$  of  $\mathrm{SL}(2, \mathbb{Z})$ . For instance, under  $S$  the functions  $\vartheta_2$  and  $\vartheta_4$  are interchanged, while  $\vartheta_3$  is invariant (ignoring the modular weights, which cancel in this expression). Using the Jacobi identity  $\vartheta_2^4 + \vartheta_4^4 = \vartheta_3^4$  we can then eliminate one theta function to obtain the action of the  $S$ -transformation on (1.35),

$$h_S(u) = \frac{2u(u - \sqrt{u^2 - 1}) - 3}{2\sqrt{2u(\sqrt{u^2 - 1} - u) + 2}}. \quad (1.36)$$

We notice that  $h_\gamma$  does not necessarily need to be rational. We can furthermore confirm that  $\mathcal{J}(h_S(u)) = \mathcal{J}(u)$ , which is expected from (1.31). This shows that  $u$  is not a modular function for  $\mathrm{SL}(2, \mathbb{Z})$ , but solving (1.31) for  $u$  instead breaks  $\mathrm{SL}(2, \mathbb{Z})$  into a subgroup.

Indeed,  $u$  can be shown to be a modular function for the congruence subgroup  $\Gamma^0(4)$  of  $\mathrm{SL}(2, \mathbb{Z})$ , which we have defined in (1.13). The group  $\Gamma^0(4)$  is generated by  $T^4$  and  $ST^{-1}S$ , and it can be checked explicitly using the formulas given in Appendix A.1 that  $u$  is invariant under these transformations. Furthermore, it is known that  $u$  is a Hauptmodul for  $\Gamma^0(4)$ , and as such it is an isomorphism  $u : \mathbb{H}/\Gamma^0(4) \rightarrow \mathbb{C}$ . The quotient  $\mathcal{F}_0 := \mathbb{H}/\Gamma^0(4)$  is the fundamental domain for  $\Gamma^0(4)$ , and a particular choice can be drawn as explained in Appendix A.3, and we plot it in Fig. 2. The function (1.35) makes precise in which way its inverse  $\tau(u)$  is multi-valued: For any given  $\tau \in \mathcal{F}_0$ , we can compute the infinite discrete set of couplings  $\Gamma^0(4) \cdot \tau$ , which all give rise to the same vacuum  $u(\tau)$  and thus to the same dynamics.

Since  $u : \mathcal{F}_0 \rightarrow \mathbb{C}$  is an isomorphism, the Coulomb branch can equivalently be understood from the fundamental domain. From (1.30) it is clear that the

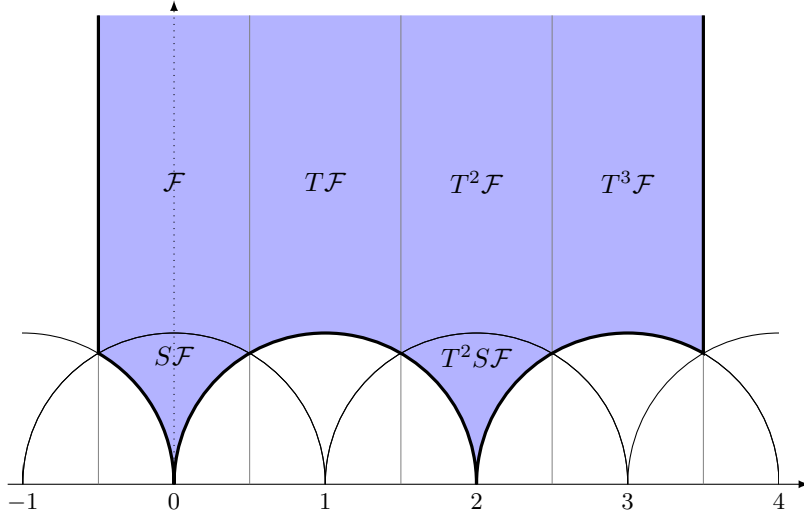


Figure 2: Fundamental domain of  $\Gamma^0(4)$ . This is the duality group of the pure  $SU(2)$  theory. The two cusps on the real line correspond to the strong coupling singularities of the gauge theory, while the cusp at  $\tau = i\infty$  corresponds to weak coupling.

theory becomes singular at precisely three points:  $u = \infty$  and  $u = \pm 1$ . At large  $\tau$ ,  $u$  is large and describes the weakly coupled region. The asymptotics and monodromy can be found from the one-loop prepotential. At  $\tau = 0$  and  $\tau = 2$ , the low-energy theory becomes singular since either the monopole or dyon, which were integrated out in the effective theory, becomes massless. These are precisely the two points where the elliptic curve (1.30) degenerates, and they correspond to  $u = -1$  and  $u = 1$ . This can be confirmed by studying the  $q$ -expansions of the  $S$  and  $T^2S$  transformations of (1.35).

The duality group  $\Gamma^0(4)$  can be equivalently found by studying the action of the monodromies on the Coulomb branch. Combining the periods  $a$  and  $a_D$  to the period vector  $\pi = (a_D, a)^T$  has an important property: They constitute a holomorphic section of a flat  $SL(2, \mathbb{Z})$  bundle, which are characterised by the monodromies around the singular points  $u = \infty$ ,  $u = \pm 1$ :

$$\mathbf{M}_\infty = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{M}_{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \mathbf{M}_{+1} = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}. \quad (1.37)$$

They satisfy the important consistency property  $\mathbf{M}_\infty = \mathbf{M}_{-1}\mathbf{M}_{+1}$ . Since the start and end point of a loop around a singular point coincide, the monodromies must leave  $u$  invariant, and the same must be true for the action of (1.37) on the coupling  $\tau$ . Indeed, the monodromies (1.37) are in  $\Gamma^0(4)$ , under which  $u$  is invariant.

In view of the fact that the monodromy matrices (1.37) generate a subgroup of  $SL(2, \mathbb{Z})$ , it is not surprising that  $u$  should be a modular function of  $\tau$ . However, from the derivation of (1.35) from (1.33) this is not obvious. In fact, the presentation here has been chosen to demonstrate that this property is

quite special and should be appreciated. One result of this thesis is that when the theory is generalised in various directions, such a correspondence

$$\text{pure } \mathcal{N} = 2 \text{ SU}(2) \text{ SYM} \quad \longleftrightarrow \quad \Gamma^0(4) \quad (1.38)$$

of the theory and its duality group can not be found in general. The arguably simplest generalisation is the inclusion of a massless hypermultiplet in the fundamental representation of  $\text{SU}(2)$ , for which the SW curve was found in [46]. In section 2.4, we give at least five explanations why such a correspondence (1.38) does not exist for that theory, and thus does not exist in general. Rather, we generalise aspects of the notion of a duality group to fundamental domains, and then the correspondence (1.38) can be made precise.

Let us instead study why the surface (1.25) is special, and gives rise to a modular function. One reason is that the elliptic surface for pure  $\text{SU}(2)$  SYM is a *modular elliptic surface*. Such surfaces were studied in [40] and classified in [38]. In fact, all rational elliptic surfaces are classified through their configuration of singular fibres. Singular fibres are the loci of elliptic surfaces where the discriminant vanishes. As discussed above, for the SW curves these are  $u = \infty$  and  $u = \pm 1$ . Every singular fibre has an associated Kodaira type, which is characterised by the order of vanishing of the Weierstraß invariants (see Appendix A.6). The singularities  $u = \pm 1$  correspond to an  $I_1$  singularity, while the singularity  $u = \infty$  corresponds to an  $I_4^*$  singularity. The Kodaira configuration of the pure  $\text{SU}(2)$  theory is thus  $(I_4^*, 2I_1)$ , which is known to be a modular elliptic surface [38].

The singularity configuration  $(I_4^*, 2I_1)$  is closely related to properties of the fundamental domain as plotted in Fig. 2: The cusp  $\tau = i\infty$  has *width* 4, while there are two cusps  $\tau = 0$  and  $\tau = 2$  of width 1. The precise definition of a cusp and its width is given in Appendix A.3. From a fundamental domain such as the one in Fig. 2, the widths can be read off from the number of copies of  $\mathcal{F}$  that taper to a cusp. Furthermore, when a Hauptmodul for the subgroup is known, then the width is given by the smallest  $\tau$ -periodicity of the  $q$ -expansion around a cusp. From (1.34) it is clear that at  $\tau = i\infty$  this is 4.

Finally, there is an algebraic explanation, which is slightly more technical. Let us assume that  $f$  is a modular function for a congruence subgroup  $\Gamma$  of  $\text{PSL}(2, \mathbb{Z})$ . Then it is known that  $f$  is the root of a polynomial over the algebraic field  $\mathbb{C}(\text{PSL}(2, \mathbb{Z}))$  of modular functions on  $\text{PSL}(2, \mathbb{Z})$ . A proof of this statement can for instance be found in [59], and is based on the following observation: If  $\Gamma$  is a congruence subgroup of  $\text{PSL}(2, \mathbb{Z})$ , then  $\text{PSL}(2, \mathbb{Z}) = \bigcup_{j=1}^n \alpha_j \Gamma$ , where  $\alpha_j \in \text{PSL}(2, \mathbb{Z})$  for  $j = 1, \dots, n$  are *coset representatives*, with  $n$  the *index* of  $\Gamma$  in  $\text{PSL}(2, \mathbb{Z})$ . We can construct a function  $g(\tau) = \prod_{j=1}^n (f(\alpha_j \tau) - X)$ , with some constant  $X$ . Since  $\text{PSL}(2, \mathbb{Z})$  transformations merely permute the coset representatives  $\alpha_j$ ,  $g$  is a modular function for  $\text{PSL}(2, \mathbb{Z})$ . If we consider this expression as a polynomial  $P(X) = \prod_{j=1}^n (f \circ \alpha_j - X) \in \mathbb{C}(\text{PSL}(2, \mathbb{Z}))[X]$ , then every coefficient must be a modular function for  $\text{PSL}(2, \mathbb{Z})$ . This shows that  $f$  is a root of the polynomial  $P(X)$  whose coefficients are modular func-

tions for  $\mathrm{PSL}(2, \mathbb{Z})$ . We discuss this aspect in more detail in Appendix A.5. What is clear from the above discussion is that the sextic equation (1.32) constructed from the SW curve is one example of such a polynomial whose roots are  $\mathrm{PSL}(2, \mathbb{Z})$ -transformations of the modular function  $u$ . We study generalisations of this aspect in great detail in section 2.

## 1.7 Topological QFT

Topological quantum field theory has been one of the most important contributions to mathematical physics in the 20th century, with a fruitful interplay between ideas from physics and mathematics [29, 60, 61]. It provides many insights into non-perturbative aspects of quantum field theory as well as low-dimensional topology [52, 62–69]. One example is Donaldson–Witten (DW) theory, which is a topological formulation of the  $\mathcal{N} = 2$  supersymmetric Yang–Mills theory on an oriented smooth four-manifold  $X$  (see [52] for an introduction). Its equivalent IR (long distance) counterpart is an abelian theory, where Seiberg–Witten geometry dictates the physics [61]. One of the most important insights is that the classical Donaldson invariants can be derived from the Seiberg–Witten solution to  $\mathcal{N} = 2$  supersymmetric Yang–Mills theory with gauge group  $\mathrm{SU}(2)$  [61, 70–73]. In the seminal paper [73], the path integral of DW theory was found as

$$Z_{\mathrm{DW}} = Z_u + Z_{\mathrm{SW}}, \quad (1.39)$$

where  $Z_{\mathrm{SW}}$  denotes the generating function of SW invariants of the four-manifold [71], while  $Z_u$  denotes the contribution to  $Z_{\mathrm{DW}}$  from the Coulomb branch of the low-energy effective  $\mathrm{U}(1)$  theory, the so-called  $u$ -plane. The  $u$ -plane and its contribution to the path integral were studied in detail in [73–76]. The  $u$ -plane integral  $Z_u$  is of particular interest since it is non-vanishing only for four-manifolds with  $b_2^+(X) \in \{0, 1\}$ . In turn, such four-manifolds are of particular interest since they are candidate topologies that probe the Coulomb branch of the theory.

Recently, interest in DW theory, and in particular the  $u$ -plane integral, was revived due to observations relating the latter for special four-manifolds to the theory of mock modular forms and harmonic Maass forms [77, 78]. For more generic compact four-manifolds, it was later reformulated in terms of the modular completion of a mock modular form [79–86].

One motivation of this thesis is to study correlation functions for topological theories other than the pure  $\mathcal{N} = 2$  SYM. Many results were already obtained in the 1990s, for instance the generalisation to higher rank gauge groups [87], massless QCD [88], superconformal theories [66, 89], extensions to non-simply connected four-manifolds [76, 90], etc. The formulation and computation of such  $u$ -plane integrals has remained a challenge, and an explicit evaluation for arbitrary four-manifolds and general 4d  $\mathcal{N} = 2$  theories has not yet been achieved.

The rest of the thesis is structured as follows. In the following section 2, we discuss the generalisation of the aspects introduced in section 1.6 to the asymptotically free  $\mathcal{N} = 2$  supersymmetric QCD with gauge group  $SU(2)$ . We argue that the correspondence (1.38) does not exist when hypermultiplets are introduced, and instead propose an alternative formulation in terms of fundamental domains that do not correspond to subgroups of  $SL(2, \mathbb{Z})$ . The obstruction is described precisely by a locus of branch points, which renders  $u(\tau)$  non-holomorphic and thus non-modular. The branch points are closely related to other Coulomb branch functions such as  $\frac{da}{du}$  or the discriminant through a generalisation of Matone’s relation. We demonstrate our claims in a plethora of examples that we work out explicitly.

In section 3, we discuss the four flavour  $SU(2)$  theory, which is superconformal up to mass terms. While often it is studied in the massless limit, allowing for generic masses gives a more intricate structure of the Coulomb branch, where the symmetry group provides an interesting permutation of special mass configurations. This allows us to find nontrivial examples of bimodular forms and vector-valued bimodular forms.

In section 4, we then move on to the  $SU(3)$  theory, where the Coulomb branch is complex two-dimensional. For pure  $\mathcal{N} = 2$  SYM, we demonstrate that there is a map from the genus two SW curve to elliptic (i.e. genus one) curves, which allows to study the Coulomb branch geometry using elliptic modular forms. For the locus where mutually local dyons become massless, we find a natural generalisation of the domain 2 for the pure  $SU(2)$  theory. For the locus containing the superconformal Argyres-Douglas vacua on the other hand, we find a fundamental domain for an Atkin-Lehner group, which is a subgroup of  $PSL(2, \mathbb{R})$  rather than  $PSL(2, \mathbb{Z})$ . This is due to the fact that this locus gives rise to a non-rational elliptic surface.

In section 5, we consider the topological twists of the theories studied in section 2 and 3, mainly focusing on the asymptotically free theories. The fundamental domains for the effective gauge coupling can be understood as integration domains for topological correlation functions. One of the key ideas of the map from the  $a$ -plane to the fundamental domains is that the integrand becomes a modular function of  $\tau$ . This allows us to prove that the integrand is single-valued under modular transformations, which is an important and non-trivial consistency check of the topological theory. We also present a formula for the evaluation of such correlation functions.

Finally, in section 6 we study the Donaldson-Witten theory on non-simply connected four-manifolds  $X$ . By adding a certain sum of  $\mathcal{Q}$ -exact operators, we show that correlation functions can be computed using mock modular forms. This generalises recent results on the case where  $b_1(X) = 0$ .

We conclude in section 7, with giving directions for future research and suggesting extensions and applications of our results to other theories. In Appendix A, we finally collect properties and definitions of modular forms and elliptic curves, which are of fundamental importance for this thesis.

## 2 Asymptotically free $\mathcal{N} = 2$ QCD

In this section, we study the duality and modularity of  $\mathcal{N} = 2$  SYM with  $N_f = 0, 1, 2, 3$  hypermultiplets in the fundamental representation. These theories have a negative one-loop beta function and thus are asymptotically free. This section is mainly based on [2], while parts of section 2.2 are based on [5] and parts of section 2.6 are based on unpublished work.

### 2.1 Introduction

A manifestation of  $S$ -duality or strong-weak coupling duality is the equivalent dynamics of a quantum field theory at distinct values of its coupling constant [19, 45, 91–93]. A natural question for such a quantum field theory is the determination of a domain for the coupling constant parametrizing inequivalent quantum field theories. In this section, we address this question for asymptotically free  $\mathcal{N} = 2$  Yang–Mills theories with gauge group  $SU(2)$  and  $N_f \leq 3$  fundamental hypermultiplets. To this end, we consider the order parameter for the Coulomb branch, which is a function of the running coupling  $\tau$  invariant under  $S$ -duality [45, 46, 94–97]. We put forward a fundamental domain  $\mathcal{F}_{N_f}$  for  $\tau$  such that the function is one-to-one. Part of our motivation is the  $u$ -plane integral [73, 74], which is a physical approach to Donaldson invariants and other topological gauge-theoretic invariants of smooth compact four-manifolds. This approach involves an integral over the Coulomb branch of the theory. Recently, the change of variables from  $u$  to  $\tau$  has been instrumental for the evaluation of the integral for generic four-manifolds [76–86, 98, 99], which suggests a potential fundamental role for this parametrisation of the Coulomb branch.

The Coulomb branches of the rank 1 theories mentioned above are complex one-dimensional, and parametrised by the Higgs vacuum expectation value  $u = \frac{1}{16\pi^2} \langle \text{Tr} \phi^2 \rangle$ ,  $\phi$  being the complex scalar of the  $\mathcal{N} = 2$  vector multiplet [46] (see [8, 47] for a review). In general, these order parameters are functions of the running coupling  $\tau$ , the masses  $m_i$  of the hypermultiplets and the dynamical scales  $\Lambda_{N_f}$  generated by the renormalisation group flow.

Compared to the pure  $SU(2)$  case, in the massive theories with  $N_f \leq 3$ , we find a number of new phenomena. To study these theories, we consider their order parameters as roots of certain degree six polynomials constructed from the Seiberg-Witten (SW) curves. These polynomials in turn encode many of the interesting structures of the Coulomb branches. For example, their ramification loci include the Argyres-Douglas (AD) theories, where the curves degenerate, as well as branch points. We show that the fundamental domain of  $u$  can be described as six or less copies of the corresponding fundamental domain of the full modular group  $SL(2, \mathbb{Z})$  as displayed in Figure 1. The cusps of these domains correspond to the singularities of the physical theory and the width of each cusp to the number of hypermultiplets becoming massless there.

The images of the  $\mathrm{SL}(2, \mathbb{Z})$  fundamental domains under the map  $u(\tau)$  provide intriguing partitions of the  $u$ -plane. See for example Figures 5, 6 and 9

Since the polynomials are order six in  $u$  it is in general not possible to find the roots, and solve for  $u$  in terms of the coefficients. Only for special configurations of the masses, e.g., equal masses in  $N_f = 2$  and one non-zero mass in  $N_f = 3$ , the polynomial splits over the field of modular functions for a congruence subgroup of  $\mathrm{SL}(2, \mathbb{Z})$ , and we can thus find explicit closed expressions for  $u$  in terms of known modular forms, reproducing and extending previous results [88, 100–103].

For generic choices of the masses, the function  $u(\tau)$  gives rise to branch points  $\tau_{\mathrm{bp}}$ , where  $u - u(\tau_{\mathrm{bp}}) \sim \sqrt{\tau - \tau_{\mathrm{bp}}} + \dots$  does not return to itself as  $\tau$  encircles  $\tau_{\mathrm{bp}}$ . While the branch points, and the inevitable branch cuts, obstruct the identification of  $\mathcal{F}_{N_f}$  as a quotient  $\Gamma \backslash \mathbb{H}$  with  $\Gamma$  a congruence subgroup, they provide a mechanism for  $\mathcal{F}_{N_f}$  to evolve as function of the mass. More precisely, the branch points move in the domain  $\mathcal{F}_{N_f}$  upon varying the masses, and the domain  $\mathcal{F}_{N_f}$  is literally cut and glued along the branch cuts. This provides a way to analyse how the domain evolves as function of the masses. We have studied this phenomena in detail in the following limits:

- *Decoupling of a hypermultiplet:*

A hypermultiplet decouples in the limit that its mass goes to infinity,  $m \rightarrow \infty$ . We demonstrate that in this situation, a branch cut disconnects (or cuts) the strong coupling cusp associated to this hypermultiplet from the rest of the domain. At the same time, the sides of the branch cut are identified to the sides of another branch cut. In this way, the strong coupling cusp is glued back to the weakly coupled cusp, near  $i\infty$ , where these branch points and cuts disappear in the limit  $m \rightarrow \infty$ . As a result, the periodicity at  $i\infty$  increases by 1 in the limit, while the cusp has disappeared from the strongly coupled region. This is displayed for  $N_f = 1$  in Figures 10.

- *Merging of local singularities:*

For a generic choice of masses, the theory with  $N_f$  hypermultiplets has  $N_f + 2$  distinct strong coupling singularities in the  $u$ -plane, where dyons become massless and the effective field theory breaks down. By tuning the masses to special values, the singularities for  $l$  mutually local dyons can merge in the  $u$ -plane. We demonstrate that such cases give rise to a cusp with width  $l > 1$  in  $\mathcal{F}_{N_f}$ . Moreover, when perturbing away from such a special value of the masses, we find that two branch cuts develop from the cusp, which disconnect the singularity in  $\mathcal{F}_{N_f}$ . This is displayed for  $N_f = 2$  in Figure 12.

- *Merging of non-local singularities (AD theories):*

The dynamics is quite different if we tune the masses to special values where singularities corresponding to non-local dyons collide in the  $u$ -



plane. Such singularities give rise to superconformal Argyres-Douglas (AD) field theories [104–109]. In such a situation, we find that two branch points in  $\mathcal{F}_{N_f}$  typically come together and annihilate at the pre-image  $\tau_{\text{AD}}$  of the AD singularity  $u_{\text{AD}}$ . The two branch cuts join at  $\tau_{\text{AD}}$  in the interior of  $\mathcal{F}_{N_f}$ , and disconnect a region from  $\mathcal{F}_{N_f}$  with the “non-local” cusps. Thus  $\mathcal{F}_{N_f}$  consists then of  $< 6$  copies of the  $\text{SL}(2, \mathbb{Z})$  fundamental domain, and the AD point is in a sense a remnant of the disconnected region. On the other hand if we take the appropriate scaling limit near the CFT point [105], we find that the disconnected region is a fundamental domain for the order parameter of the AD theory. If no other branch points remain in  $\mathcal{F}_{N_f}$ , the order parameters become modular functions for a congruence subgroup.

Let us briefly return to the  $u$ -plane integral. The change of variables from  $u$  to  $\tau$ , gives rise to the factor  $du/d\tau$  in the integrand. Interestingly,  $du/d\tau$  can be expressed in case of the  $N_f = 0$  theory in terms of the discriminant  $\Delta$  and  $du/da$  [73], which is a consequence of a relation between the prepotential and  $u$  [100, 110]. Up to numerical constants,  $\Delta$  and  $du/da$  are precisely the two gravitational couplings of the topological theory [72, 111], such that  $du/d\tau$  is naturally included. Extending previous work on the massless  $N_f \leq 3$  [88], we derive a further generalisation for all cases  $N_f \leq 4$  with generic masses. We also discuss how this relation encodes interesting information on the special points of the Coulomb branch, specifically the branch points.

## 2.2 Fundamental domains

In this section, we develop techniques to determine a fundamental domain for the effective coupling of asymptotically free  $\mathcal{N} = 2$   $\text{SU}(2)$  SQCD.

### 2.2.1 The SW solutions

We recall a few essential aspects of the SW solutions for these theories, which we use to analyse  $u$  as function of  $\tau$ . The gauge group  $\text{SU}(2)$  is spontaneously broken to  $\text{U}(1)$  on the Coulomb branch. The order parameter for this branch is the vev  $u$ , defined as

$$u = \frac{1}{16\pi^2} \langle \text{Tr } \phi^2 \rangle_{\mathbb{R}^4} \in \mathcal{B}_{N_f}, \quad (2.1)$$

where the trace is in the 2-dimensional representation of  $\text{SU}(2)$ . Topologically,  $\mathcal{B}_{N_f}$  is the complex plane  $\mathbb{C}$  minus  $2 + N_f$  singular points (for generic masses).

The scalar field related to the photon in the low energy effective field theory is  $a$ , while  $a_D$  is related to the dual photon. The SW solution identifies these fields as periods of a specific differential  $\lambda$  over two dual cycles,  $A$  and  $B$ , of an elliptic curve with complex structure  $\tau$ ,

$$a = \int_{\gamma} \lambda, \quad a_D = \int_{\gamma_D} \lambda. \quad (2.2)$$

To list the SW curves of the theories with  $N_f \leq 3$  hypermultiplets, let  $\Lambda_{N_f}$  be the scale of the theory with  $N_f$  hypermultiplets, and  $m_j$ ,  $j = 1, \dots, N_f$  be the masses of the hypermultiplets. The SW curves of the theories are given by [46]<sup>5</sup>

$$\begin{aligned}
N_f = 0: \quad & y^2 = x^3 - ux^2 + \frac{1}{4}\Lambda_0^4 x, \\
N_f = 1: \quad & y^2 = x^2(x - u) + \frac{1}{4}m\Lambda_1^3 x - \frac{1}{64}\Lambda_1^6, \\
N_f = 2: \quad & y^2 = (x^2 - \frac{1}{64}\Lambda_2^4)(x - u) + \frac{1}{4}m_1 m_2 \Lambda_2^2 x - \frac{1}{64}(m_1^2 + m_2^2)\Lambda_2^4, \\
N_f = 3: \quad & y^2 = x^2(x - u) - \frac{1}{64}\Lambda_3^2(x - u)^2 - \frac{1}{64}(m_1^2 + m_2^2 + m_3^2)\Lambda_3^2(x - u) \\
& + \frac{1}{4}m_1 m_2 m_3 \Lambda_3 x - \frac{1}{64}(m_1^2 m_2^2 + m_2^2 m_3^2 + m_1^2 m_3^2)\Lambda_3^2.
\end{aligned} \tag{2.3}$$

The family of SW curves are Jacobian rational elliptic surfaces with singular fibres [39, 40, 114, 115]. Rational in this context means that  $g_2$  and  $g_3$  are polynomials in  $u$  of degree at most 4 and 6, respectively [116].

Decoupling a hypermultiplet corresponds to the following double scaling limit [117]

$$m_j \rightarrow \infty, \quad \Lambda_{N_f} \rightarrow 0, \quad m_j \Lambda_{N_f}^{4-N_f} = \Lambda_{N_f-1}^{4-(N_f-1)} \tag{2.4}$$

One can directly decouple more than one hypermultiplet, where the scales of the low energy theories are defined as

$$\Lambda_0^2 = m\Lambda_2, \quad \Lambda_0^4 = m^3\Lambda_3, \quad \Lambda_1^3 = m^2\Lambda_3, \tag{2.5}$$

and  $m$  is the equal mass of the hypermultiplets being decoupled. These curves are constructed in such a way that their mathematical discriminants will, up to an overall normalisation, correspond to the *physical discriminant*. This we define as the monic polynomial,

$$\Delta_{N_f} := \prod_{i=1}^{N_f+2} (u - u_i), \tag{2.6}$$

with  $u_i$  being singular points of the effective theory, where hypermultiplets become massless. It is a polynomial of degree  $\deg \Delta_{N_f} = N_f + 2$  in  $u$ .<sup>6</sup> To see this, we bring the SW curves (2.3) into Weierstraß form by shifting  $x \rightarrow x + \frac{u}{3} + \frac{\Lambda_3^2}{192}\delta_{3,N_f}$ , and rescaling  $y \rightarrow y/2$ ,

$$\mathcal{W}: \quad y^2 = 4x^3 - g_2 x - g_3, \tag{2.7}$$

<sup>5</sup>There are other formulations of the SW curve. For example the class  $\mathcal{S}$  form is  $x^2 = p_{N_f}(z, u, \Lambda_{N_f}, \mathbf{m})$  [20, 112, 113]. This has the advantage that the SW differential is canonically determined as  $\lambda = x dz$ . The analysis in the present section still holds for these formulations.

<sup>6</sup>One important note is that in [45] another convention is used for the curve of the pure theory. This gives the duality group  $\Gamma(2)$  rather than  $\Gamma^0(4)$  as in the above. The  $\Gamma(2)$ -convention, however, turns out to not be suitable for the discussion in this section due to multiplicities of the singularities of the curve.

where  $g_2 = g_2(u, \mathbf{m}, \Lambda_{N_f})$  and  $g_3 = g_3(u, \mathbf{m}, \Lambda_{N_f})$  are polynomials in  $u$ ,  $\mathbf{m} = (m_1, \dots, m_{N_f})$  and the scale  $\Lambda_{N_f}$ . The discriminant  $\Delta_{N_f}$  is unchanged for this change of variables, and equals

$$\Delta_{N_f} = (-1)^{N_f} \Lambda_{N_f}^{2N_f-8} (g_2^3 - 27g_3^2), \quad (2.8)$$

where the last factor is the ‘‘mathematical’’ discriminant. The functions  $g_2$  and  $g_3$  can be combined to an absolute invariant  $\mathcal{J}$ ,

$$\mathcal{J} = 12^3 \frac{g_2^3}{g_2^3 - 27g_3^2}. \quad (2.9)$$

As opposed to  $g_2$  and  $g_3$ ,  $\mathcal{J}$  is invariant under admissible changes of variables. Two curves are isomorphic if and only if they have the same absolute invariant  $\mathcal{J}$ . Since  $g_2(u, \mathbf{m}, \Lambda)$  and  $g_3(u, \mathbf{m}, \Lambda)$  are polynomial functions of  $u$ ,  $\mathbf{m}$  and  $\Lambda$  for the SW curves,  $\mathcal{J}$  is naturally a rational function  $\mathcal{J}(u, \mathbf{m}, \Lambda)$  of these variables. On the other hand, the modular Weierstraß form expresses  $\mathcal{J}$  in terms of the complex structure  $\tau$ , namely as the modular  $j$ -invariant  $j(\tau)$  (see (A.9) for a definition).

$$\mathcal{J}(u, \mathbf{m}, \Lambda) = j(\tau). \quad (2.10)$$

This allows to obtain  $u$  as function of  $\tau$ , which is physically the effective coupling constant. Cusps are points where  $j(\tau) = \infty$ , which correspond to  $\tau \in \{i\infty\} \cup \mathbb{Q}$ . The  $j$ -function has fundamental domain  $\mathcal{F} = \text{SL}(2, \mathbb{Z}) \backslash \mathbb{H}$ , which is typically taken to be the key-hole fundamental domain displayed in Figure 1. In other words, the function  $j : \mathcal{F} \rightarrow \mathbb{C}$  is a bijective map.

### 2.2.2 Partitioning the upper half-plane

We are interested in determining the fundamental domains  $\mathcal{F}_{N_f}$  for the effective coupling  $\tau$  for a theory with  $0 \leq N_f < 4$ . Let us consider  $u$  as a function,

$$u : \mathbb{H} \longrightarrow \mathcal{B}_{N_f}, \quad (2.11)$$

and study the analytic properties of this map. We will discuss later the dependence of  $\mathcal{F}_{N_f}$  on the masses  $\mathbf{m}$ , which we will make manifest in the notation as  $\mathcal{F}_{N_f}(\mathbf{m})$  or more compactly  $\mathcal{F}(\mathbf{m})$ . We find that for  $N_f \geq 1$  and generic masses the duality group does not act on  $\tau$  by fractional linear transformations. This prevents us from defining a fundamental domain as is customary for a congruence subgroup  $\Gamma$  of  $\text{SL}(2, \mathbb{Z})$ : For any point  $\tau \in \mathbb{H}$  there exists a  $g \in \Gamma$  such that  $g \cdot \tau \in \Gamma \backslash \mathbb{H}$ , and no two distinct points  $\tau, \tau'$  in  $\Gamma \backslash \mathbb{H}$  are equivalent to each other under  $\Gamma$ . Rather, we can compare if points  $\tau, \tau'$  are equivalent under (2.11): If we define the equivalence relation

$$\tau \sim \tau' \iff u(\tau) = u(\tau'), \quad (2.12)$$

then the quotient set  $\mathbb{H}/\sim$  is a fundamental domain  $\mathcal{F}_{N_f}$  for the function  $u$ . Upon plotting  $\mathcal{F}_{N_f}$  as a domain in  $\mathbb{H}$ , we will have to introduce identifications along co-dimension 1 segments as for  $\mathcal{F}$  in Figure 1.

To determine  $\mathcal{F}_{N_f}$ , we bring (2.9) into a more convenient form. We multiply (2.9) by  $\Delta_{N_f}$  and bring all terms to one side. This gives the sextic polynomial,

$$\begin{aligned} P_{N_f}(X) &:= (g_2(X, \mathbf{m}, \Lambda)^3 - 27g_3(X, \mathbf{m}, \Lambda)^2) j - 12^3 g_2(X)^3 \\ &= a_6 X^6 + a_5 X^5 + \dots + a_1 X + a_0, \end{aligned} \quad (2.13)$$

where the coefficients  $a_i = a_i(\mathbf{m}, \Lambda, j)$  are polynomial functions of  $\mathbf{m}$ ,  $\Lambda$ , and the  $j$  function,  $a_i(\mathbf{m}, \Lambda, j) \in \mathbb{C}[\mathbf{m}, \Lambda, j]$ . The polynomials (2.13) can thus be viewed as polynomials over the field  $\mathbb{C}[\mathbf{m}, \Lambda, j]$ . As discussed in Appendix A.5, roots to such polynomials can be considered as algebraic modular forms, generalising the well-studied class of classical modular forms for congruence subgroups of  $\mathrm{PSL}(2, \mathbb{Z})$ .

We see that (2.9) is equivalent to  $P_{N_f}(u) = 0$  for  $\Delta_{N_f} \neq 0$ , or in other words, away from the singular locus of the theory. The roots of  $P_{N_f}$  can therefore be identified with the order parameter of the corresponding SW curve. Recall that we can assign  $U(1)_{\mathcal{R}}$  charges  $[u : m_i : x : y] = [4 : 2 : 4 : 6]$  to the quantities of the Seiberg-Witten curves [46]. Since  $g_2$  and  $g_3$  are polynomials in  $u$  by construction, by bringing the SW curves to the Weierstraß form and using that  $[u] = 4$  we have that the degrees of  $g_2$  and  $g_3$  as polynomials in  $u$  must be  $\deg(g_2) = 2$  and  $\deg(g_3) = 3$ . Therefore,  $P_{N_f}$  is a sextic polynomial in  $X$ .

For generic masses  $\mathbf{m}$ , the sextic equation  $P_{N_f} = 0$  gives rise to  $n = 6$  different solutions as functions of  $j$ , while for special choices of  $\mathbf{m}$ , such as those giving rise to superconformal (AD) theories, we have  $2 \leq n \leq 4$  different  $j$ -dependent solutions and  $6 - n$   $j$ -independent solutions. Since  $j : \mathcal{F} \rightarrow \mathbb{C}$  is an isomorphism, the  $n \leq 6$  solutions provide a multi-valued ( $n$ -valued) function over  $\mathcal{F}$ .

To obtain  $u$  as a single-valued function of the effective coupling, we choose a different copy of  $\mathcal{F}$  for each of the  $n \leq 6$  branches, and appropriately identify the boundaries of these domains. These are related to  $\mathcal{F}$  by an element of  $\mathrm{SL}(2, \mathbb{Z})$ , and their union is

$$\mathcal{F}_{N_f} = \bigcup_{j=1}^n \alpha_j \mathcal{F}, \quad (2.14)$$

with  $\alpha_j \in \mathrm{SL}(2, \mathbb{Z})$ . A priori, there is no canonical choice for the  $\alpha_j$ , they are determined up to the action of the duality group of the theory. However, some choices are more natural than others. If we demand that  $\mathcal{F}_{N_f}$  is connected and take  $\alpha_1 = \mathbb{1} \in \mathrm{SL}(2, \mathbb{Z})$ , there is only a finite number of choices for  $\mathcal{F}_{N_f}$ . In some cases,  $\mathcal{F}_{N_f}$  is a modular curve  $\Gamma \backslash \mathbb{H}$  for a congruence subgroup  $\Gamma \subseteq \mathrm{SL}(2, \mathbb{Z})$ . In such cases,  $n$  equals the index of  $\Gamma$  in  $\mathrm{SL}(2, \mathbb{Z})$  [118] (see also Appendix A.3 for the corresponding definitions for modular curves). For later use, we define the set of  $\alpha_j$  as  $\mathcal{C}_{N_f} = \{\alpha_j, j = 1, \dots, n\}$ .

For *generic* masses,  $n = 6$  and  $\mathcal{F}_{N_f}$  has  $3 + N_f$  cusps, corresponding to weak coupling  $\tau \rightarrow i\infty$  and the  $2 + N_f$  singularities of the theory. We find the widths

of the cusps by expanding  $j(\tau) = \mathcal{J}(u, \mathbf{m}, \Lambda_{N_f})$  for  $\tau$  near the cusp. For general  $N_f \in \{0, 1, 2, 3\}$ , the cusp at infinity has width  $h_\infty = 4 - N_f$ . This is because  $q^{-1} \sim j(\tau) = \mathcal{J} \sim u^{4-N_f}$ , which implies  $u(\tau) \sim q^{-\frac{1}{4-N_f}}$  (where  $q = e^{2\pi i\tau}$ ). Thus for large  $\tau$ ,  $u(\tau)$  is invariant under  $T^{4-N_f}$ , where  $T : \tau \mapsto \tau + 1$ . Near any singularity  $u_s$ , it is clear that  $q^{-1} \sim \frac{1}{(u-u_s)^{n_s}}$ , where  $n_s$  is the multiplicity of the singularity. Similarly, near  $u_s$  one finds  $u(\tau) - u_s \sim q^{\frac{1}{n_s}}$ . Locally, the function  $u(\tau)$  has period  $h_s$ , giving the width  $h_s$  of the cusp. The widths of all cusps then add up to 6,

$$h_\infty + \sum_s h_s = 6. \quad (2.15)$$

As mentioned above, the equation  $P_{N_f} = 0$  gives six different solutions for  $u$ . A natural question that then arises is which of these six to use as our  $u$ . In some sense this is of course arbitrary, all of them correspond to the order parameter  $u$  simply expressed in different duality frames. On the other hand, the most natural solution is the one corresponding to the weak coupling duality frame where  $|u|$  is large for  $\tau \rightarrow i\infty$ . Since the width of the cusp at infinity is  $4 - N_f$  we see that there is still some ambiguity in this choice as long as  $N_f < 3$ , but for  $N_f = 3$  there is exactly one choice. We show in section 2.6 that this has  $u \rightarrow -\infty$  for  $\tau \rightarrow i\infty$ , and it turns out that this choice can be taken for all  $N_f \leq 3$  theories, and is preserved under the decoupling of hypermultiplets, we therefore make this choice throughout. Note that this sign differs from the conventional choice in the literature [46, 88, 102].

Different mass configurations can give different decompositions of 6. When singularities merge, their cusps are identified under the duality group and their widths add up. Moreover a cusp moves from the real axis to infinity upon decoupling of a matter multiplet.

For *special* choices of the masses, not all singularities correspond to cusps  $i\infty$  or the real line; also singularities in the interior of the upper half-plane can occur. The theories at these points are of superconformal or Argyres-Douglas type, and the widths of all cusps add up to  $n$ .

Yet another aspect of the parametrisation by  $\tau$  is that for special values of  $\tau$  in the interior of  $\mathcal{F}$ , otherwise distinct solutions can coincide. These are branch points of the solutions, where the function  $u(\tau)$  ceases to be meromorphic in  $\tau$ . The branch points in  $\mathcal{F}_{N_f}$  emanate a branch cut. We will discuss these aspects in more detail in section 2.2.3.

For generic masses the equation  $P_{N_f}(X) = 0$  furthermore defines a Riemann surface, which is a 6-fold ramified covering over the classical modular curve  $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H}$  [119]. On this Riemann surface, any root  $u$  forms a *meromorphic* map to the Coulomb branch. It would be interesting to study the topology of these surfaces in more detail. See also Appendix A.5 and [120].

Even if the duality group is not a congruence subgroup of  $\mathrm{SL}(2, \mathbb{Z})$ , there is a procedure to find closed expressions for the order parameters in special

cases. The sextic equation (2.13) for fixed masses  $\mathbf{m}$  and scale  $\Lambda$  can be viewed as a polynomial over the algebraic field  $\mathbb{C}(\Gamma)$  of modular functions on  $\Gamma = \text{SL}(2, \mathbb{Z})$ . Such nontrivial polynomials define field extensions over  $\mathbb{C}(\Gamma)$ . By the fundamental theorem of Galois theory, there is a one-to-one correspondence between the Galois group of the field extension and its intermediate fields. Intermediate fields can be obtained by adjoining roots of the polynomial to the base field. Since  $P_{N_f}(X)$  is a sextic polynomial, for generic masses  $\mathbf{m}$  it is not possible to find exact expressions for the roots. However, if one of the intermediate fields is known, the polynomial factors over the intermediate field into products of polynomials of lower degree. If the resulting degree is less than or equal to 4, there are closed formulas for the roots.

We find below that in many cases, such as massive  $N_f = 2$  and 3 with one mass parameter,  $\mathbb{C}(\Gamma(2))$  for the principal congruence subgroup  $\Gamma(2)$  (see Appendix A.1) is an intermediate field. Since the function  $\lambda = \frac{\vartheta_2^4}{\vartheta_3^4}$  is a Hauptmodul for the genus 0 congruence subgroup  $\Gamma(2)$ , it is the root of a polynomial of degree  $[\Gamma : \Gamma(2)] = 6$  over  $\mathbb{C}(\Gamma)$ . More precisely, there exists a rational function  $\mathcal{R}$  with the property that  $\mathcal{R}(\lambda(\tau)) = j(\tau)$ . It is given by

$$\mathcal{R}(p) = 2^8 \frac{(1 + (p-1)p)^3}{(p-1)^2 p^2}. \quad (2.16)$$

Instead of solving  $\mathcal{J}(u, \mathbf{m}, \Lambda) = j(\tau)$  we can then rather solve  $\mathcal{J}(u, \mathbf{m}, \Lambda) = \mathcal{R}(\lambda(\tau))$ . If  $\mathbb{C}(\Gamma(2))$  is an intermediate field, the sextic equation corresponding to this equation factors over  $\mathbb{C}(\Gamma(2))$ . In massive  $N_f = 2, 3$  we find that it factors into three quadratic polynomials with coefficients depending on  $\lambda$ , which can be easily solved analytically. Such rational relations between the  $j$ -invariant and Hauptmoduln exist for any genus 0 congruence subgroup, which are classified. They allow to invert the equation  $\mathcal{J}(u, \mathbf{m}, \Lambda) = j(\tau)$  for a large class of mass parameters, as we demonstrate in the following sections. See also [121–123].

### 2.2.3 Ramification locus

The covering  $\mathcal{F}_{N_f}(\mathbf{m}) \rightarrow \mathcal{B}_{N_f}$  is not 1-to-1 on a discrete subset, namely at points of  $\mathcal{F}_{N_f}(\mathbf{m})$  where the discriminant  $D(P_{N_f})$  vanishes.<sup>7</sup> In all cases,  $N_f = 0, 1, 2, 3$ , we find that the discriminant of  $P_{N_f}$  factorises as

$$D(P_{N_f}) = j^4 (j - 1728)^3 (D_{N_f}^{\text{AD}})^3 D_{N_f}^{\text{bp}}. \quad (2.17)$$

We discuss each of the three factors:

#### *The $\mathbf{m}$ -independent factor*

<sup>7</sup>The discriminant of a polynomial  $p(X) = X^n + a_{n-1}X^{n-1} + \dots + a_1X + a_0 = \prod_{j=1}^n (X - r_j)$  is defined as  $D(p) = \prod_{i < j} (r_i - r_j)^2$ , in particular it vanishes if and only if two roots coincide. Since we are interested in finding the zeros of  $D(p)$ , we are not careful about overall normalisation factors.

The factor  $j^4(j-1728)^3$  is independent of the masses  $\mathbf{m}$  and can be understood from (2.13). It is immediate that when  $j = 12^3$ , every root of  $P_{N_f}$  has multiplicity at least 2, and if  $j = 0$  every root has multiplicity at least 3. On  $\mathbb{H}$  this occurs whenever  $\tau \in \mathrm{SL}(2, \mathbb{Z}) \cdot i$  or  $\tau \in \mathrm{SL}(2, \mathbb{Z}) \cdot \omega_3$ , with  $\omega_j = e^{2\pi i/j}$ . On the modular curve  $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H}$ , these orbits collapse to a point and in fact the covering  $\pi$  is ramified only over  $\{i\infty, i, \alpha\}$ , or  $j \in \{0, 1728, \infty\}$ , respectively. This resembles the *Belyi functions*, which are holomorphic maps from a compact Riemann surface to  $\mathbb{P}^1(\mathbb{C})$  ramified over precisely these three points [39, 124]. They can be described combinatorially by so-called *dessins d'enfants*. Such dessins have also appeared in the context of SW theory [125–127]. For generic masses, the SW family of curves do not satisfy this definition, as there are additional ramification points.

The polynomial  $D_{N_f}^{\mathrm{AD}}$

The factor  $D_{N_f}^{\mathrm{AD}}$  corresponds to Argyres-Douglas (AD) loci, where two or more singularities coincide [104, 105]. More precisely, the zero locus of  $D_{N_f}^{\mathrm{AD}}$  corresponds to the masses for which the Coulomb branch contains AD points. To see this, recall that the AD points correspond to

$$g_2(u, \mathbf{m}, \Lambda) = g_3(u, \mathbf{m}, \Lambda) = 0. \quad (2.18)$$

Since  $g_2$  and  $g_3$  are polynomials in  $u$  of degrees 2 and 3, respectively, we can eliminate  $u$  from the above equations and characterise  $\mathcal{L}_{N_f}^{\mathrm{AD}}$  as the zero locus of a polynomial  $D_{N_f}^{\mathrm{AD}}$  in  $\mathbf{m}$ ,

$$\mathcal{L}_{N_f}^{\mathrm{AD}} = \{\mathbf{m} \in \mathbb{C}^{N_f} \mid D_{N_f}^{\mathrm{AD}}(\mathbf{m}) = 0\}. \quad (2.19)$$

These are precisely the polynomials appearing in (2.17). From the SW curves we can easily find that they are given by

$$\begin{aligned} D_0^{\mathrm{AD}} &= 1, \\ D_1^{\mathrm{AD}} &= 27\Lambda_1^3 - 64m^3, \\ D_2^{\mathrm{AD}} &= \Lambda_2^6 - 12m_1m_2\Lambda_2^4 + 3(9m_1^4 + 9m_2^4 - 2m_1^2m_2^2)\Lambda_2^2 - 64m_1^3m_2^3, \\ D_3^{\mathrm{AD}} &= \Lambda_3^9 - 12\widetilde{M}_2\Lambda_3^7 + 168\widetilde{M}_3\Lambda_3^6 - 174\widetilde{M}'_4\Lambda_3^5 + 48\widetilde{M}_4\Lambda_3^5 \\ &\quad + 168\widetilde{M}_2\widetilde{M}_3\Lambda_3^4 - 372\widetilde{M}_3^2\Lambda_3^3 + 24\widetilde{M}'_6\Lambda_3^3 - 64\widetilde{M}_6\Lambda_3^3 \\ &\quad - 24\widetilde{M}_3\widetilde{M}'_4\Lambda_3^2 + 96\widetilde{M}_3\widetilde{M}_4\Lambda_3^2 + 6\widetilde{M}_2\widetilde{M}_3^2\Lambda_3 - 27\widetilde{M}'_8\Lambda_3 + 8\widetilde{M}_3^3, \end{aligned} \quad (2.20)$$

where for  $N_f = 3$  we have defined the symmetric combinations

$$\begin{aligned} \widetilde{M}_{2k} &= 2^{6k} \sum_{j=1}^3 m_j^{2k}, & \widetilde{M}_3 &= 2^9 \prod_{j=1}^3 m_j, \\ \widetilde{M}'_4 &= 2^{12} \sum_{i<j} m_i^2 m_j^2, & \widetilde{M}'_6 &= 2^{18} \sum_{i \neq j} m_i^2 m_j^4, & \widetilde{M}'_8 &= 2^{24} \sum_{i<j} m_i^4 m_j^4. \end{aligned} \quad (2.21)$$

The type of singularity that appears for specific masses on these loci are found by studying the order of vanishing of  $g_2$ ,  $g_3$  and  $\Delta$  according to the Kodaira

classification,

$$\begin{aligned}
II : \quad & \text{ord}(g_2, g_3, \Delta) = (1, 1, 2) \text{ or } (2, 1, 2), \\
III : \quad & \text{ord}(g_2, g_3, \Delta) = (1, 2, 3), \\
IV : \quad & \text{ord}(g_2, g_3, \Delta) = (2, 2, 4).
\end{aligned}
\tag{2.22}$$

See Appendix A.6 for more details. The zero loci of the AD polynomials can be understood as codimension 1 loci in the space  $\mathbb{C}^{N_f} \ni \mathbf{m}$  [105]. For  $N_f = 3$  such a locus is shown in Fig. 3. Argyres-Douglas loci are studied for a more general class of SW theories in [108].

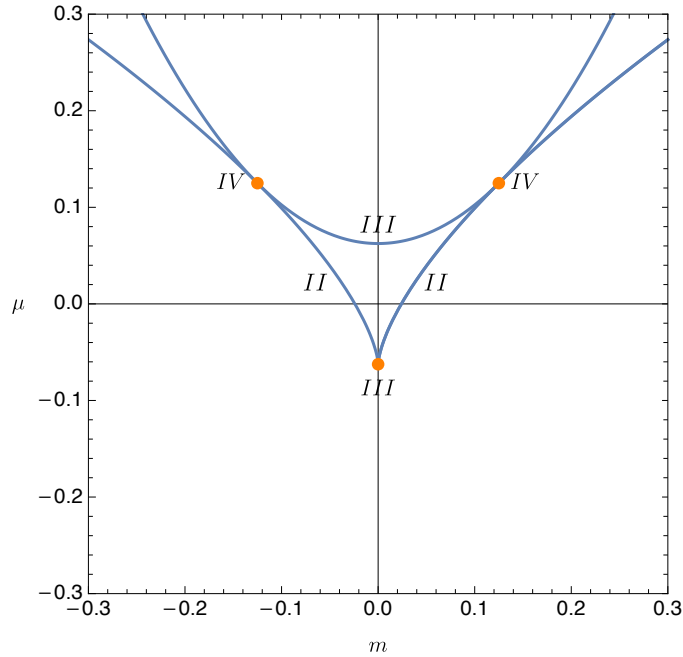


Figure 3: The AD locus  $\mathcal{L}_3^{\text{AD}}$  for  $N_f = 3$  with masses  $\mathbf{m} = (m, m, \mu)$  in the real  $(m, \mu)$ -plane, with units  $\Lambda_3 = 1$ . It is a union of three smooth lines, two of them generically describing type *II* AD points and the third one type *III*. The two *II* lines touch at a *III* point, while both *II* lines touch the *III* line in a type *IV* AD point.

In section 2.2.2, we argued that the widths of the different cusps of the  $\text{SU}(2)$  theories always add up to  $n \leq 6$ . We will now argue that  $n < 6$  and only if  $\mathbf{m}$  is a zero  $\mathbf{m}_{\text{AD}}$  of  $D_{N_f}^{\text{AD}}$ . It is possible that some zero of  $\Delta$  is also a zero of  $g_2$ . Then the index is given by the degree of the numerator of  $j$ , which can be smaller than 6. In Sections 2.4.2, 2.5.3 and 2.6.4-2.6.6 we study a few examples of AD theories appearing in the  $N_f = 1, 2, 3$  theories, and demonstrate that the curve degenerates to Kodaira types *II*, *III* or *IV*. Each singularity type is not exclusive to a specific number of flavours, but appears on the discriminant divisor of the higher  $N_f$  theories as well [105]. See Sections 2.6.5 and 2.6.6 for two explicit examples of this. The three types of AD theories corresponds to 2, 3 or 4 mutually non-local states becoming massless at the AD point. The cusps corresponding to the non-local states are disconnected from the rest of the domain, and the branch points collide at an



elliptic point of the duality group. As a result, the index is reduced by  $\text{ord } \Delta$ , which equals the number of mutually non-local states becoming massless, i.e., 2, 3, and 4 for the theories *II*, *III* and *IV*, respectively. Note that the order of vanishing of the discriminant may be larger than zero for ordinary singularities as well, so it is not enough to simply subtract  $\text{ord } \Delta$  from six to get the index right but rather we should subtract the number of mutually non-local states becoming massless at each cusp,

$$n = 6 - \# (\text{mutually non-local massless dyons}). \quad (2.23)$$

This is because for the index to reduce it is necessary for  $g_2$  and  $\Delta$  to have a common root, such that due to (2.8) it is also a root of  $g_3$  and because of (2.18) therefore an AD point. In the limit  $\mathbf{m} \rightarrow \mathbf{m}_{\text{AD}}$ , the  $6 - n$  copies of  $\mathcal{F}_{N_f}(\mathbf{m})$  corresponding to the regular singularities are removed from the fundamental domain. We have also found mass configurations whose corresponding Coulomb branch contains *two* (type II) AD points. The correspondence (2.23) nevertheless holds, for a similar argument as presented above.

*The polynomial  $D_{N_f}^{\text{bp}}$*

The last factor  $D_{N_f}^{\text{bp}}$  corresponds to branch points. These are values of  $j$  for which two solutions of  $P_{N_f}(X) = 0$  coincide, such that the map  $u : \mathcal{F}_{N_f}(\mathbf{m}) \rightarrow \mathcal{B}_{N_f}$  is not 1-to-1 on these points. The identifications are different from the multiple images of  $\mathcal{F}$  in  $\mathcal{B}_{N_f}$ , which identify the images of the boundary of  $\mathcal{F}$ ,  $\alpha_j(\partial\mathcal{F})$ , in  $\mathcal{F}_{N_f}(\mathbf{m})$ .

The  $D_{N_f}^{\text{bp}}$  are explicitly given by

$$\begin{aligned} D_0^{\text{bp}} &= 1, \\ D_1^{\text{bp}} &= 27j\Lambda_1^6 - 27 \cdot 2^{14} m^3 \Lambda_1^3 + 2^{20} m^6, \\ D_2^{\text{bp}} &= (m_1^2 - m_2^2)^2 j^2 \Lambda_2^8 - 128 \Lambda_2^4 (216(m_1^8 + m_2^8) - 288m_1^2 m_2^2 (m_1^4 + m_2^4) \\ &\quad + 16m_1^4 m_2^4 + 240m_1^3 m_2^3 \Lambda_2^2 - 72m_1 m_2 (m_1^4 + m_2^4) \Lambda_2^2 + 9(m_1^4 + m_2^4) \Lambda_2^4 \\ &\quad - 42m_1^2 m_2^2 \Lambda_2^4 - 2m_1 m_2 \Lambda_2^6) j + 2^{12} (16m_1 m_2 - \Lambda_2^2)^3 P_2^{\text{AD}}, \end{aligned} \quad (2.24)$$

and we define  $\mathcal{L}_{N_f}^{\text{bp}}$  as the zero locus of  $D_{N_f}^{\text{bp}}$ . The expression for  $D_3^{\text{bp}}$  for generic masses is very long so we do not write it out here, but we can note that it has degree three in  $j$ . For later reference we write it out for two special mass configurations

$$\begin{aligned} D_3^{\text{bp}}(m, m, m) &= 432m^4 \Lambda_3^2 j + (8m - \Lambda_3)^2 (16m + \Lambda_3)^3 (64m + \Lambda_3), \\ D_3^{\text{bp}}(m, 0, 0) &= 16m^4 \Lambda_3^2 j + (8m - \Lambda_3)^3 (8m + \Lambda_3)^3. \end{aligned} \quad (2.25)$$

To show that the zero locus of these polynomials really correspond to branch points we will need some specific details of the corresponding theory and we therefore hold off on this discussion until the respective sections below. We can, however, note that by solving  $D_{N_f}^{\text{bp}} = 0$  for  $j$  and plugging it into (2.13) we get the corresponding solutions for  $u$ . For example, in  $N_f = 1$  we find

$u = \frac{4}{3}m^2$  and as we will see, away from  $m = m_{\text{AD}} = \frac{3}{4}\Lambda_1$ , this is not part of the discriminant of the curve and therefore does not correspond to a physical singularity of the theory. We denote a branch point of  $u$  in  $\mathcal{F}_{N_f}$  by  $\tau_{\text{bp}}$ , and its image in  $\mathcal{B}_{N_f}$  as  $u_{\text{bp}}$ . As explained in Section 2.3.3, for generic masses there are two branch points  $\tau_{\text{bp}}$  and  $\tau'_{\text{bp}}$  with image  $u_{\text{bp}} = u(\tau_{\text{bp}}) = u(\tau'_{\text{bp}})$ . Since their image in  $\mathcal{B}_{N_f}$  is the same, the points  $\tau_{\text{bp}}$  and  $\tau'_{\text{bp}}$  are identified in  $\mathcal{F}_{N_f}$ , even though they appear as distinct points in plots of  $\mathcal{F}_{N_f}$  in  $\mathbb{H}$ . A branch cut emanates from each branch point; there can be a single cut connecting both branch points, or two separate cuts which go to either  $i\infty$  or to the real axis.

#### *Mutually local singularities*

As the masses are tuned, some of the singularities on the Coulomb branch can collide. If we consider  $\Delta_{N_f}$  as a polynomial in  $u$ , its discriminant  $D(\Delta_{N_f})$  vanishes if and only if two roots coincide. It is straightforward to show that for  $N_f \leq 3$ ,

$$D(\Delta_{N_f}) = \left(D_{N_f}^{\text{AD}}\right)^3 \prod_{i < j} (m_i - m_j)^2 (m_i + m_j)^2. \quad (2.26)$$

This factorises the locus in mass space where singularities collide into two orthogonal components: The first component is the Argyres-Douglas (AD) locus given by the polynomial equation  $D_{N_f}^{\text{AD}} = 0$ , where mutually *non-local* singularities collide [105, 128]. The other component is characterised by the equations  $m_i = \pm m_j$ , and one can check that this gives rise to mutually *local* singularities colliding. Here, the flavour symmetry gets enhanced and a Higgs branch opens up [46].

Given a mass configuration  $\mathbf{m} = (m_1, \dots, m_{N_f})$ , we can denote by  $k_l$  the weight (or multiplicity) of the  $l$ -th singularity, and by  $\mathbf{k}(\mathbf{m}) = (k_1, k_2, \dots)$  the vector of those weights. Since the Coulomb branch  $\mathcal{B}_{N_f}(\mathbf{m})$  contains  $2 + N_f$  singularities aside from weak coupling  $u = \infty$ , it is clear that  $\mathbf{k}(\mathbf{m})$  provides a partition of  $2 + N_f$ . This in turn partitions the mass space  $\mathbb{C}^{N_f} \ni \mathbf{m}$  into finitely many regions where  $\mathbf{k}(\mathbf{m})$  is locally constant. As an example, in Fig. 4 we plot the contours of (2.26) for  $N_f = 2$  in the real  $\mathbf{m} = (m_1, m_2)$  plane.

The possible singularity structures of the rational elliptic curves (2.3) are classified in Persson's list of allowed configurations of singular fibres [43, 44]. From Kodaira's classification, it follows that any solution to (2.26) gives rise to a singularity on the Coulomb branch of Kodaira type  $I_k$ ,  $II$ ,  $III$ , or  $IV$ . As described in [2, 129], the solutions to  $0 = D_{N_f}^{\text{AD}}$  give rise to AD points of Kodaira type  $II$ ,  $III$  and  $IV$ . The second component  $0 = \prod_{i < j} (m_i - m_j)(m_i + m_j)$  can be studied in more detail. These are  $2(N_f - 1)$  independent equations. Whenever one of the factors vanishes, the SW surface contains an  $I_k$  singularity with  $k \geq 2$ . For  $N_f = 2$ , the only possibility is  $I_2$ , while for  $N_f = 3$  singularities of type  $I_2$ ,  $I_3$  and  $I_4$  are possible. The point in the Coulomb branch  $\mathcal{B}_{N_f}$  corresponding to an  $I_k$  singularity with  $k \geq 2$  intersects with a Higgs branch of quaternionic dimension  $k - 1 \geq 1$  [46]. Further merging these  $I_k$  singularities with a mutually non-local singularity does not affect the

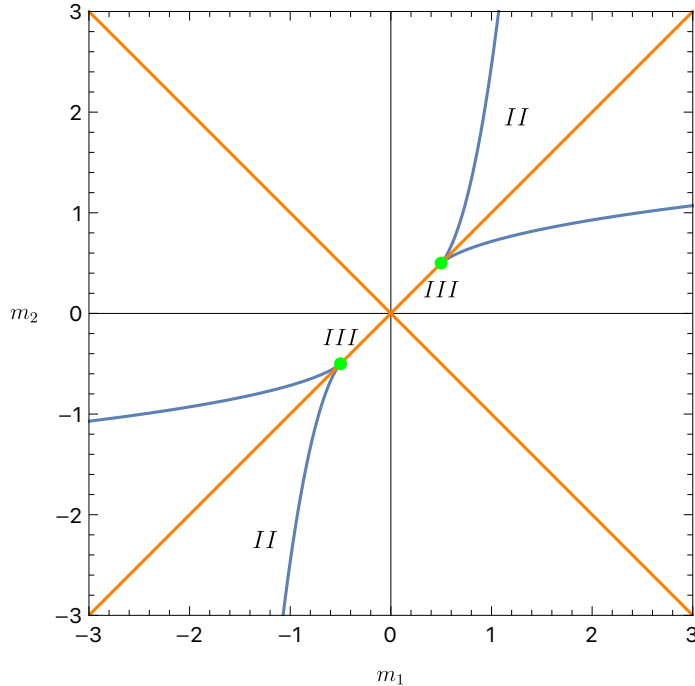


Figure 4: Partitioning of the real  $\mathbf{m} = (m_1, m_2)$  plane in  $N_f = 2$ , in units of  $\Lambda_2 = 1$ . On the AD component the Coulomb branch  $\mathcal{B}_2(\mathbf{m})$  contains an AD point of Kodaira type II (blue) or III (green). On the other component (orange), mutually local singularities collide. If  $\mathbf{m}$  is varied along a continuous path that does not cross the partitioning  $0 = D(\Delta_{N_f})$ , the weight vector  $\mathbf{k}(\mathbf{m})$  is constant.

Higgs branch, such that the points with a *III* or *IV* singularity also intersect with a Higgs branch of quaternionic dimension one or two, respectively, while the points with AD theories of type *II* do not intersect with any Higgs branch.

#### *The genus of $\mathcal{F}_{N_f}(\mathbf{m})$*

For special choices of the masses  $\mathbf{m}$ ,  $\mathcal{F}(\mathbf{m})$  coincides with the modular curve  $X(\Gamma)$  for a congruence subgroup  $\Gamma \in \text{SL}(2, \mathbb{Z})$ . Then the genus of  $\mathcal{F}(\mathbf{m})$  is given by that of  $X(\Gamma)$ , for which there is the formula (A.40) in terms of the index  $n$ , the number of elliptic points  $\varepsilon_2$  and  $\varepsilon_3$  and the number of cusps  $\varepsilon_\infty$ . In all examples of such masses  $\mathbf{m}$  discussed below, we find that  $\mathcal{F}(\mathbf{m})$  is a genus zero Riemann surface. In the presence of branch points in  $\mathcal{F}(\mathbf{m})$ , Equation (A.40) needs to be modified. First, we note that for an AD theory,  $\tau_{\text{AD}}$  corresponds to an elliptic point. In fact, in all AD cases studied here, (2.23) can be expressed as<sup>8</sup>

$$n = 6 - 2\varepsilon_3 - 3\varepsilon_2, \quad (2.27)$$

For  $\varepsilon_\infty$  there is no simple formula since for example it is not unique in some limit  $\mathbf{m} \rightarrow \mathbf{m}_{\text{AD}}$ , but rather depends on the direction in mass space from which  $\mathbf{m}_{\text{AD}}$  is approached. As the map  $u : \mathcal{F}(\mathbf{m}) \rightarrow \mathcal{B}_{N_f}$  is between Riemann surfaces  $\mathcal{F}(\mathbf{m})$  and  $\mathcal{B}_{N_f}$ , we can consider the Riemann-Hurwitz formula (A.39), which relates their genera  $g$ . The inverse map  $\tau : \mathcal{B}_{N_f} \rightarrow \mathcal{F}(\mathbf{m})$  can be defined

<sup>8</sup>The type IV AD point can be viewed as a collision of two elliptic fixed points of period 3.

through  $\tau = \frac{da_D}{da} = \frac{da_D/du}{da/du}$ , with the periods  $a, a_D$  given by (2.2). The dependence of  $\tau$  on  $u$  is holomorphic everywhere [41, 42]. Then (A.39) for the inverse map implies that  $0 = g_{\mathcal{B}_{N_f}} \geq g_{\mathcal{F}(\mathbf{m})}$ , such that

$$g_{\mathcal{F}(\mathbf{m})} = 0. \quad (2.28)$$

Applying this to the Riemann-Hurwitz formula for the ramified covering  $\mathcal{F}(\mathbf{m}) \rightarrow \mathcal{F}$ , we find the number of distinct branch points on  $\mathcal{F}(\mathbf{m})$  for arbitrary  $\mathbf{m}$  as

$$\sum_{\tau_{\text{bp}} \in \mathcal{F}(\mathbf{m})} (e_{\tau_{\text{bp}}} - 1) = \varepsilon_{\infty} - 3 + \varepsilon_2 + \varepsilon_3. \quad (2.29)$$

This shows that  $\mathcal{F}(\mathbf{m})$  is a Riemann sphere with  $\varepsilon_{\infty}$  cusps,  $\varepsilon_2, \varepsilon_3$  elliptic points of periods 2 and 3 and  $\varepsilon_{\infty} - 3 + \varepsilon_2 + \varepsilon_3$  branch points. As an example, in massless  $N_f = 1$  (see section 2.4.1) we have  $\varepsilon_{\infty} = 1 + 3 = 4$ , while all singularities are on  $\mathbb{Q}$  and thus  $\varepsilon_2 = \varepsilon_3 = 0$ . There is one branch point in  $\mathcal{F}(0)$ , which agrees with (2.29).

#### 2.2.4 Partitioning the $u$ -plane

An approach to better understand the  $u$ -plane geometry is to study the partitions that the map  $u : \mathcal{F}_{N_f} \rightarrow \mathcal{B}_{N_f}$  produces on the  $u$ -plane  $\mathcal{B}_{N_f}$ . Let us study the union (2.14). Now since  $u(\mathcal{F}_{N_f}) = \mathcal{B}_{N_f}$ , it is natural to ask what

$$\mathcal{T}_{\mathbf{m}} = u \left( \bigcup_{j=1}^n \alpha_j \partial \mathcal{F} \right) \subseteq \mathcal{B}_{N_f} \quad (2.30)$$

describes. The insight is that while  $j : \mathcal{F} \rightarrow \mathbb{C}$  is an isomorphism, it surjects the boundary onto a half-line,

$$j(\partial \mathcal{F}) = (-\infty, 12^3] \subseteq \mathbb{R} \subseteq \mathbb{C}. \quad (2.31)$$

This is straightforward to prove. On the half-lines  $i[\frac{\sqrt{3}}{2}, \infty)$  the  $q$ -series of  $j$  is an alternating series with the same Fourier coefficients as  $j$  and therefore real. On the arc  $\{e^{\varphi i} \mid \varphi \in (\frac{\pi}{3}, \frac{2\pi}{3})\}$  the complex conjugate of  $j(e^{\varphi i})$  is equal to the value of  $j$  at the  $S$ -transform of  $e^{\varphi i}$  and therefore equal to  $j(e^{\varphi i})$ .

The only other region in  $\mathcal{F}$  where  $j$  is real are the  $\text{SL}(2, \mathbb{Z})$  images of the half-line  $i[1, \infty)$  on the imaginary axis. We can directly apply this to the SW curves, whose  $j$ -invariant  $\mathcal{J}(u, \mathbf{m}, \Lambda)$  is identified with  $j(\tau)$ . The partitioning is then

$$\mathcal{T}_{\mathbf{m}} = \{u \in \mathcal{B}_{N_f} \mid \mathcal{J}(u, \mathbf{m}, \Lambda_{N_f}) \in (-\infty, 12^3]\}. \quad (2.32)$$

It is included in the level set  $\text{Im } \mathcal{J} = 0$ . Let us therefore study the curves

$$\text{Im } \mathcal{J}(u, \mathbf{m}, \Lambda_{N_f}) = 0, \quad (2.33)$$

which contrary to (2.32) are algebraic curves. It turns out that some of the components of this equation do not belong to the partitioning (2.32), and it

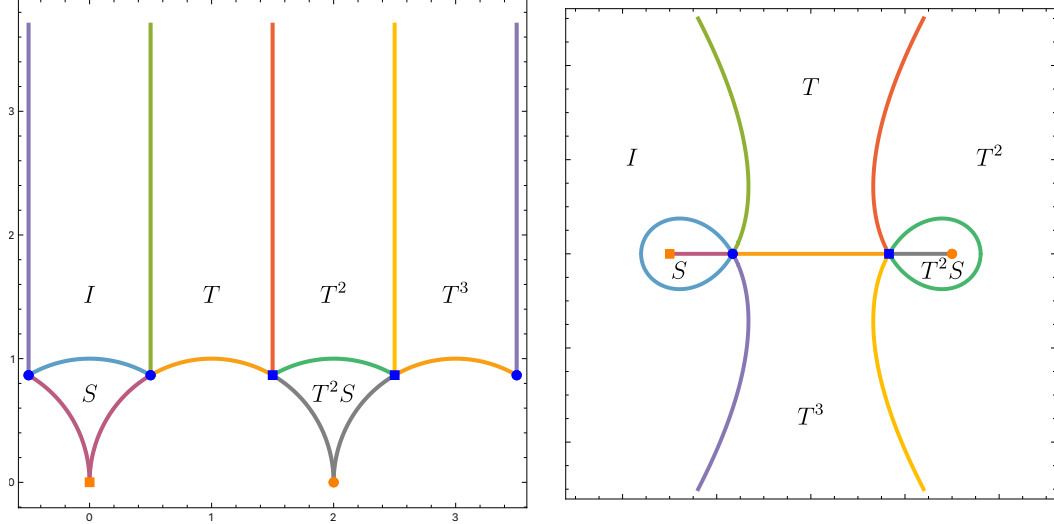


Figure 5: Identification of the components of the partitioning  $\mathcal{T}$  in the pure theory. The  $u$ -plane  $\mathcal{B}_0$  is partitioned into 6 regions  $u(\alpha\mathcal{F})$ , with the  $\alpha \in \text{SL}(2, \mathbb{Z})$  given in both pictures.

is clear that they correspond to components of curves with  $j > 12^3$ . Due to the imaginary part, it is instructive to choose coordinates  $u/\Lambda_{N_f}^2 = x + iy$ . The equations (2.33) are straightforward to compute in terms of zero-loci of polynomials in  $x$  and  $y$ . For fixed  $\mathbf{m}$ , they define algebraic varieties

$$T_{\mathbf{m}}(x, y) = 0. \quad (2.34)$$

More specifically, they are an  $N_f$ -parameter family of affine algebraic plane curves. For the pure  $N_f = 0$  theory, one finds

$$T_0 = xy(81 - 288x^2 + 336x^4 - 128x^6 + 288y^2 - 352x^2y^2 - 128x^4y^2 + 336y^4 + 128x^2y^4 + 128y^6). \quad (2.35)$$

The identification of this partitioning of the  $u$ -plane for the pure theory is shown in Figure 5. The defining equations can be computed in full generality for any  $N_f$ , but they are rather lengthy: The polynomials  $T_{\mathbf{m}}$  for generic masses have total degree  $8 + N_f$ . For generic real masses, the polynomials  $T_{\mathbf{m}}$  have 30, 131, and 1081 terms in  $N_f = 1, 2$  and  $3$ , respectively. If we allow the masses to be complex, we can decompose  $m_i = \text{Re } m_i + i\text{Im } m_i$  and the  $T_{\mathbf{m}}$  are then polynomials in  $x, y, \text{Re } m_i$  and  $\text{Im } m_i$ . For generic (complex) masses in  $N_f = 1, 2$  and  $3$ ,  $T_{\mathbf{m}}$  has 93, 1310 and 48754 terms, respectively.

The polynomials  $T_{\mathbf{m}}$  are in general reducible. For instance, for  $\mathbf{m} = (m, m)$  and  $\mathbf{m} = (m, 0, 0)$ ,  $T_{\mathbf{m}}$  factors into multiple nontrivial polynomials. It is straightforward to check that  $T_{\mathbf{m}}$  for given  $N_f$  flows into  $T_{\mathbf{m}}$  for  $N_f - 1$  by decoupling one hypermultiplet. This allows to study the decoupling procedure of the fundamental domains in detail.

The partitioning  $\mathcal{T}_{\mathbf{m}}$  is a finite union of smooth curves that intersect. The tessellation of  $\mathbb{H}$  in  $\text{SL}(2, \mathbb{Z})$  images of  $\mathcal{F}$ ,

$$\mathcal{T}_{\mathbb{H}} = \bigcup_{\alpha \in \text{SL}(2, \mathbb{Z})} \alpha(\partial\mathcal{F}) = \{\tau \in \mathbb{H} \mid j(\tau) \leq 12^3\}, \quad (2.36)$$

has intersection points  $\tau \in \text{SL}(2, \mathbb{Z}) \cdot e^{\frac{\pi i}{3}}$ , where  $j(\tau) = 0$ . From (2.10) we see that these intersection points correspond to  $\mathcal{J}(u, \mathbf{m}, \Lambda) = 0$ , whose only solutions are given by  $g_2(u, \mathbf{m}, \Lambda) = 0$  (see (2.9)). Since  $g_2$  is a polynomial in  $u$  of degree 2 for all curves (2.3), there are at most two intersection points in  $\mathcal{T}_m$  corresponding to  $\mathcal{J} = 0$ . As  $g_2$  is strictly quadratic, there is also always at least one such point. We find below that when the branch points (as introduced in section 2.2.3) belongs to  $\mathcal{T}_m$ , they give further intersection points of  $\mathcal{T}_m$ .

One can study how the partitioning is deformed upon varying the masses. For the cases where the branch points belong to  $\mathcal{T}_m$ , the complex  $u$ -plane is generically partitioned into 6 regions. When going to the AD locus two or more of these regions shrink to a point together with at least one branch point. At precisely  $m = m_{\text{AD}}$ , the  $u$ -plane is then partitioned into  $\leq 4$  regions, giving an explanation for the discontinuous decrease in the index in the limit  $m \rightarrow m_{\text{AD}}$ . This can also be understood directly from the polynomials  $T_m(x, y)$ . For instance, at the point  $m = m_{\text{AD}} = \frac{3}{4}\Lambda_1$  in  $N_f = 1$ , the polynomial  $T_{m_{\text{AD}}}(x, y)$  contains a factor  $9 - 24x + 16x^2 + 16y^2$ . Its zero locus in  $\mathbb{R}^2$  is just a point  $x + iy = \frac{3}{4} = u_{\text{AD}}/\Lambda_1^2$ , while the massive deformation away from  $m_{\text{AD}}$  describes a curve that encloses a region. For  $u_{\text{bp}} \notin \mathcal{T}_m$  one needs to cut and glue interior points of different regions and the  $u$ -plane is therefore partitioned into less than 6 regions. See for example Fig. 11.

### 2.3 Matone's relation

In pure  $\mathcal{N} = 2$  supersymmetric gauge theories, there is a relation between instanton corrections of the prepotential  $F$  to instanton corrections of  $u \sim \langle \text{Tr}\phi^2 \rangle$  (2.1). For the pure SU(2) gauge theory, it has been found in [110] that

$$u = \pi i(2F - a\partial_a F). \quad (2.37)$$

In combination with the Picard-Fuchs differential equations for the periods  $a$  and  $a_D$  (see [96, (2.5)] or [110, (30)]), this gives an explicit recursion relation [100, (8)]

$$\partial_a^3 F = \frac{\pi^2}{4} \frac{(a\partial_a^2 F - \partial_a F)^3}{(1 + \pi^2(2F - a\partial_a F))^2} \quad (2.38)$$

for the prepotential  $F$ . Using (2.37), one may write this relation as

$$\frac{du}{d\tau} = -4\pi i \Delta \left( \frac{da}{du} \right)^2, \quad (2.39)$$

where  $\Delta = u^2 - \Lambda_0^4$  is the physical discriminant. In order to see this, one can differentiate (2.37) w.r.t.  $a$  to find  $\frac{du}{da} = \pi i(a_D - a\tau)$ . In the following, we will call relations such as (2.39) *Matone's relations* rather than (2.37), (2.38) or equivalent equations.

One can furthermore differentiate (2.37) w.r.t.  $u$  to find  $aa'_D - a'a_D = \frac{i}{\pi}$ , where  $' = \partial_u$ . By computing  $\frac{d\tau}{du} = \partial_u \partial_a^2 F$ , this identity together with the Picard-Fuchs equations implies (2.39). Due to its central importance,

generalisations of (2.37) to many theories have been found. For  $SU(N)$  SQCD it reads

$$2F = \left( \Lambda \partial_\Lambda + \sum_j m_j \partial_{m_j} + \sum_j a_j \partial_{a_j} \right) F, \quad (2.40)$$

which can be interpreted as an RG flow of the prepotential,  $\Lambda \partial_\Lambda F \sim u$  [130–132]. By combining  $\boldsymbol{\pi} = (\Lambda, \mathbf{m}, \mathbf{a})^\top$  into a vector, the prepotential satisfies the differential equation  $\boldsymbol{\pi}^\top \partial_{\boldsymbol{\pi}} F = 2F$ . This Euler equation characterises the homogeneity of the prepotential. Its relations to supergravity [131, 133], the  $\Omega$ -background [134, 135], integrability [136–138] and 5 dimensions [139, 140] are well-known.

In  $\mathcal{N} = 2$   $SU(2)$  SYM, all ingredients of (2.39) are classically modular, while the prepotential  $F$  is not. This is easy to see from the above relations: It involves the period  $a_D$ , which unlike  $a$  is not modular since it depends explicitly non-modularly on  $\tau$ . Picard-Fuchs solutions of massive  $SU(2)$  SQCD have been studied in [141–143], where generalisations of (2.39) for the massless  $N_f = 1, 2, 3$  theories have been obtained. By including massless hypermultiplets, (2.39) receives interesting corrections [88]: The physical discriminant  $\Delta$  is divided by another polynomial in  $u$ . For massless  $N_f = 2, 3$  it divides  $\Delta$ , while it does not for  $N_f = 1$ . In this Section, we derive a generalisation of (2.39) for  $N_f = 1, 2, 3, 4$  with generic masses and give an explanation of these denominators.

Another motivation comes from the topologically twisted theory [61], where correlation functions of the topological theory on a four-manifold  $X$  can be computed as integrals over the  $u$ -plane [73]. The coupling of the low energy effective theory to topological invariants of the background gravitational field is encoded by a holomorphic function  $A^\chi B^\sigma$ , where  $\chi$  and  $\sigma$  are the Euler characteristic and the signature of  $X$ . For pure  $SU(2)$  SYM, the functions  $A$  and  $B$  are  $A = \left(\Delta \frac{d\tau}{du}\right)^{\frac{1}{4}}$  and  $B = \Delta^{\frac{1}{8}}$  [72].<sup>9</sup> The function  $B^8$  can in fact be viewed as a definition of the physical discriminant [144]. Due to the pure  $SU(2)$  Matone relation (2.39), it was realised in [73] that  $A = \left(\frac{du}{da}\right)^{\frac{1}{2}}$ . If we include hypermultiplets, (2.39) is modified and this argument does not work anymore. Instead, it was argued [73] that  $A = \left(\frac{du}{da}\right)^{\frac{1}{2}}$  is in fact correct when including hypermultiplets. However, this bases on the assumption that  $A^\chi$  is a holomorphic modular form of weight  $-\frac{\chi}{2}$ , and has neither zeros nor poles anywhere on the  $u$ -plane. In  $\mathcal{N} = 2$  SQCD, these statements are not quite true. We will discuss these issues in section 5.

In this Section, we derive a generalisation of (2.39) for massive  $N_f = 1, 2, 3$ . Section 2.3.1 derives expressions for  $da/du$  and  $\Delta_{N_f}$  as functions of  $\tau$ . Section 2.3.2 derives Matone’s relation (2.52) for generic  $N_f \leq 3$ .

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<sup>9</sup>The prefactors were determined in [73] for pure  $SU(2)$  and in [111] for the asymptotically free  $SU(2)$  theories. We ignore them in the following.

### 2.3.1 Periods and Weierstraß form

We proceed by deriving an expression for  $da/du$ . To this end, recall that  $a$  is given as a period integral (2.2), and that the derivative of the SW differential  $\lambda$  to  $u$  is holomorphic [46]. Therefore, we can express  $da/du$  in terms of the variables  $x$  and  $y$  of (2.7)

$$\frac{da}{du} = \frac{\sqrt{2}}{4\pi} \int_{\gamma} \frac{dx}{y}, \quad (2.41)$$

where  $\gamma$  is one of the cycles of the elliptic curve. To determine this quantity for the theories with  $N_f \leq 3$ , we map the curve  $\mathcal{W}$  to the modular Weierstraß form  $\widetilde{\mathcal{W}}$ ,  $A : \mathcal{W} \rightarrow \widetilde{\mathcal{W}}$ . See for example [145, Section 7.1]. The curve  $\widetilde{\mathcal{W}}$  reads

$$\widetilde{\mathcal{W}} : \quad \tilde{y}^2 = 4\tilde{x}^3 - \tilde{g}_2\tilde{x} - \tilde{g}_3, \quad (2.42)$$

with the variables related by the map  $A$  as

$$A : \begin{cases} \tilde{x} = \alpha^2 x = \wp(z), \\ \tilde{y} = \alpha^3 y = \wp'(z), \\ \tilde{g}_2 = \alpha^4 g_2, \\ \tilde{g}_3 = \alpha^6 g_3, \end{cases} \quad (2.43)$$

where  $\wp$  is the Weierstraß function and  $z \in \mathbb{C}$  a coordinate on the curve. Since  $\widetilde{\mathcal{W}}$  is the modular Weierstraß curve, the variables  $\tilde{g}_2$  and  $\tilde{g}_3$  equal

$$\tilde{g}_2 = \frac{4\pi^4}{3} E_4, \quad \tilde{g}_3 = \frac{8\pi^6}{27} E_6, \quad (2.44)$$

with  $E_k$  the Eisenstein series defined in (1.7). We note that the variables for  $\mathcal{W}$  (2.7) have weight 0 under modular transformations, while in (2.42) the weights are  $\text{wt}(\alpha, \tilde{y}, \tilde{x}, \tilde{g}_2, \tilde{g}_3) = (1, 3, 2, 4, 6)$ . Using the two equations for  $\tilde{g}_2$  and  $\tilde{g}_3$ , we can solve for  $u$  and  $\alpha$ . The relation

$$\alpha = \frac{\sqrt{2}\pi}{3} \sqrt{\frac{g_2 E_6}{g_3 E_4}}, \quad (2.45)$$

will be particularly useful for us in the next subsection. This relation can also be derived using Picard-Fuchs equations [146].

Now it is straightforward to determine  $da/du$  (2.41) using the Weierstraß representation of  $(\tilde{x}, \tilde{y})$ ,

$$\frac{da}{du} = \frac{\sqrt{2}\alpha}{4\pi} \int_{\tilde{\gamma}} \frac{d\tilde{x}}{\tilde{y}} = \frac{\sqrt{2}\alpha}{4\pi}, \quad (2.46)$$

where  $\tilde{\gamma}$  is the image of the  $\gamma$  under the map  $A$ , with the variable  $z$  of  $\tilde{x}(z)$  changing from 0 to 1.

We continue by studying the discriminants of  $\mathcal{W}$  and  $\widetilde{\mathcal{W}}$ . Using  $E_4^3 - E_6^2 = 12^3 \eta^{24}$  with  $\eta$  as in (A.18), we find for the discriminant of  $\widetilde{\mathcal{W}}$ ,  $\tilde{\Delta} = (2\pi)^{12} \eta^{24}$ . The discriminant of  $\mathcal{W}$ ,  $\Delta_{N_f}$  (2.8), on the other hand is a polynomial in  $u$ ,  $\mathbf{m}$  and  $\Lambda$  and therefore has weight 0. The two discriminants are related by

$$\tilde{\Delta} = \alpha^{12} (-1)^{N_f} \Lambda_{N_f}^{8-2N_f} \Delta_{N_f}, \quad (2.47)$$



or substituting  $\alpha$  in terms of  $da/du$  (2.46),

$$\eta^{24} = 2^6 (-1)^{N_f} \Lambda_{N_f}^{2(4-N_f)} \left( \frac{da}{du} \right)^{12} \Delta_{N_f}, \quad (2.48)$$

which holds for  $0 \leq N_f \leq 3$ . Similar expressions exist for  $N_f = 4$  and  $\mathcal{N} = 2^*$  [85].

Let us consider the case that  $\mathcal{W}$  or  $\widetilde{\mathcal{W}}$  is singular. The curve  $\widetilde{\mathcal{W}}$  is only singular at the cusps  $\tau \in \{i\infty\} \cup \mathbb{Q}$ , since  $\tilde{\Delta} \sim \eta^{24}$  vanishes at the cusps and is non-vanishing for  $\tau$  in the interior of  $\mathbb{H}$ . From (2.47) we see that, at the cusps of  $\widetilde{\mathcal{W}}$  either  $da/du$  or  $\Delta_{N_f}$  must vanish. On the other hand, for  $\tau$  in the interior of  $\mathbb{H}$ ,  $\tilde{\Delta}$  is non-vanishing. This means that, if  $\mathcal{W}$  is singular ( $\Delta_{N_f} = 0$ ) for such values of  $\tau$ ,  $da/du$  should diverge. This is exactly what happens at the AD points,

$$\frac{du}{da}(\tau_{\text{AD}}) = 0, \quad \Delta_{N_f}(u(\tau_{\text{AD}})) = 0, \quad \tau_{\text{AD}} \in \mathbb{H}. \quad (2.49)$$

We can further note that  $\frac{du}{da}(\tau) = 0$  is true also for singularities that are cusps and not elliptic points, i.e.,  $\Delta_{N_f} = 0$  for  $\tau \in \mathbb{Q}$ . This is because if  $u$  is not an elliptic point then  $g_2 \neq 0$  and  $g_3 \neq 0$ , since otherwise, from  $\Delta = g_2^3 - 27g_3^2$ , both would be zero, giving an elliptic point. Then, from (2.46) we have that  $\left(\frac{du}{da}\right)^2$  is proportional to  $\frac{E_4}{E_6}$ . This is a meromorphic modular form of weight  $-2$  for  $\text{SL}(2, \mathbb{Z})$ , and one can show using modular transformations that it vanishes on  $\mathbb{Q}$ . Therefore, we have that  $\Delta_{N_f} = 0$  implies  $\frac{du}{da} = 0$ .

### 2.3.2 Matone's relation

We will now give a generalisation of (2.39) that holds also for the massive  $N_f = 1, 2, 3$  theories. Let us denote by  $'$  the derivative with respect to  $u$  keeping  $\mathbf{m}$  and  $\Lambda_{N_f}$  fixed. The derivative with respect to  $\tau$  is always given explicitly. From the explicit expression for  $j$  as function of  $\tau$  (A.9), it is easy to check that  $\frac{d}{d\tau} j = -2\pi i \frac{E_6}{E_4} j$ . Using the chain rule and (2.10), we can express this as  $\frac{d}{d\tau} \mathcal{J} = \mathcal{J}' \frac{du}{d\tau}$ . This gives the first important identity,

$$\frac{du}{d\tau} = -2\pi i \frac{E_6}{E_4} \frac{\mathcal{J}}{\mathcal{J}'}, \quad (2.50)$$

which holds for any SW curve. From (2.9) we can compute  $\mathcal{J}'$  in terms of  $g'_2$  and  $g'_3$ . Using relations (2.45) and (2.46), we can substitute  $E_6/E_4$  in terms of  $g_2, g_3$  and  $da/du$ . This gives the exact relation

$$\frac{du}{d\tau} = -72\pi i \frac{g_3}{g_2} \frac{\mathcal{J}}{\mathcal{J}'} \left( \frac{da}{du} \right)^2 = -\frac{8\pi i}{3} \frac{g_2^3 - 27g_3^2}{2g_2g'_3 - 3g'_2g_3} \left( \frac{da}{du} \right)^2. \quad (2.51)$$

An analogous formula for five-dimensional gauge theories was derived from the Picard-Fuchs perspective in [147, Eq. (4.23)]. Both factors on the rhs are only relative invariants, but their product is an absolute invariant of the curve  $\mathcal{W}$ .

The numerator on the rhs is proportional to the physical discriminant. The equation has modular weight 2, since both  $\frac{du}{d\tau}$  and  $\left(\frac{da}{du}\right)^2$  are of weight 2.

For  $0 \leq N_f \leq 3$ , we can compute the corresponding  $g_i$ , and one can rewrite (2.51) as<sup>10</sup>

$$\frac{du}{d\tau} = -\frac{16\pi i}{4 - N_f} \frac{\Delta_{N_f}}{P_{N_f}^M} \left(\frac{da}{du}\right)^2, \quad (2.52)$$

where we substituted (2.8) for  $\Delta_{N_f}$ , and defined the polynomial  $P_{N_f}^M$ ,

$$P_{N_f}^M = \frac{6}{4 - N_f} (-1)^{N_f} \Lambda_{N_f}^{2N_f - 8} (2g_2g_3' - 3g_2'g_3). \quad (2.53)$$

The normalisation is chosen such that  $P_{N_f}^M$  is a monic polynomial. Explicit computation gives,

$$\begin{aligned} P_0^M &= 1, \\ P_1^M &= u - \frac{4}{3}m_1^2, \\ P_2^M &= u^2 - \frac{3}{2}(m_1^2 + m_2^2)u + 2m_1^2m_2^2 + \frac{1}{8}m_1m_2\Lambda_2^2 - \frac{1}{64}\Lambda_2^4, \\ P_3^M &= u^3 - 2M_2u^2 + \left(3M_4' + \frac{3}{4}M_3\Lambda_3 - \frac{1}{64}M_2\Lambda_3^2\right)u + \frac{1}{256}M_3\Lambda_3^3 \\ &\quad - \frac{1}{4}M_2M_3\Lambda_3 + \frac{1}{32}(M_4 - M_4')\Lambda_3^2 - 4M_3^2, \end{aligned} \quad (2.54)$$

where we defined

$$\begin{aligned} M_2 &= m_1^2 + m_2^2 + m_3^2, & M_3 &= m_1m_2m_3, \\ M_4 &= m_1^4 + m_2^4 + m_3^4, & M_4' &= \sum_{i < j} m_i^2m_j^2. \end{aligned} \quad (2.55)$$

We note that these polynomials appear in the Picard-Fuchs equations for the periods of these theories and their zeros give regular singular points of the differential equations [141, 143].<sup>11</sup>

### 2.3.3 Branch points

An important difference between  $N_f = 0$  and  $N_f > 0$  are the poles where  $P_{N_f}^M$  vanishes. To understand these poles as well as zeros of  $du/d\tau$ , note that at such points the change of variables between  $u$  and  $\tau$  is ill-defined. We have seen earlier that the change of variables is ill-defined at the points where the discriminant  $D(P_{N_f})$  (2.17) vanishes. Indeed if we substitute  $\mathcal{J}(u, \mathbf{m}, \Lambda_{N_f})$  for  $j(\tau)$  in  $D_{N_f}^{\text{bp}}$ ,  $P_{N_f}^M$  factors out.

The reason for this is the following. The discriminant of a polynomial  $p$  vanishes if and only  $p$  has a double root. It can be computed as the *resultant* of

<sup>10</sup>We can in fact perform the same computations in the case of  $N_f = 4$ , leading to a similar formula.

<sup>11</sup>The identity (2.52) does in fact not depend on the specific form of the SW curves. Given a Jacobian rational elliptic surface, let  $\omega = \int_\gamma \frac{dx}{y}$  be the period of the Néron differential on the elliptic curve. Then  $\frac{du}{d\tau} = \frac{1}{3\pi i} \omega^2 \Delta / (2g_2g_3' - 3g_2'g_3)$ , with  $u$  a coordinate on  $\mathbb{P}^1(\mathbb{C})$ .

the polynomial and its formal derivative,  $D(p) \sim \text{Res}_X(p, p')$  (see also [106]).<sup>12</sup> The zero locus  $D(P_{N_f}) = 0$  of  $P_{N_f}(X)$  is then given by the solutions to the two equations  $P_{N_f}(X) = 0$  and  $P_{N_f}'(X) = 0$ . Since  $\Delta_{N_f} \neq 0$ , all solutions can be found by solving the former for  $j$  and inserting into the latter. It is straightforward to show that this gives

$$\frac{g_2^2 g_3}{\Delta_{N_f}} P_{N_f}^M = 0, \quad (2.56)$$

which provides the decomposition (2.17): If  $g_2 = 0$  but  $g_3 \neq 0$ , then  $j = 0$ . If  $g_3 = 0$  but  $g_2 \neq 0$ , then  $j = 12^3$ . If both  $g_2 = g_3 = 0$ , we are in  $\mathcal{L}_{N_f}^{\text{AD}} \subseteq \mathcal{L}_{N_f}^\Delta$ . Now since the sextic equation is only well-defined away from the physical discriminant locus  $\mathcal{L}_{N_f}^\Delta$  where  $\Delta_{N_f} = 0$ , the true branch point locus  $\mathcal{L}^{\text{bp}}$  is the difference of the Matone locus  $\mathcal{L}_{N_f}^M = \{u \in \mathcal{B}_{N_f} | P_{N_f}^M = 0\}$  and the discriminant locus,

$$\mathcal{L}_{N_f}^{\text{bp}} = \mathcal{L}_{N_f}^M \setminus \mathcal{L}_{N_f}^\Delta. \quad (2.57)$$

On the Coulomb branch with  $N_f$  hypermultiplets there are generically  $2 + N_f$  distinct singular points. For special mass configurations  $\mathbf{m}$ , some singularities can collide. Then  $\Delta_{N_f}$  has a double root. From above it is clear that this is equivalent to  $D(\Delta_{N_f}) = 0$ , which in turn is equivalent to  $\Delta_{N_f} = 0$  and  $\Delta'_{N_f} = 0$ . We can again solve the former for  $g_2$  and  $g_3$  and insert into the latter to obtain  $P_{N_f}^M \sim \frac{g_2}{g_3} \Delta_{N_f} = 0$ . This implies that whenever  $\Delta_{N_f}$  has a double root, it is also a root of  $P_{N_f}^M$ . It is also observed in all examples below. To be more precise, if  $\Delta_{N_f}$  contains a root of  $d > 1$ -th order, then  $\Delta'_{N_f}$  has the same root but with multiplicity  $d - 1$ . The excess factors can be extracted by the operation  $\text{gcd}(\Delta_{N_f}, \Delta'_{N_f})$ , where  $\text{gcd}$  is the polynomial greatest common divisor. The multiple roots are removed from the discriminant by the square-free factorisation<sup>13</sup>

$$\widehat{\Delta}_{N_f} = \frac{\Delta_{N_f}}{\text{gcd}(\Delta_{N_f}, \Delta'_{N_f})}. \quad (2.58)$$

This *reduced* discriminant  $\widehat{\Delta}_{N_f}$  has single roots only, concretely we map  $\prod_s (u - u_s)^{n_s}$  to  $\prod_s (u - u_s)$ . This quantity is also of importance for determining gravitational couplings to Seiberg-Witten theory [148]. One can show that  $\text{gcd}(\Delta_{N_f}, \Delta'_{N_f})$  always divides  $P_{N_f}^M$ , such that

$$\widehat{P}_{N_f}^M := \frac{\widehat{\Delta}_{N_f}}{\Delta_{N_f}} P_{N_f}^M \quad (2.59)$$

<sup>12</sup>The resultant of two polynomials over a commutative ring is a polynomial of their coefficients which vanishes if and only if the polynomials have a common root. It can be computed as the determinant of their Sylvester matrix.

<sup>13</sup>The polynomial  $\text{gcd}$  is unique only up to multiplication with invertible constants, we choose it such that  $\widehat{\Delta}_{N_f}$  is again monic.

is in fact a polynomial. The branch point equation (2.56) is then equivalent to  $\widehat{P}_{N_f}^M / \widehat{\Delta}_{N_f} = 0$ , which reduces to

$$\widehat{P}_{N_f}^M = 0. \quad (2.60)$$

The Matone relation thus always takes the form

$$\frac{du}{d\tau} = -\frac{16\pi i}{4 - N_f} \frac{\widehat{\Delta}_{N_f}}{\widehat{P}_{N_f}^M} \left( \frac{da}{du} \right)^2, \quad (2.61)$$

where both  $\widehat{\Delta}_{N_f}$  and  $\widehat{P}_{N_f}^M$  are polynomials. In the subsequent sections we show explicitly that the roots of the denominator (2.60) are precisely the branch points. We note that for generic masses the form (2.61) does not differ from (2.52), because  $\widehat{\phantom{x}}$  is trivial when all roots are distinct.

As argued above, AD points correspond to points  $\tau_{\text{AD}}$  in the upper half-plane. Since they lie on the discriminant locus, we exclude them to define the sextic polynomial  $P_{N_f}$ . We will discuss in more detail below that, if the masses approach the AD locus, a branch point in the  $u$ -plane collides with two mutually non-local singularities forming the AD point. The branch point under consideration lifts, while the  $N_f - 1$  other branch points remain for a generic point on the AD mass locus  $\mathcal{L}_{N_f}^{\text{AD}}$ . Thus for a generic point on the AD mass locus, AD points are not branch points of  $u(\tau)$ . A non-generic example is the most symmetric AD theory, the  $IV$  fibre in  $N_f = 3$ , discussed in more detail in Section 2.6.4. For this theory,  $\tau_{\text{AD}}$  corresponds to a singular point of the theory as well as a branch point. As a result, the domain for  $\tau$  does not correspond to that of a congruence subgroup of  $\text{SL}(2, \mathbb{Z})$ .

Since any branch point  $\tau_{\text{bp}}$  induces a non-trivial monodromy,  $u$  does not have a regular Taylor series at such a point. For instance, if the  $u$ -plane contains one branch point  $u_{\text{bp}} = u(\tau_{\text{bp}})$ , then we have  $u(\tau) - u_{\text{bp}} = \mathcal{O}(\sqrt{\tau - \tau_{\text{bp}}})$  as  $\tau \rightarrow \tau_{\text{bp}}$ . If the leading coefficient is nonzero, then  $\frac{du}{d\tau}$  diverges at  $\tau_{\text{bp}}$ . Away from the discriminant locus, this can be understood from (2.61): From (2.46) we see that  $\frac{da}{du}$  is regular and nonzero at a branch point, since none of  $g_2$ ,  $g_3$ ,  $E_4$  and  $E_6$  diverge or vanish. Thus the zeros of the denominator  $\widehat{P}_{N_f}^M$  correspond to the singular points of  $\frac{du}{d\tau}$ , as observed.

This can also be seen directly from the  $\mathcal{J}$ -invariant of the SW curve. It is easy to show that

$$\mathcal{J}' = 36^3 \frac{g_2^2 g_3}{\Delta_{N_f}^2} P_{N_f}^M, \quad (2.62)$$

which due to (2.56) vanishes at any branch point  $u_{\text{bp}}$ . Since for fixed mass and scale  $\mathcal{J}(u)$  is rational in  $u$ , it is a meromorphic function on  $\mathcal{B}_{N_f}$ . Away from the discriminant locus it thus has a Taylor series around  $u_{\text{bp}}$ , where the linear coefficient is missing. We therefore find

$$\mathcal{J}(u) - \mathcal{J}(u_{\text{bp}}) = \mathcal{O}((u - u_{\text{bp}})^{n_{\text{bp}}}), \quad (2.63)$$

with  $n_{\text{bp}} \geq 2$ . Now we identify  $\mathcal{J}(u) = j(\tau)$ , which relates the power series of  $u$  and  $\tau$ . For a generic  $\tau \in \mathbb{H}$ ,  $j$  has a regular Taylor series at  $\tau$  with non-zero linear coefficient. However if  $\tau$  is in the  $\text{SL}(2, \mathbb{Z})$ -orbit of  $i$  or  $e^{\frac{\pi i}{3}}$ ,  $j$  has a zero of order 2 or 3. Let  $n_{\tau_{\text{bp}}} \in \{1, 2, 3\}$  be this number for a given branch point  $\tau_{\text{bp}} \in \mathbb{H}$ . Then  $\mathcal{J}(u) - \mathcal{J}(u_{\text{bp}}) = \mathcal{O}((\tau - \tau_{\text{bp}})^{n_{\tau_{\text{bp}}}})$ , such that from (2.63) we conclude

$$u(\tau) - u_{\text{bp}} = \mathcal{O}\left((\tau - \tau_{\text{bp}})^{n_{\tau_{\text{bp}}}/n_{\text{bp}}}\right), \quad (2.64)$$

where the leading coefficient is strictly non-zero. From this we see that the branch point  $\tau_{\text{bp}}$  does not necessarily correspond to an  $n_{\text{bp}}$ -th root, but since the ratio can cancel  $\tau_{\text{bp}}$  rather corresponds to a branch point of order

$$\frac{n_{\text{bp}}}{\text{gcd}(n_{\text{bp}}, n_{\tau_{\text{bp}}})}. \quad (2.65)$$

It is difficult to compute this integer for a generic branch point, however in all examples discussed below it is equal to 2, which corresponds to a *square* root.

If the number  $\frac{n_{\tau_{\text{bp}}}}{n_{\text{bp}}} \in \mathbb{Q} \setminus \mathbb{Z}$  is larger than 1, then it is clear that  $\frac{du}{d\tau}(\tau_{\text{bp}}) = 0$ . Conversely, if  $\frac{n_{\tau_{\text{bp}}}}{n_{\text{bp}}} < 1$  then  $\frac{du}{d\tau}(\tau_{\text{bp}}) = \infty$ . We thus see that any branch point has the property that  $\frac{du}{d\tau}$  diverges or vanishes, such that the change of variables from the  $u$ -plane to the  $\tau$ -plane is not well-defined.

The branch point locus also allows to find the effective coupling at the AD points. In the limit where the masses approach the AD locus,  $\mathbf{m} \rightarrow \mathbf{m}_{\text{AD}}$ , the AD point  $u_{\text{AD}}$  is the point where branch points  $u_{\text{bp}}$  in the  $u$ -plane merges with mutually non-local singularities. While away from  $\mathbf{m}_{\text{AD}}$  the effective coupling  $\tau$  of the singularities remain as distinct cusps on the real line, the branch points move along certain paths inside  $\mathbb{H}$ . In an AD limit  $\mathbf{m} \rightarrow \mathbf{m}_{\text{AD}}$ , a number of pairs of branch points,  $\tau_{\text{bp}}$  and  $\tau'_{\text{bp}}$ , coincide at the intersection of copies of  $\mathcal{F}$ , and the branch cut will then disconnect regions from  $\mathcal{F}_{N_f}$ . The effective coupling of the AD point  $\tau_{\text{AD}}$  is therefore given by that of the merged branch points. This is an efficient way to determine  $\tau_{\text{AD}}$ , which otherwise can only be found by inverting modular functions. Moreover, if the duality group is a congruence subgroup of  $\text{SL}(2, \mathbb{Z})$ ,  $\tau_{\text{AD}}$  corresponds to an elliptic point of the duality group.

## 2.4 The $N_f = 1$ curve

To make the above discussions more concrete we will now go on to study some specific examples. We will start by including one hypermultiplet. The  $N_f = 1$  theory has been discussed in some detail in [103, 149–152].

In the massive  $N_f = 1$  theory, there are three (in general) distinct strong coupling singularities where a hypermultiplet becomes massless. These remain at distinct points in the massless limit, while for special values of the mass two of them can merge into AD points. To analyse the  $N_f = 1$  theory we will start by restricting to the massless case and then go to an AD mass. Here we can find closed expressions for  $u$  in terms of well-known modular forms. Only in

the AD case does the theory turn out to be modular. In the end we can use the knowledge gained from these cases to draw some conclusions of the general massive case.

### 2.4.1 The massless theory

Let us begin with the massless  $N_f = 1$  theory. Using the procedure outlined in Sec. 2.2.2 we find [101]

$$\begin{aligned} \frac{u(\tau)}{\Lambda_1^2} &= -\frac{3}{2^{\frac{7}{3}}} \frac{\sqrt{E_4(\tau)}}{\sqrt[3]{E_4(\tau)^{\frac{3}{2}} - E_6(\tau)}} \\ &= -\frac{1}{16} (q^{-1/3} + 104q^{2/3} - 7396q^{5/3} + \mathcal{O}(q^{8/3})), \end{aligned} \quad (2.66)$$

where we again have made the choice of solution consistent with our convention, such that  $u \rightarrow -\infty$  for  $\tau \rightarrow i\infty$ . This function also appears as an order parameter in pure SU(3) SW theory (see section 4) as well as in the description of certain elliptically fibred Calabi-Yau spaces [153]. The singularities of the curve are  $\frac{u^3}{\Lambda_1^6} = -\frac{3^3}{2^8}$ . They are associated with states of charges (1, 0), (1, 1) and (1, 2) becoming massless. The global  $\mathbb{Z}_3$  symmetry acts as  $T^{-1} : u(\tau - 1) = \omega_3 u(\tau)$ , with  $\omega_j = e^{\frac{2\pi i}{j}}$ .

By restricting to the imaginary axis, we can perform the  $S$ -transformation. For this, let  $\tau = i\beta$  with  $\beta > 0$ . We have that  $E_4(i/\beta) = (i\beta)^4 E_4(i\beta) = \beta^4 E_4(i\beta)$ . Taking the square root is unambiguous since  $E_4$  is real on the imaginary axis and  $\beta^4$  is positive. This gives  $E_4^{\frac{1}{2}}(i/\beta) = \beta^2 E_4^{\frac{1}{2}}(i\beta)$ . On the other hand for  $E_6$  we have  $E_6(\tau) = (i\beta)^6 E_6(i\beta) = -\beta^6 E_6(i\beta)$ . This implies that the relative sign of  $E_6$  flips, and it holds for  $\tau_D \in i\mathbb{R}_{>0}$  that

$$\begin{aligned} \frac{u_D(\tau_D)}{\Lambda_1^2} &= -\frac{3}{2^{\frac{7}{3}}} \frac{E_4(\tau_D)^{\frac{1}{2}}}{\left(E_4(\tau_D)^{\frac{3}{2}} + E_6(\tau_D)\right)^{\frac{1}{3}}} \\ &= \frac{3}{2^{\frac{8}{3}}} (1 + 144q - 3456q^2 + 596160q^3 + \mathcal{O}(q^4)). \end{aligned} \quad (2.67)$$

With the  $\mathbb{Z}_3$  symmetry  $u(\tau - 1) = \omega_3 u(\tau)$  this confirms the strong coupling singularities given above.

The monodromies on the massless  $N_f = 1$   $u$ -plane are [46]

$$\begin{aligned} M_1 &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = STS^{-1}, \\ M_2 &= \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = (TS)T(TS)^{-1}, \\ M_3 &= \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix} = (T^2S)T(T^2S)^{-1}, \\ M_\infty &= \begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix} = PT^{-3}, \end{aligned} \quad (2.68)$$

where  $P = S^2 = -\mathbb{1}$ . Note that these matrices generate the full  $\text{SL}(2, \mathbb{Z})$  modular group rather than a (congruence) subgroup. Indeed, as fractional linear transformations acting on the complex structure through their matrix

representations, they do not leave  $u$  invariant. However, we can consider these matrices as compositions of paths in the fundamental domain, and as such they do leave  $u$  invariant. To make the connection to the discussion in [1] more direct we can note that by using another choice of homology basis in the present case we can construct a different set of monodromies, see for example [117], which exactly coincides with the ones listed for the  $SU(3)$  case of [1].

Since  $E_4$  has a simple zero at  $\tau = \omega_3$  (and  $SL(2, \mathbb{Z})$ -images),  $u(\tau)$  has a branch point at  $\tau_{\text{bp}} = \omega_3$ . The function  $u(\tau)$  does not possess a Taylor series at  $\tau_{\text{bp}}$  and is therefore not holomorphic at  $\tau_{\text{bp}}$ . Since  $u(\tau_{\text{bp}}) = 0$ , the branch point of  $u(\tau)$  indeed agrees with what is found in (2.24). Since  $u$  is not holomorphic on  $\mathbb{H}$ , it can also not be classically modular. Another reason why  $u$  is not modular is the following. If we define  $x := -16 \frac{u}{\Lambda_1^2} = q^{-\frac{1}{3}} + \mathcal{O}(q^{\frac{2}{3}})$ , then one can read off from the curve that  $\mathcal{J} = x^6/(x^3 - 432)$ . This implies that  $u$  should be a Hauptmodul of an index 6 subgroup of  $SL(2, \mathbb{Z})$  with width  $h(\infty) = 3$  and width decomposition  $6 = 3 + 1 + 1 + 1$  (see (A.37)). From the classification of index 6 groups in Table 4 we see that such a subgroup of  $SL(2, \mathbb{Z})$  does not exist. In fact, no index 6 subgroup of  $SL(2, \mathbb{Z})$  with 4 cusps exists. This distinguishes massless  $N_f = 1$  from massless  $N_f = 0, 2, 3$ , where the duality groups are congruence subgroups isomorphic to  $\Gamma^0(4)$  [101].

From (2.66) one finds

$$\frac{du}{d\tau} = \frac{\pi i \Lambda_1^2}{2^{\frac{7}{3}}} \frac{E_4^{\frac{3}{2}} + E_6}{E_4^{\frac{1}{2}} (E_4^{\frac{3}{2}} - E_6)^{\frac{1}{3}}}, \quad \frac{da}{du} = \frac{i (E_4^{\frac{3}{2}} - E_6)^{\frac{1}{6}}}{2^{\frac{1}{3}} \sqrt{3} \Lambda_1}. \quad (2.69)$$

We can explicitly check that these satisfy Matone's relation, (2.52), for massless  $N_f = 1$ ,

$$\frac{du}{d\tau} = -\frac{16\pi i}{3} \frac{\Delta}{u} \left( \frac{da}{du} \right)^2. \quad (2.70)$$

The fundamental region

$$\mathcal{F}_1(0) = \bigcup_{\ell=0}^2 T^\ell \mathcal{F} \cup T^\ell S\mathcal{F} \quad (2.71)$$

as in (2.14) was obtained in [1]. It is shown in Fig. 6, together with its image under  $u$  to the  $u$ -plane. We stress that (2.71) can not be written as  $G \backslash \mathbb{H}$  for any subgroup  $G \subseteq SL(2, \mathbb{Z})$ .

In the massless  $N_f = 1$  theory, the partitioning (2.32) is contained in the algebraic plane curve  $T_{(0)}(x, y) = 0$ , where  $\frac{u}{\Lambda_1^2} = x + iy$  and

$$T_{(0)}(x, y) = y(3x^2 - y^2)(27x^3 + 128x^6 - 81xy^2 + 384x^4y^2 + 384x^2y^4 + 128y^6). \quad (2.72)$$

The first two factors of  $T_{(0)}(x, y)$  contains also values which correspond to  $j > 12^3$  and they need to be sufficiently truncated. The identification of the algebraic curve with the partitioning of  $\mathbb{H}$  is immediate from Fig. 6.

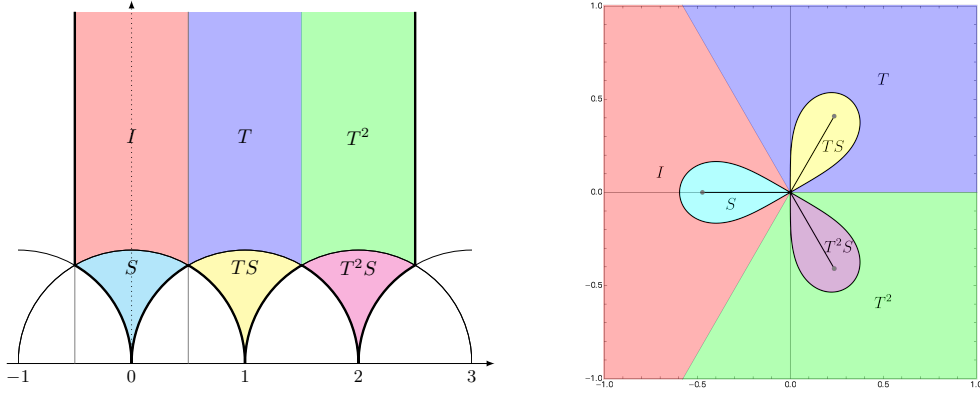


Figure 6: Left: Proposal of a fundamental region for massless  $N_f = 1$ . It is clear that it can not be a fundamental domain of any  $\text{SL}(2, \mathbb{Z})$  subgroup if we identify the sides,  $\tau \sim \tau + 3$ . This is because the lower boundary is not given by a union of half-disks. If we do *not* identify the sides then the picture is in fact a fundamental domain for  $\Gamma^0(4)$ . This is because there is an element in  $\Gamma^0(4)$  which maps  $i\infty \mapsto 1$ . Right: Plot of the massless  $N_f = 1$   $u$ -plane as the union of the images of  $u$  under the six  $\text{SL}(2, \mathbb{Z})$  images of  $\mathcal{F}$  as in (2.71), in units of  $\Lambda_1^2 = 1$ . In particular, the function  $u$  is surjective onto  $\mathbb{C}$  on this domain. The singular points sit neatly in the interior of the strong coupling regions  $u(S^\ell \mathcal{F})$ ,  $\ell = 0, 1, 2$ . The origin  $u = 0$  is  $\tau = \omega_3 + \mathbb{Z}$  as discussed above, and bounds all six domains as is clear from the left figure.

#### 2.4.2 Type II AD mass

In section 2.2.3 we saw that the  $N_f = 1$  theory has AD fixed points in its moduli space. To study these, we fix the mass to be one of the AD values,  $m \rightarrow m_{\text{AD}} = \frac{3}{4}\Lambda_1$ , specified by the zero locus of the AD polynomials (2.20). Two mutually non-local singularities now collide at the AD point  $u = u_{\text{AD}} = \frac{3}{4}\Lambda_1^2$  while the third one simplifies to  $u_0 = -\frac{15}{16}\Lambda_1^2$ , such that the discriminant reads

$$\Delta = (u - u_{\text{AD}})^2(u - u_0). \quad (2.73)$$

From the curve we now find

$$\frac{u(\tau)}{\Lambda_1^2} = -\frac{f_{3\text{B}}(\tau) + 15}{16}, \quad (2.74)$$

where  $f_{3\text{B}}$  is the McKay-Thompson series of class 3B for the Monster group [36] (see Appendix A.4 for an overview of McKay-Thompson series),

$$\begin{aligned} f_{3\text{B}}(\tau) &= \left( \frac{\eta(\frac{\tau}{3})}{\eta(\tau)} \right)^{12} = 27 \left( \frac{b_{3,0}(\frac{\tau}{3})}{b_{3,1}(\frac{\tau}{3})} \right)^3 - 27 \\ &= q^{-\frac{1}{3}} - 12 + 54q^{\frac{1}{3}} - 76q^{\frac{2}{3}} - 243q + 1188q^{\frac{4}{3}} - 1384q^{\frac{5}{3}} + \mathcal{O}(q^2). \end{aligned} \quad (2.75)$$

The Dedekind  $\eta$ -function is defined in (A.18), while the functions  $b_{3,j}$  are defined in (A.14). We are again careful to choose the solution for  $u$  with the consistent asymptotics. Substitution of the  $q$  series (2.75) in (2.74) reproduces the  $q$ -series based on [103, Eq. (4.93)].

Using Theorem 1 in Appendix A.1, we find that  $\tau \mapsto f_{3\text{B}}(3\tau)$  is a classical modular function for  $\Gamma_0(3)$  and therefore  $u$  is a modular function for  $\Gamma^0(3)$ . A



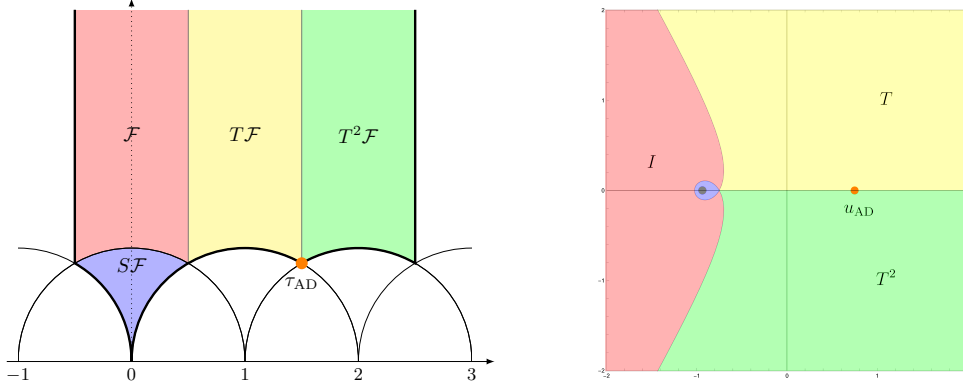


Figure 7: Left: Fundamental domain of  $\Gamma^0(3)$ , the duality group of  $N_f = 1$  with  $m = m_{\text{AD}}$ . The AD point  $\tau_{\text{AD}} = \sqrt{3}\omega_{12}$  is highlighted. Right: Plot of the  $N_f = 1$   $u$ -plane with AD mass as the union of the images of  $u$  under the  $6 - 2 = 4$   $\text{SL}(2, \mathbb{Z})$  images of  $\mathcal{F}$  forming  $\Gamma^0(3) \backslash \mathbb{H}$ , as in the left figure. The complex plane can clearly be covered by 4 triangles. There is only one strong coupling region, which is the circular region. It contains  $u_0$  in its interior. The AD point (orange) lies on the boundary of  $T^{\pm 1}\mathcal{F}$ , as is clear from the left figure. The areas with the same colours are mapped to each other in the two figures.

fundamental domain of  $\Gamma^0(3)$  is

$$\mathcal{F}_1(m_{\text{AD}}) = \bigcup_{\ell=0}^2 T^\ell \mathcal{F} \cup S\mathcal{F}. \quad (2.76)$$

This is shown in Fig. 7 together with the map to the  $u$ -plane. The cusps are  $i\infty$  and 0, with widths 3 and 1, respectively. We take from [154, Table 4.1] that  $\Gamma^0(3)$  has an elliptic fixed point of order 3.

Using the transformation properties of the  $\eta$ -function (A.19) it is straightforward to show that the locations of the singularities of (2.73) in the  $\tau$ -plane are given by ( $\omega_j = e^{2\pi i/j}$ )

$$u(i\infty) = \infty, \quad u(0) = u_0, \quad u(\sqrt{3}\omega_{12}) = u_{\text{AD}}, \quad (2.77)$$

where the proper limits are understood. The AD point  $\sqrt{3}\omega_{12}$  is stabilised by the order 3 element  $\begin{pmatrix} -1 & 3 \\ -1 & 2 \end{pmatrix} \in \Gamma^0(3)$ , and it is therefore the order 3 elliptic fixed point of  $\Gamma^0(3)$ . Comparing the locations to the massless case we see that the regular singularity  $u_0$  has stayed on  $\tau = 0$ , while, contrary to the massless case, the cusps with the two mutually non-local singularities are disconnected (or cut) from the domain for massless  $N_f = 1$ , and leaves as remnant the point  $\tau_{\text{AD}}$  into the interior of  $\mathbb{H}$ . This procedure also reduces the index of the solution: Indeed, from (A.32) we compute that  $\text{ind } \Gamma^0(3) = 4$ , where the  $6 - 4 = 2$  AD points do not contribute since they are not cusps. This can also be seen from the fact that

$$j = \frac{(f_{3\text{B}} + 3)^3 (f_{3\text{B}} + 27)}{f_{3\text{B}}}. \quad (2.78)$$

Indeed, since  $\mathcal{J} = 1728 \frac{g_2^3}{\Delta}$ , a common factor  $(f_{3\text{B}} + 27)^2$  of  $g_2^3$  and  $\Delta$  has cancelled. The last factor  $f_{3\text{B}} + 3 = 0$  in (2.78) implies  $j(\tau) = 0$  and therefore

$\tau = \omega_3 \pmod{\text{SL}(2, \mathbb{Z})}$ . In fact, it corresponds to  $u(\omega_3) = u(\omega_3 + 1) = -\frac{3}{4}\Lambda_1^2$  and it is just a regular point in the  $u$ -plane. We can also read off from this that  $\text{ord}(g_2, g_3, \Delta) = (1, 1, 2)$  and therefore the AD theory in  $N_f = 1$  is according to Table 5 a type  $II$  singular fibre [155].

We can also study more characteristic functions of the theory with the AD mass. Using Appendix A.1, we can differentiate (2.74) to find

$$\frac{du}{d\tau}(\tau) = \frac{\pi i \Lambda_1^2}{24} f_{3B}(\tau) b_{3,0}(\frac{\tau}{3})^2. \quad (2.79)$$

This implies that  $\frac{du}{d\tau}$  is a modular form of weight 2 for  $\Gamma^0(3)$ , without phases. One can also show that

$$\frac{da}{du}(\tau) = \frac{\sqrt{2}i}{\sqrt{27}\Lambda_1} \sqrt{\frac{b_{3,1}(\frac{\tau}{3})^3}{b_{3,0}(\frac{\tau}{3})}}. \quad (2.80)$$

An expression for  $da/du$  in terms of  ${}_2F_1$  was given in [156, Eq. (4.13)].

The  $q$ -expansion of  $\frac{da}{du}$  has growing denominators, and therefore  $\frac{da}{du}$  is not a modular form of weight 1 for  $\Gamma^0(3)$ . However, it is straightforward to check that  $(\frac{da}{du})^2$  is a modular form of weight 2 for  $\Gamma^0(3)$ . We thus find the Matone relation

$$\frac{du}{d\tau} = -\frac{16\pi i}{3} \widehat{\Delta} \left( \frac{da}{du} \right)^2, \quad (2.81)$$

where  $\widehat{\Delta}$  denotes the reduced discriminant. This is consistent with (2.52).

The monodromies can be found from the ones of the massless theory (2.68),

$$M_1 = STS^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad M_{\text{AD}} = M_2 M_3 = T^2 (ST)^{-1} T^{-2} = \begin{pmatrix} -1 & 3 \\ -1 & 2 \end{pmatrix}. \quad (2.82)$$

They generate the duality group  $\Gamma^0(3)$  and give the large  $u$  monodromy  $M_1 M_{\text{AD}} = PT^{-3}$ . Furthermore,  $M_1$  stabilises  $\tau = 0$  and  $M_{\text{AD}}$  stabilises the AD point  $\tau_{\text{AD}} = \sqrt{3}\omega_{12}$ . We have that  $M_{\text{AD}}^6 = \mathbb{1}$  and therefore  $\tau_{\text{AD}}$  is indeed an elliptic fixed point. The AD monodromy is conjugate to  $(ST)^{-1}$ , which fixes  $\tau = \omega_3$ . Since  $\tau_{\text{AD}} = \omega_3 + 2$ , this gives a path in  $\tau$ -space.

### 2.4.3 Generic real mass

By turning on a generic real mass, the singularities do not split compared to the massless case since there are already  $N_f + 2 = 3$  singularities. Therefore, the fundamental domain of the massive theory should look similar to the massless one of Fig. 6, but we now need to consider the presence of branch points and cuts in more detail. We will discuss this and the limits to the pure theory as well as the theory with the AD mass now.

For generic mass we have not been able to find a closed expression for  $u$  as a function of  $\tau$ . By expanding  $\mathcal{J}(u, m, \Lambda_1)$  and inverting the series we can,

however, get an expansion of  $u$  for the general massive theory near any cusp. For example, the expansion near  $\tau = i\infty$  reads ( $\mu = \frac{m}{\Lambda_1}$ )

$$\frac{u(\tau)}{\Lambda_1^2} = -\frac{1}{16}q^{-\frac{1}{3}} - \frac{1}{3}\mu^2 + \left(\frac{32}{9}\mu^4 - 6\mu\right)q^{\frac{1}{3}} - \left(\frac{5120}{81}\mu^6 - \frac{160}{3}\mu^3 + \frac{13}{2}\right)q^{\frac{2}{3}} + \mathcal{O}(q), \quad (2.83)$$

where we are careful to choose the expansion such that  $u \rightarrow -\infty$  for consistency with our conventions. It is easy to see that this reproduces the earlier expansions, (2.66) and (2.74), in the respective limits.

The branch point locus is given by the zero locus of (2.24). By calculating  $\mathcal{J}(u, m, \Lambda_1)$  from the curve and plugging it into the polynomial  $D_1^{\text{bp}}$  we find that the zero of the linear polynomial is  $u = u_{\text{bp}} = \frac{4}{3}m^2$ , and we recognise that this is the polynomial appearing in the denominator of the generalised Matone relation, (2.54), such that  $\frac{du}{d\tau}$  diverges here. In the massive  $N_f = 2, 3$  theories, where the theories can be studied in detail, we argue that it corresponds to two branch points in the closure of the fundamental domain, which are connected by a branch cut. Motivated by these analyses, we can draw the two branch point loci for positive mass. It is given in Fig. 8. For  $m = 0$ , the branch point is located at the origin  $u = 0$ . At the AD point, they collide, the branch cut vanishes and the order parameter becomes holomorphic, and even modular. In the  $m \rightarrow \infty$  limit, the branch points also move to infinity.

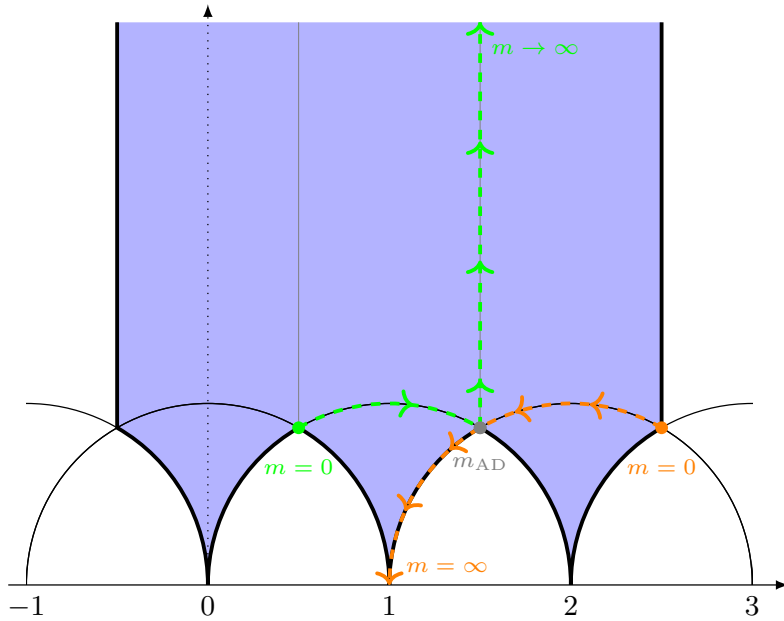


Figure 8: Conjectured paths of the branch points in the fundamental domain of the massive  $N_f = 1$  theory.

We can also confirm this from the analysis in section 2.3.3. By expanding  $\mathcal{J}(u) - \mathcal{J}(u_{\text{bp}})$  around  $u_{\text{bp}}$  for generic mass  $m$ , the linear coefficient is zero. The  $(u - u_{\text{bp}})^2$  coefficient vanishes if and only if either  $m = 0$  or  $m = m_i := \frac{3}{4\sqrt[3]{2}}$ . For  $m = 0$  we have  $\tau_{\text{bp}} \sim \omega_3$ , such that  $n_{\omega_3} = 3$  in the notation of section

**2.3.3.** Furthermore,  $n_{\text{bp}} = 6$ , such that the order of the branch point (2.65) is the denominator of the reduced fraction  $\frac{3}{6}$ , namely 2. Since  $E_4$  has a simple zero at  $\tau_{\text{bp}}$ , this agrees with (2.66) having a *square* root.

From Fig. 8 we see that the branch point loci pass through  $\tau_{\text{bp}} = 1 + i$  where  $m = m_i$ , such that  $n_i = 2$ . Furthermore we find  $n_{\text{bp}} = 4$ , and thus the order of the branch point is 2. Since  $\frac{2}{4} = \frac{1}{2}$ , it is indeed again the branch point of a square root.

For any other mass  $m \in \mathbb{R}_{\geq 0} \setminus \{0, m_i, m_{\text{AD}}\}$  we have  $n_{\text{bp}} = 2$  while  $n_{\tau_{\text{bp}}} = 1$ , such that the branch point is again of order 2. This demonstrates that the loci in Fig. 8 are complete: there is a single branch point on the Coulomb branch  $\mathcal{B}_1$ , and for any mass there are two branch points of a square root in  $\mathbb{H}$ , which are connected by a single branch cut. It also implies that if an expression such as (2.66) existed for generic mass, while it could contain higher roots of modular forms, they can never have zeros in  $\mathbb{H}$  (as is the case also for  $m = 0$ ).

We can study the partition of the  $u$ -plane provided by (2.32) in detail. For  $m \neq m_{\text{AD}}$  the  $u$ -plane is partitioned into six regions, whose union of boundary pieces is included in the algebraic curve given by the zero locus (2.34) of  $T_1$ , where  $T_1 = y\tilde{T}_1$  and

$$\begin{aligned} \tilde{T}_1 = & 972\mu^4 + 8192\mu^2x^7 - 21504\mu^3x^5 - 12096\mu^2x^4 + 18432\mu^4x^3 + 1296\mu^3x^2 \\ & + 16128\mu x^6 + 1944\mu x^3 + 8192\mu^2x^5y^2 + 22528\mu^3x^3y^2 - 8192\mu^2x^3y^4 - 3456\mu^2x^2y^2 \\ & + 2304\mu x^4y^2 - 6912\mu x^2y^4 - 16384x^6y^2 - 12288x^4y^4 + 4320x^3y^2 - 6144x^8 \\ & - 1296x^5 - 5184\mu^5x - 729\mu^2x - 18432\mu^4xy^2 - 21504\mu^3xy^4 - 8192\mu^2xy^6 \\ & - 1944\mu xy^2 - 1296xy^4 + 4752\mu^3y^2 + 8640\mu^2y^4 + 6912\mu y^6 + 2048y^8. \end{aligned} \tag{2.84}$$

Since the AD point  $m = m_{\text{AD}}$  corresponds to a phase transition, we have to study the two cases  $m < m_{\text{AD}}$  and  $m > m_{\text{AD}}$  separately.

### The case $m < m_{\text{AD}}$

From Fig. 8 we can take the location of the branch points. There is one singularity  $u_1$  on the negative real line, and the other two are complex conjugates (as  $\Delta_1$  is a real polynomial). Using the definition (2.32), it is straightforward to show that not all of  $y = 0$  lies in  $\mathcal{T}_1$ , but rather only the real half-line with  $u \geq u_1$ . Furthermore, the lines truncate at the singularities. On the upper-half plane, the branch points can be viewed as endpoints of branch cuts coming from  $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$  and  $\tau = \frac{5}{2} + \frac{\sqrt{3}}{2}i$ . See Fig. 9. From this it is straightforward to see how taking the massless limit gives back Fig. 6.

### The case $m > m_{\text{AD}}$

At  $m = m_{\text{AD}}$  two singularities collide, and  $\Delta_1$  has a double root. Since  $\Delta_1$  is a real polynomial and depends smoothly on  $m$ , the two singularities which are complex for  $m < m_{\text{AD}}$  are real for  $m > m_{\text{AD}}$ . There is no meaningful identification of the singular points when going through  $m = m_{\text{AD}}$ , however for large  $m$  there is a distinguished singularity  $u_*$  that diverges. We can make

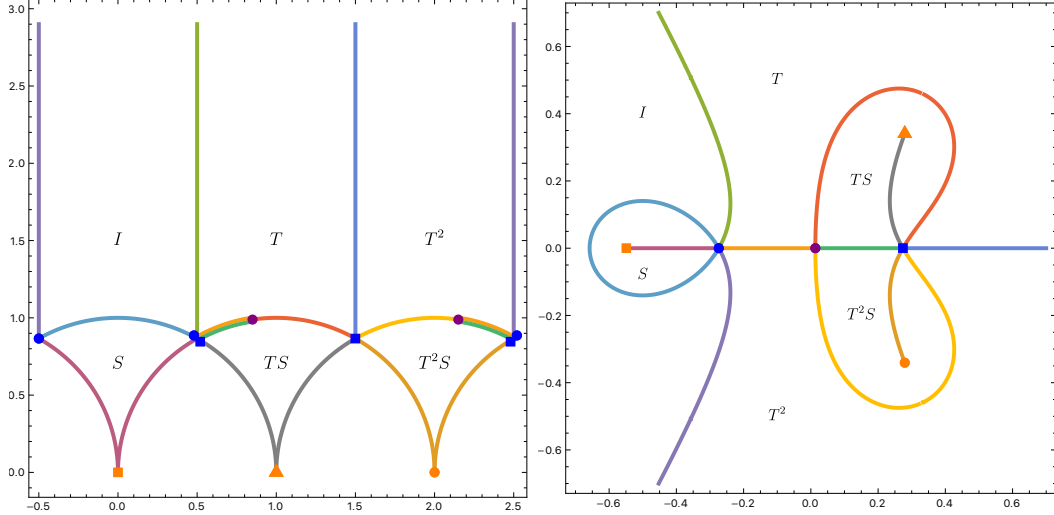


Figure 9: Identification of the components of the partitioning  $\mathcal{T}_m$  in  $N_f = 1$  for  $\mu < \mu_{\text{AD}}$ , here for the choice  $\mu = \frac{1}{10}$ . The  $u$ -plane  $\mathcal{B}_1$  is partitioned into 6 regions  $u(\alpha\mathcal{F})$ , with the  $\alpha \in \text{SL}(2, \mathbb{Z})$  given in both pictures. The branch point (purple) identifies two points on  $\partial\mathcal{F}_1(m)$ . The green/orange arcs are separated by a branch cut, and the opposite ends are glued with the corresponding identical colour on the other side. The blue points correspond to  $j = \mathcal{J} = 0$  and form the intersection points of  $\mathcal{T}_1$ .

the choice of  $\mathcal{F}_m$  suitable for the limit  $m \rightarrow \infty$ , where we should obtain Fig. 2. By studying the dependence of the partition of the  $u$ -plane on the mass, one finds that  $u_*$  is bounded by a region whose area grows as  $m \rightarrow \infty$ . It squeezes into  $T\mathcal{F}$  and  $T^3\mathcal{F}$  and becomes  $T^2\mathcal{F}$  in the limit  $m \rightarrow \infty$ . However, as we want to put the singularities on the real line we need it to touch this axis for finite  $m > m_{\text{AD}}$ . In order to find the corresponding fundamental domain, we can glue parts of the boundary  $\partial\mathcal{F}_1$ , such that it not only agrees with the geometry of the partition of the  $u$ -plane, but also the decoupling procedure is inherent. See Fig. 10.

#### 2.4.4 Generic complex mass

We can also consider a generic complex mass. The locus of AD masses (2.20) is then real codimension 2. In fact, it is just  $\omega_3 m_{\text{AD}}$ , with  $\omega_3$  a cube root of unity. If  $m$  is not any of these three values, the corresponding Coulomb branch has three distinct singularities.

We can decompose  $m = a + ib$ , and  $T_m$  is then a polynomial in  $a$ ,  $b$ ,  $x$  and  $y$ . From (2.24) we see that if  $m$  is complex, then  $j(\tau_{\text{bp}}) = \mathcal{J}(u_{\text{bp}})$  is also complex, such that  $\tau_{\text{bp}}$  is generically an interior point of  $\mathcal{F}$  or an  $\text{SL}(2, \mathbb{Z})$  copy thereof. The branch cuts most conveniently run from such branch points to the intersection points of the curves, where  $\mathcal{J}(u) = j(\tau) = 0$ . From (2.9) it is clear that they correspond to the two solutions of  $g_2(u) = 0$ . We plot the partitioning of the  $u$ -plane with the branch cuts for an imaginary mass in Fig. 11.

Due to the fact that  $u_{\text{bp}} \notin \mathcal{T}_1$ , the branch cuts run to the interior of the

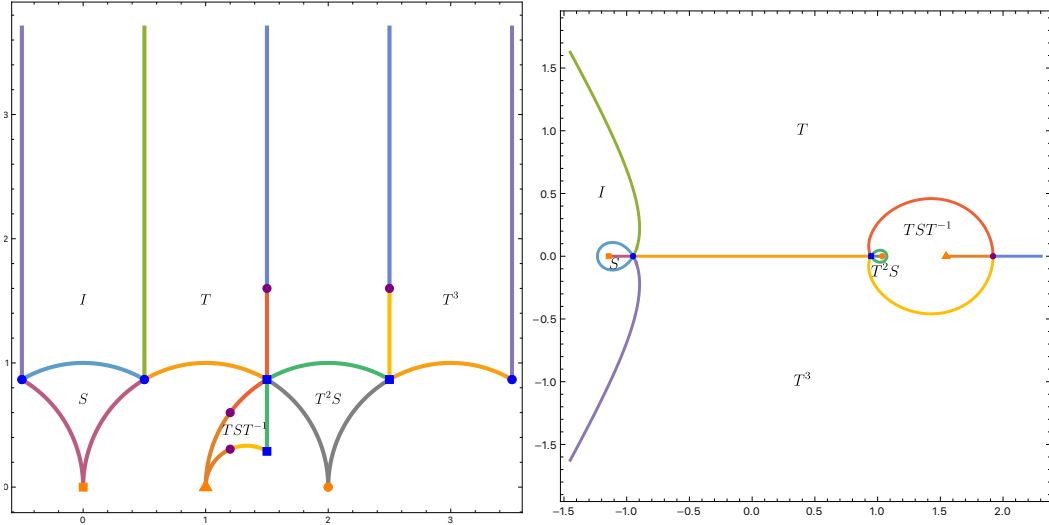


Figure 10: Identification of the components of the partitioning  $\mathcal{T}_m$  in  $N_f = 1$  for  $\mu > \mu_{AD}$ , here for the choice  $\mu = \frac{6}{5}$ . The  $u$ -plane  $\mathcal{B}_1$  is partitioned into 6 regions  $u(\alpha_j\mathcal{F})$ , with the  $\alpha_j \in \text{SL}(2, \mathbb{Z})$  given in both pictures. The branch point (purple) identifies four points on  $\partial\mathcal{F}_1(m)$ .

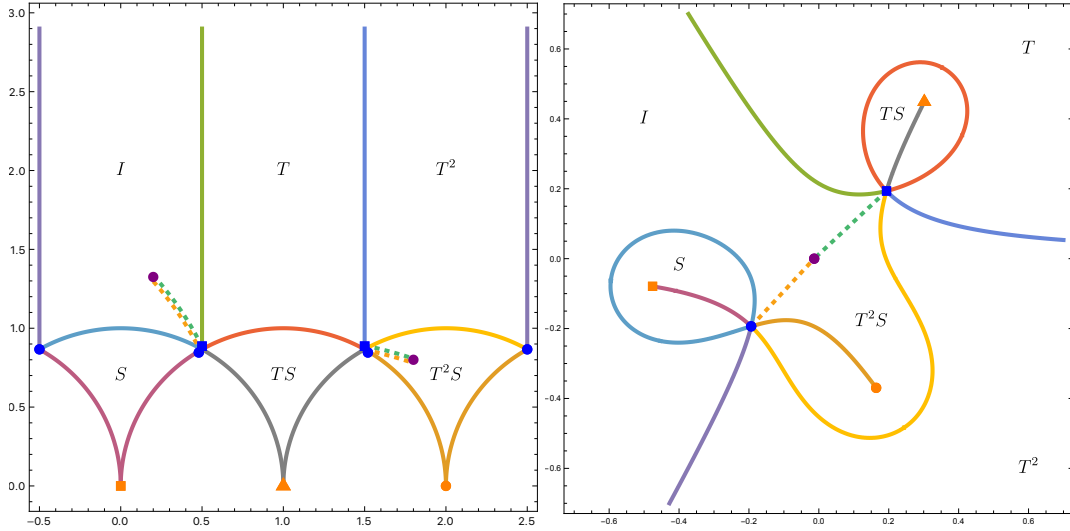


Figure 11: Identification of the components of the partitioning  $\mathcal{T}_m$  in  $N_f = 1$  for a complex mass  $\mu \in \mathbb{C}$ , here for the choice  $\mu = \frac{i}{10}$ . The  $u$ -plane  $\mathcal{B}_1$  is partitioned into 5 regions, which is due to the fact that  $u(\mathcal{F})$  and  $u(T^2S\mathcal{F})$  are glued at the branch cuts (dashed). The purple dot is the branch point in  $N_f = 1$ .

$\alpha_j \mathcal{F}$ . The four sides of the two cuts are pairwise identified, which makes points on the branch cut smooth points on the Riemann surface. This identification glues different regions  $u(\alpha_j \mathcal{F})$  together, in this case  $u(\mathcal{F})$  and  $u(T^2 S \mathcal{F})$ . This is clearly visible in Fig. 11, where the dashed lines in the  $u$ -planes are the images of the branch cuts in  $\mathcal{F}_m$ , and they do *not* belong to the partitioning  $\mathcal{T}_1$ . As a result, the  $u$ -plane is partitioned into five and not six components. This is not in contradiction with section 2.2 because the fundamental domain  $\mathcal{F}_1(m)$  is still a union of six copies of  $\mathcal{F}$ : the cutting and glueing along the branch cuts is an additional feature of the domain.

## 2.5 The $N_f = 2$ curve

Let us now move on to discuss the theory with two hypermultiplets. This theory has four strong coupling singularities where massless hypermultiplets appear. For general masses they are distinct points while for special mass configurations one or more singularities can collide.

The equal mass case  $\mathbf{m} = (m, m)$  has been studied in [157]. Its modular properties, fundamental domains, and mass limits are studied in great detail in [2]. It is also conceptually and computationally equivalent to the  $N_f = 3$  theory with  $\mathbf{m} = (m, 0, 0)$ , which we discuss in section 2.6.1. For these reasons, we omit the discussion here and refer the reader to section 5.1 of [2].

### 2.5.1 Two distinct masses

In the generic case, the two masses are distinct. As in  $N_f = 1$ , we can expand and invert the  $\mathcal{J}$ -invariant for large  $u$  to find the series ( $\mu_i = \frac{m_i}{\Lambda_2}$ )

$$\begin{aligned} \frac{u(\tau)}{\Lambda_2^2} = & -\frac{1}{64}q^{-\frac{1}{2}} - \frac{1}{2}(\mu_1^2 + \mu_2^2) + \left(24(\mu_1^4 + \mu_2^4) + 16\mu_1^2\mu_2^2 - 32\mu_1\mu_2 - \frac{5}{16}\right)q^{\frac{1}{2}} \\ & -128(\mu_1^2 + \mu_2^2)(16(\mu_1^4 + \mu_2^4) - 14\mu_1\mu_2 + 1)q + \mathcal{O}(q^{\frac{3}{2}}). \end{aligned} \quad (2.85)$$

The double singularity  $u_*$  in the equal mass case now splits into two distinct singularities,  $u_*^\pm$ . Due to the locus of masses giving rise to  $u$ -planes with AD points, it is difficult to give a fundamental domain  $\mathcal{F}_2(\mathbf{m})$  for any choice of  $\mathbf{m} = (m_1, m_2)$ . From (2.24) it is clear that there are two distinct branch points in  $\mathcal{B}_2$ . When both  $m_1$  and  $m_2$  are real and small, i.e. have not made a phase transition compared to  $\mathbf{m} = 0$ , one branch point  $u_{\text{bp},1}$  belongs to  $\mathcal{T}_m$ , while the other  $u_{\text{bp},2}$  does not. However,  $\mathcal{J}(u_{\text{bp},2}) = j(\tau_{\text{bp},2}) \in \mathbb{R}$  is also real but larger than  $12^3$ . A natural choice of branch cuts is along the tessellation  $\{\tau \in \mathbb{H} \mid j(\tau) \in \mathbb{R}\}$ , which aside from (2.36) contains the  $\text{SL}(2, \mathbb{Z})$  images of the positive imaginary axis. The plot of the partitioning  $\mathcal{T}_m$  shows a feature found already in  $N_f = 1$  with a complex mass (see section 2.4.4): The  $u$ -plane is partitioned into only 5 regions, which is due to two regions  $u(\alpha_j \mathcal{F})$  being glued along pairs of branch cuts (see Fig. 12). The splitting of  $u_*$  into two distinct singularities in this case does not require the two regions  $T S \mathcal{F}$  and

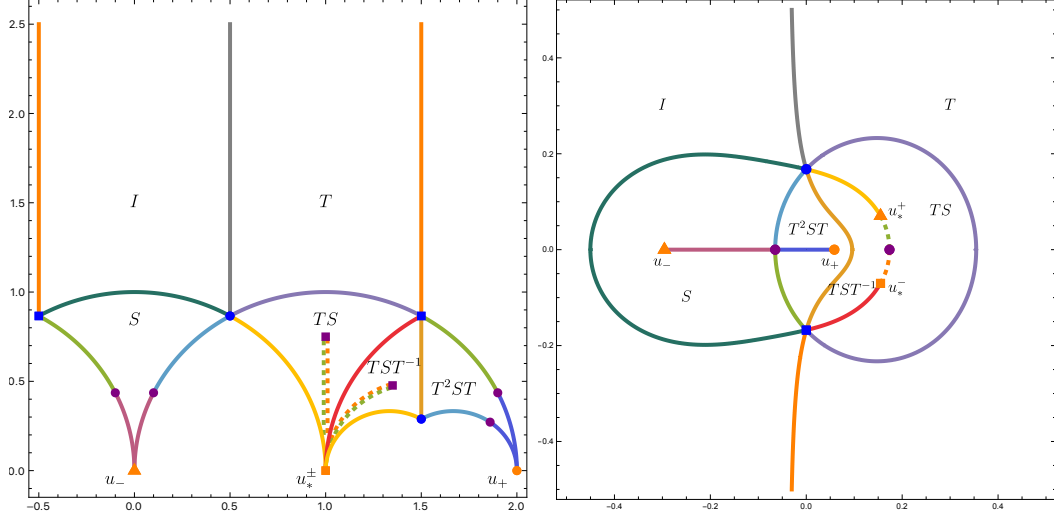


Figure 12: Identification of the components of the partitioning  $\mathcal{T}_{(m_1, m_2)}$  in  $N_f = 2$  for the particular choice  $\mu_1 = \frac{1}{10}$  and  $\mu_2 = \frac{1}{4}$ . The  $u$ -plane  $\mathcal{B}_2$  is naively partitioned into six regions  $u(\alpha\mathcal{F})$ , with the  $\alpha \in \text{SL}(2, \mathbb{Z})$  given in both pictures. Two regions  $u(TS\mathcal{F})$  and  $u(TST^{-1}\mathcal{F})$  are however glued along the pairs of branch cuts (dotted), running from the two singular points  $u_*^{\pm}$  (orange, square) to the branch point  $\tau_{\text{bp}, 2}$  (purple, square). They do not belong to the partitioning  $\mathcal{T}_m$ . A natural choice for the branch cut is along the lines where  $j(\tau)$  is real.

$TST^{-1}\mathcal{F}$  to taper to distinct cusps, as we have that both  $TS, TST^{-1} : i\infty \mapsto 1$ . The two singularities are rather split due to the branch cut, and the limit of  $u(\tau)$  as  $\tau \rightarrow 1$  depends on the path from which  $\tau = 1$  is approached. This is different from  $u_+ \neq u_-$ , where the boundary pieces near the cusps are not identified.

## 2.5.2 The massless theory

When we go to the massless theory we now find

$$\begin{aligned} \frac{u(\tau)}{\Lambda_2^2} &= -\frac{1}{8} \frac{\vartheta_3(\tau)^4 + \vartheta_4(\tau)^4}{\vartheta_2(\tau)^4} = -\frac{1}{8} - \frac{1}{64} \left( \frac{\eta(\frac{\tau}{2})}{\eta(2\tau)} \right)^8 \\ &= -\frac{1}{64} (q^{-1/2} + 20q^{1/2} - 62q^{3/2} + 216q^{5/2} + \mathcal{O}(q^{7/2})). \end{aligned} \quad (2.86)$$

This function is the completely replicable function of class 4C and is a Hauptmodul for  $\Gamma(2)$  [56–58]. The physical discriminant becomes  $\Delta = (u + \frac{\Lambda_2^2}{8})^2 (u - \frac{\Lambda_2^2}{8})^2$ . The two cusps correspond to  $u(0) = -\frac{\Lambda_2^2}{8}$  and  $u(1) = +\frac{\Lambda_2^2}{8}$ . They are associated with the particles of charges  $(1, 0)$  and  $(1, 1)$  becoming massless.

A fundamental domain for  $\Gamma(2)$  is given by

$$\mathcal{F}_2(0, 0) = \mathcal{F} \cup T\mathcal{F} \cup S\mathcal{F} \cup TS\mathcal{F} \cup ST^{-1}\mathcal{F} \cup TST\mathcal{F} \quad (2.87)$$

and is plotted in Fig. 13 together with the map to the  $u$ -plane. This picture gives rise to the *dessin d'enfant* of the  $j$ -invariant [158, Fig. 6], as  $u$  is a linear function of the modular  $\lambda$ -invariant, which has critical points  $\lambda = 0, 1, \infty$ .



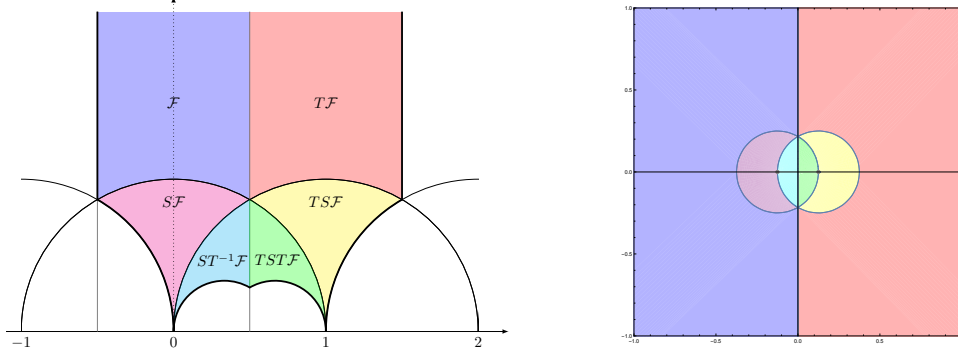


Figure 13: Left: Fundamental domain of  $\Gamma(2)$ . This is the duality group of massless  $N_f = 2$ . All three cusps  $\{i\infty, 0, 1\}$  have width 2. Right: Plot of the massless  $N_f = 2$   $u$ -plane as the union of the images of  $u$  under the  $\text{ind}\Gamma(2) = 6 \text{SL}(2, \mathbb{Z})$  images of  $\mathcal{F}$ . Here, we use the decomposition  $\Gamma(2)\backslash\mathbb{H} = \bigcup_{k,\ell=0}^1 T^\ell S^k \mathcal{F} \cup ST^{-1}\mathcal{F} \cup TSTF$ . There is a  $\mathbb{Z}_2$  symmetry which acts by  $u \mapsto -u$ . The singularities  $\tau = 0, 1$  are both touched by two triangles each.

### 2.5.3 Type III AD mass

If we choose  $m_1 = m_2 = m_{\text{AD}} = \frac{1}{2}\Lambda_2$ , we find a  $u$ -plane with an AD theory of type III located at  $u = u_{\text{AD}} = \frac{3}{8}\Lambda_2^2$  [105]. Three singularities collide in this point, while one remains at  $u_0 = -\frac{5}{8}\Lambda_2^2$ . The discriminant now takes the form

$$\Delta = (u - u_{\text{AD}})^3(u - u_0). \quad (2.88)$$

Using  $\Gamma(2)$  as an intermediate field of the sextic equation, we can show that

$$\frac{u(\tau)}{\Lambda_2^2} = -\frac{f_{2\text{B}}\left(\frac{\tau}{2}\right) + 40}{64} = -\frac{1}{64} \left( q^{-1/2} + 16 + 276q^{1/2} - 2048q + \mathcal{O}(q^{3/2}) \right), \quad (2.89)$$

where  $f_{2\text{B}}$  is defined as

$$f_{2\text{B}}(\tau) = \left( \frac{\eta(\tau)}{\eta(2\tau)} \right)^{24} = 256 \frac{\vartheta_3(\tau)^4 \vartheta_4(\tau)^4}{\vartheta_2(\tau)^8}, \quad (2.90)$$

and it is the McKay-Thompson series of class 2B [56–58]. It is a Hauptmodul for  $\Gamma_0(2)$ . Therefore,  $u$  is a modular function for  $\Gamma^0(2)$ . A fundamental domain of  $\Gamma^0(2)$  is

$$\mathcal{F}_2(\mathbf{m}_{\text{AD}}) = \mathcal{F} \cup T\mathcal{F} \cup S\mathcal{F} \quad (2.91)$$

and is shown in Fig. 14. It has index 3 in  $\text{PSL}(2, \mathbb{Z})$ , since three mutually non-local singularities have collided. This can also be seen from the fact that the curve reads

$$j(\tau) = \frac{(f_{2\text{B}}(\frac{\tau}{2}) + 16)^3}{f_{2\text{B}}(\frac{\tau}{2})}. \quad (2.92)$$

One has that  $u(\tau_{\text{AD}}) = u_{\text{AD}}$  whenever  $f_{2\text{B}}(\frac{\tau_{\text{AD}}}{2}) = -64$ , whose solution locus intersects with our choice of  $\Gamma^0(2)\backslash\mathbb{H}$  in  $\tau_{\text{AD}} = 1 + i$ . This can be proven from the  $S$ -transformation of the Dedekind  $\eta$  function. It is also easy to check

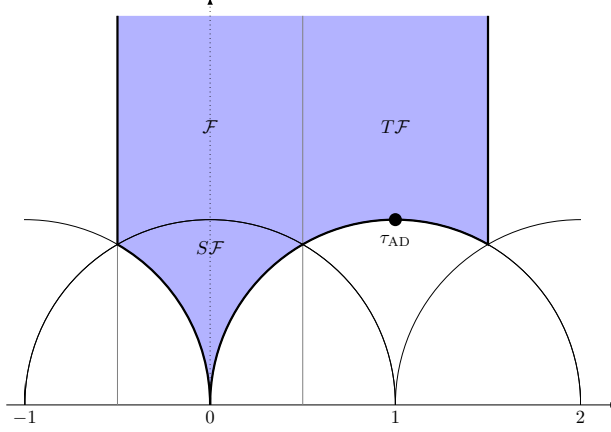


Figure 14: Fundamental domain of  $\Gamma^0(2)$ . This is the duality group of  $N_f = 2$  with masses  $\mathbf{m} = \frac{1}{2}(\Lambda_2, \Lambda_2)$ . The AD point corresponds to the elliptic fixed point  $\tau_{\text{AD}} = 1 + i$ .

that  $u(0) = u_0$ . Taking the proper limits from  $N_f = 2$  [2] we directly find  $\frac{du}{d\tau}$  as well as  $\frac{da}{du}$  and we can check that they satisfy the Matone relation

$$\frac{du}{d\tau} = -\frac{16\pi i}{2} \hat{\Delta} \left( \frac{da}{du} \right)^2, \quad (2.93)$$

consistent with (2.52). Both branch points of the  $N_f = 2$  theory have collided along with the singularities where mutually non-local states become massless. The monodromies are

$$M_0 = STS^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad M_{\text{AD}} = TS^{-1}T^{-1} = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}, \quad (2.94)$$

and they satisfy  $M_0 M_{\text{AD}} = M_\infty$  with  $M_\infty = PT^{-2}$ . Furthermore,  $M_{\text{AD}}^4 = \mathbb{1}$ , such that  $\tau_{\text{AD}}$  indeed is an elliptic fixed point of  $\Gamma^0(2)$ . The AD monodromy is conjugate to  $S^{-1}$ , which fixes  $\tau = i$ . Since  $\tau_{\text{AD}} = i + 1$ , this gives a path in  $\tau$ -space.

## 2.6 The $N_f = 3$ curve

We will start by discussing the  $N_f = 3$  theory with one non-zero mass,  $\mathbf{m} = (m, 0, 0)$ , where we can find an explicit expression for  $u$  in terms of Jacobi theta functions. After this we discuss the generic mass case, the massless theory and a number of theories with specific AD masses.

### 2.6.1 One non-zero mass

For the general theory it turns out to be complicated to find closed expressions for  $u$ , but if we only keep one non-zero mass,  $\mathbf{m} = (m, 0, 0)$ , we can make more progress. Four of the strong coupling singularities now merge in pairs of two and the physical discriminant becomes

$$\Delta = (u - u_+)^2 (u - u_-)^2 (u - u_*), \quad (2.95)$$

with

$$u_{\pm} = \pm \frac{m\Lambda_3}{8}, \quad u_* = \frac{\Lambda_3^2}{2^8} + m^2. \quad (2.96)$$

There are two AD points at  $m = m_{\text{AD}} = \pm \frac{1}{16}\Lambda_3$  and  $u = u_{\text{AD}} = \frac{1}{128}\Lambda_3^2$  where either  $u_+$  or  $u_-$  merges with  $u_*$  to give a type *III* singular fibre. We now find that the sextic equation for  $u$  again splits over the intermediate field  $\Gamma(2)$ . In this case there is only one solution that has  $|u| \rightarrow \infty$  for  $\tau \rightarrow i\infty$ , and as mentioned before this has  $u \rightarrow -\infty$ . This is then the reason why we have persistently chosen this convention in all other cases, to make the decoupling limits from  $N_f = 3$  consistent. We find that

$$\begin{aligned} \frac{u}{\Lambda_3^2} &= -\frac{2\vartheta_3^4\vartheta_4^4 + (\vartheta_3^4 + \vartheta_4^4)\sqrt{f_3}}{64\vartheta_2^8} \\ &= -\frac{1}{2^{12}} \left( \frac{1}{q} + (-8 + 4096\mu^2) + 4(5 + 32768\mu^2 - 4194304\mu^4)q + \mathcal{O}(q^2) \right), \end{aligned} \quad (2.97)$$

where we have defined  $f_3 = \frac{64m^2}{\Lambda_3^2}\vartheta_2^8 + \vartheta_3^4\vartheta_4^4$  and  $\mu = \frac{m}{\Lambda_3}$ . Due to the appearance of the square root in (2.97)  $u$  is not holomorphic, and similarly to the  $N_f = 1$  case there will be branch points in the fundamental domain. They are given by

$$j^{\text{bp}}(m) = \frac{(\Lambda_3 - 8m)^3(\Lambda_3 + 8m)^3}{16m^4\Lambda_3^2}. \quad (2.98)$$

By plugging in the expression for  $\mathcal{J}$  in terms of  $u$  we find that the branch point lies at  $u_{\text{bp}} = 2m^2$  in the  $u$ -plane, as is also found by studying the Matone polynomial (2.54). We can also use known relations between the  $j$ -invariant and theta functions to check that (2.98) coincides with  $f_3 = 0$ , such that the branch point of  $u$  is that of the square root in (2.97).

We see that the branch point of the square root is  $f_3(\tau_0) = 0$ . Near  $\tau_0$ , the expansion of  $f_3$  reads  $f_3(\tau) = (\tau - \tau_0)h(\tau)$ , where  $h(\tau)$  is holomorphic near  $\tau_0$  and  $h(\tau_0) \neq 0$ . Then one branch of the square root reads  $\sqrt{f_3(\tau)} = \sqrt{\tau - \tau_0}\sqrt{h(\tau)}$ . Now since  $h(\tau_0) \neq 0$ , we have that  $\tau \mapsto \sqrt{h(\tau)}$  is nonzero and in fact holomorphic in a neighbourhood of  $\tau_0$ . However,  $\tau \mapsto \sqrt{\tau - \tau_0}$  is strictly non-holomorphic at  $\tau_0$ . This proves that  $u$  is not holomorphic at  $\tau_0$ .

It is straightforward to calculate other interesting quantities explicitly from (2.97), including

$$\begin{aligned} \frac{da}{du} &= \frac{2\sqrt{2}i}{\Lambda_3} \frac{\vartheta_2^4}{\sqrt{\vartheta_3^4 + \vartheta_4^4 + 2\sqrt{f_3}}} \\ \frac{du}{d\tau} &= \pi i \Lambda_3^2 \vartheta_3^4 \vartheta_4^4 \frac{2^8 \mu^2 \vartheta_2^8 + \vartheta_3^8 + 6\vartheta_3^4 \vartheta_4^4 + \vartheta_4^8 + 4(\vartheta_3^4 + \vartheta_4^4)\sqrt{f_3}}{2^7 \vartheta_2^8 \sqrt{f_3}}. \end{aligned} \quad (2.99)$$

We can explicitly check that they satisfy Matone's relation (2.52),

$$\frac{du}{d\tau} = -16\pi i \frac{\widehat{\Delta}}{u - u_{\text{bp}}} \left( \frac{da}{du} \right)^2. \quad (2.100)$$

On the rhs, the double singularities  $u_+$  and  $u_-$  have cancelled, while, as discussed in section 2.3, the branch point  $u_{\text{bp}} = 2m^2$  remains in the denominator.

## Fundamental domain

A fundamental domain can be found in the following way. The six roots of the sextic equation give the six cusp expansions. They can be canonically normalised to match the form of the expansion (2.97) at  $\infty$ . Then instead of studying which transformations give the right values at the cusps, we can take the cusp expansions and try to find maps  $\alpha_j \in \text{SL}(2, \mathbb{Z})$  that take  $u(\tau)$  to the functions under study. Due to the square root, this is quite subtle. We furthermore need to take  $m$  as generic, such that the square root does not resolve. This allows to find the maps  $\alpha_j \in \text{SL}(2, \mathbb{Z})$ , which give the fundamental domain

$$\mathcal{F}_3(m, 0, 0) = \mathcal{F} \cup S\mathcal{F} \cup ST^{-1}\mathcal{F} \cup TST\mathcal{F} \cup TST^2\mathcal{F} \cup TST^2S\mathcal{F}, \quad (2.101)$$

shown in Fig. 15. It is valid for all masses  $m$  that do not allow the square root to resolve. We prove below that this does not happen unless  $m = 0$  or  $m = m_{\text{AD}} = \frac{1}{16}\Lambda_3$ .

Let us also study the paths of branch points in the fundamental domain. Similarly as in massive  $N_f = 1, 2$  [2], we analyse the critical values of (2.98). We have that  $\lim_{m \searrow 0} j^{\text{bp}}(m) = +\infty$ ,  $j^{\text{bp}}(m_{\text{AD}}) = 12^3$ ,  $j^{\text{bp}}(\frac{\Lambda_3}{8}) = 0$  and  $\lim_{m \rightarrow \infty} j^{\text{bp}}(m) = -\infty$ . It is easy to show that  $j^{\text{bp}} : (0, \infty) \rightarrow \mathbb{R}$  is monotonically decreasing and therefore injective.

Since  $u_{\text{bp}} = 2m^2$ , we have that at  $m = 0$  the branch points coincides with  $u_+$  and  $u_-$ . At the AD point,  $m = \frac{\Lambda_3}{16}$ , it collides along with  $u_*$  and  $u_+$ . Finally, for  $m \rightarrow \infty$  it diverges, just as  $u_*$  does. This fixes the points  $\tau = 0$  and  $\tau = 1$  for  $m = 0$ ,  $\tau = \tau_{\text{AD}} = \frac{1}{2} + \frac{i}{2}$  for  $m = m_{\text{AD}}$  and  $\tau = \frac{1}{2}$  or  $i\infty$  for  $m = \infty$ . The simplest curves connecting these three points are quarter-circles with radius  $\frac{1}{2}$  around  $\tau = \frac{1}{2}$  starting from either  $\tau = 0$  or  $\tau = 1$ , followed by a vertical path from  $\frac{1}{2} + \frac{i}{2}$  to either  $\frac{1}{2}$  or  $i\infty$ .

The fundamental domain together with the path of the branch points found from the above considerations is shown in Fig. 15.

The various checks of the branch points paths are analogous to  $N_f = 1, 2$ . We can plot  $j(\tau)$  along these curves and find that it has the same global properties and critical points as (2.98). The intermediate value  $j(\tau) = 0$  corresponds to  $m = \frac{\Lambda_3}{8}$  and  $\tau = \frac{1}{2} + \frac{1}{2\sqrt{2}}i$ , which is in the  $\text{SL}(2, \mathbb{Z})$ -orbit of  $\omega_3$ . Along the branch point locus,  $u$  simplifies to

$$u(\tau_{\text{bp}}) = -\frac{f_{2\text{B}}(\tau_{\text{bp}})}{2^{13}}, \quad (2.102)$$

with  $f_{2\text{B}}$  given by (2.90). On the paths in Fig. 15 this function behaves precisely as  $u_{\text{bp}}(m) = 2m^2$ .

## Partitioning of the $u$ -plane

Finally, we can study the partitioning that the domain (2.101) induces on the  $u$ -plane under the map (2.97). As studied in section 2.2.4, the partitioning

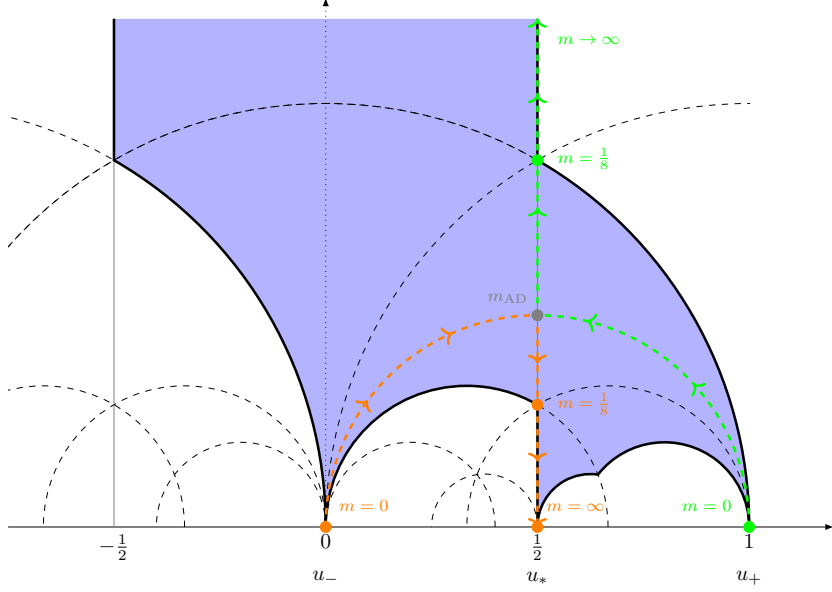


Figure 15: Fundamental domain for  $N_f = 3$  with  $\mathbf{m} = (m, 0, 0)$ . The dashed lines corresponds to the path of the branch points on the positive ray from massless to infinite mass.

is contained in a real algebraic plane curve, which is given by the equation  $\text{Im } \mathcal{J}(u, \mathbf{m}, \Lambda_3) = 0$ . For generic  $\mu = \frac{m}{\Lambda_3}$  where  $\mathbf{m} = (m, 0, 0)$ , we can compute it as the zero locus of the polynomial

$$\begin{aligned}
T_{(m,0,0)} = & y(2\mu^2 + 128x^2 - 256\mu^2x - x + 128y^2) \\
& \times (10616832\mu^8 + 147456\mu^6 + 672\mu^4 + \mu^2 + 219902325552x^8 - 137438953472\mu^2x^6 \\
& + 8796093022208x^6y^2 - 25769803776x^6 + 51539607552\mu^2x^5 + 1275068416x^5 \\
& - 22548578304\mu^4x^4 - 2617245696\mu^2x^4 + 13194139533312x^4y^4 - 137438953472\mu^2x^4y^2 \\
& - 77309411328x^4y^2 - 12976128x^4 + 1610612736\mu^4x^3 + 2097152\mu^2x^3 + 103079215104\mu^2x^3y^2 \\
& + 2550136832x^3y^2 + 49152x^3 - 301989888\mu^6x^2 + 37748736\mu^4x^2 + 159744\mu^2x^2 \\
& + 8796093022208x^2y^6 + 137438953472\mu^2x^2y^4 - 77309411328x^2y^4 - 53687091200\mu^4x^2y^2 \\
& - 2550136832\mu^2x^2y^2 - 9175040x^2y^2 - 64x^2 - 37748736\mu^6x - 344064\mu^4x - 768\mu^2x \\
& + 51539607552\mu^2xy^4 + 1275068416xy^4 + 536870912\mu^4xy^2 - 18874368\mu^2xy^2 - 16384xy^2 \\
& + 219902325552y^8 + 137438953472\mu^2y^6 - 25769803776y^6 - 22548578304\mu^4y^4 \\
& + 67108864\mu^2y^4 + 3801088y^4 + 301989888\mu^6y^2 + 16777216\mu^4y^2 + 102400\mu^2y^2 + 64y^2).
\end{aligned} \tag{2.103}$$

The second factor on the rhs gives a circle on the  $x + iy = \frac{u}{\Lambda_3^2}$ -plane with radius  $|\mu^2 - \frac{1}{28}|$  and centre  $(x, y) = (\mu^2 + \frac{1}{28}, 0)$ . By tuning the mass  $\mu$  from 0 to  $\infty$ , one passes through the AD point  $\mu = \frac{1}{24}$  where the radius of the circle shrinks to 0. For this mass, three regions defined through  $T_{(m,0,0)} = 0$  collapse to a point  $x + iy = \frac{u_{\text{AD}}}{\Lambda_3^2} = \frac{1}{128}$ , which is the only root over  $\mathbb{R}^2$  of the quadratic polynomial. This gives further evidence that the domain 15 is in fact correct for all  $\mu \in (0, \infty) \setminus \{\frac{1}{16}\}$ .

### Limits to zero, AD and infinite mass

As in the  $\mathbf{m} = (m, m)$   $N_f = 2$  theory, there are three interesting limits:  $m \rightarrow 0$ ,  $m \rightarrow \infty$  and  $m \rightarrow m_{\text{AD}}$ . In the massless limit, we aim to recover  $\Gamma_0(4) \backslash \mathbb{H}$ . This is not difficult to see: Under  $\Gamma_0(4)$ , we can identify  $TST^2SF$

with  $ST^{-2}S\mathcal{F}$ , and similarly  $TST^2\mathcal{F}$  with  $ST^{-2}\mathcal{F}$ , since the transition maps are in  $\Gamma_0(4)$ . This gives precisely Fig. 17.

By decoupling the massive hypermultiplet, the theory flows to massless  $N_f = 2$ . We find that  $u_* \rightarrow \infty$ , while  $u_{\pm} \rightarrow \pm \frac{\Lambda_3^2}{8}$ . From section 2.5.2 it is clear that the singularities  $u_{\pm}$  do not move in  $\tau$ -space. The cusp region  $TST^2S\mathcal{F}$  is identified with  $T\mathcal{F}$  under the duality group  $\Gamma(2)$  of massless  $N_f = 2$ . Moreover, the remaining differing triangle  $TST^2\mathcal{F}$  can be mapped to  $T\mathcal{F}$  using  $\Gamma(2)$ . This then gives precisely  $\Gamma(2)\backslash\mathbb{H}$  as in Fig. 13.

Finally, in the limit  $m \rightarrow m_{\text{AD}} = \frac{1}{16}\Lambda_3$  the singularities  $u_*$  and  $u_+$  collide. Since they drop out of the curve, we should remove all regions near those cusps. In Fig. 15 we can remove the triangles  $TST\mathcal{F}$ ,  $TST^2\mathcal{F}$  and  $TST^2S\mathcal{F}$ , after which the index 3 group  $\Gamma_0(2)$  remains. This is precisely what is found as the duality group of the  $\mathbf{m} = (m_{\text{AD}}, 0, 0)$  theory, as shown in Fig. 14. The pre-image of the merged non-local singularities  $u_{\text{AD}}$  is the point  $\tau_{\text{AD}}$ , which lies in the interior of  $\mathbb{H}$  and corresponds to the point where the branch points have collided.

## 2.6.2 Generic masses

For generic masses  $\mathbf{m} = (m_1, m_2, m_3)$ , the order parameter reads

$$\begin{aligned} \frac{u}{\Lambda_3^2} = & -\frac{1}{2^{12}} \left( \frac{1}{q} + (-8 + 4096M_2) \right. \\ & \left. + 4(5 + 32768M_2 + 3670016M_3 - 4194304M_4 - 4194304M'_4)q + \mathcal{O}(q^2) \right), \end{aligned} \quad (2.104)$$

where the coefficients  $M_i$  are the symmetric polynomials defined in (2.55) for the variables  $\frac{m_i}{\Lambda_3}$ . There are five generally distinct singular points.

Due to the  $N_f = 3$  distinct branch points on the  $u$ -plane, the fundamental domain for a given mass  $\mathbf{m}$  has an intricate web of branch cuts. Furthermore, the fundamental domains  $\mathcal{F}_3(\mathbf{m})$  change as  $\mathbf{m}$  passes through  $\mathcal{L}_3^{\text{AD}}$  (see Fig. 3). A fundamental domain  $\mathcal{F}_3(\mathbf{m})$  can also change when  $\mathbf{m}$  is varied such that  $\Delta_3$  has any double root, and when branch points in  $\mathbb{H}$  pass through the  $\text{SL}(2, \mathbb{Z})$  tessellation  $\mathcal{T}_{\mathbb{H}}$  (2.36).

For any given mass  $\mathbf{m}$  one easily computes  $T_3$  from (2.33), and truncates the plot of the level set to the region where  $\mathcal{J}(u, \mathbf{m}, \Lambda_3) \leq 12^3$ . The branch points  $u_{\text{bp}}$  are the zeros of  $P_3^{\text{M}}$  (2.54). On the upper half-plane  $\mathbb{H}$ , a branch point  $\tau_{\text{bp}}$  is any of the  $\text{SL}(2, \mathbb{Z})$ -images of  $(j|_{\mathcal{F}})^{-1}(\mathcal{J}(u_{\text{bp}}))$ . When  $\mathcal{J}(u_{\text{bp}}) \leq 12^3$ , then obviously  $\tau_{\text{bp}} \in \mathcal{T}_{\mathbb{H}}$ . If  $\mathcal{J}(u_{\text{bp}}) > 12^3$ , then  $\tau_{\text{bp}} \in \text{SL}(2, \mathbb{Z}) \cdot i\mathbb{R}_{>0}$ . Lastly, if  $\mathcal{J}(u_{\text{bp}}) \notin \mathbb{R}$ , then  $\tau_{\text{bp}}$  is an interior point of an  $\text{SL}(2, \mathbb{Z})$  copy of  $\mathcal{F}$ .

In Fig. 16 we plot the  $u$ -plane and corresponding fundamental domain for three distinct masses. The five distinct singular points are partitioned into five regions  $u(\alpha\mathcal{F})$ , where two of them are glued by branch cuts.

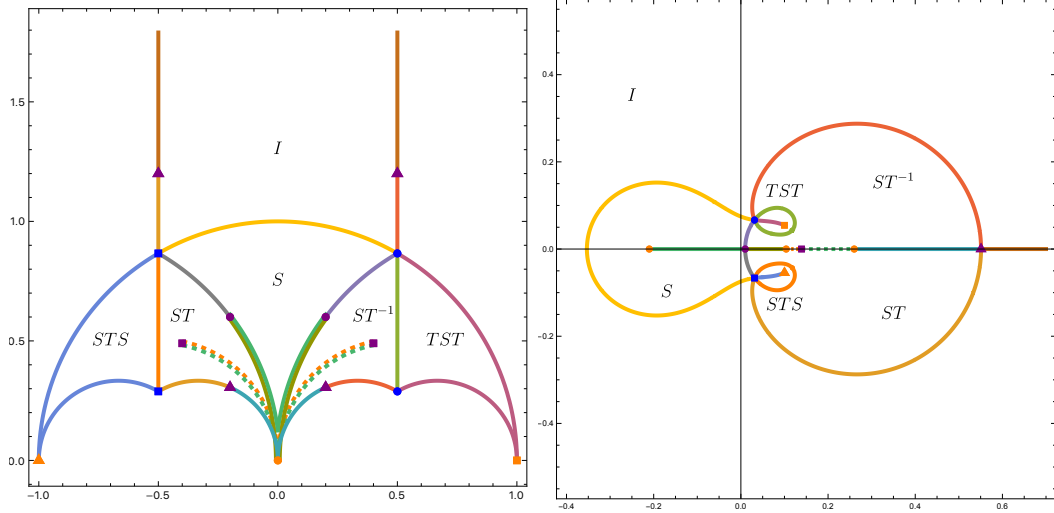


Figure 16: Identification of the components of the partitioning  $\mathcal{T}_{(m_1, m_2, m_3)}$  in  $N_f = 3$  for the particular choice  $\mu_1 = \frac{1}{10}$ ,  $\mu_2 = \frac{3}{10}$  and  $\mu_3 = \frac{5}{10}$ . The  $u$ -plane  $\mathcal{B}_3$  is partitioned into five regions  $u(\alpha\mathcal{F})$ , as those for  $ST^{-1}$  and  $ST$  are glued by two branch cuts. The fundamental domain is given by six copies of  $\mathcal{F}$ , with three pairs of branch points (purple). Two branch points (triangle and disk) lie on  $\mathcal{T}_3$ . The third (square) lies in the interior and glues the copies  $ST\mathcal{F}$  and  $ST^{-1}\mathcal{F}$ . A natural choice for the branch cuts not lying in  $\mathcal{T}_3$  (dashed) is on the real axis in the  $u$ -plane, for which the path in  $\mathbb{H}$  is along the tessellation  $SL(2, \mathbb{Z}) \cdot i\mathbb{R}_{>0}$ . The singularity at  $\tau = 0$  does not have width 3 as apparent, since due to the branch cuts  $u(\tau)$  assumes three different values depending on which side of the branch cut  $\tau = 0$  is approached from.

### 2.6.3 The massless theory

When sending  $m \rightarrow 0$  from above we find

$$\begin{aligned} \frac{u(\tau)}{\Lambda^2} &= -\frac{1}{64} \frac{\vartheta_3(\tau)^2 \vartheta_4(\tau)^2}{(\vartheta_3(\tau)^2 - \vartheta_4(\tau)^2)^2} = -\frac{1}{2^{12}} \left( \frac{\eta(\tau)}{\eta(4\tau)} \right)^8 \\ &= -\frac{1}{2^{12}} (q^{-1} - 8 + 20q - 62q^3 + \mathcal{O}(q^{9/2})). \end{aligned} \quad (2.105)$$

It is the completely replicable function of class 4C and a Hauptmodul for  $\Gamma_0(4)$  [56–58]. The physical discriminant is  $\Delta = u^4(u - \frac{\Lambda^2}{28})$ , and one finds that the singularities are located at  $u(0) = 0$  and  $u(\frac{1}{2}) = \frac{\Lambda^2}{28}$ . At  $\tau = 0$  a dyon with charge  $(1, 0)$  becomes massless, while at  $\tau = \frac{1}{2}$  one finds instead that a dyon with charge  $(2, 1)$  becomes massless. The massless  $N_f = 3$   $u$ -plane has no global symmetries.

A choice of fundamental domain for  $\Gamma_0(4)$  is

$$\mathcal{F}_3(0, 0, 0) = \mathcal{F} \cup S\mathcal{F} \cup ST\mathcal{F} \cup ST^{-1}\mathcal{F} \cup ST^{-2}\mathcal{F} \cup ST^{-2}S\mathcal{F}, \quad (2.106)$$

and is shown in Fig. 17.

### 2.6.4 Type IV AD mass

As illustrated in Fig. 3, on the generic mass  $N_f = 3$   $u$ -plane, there is not only the IV AD point but also a variety of III and II points. We will give a few

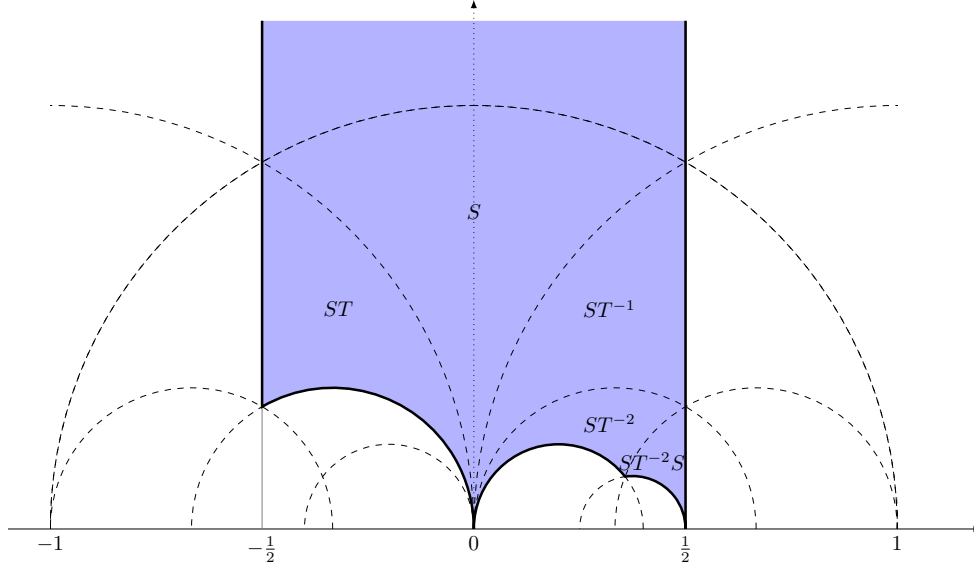


Figure 17: Fundamental domain of  $\Gamma_0(4)$ , the duality group of massless  $N_f = 3$ . The cusp at  $\tau = 0$  has width four, while the cusp at  $\tau = \frac{1}{2}$  has width one.

explicit examples of the  $u$ -plane of the theories with masses tuned to these specific values, starting with the most symmetric case.

For  $\mathbf{m} = (m, m, m)$  and  $m = \frac{1}{8}\Lambda_3$ , four mutually non-local singularities collide in  $u_{\text{AD}} = \frac{1}{32}\Lambda_3^2$ . The remaining singularity is  $u_0 = -\frac{19}{2^8}\Lambda_3^2$  and never collides with the other four. The physical discriminant is  $\Delta = (u - u_0)(u - u_{\text{AD}})^4$ . One finds that

$$\frac{u}{\Lambda_3^2} = -\frac{j^* + 304}{2^{12}}, \quad (2.107)$$

where

$$\begin{aligned} j^* &= 432 \frac{\sqrt{j} + \sqrt{j - 1728}}{\sqrt{j} - \sqrt{j - 1728}} = 432 \frac{E_4^{\frac{3}{2}} + E_6}{E_4^{\frac{3}{2}} - E_6} \\ &= \frac{1}{q} - 120 + 10260q - 901120q^2 + 91676610q^3 + O(q^4) \end{aligned} \quad (2.108)$$

is the Ramanujan-Sato series of level 1 [159–161].<sup>14</sup>

Inverting (2.108) we find

$$j = \frac{(j^* + 432)^2}{j^*}. \quad (2.109)$$

Using this and a discussion similar to the massless  $N_f = 1$  case for the transformations of  $j^*$  we find that the singularities are located at ( $\omega_j = e^{2\pi i/j}$ )

$$u(i\infty) = \infty, \quad u(\omega_3) = u_{\text{AD}}, \quad u(0) = u_0. \quad (2.110)$$

<sup>14</sup>The Ramanujan-Sato series generalise Ramanujan's formula for  $\frac{1}{\pi}$  as a series of quotients of modular forms. They exist for level 1 up to 11 and beyond. The level 1 series is the only one whose generating function can not be expressed by an  $\eta$ -quotient [162].



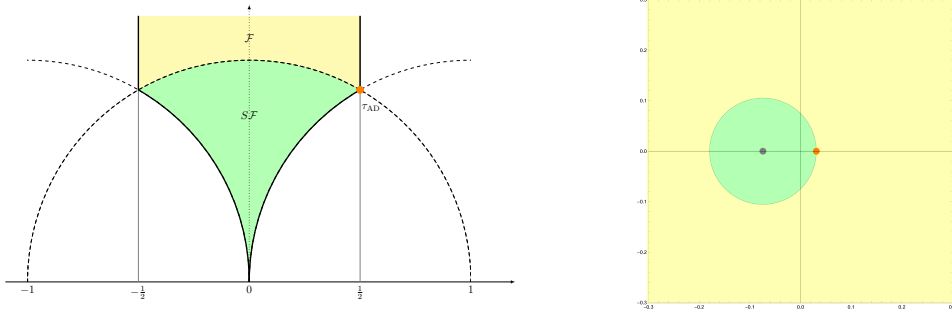


Figure 18: Left: Proposed fundamental domain of the duality group  $\Gamma_{j^*}$  of the  $N_f = 3$   $\mathbf{m} = \frac{1}{8}(\Lambda_3, \Lambda_3, \Lambda_3)$  theory. It is a region with index  $6 - 4 = 1 + 1$ , elliptic fixed point  $\omega_3$  and cusp at 0. This uniquely fixes it. Right: Plot of the  $u$ -plane for the corresponding theory as the union of the images of  $u$  under the  $6 - 4 = 2$   $\text{SL}(2, \mathbb{Z})$  images of  $\mathcal{F}$  as in the left figure. The complex plane can clearly be covered by 2 triangles. This demonstrates that the proposal of the left figure is indeed a fundamental domain for (2.107). The weak coupling region is covered by  $u(\mathcal{F})$ . The strong coupling region  $u(S\mathcal{F})$  contains the singular point  $u_0$  in its interior. The AD point (orange) lies on the boundary.

We can read off from (2.109) that  $\text{ord}(g_2, g_3, \Delta) = (2, 2, 4)$  at the AD point, such that according to Table 5 we indeed have a singular fibre of Kodaira type *IV* [155].

From (2.109) we read off that the duality group  $\Gamma_{j^*}$  has index 2, which is consistent with the previous cases in  $N_f = 1, 2$  in that a factor of  $(u - u_{\text{AD}})^4$  has cancelled from  $g_2^3$  and  $\Delta$ , and therefore does not contribute to the index. The fundamental region of  $u$  is therefore of index 2 with 0,  $\omega_3$  and  $i\infty$  on its boundary. However, there is no index 2 subgroup of  $\text{SL}(2, \mathbb{Z})$  with two distinct cusps [154, Table 4.1].<sup>15</sup> This agrees with the fact that (2.107) is not a classical modular form and the monodromy group does not promote to a modular group since its action on  $u$  is not associative (see section 2.4.1). We can nevertheless propose a fundamental region

$$\mathcal{F}_3(\mathbf{m}_{\text{AD}}) = \mathcal{F} \cup S\mathcal{F}, \quad (2.111)$$

see Figure 18.

The monodromies are found by consistency,

$$m_0 = STS^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad m_{\text{AD}} = TS^{-1} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \quad (2.112)$$

and are unique in  $\text{SL}(2, \mathbb{Z})$ .<sup>16</sup> They fix  $\tau = 0$  and  $\tau_{\text{AD}} = e^{\pi i/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ , respectively, and produce the large  $u$  monodromy  $m_0 m_{\text{AD}} = PT^{-1}$ . Just as in the massless  $N_f = 1$  case, we note that as matrices they do not form a

<sup>15</sup>In fact, there is exactly one index 2 subgroup of  $\text{SL}(2, \mathbb{Z})$  and it has only one cusp of width two. This group is sometimes referred to as  $\Gamma_0(1)^*$  and is generated by  $TS$  and  $T^2$ , and the Hauptmodul is given by  $\sqrt{j - 1728} = 8 \frac{(\vartheta_2^4 + \vartheta_3^4)(\vartheta_3^4 + \vartheta_4^4)(\vartheta_4^4 - \vartheta_2^4)}{\vartheta_2^4 \vartheta_3^4 \vartheta_4^4}$ .

<sup>16</sup>The overall signs are fixed in the following way. The large  $u$  monodromy is  $PT^{-1}$ . The monodromy at  $m_0$  is oriented such that it conjugates to  $T$ . This fixes the sign of  $m_{\text{AD}}$  from the below relation.

congruence subgroup but instead generate the whole of  $\mathrm{SL}(2, \mathbb{Z})$ , since  $m_\infty = PT^{-1}$  and  $Tm_0T = S$ . However,  $u$  is not invariant under  $S$ .

### 2.6.5 Type III AD mass

In the single mass case  $\mathbf{m} = (m, 0, 0)$  with  $m_{\mathrm{AD}} = \frac{1}{16}\Lambda_3$ , the  $N_f = 3$  curve has an AD point at  $u_{\mathrm{AD}} = \frac{1}{128}\Lambda_3^2$ . The physical discriminant is  $\Delta = (u - u_{\mathrm{AD}})^3(u - u_0)^2$  with  $u_0 = -\frac{1}{128}\Lambda_3^2$ , which is  $-u_{\mathrm{AD}}$  by coincidence. One easily finds

$$\frac{u(\tau)}{\Lambda_3^2} = -\frac{f_{2\mathrm{B}}(\tau) + 32}{2^{12}}, \quad (2.113)$$

with  $f_{2\mathrm{B}}$  defined in (2.90). This fits nicely into the description as  $\mathrm{ind} \Gamma_0(2) = 3$  is equal to  $6 - 3 = 3$ , being the number of mutually non-local singularities collided at  $u_{\mathrm{AD}}$ . We find that  $u(\tau) = u_0$  is equivalent to  $f_{2\mathrm{B}}(\tau) = 0$ , and one can easily show that  $f_{2\mathrm{B}}$  vanishes at the cusp  $\tau = 0$ . Using the  $S$ -transformation of  $\eta$ , we can show that  $\tau_{\mathrm{AD}} = \frac{1}{2} + \frac{i}{2}$ . In terms of the Hauptmodul, the curve reads  $j = \frac{(f_{2\mathrm{B}} + 256)^3}{f_{2\mathrm{B}}^2}$ . This shows that the AD point is indeed a type III singularity. It also follows that  $j(\tau_{\mathrm{AD}}) = 12^3$  and that  $\tau_{\mathrm{AD}}$  is in the  $\mathrm{SL}(2, \mathbb{Z})$  orbit of  $i$ . The duality group  $\Gamma_0(2)$  is generated by  $g_1 = T$  and  $g_2 = ST^2S$ . The AD point is stabilised by  $g_1g_2 \in \Gamma_0(2)$ , which makes it an elliptic fixed point. A fundamental domain for  $u$  is given in Figure 19. The effective coupling at the AD point is explained through the fact that the  $N_f = 3$  branch point collides along with the three mutually non-local singularities in  $u_{\mathrm{AD}}$ , and the two branch points on the upper half-plane as drawn in Fig. 15 collide at  $\tau_{\mathrm{AD}}$  for  $m = m_{\mathrm{AD}}$ .

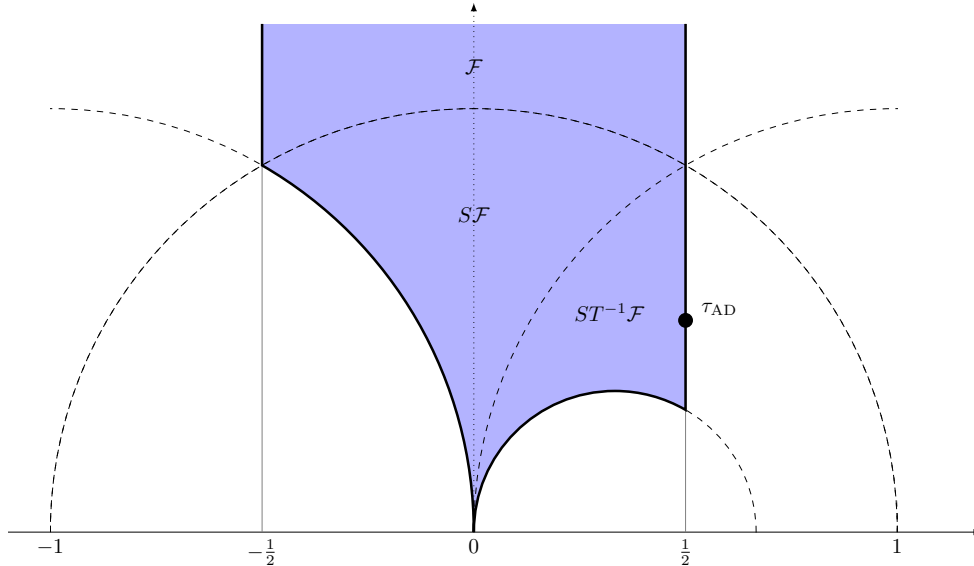


Figure 19: Fundamental domain of  $\Gamma_0(2)$ . This is the duality group of  $N_f = 3$  with mass  $\mathbf{m} = \frac{1}{16}(\Lambda_3, 0, 0)$ . The AD point is the elliptic fixed point of the domain, located at  $\tau_{\mathrm{AD}} = \frac{1}{2} + \frac{i}{2}$ .

The monodromies are

$$M_0 = ST^2S^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad M_{\text{AD}} = (TS)^{-1}S^{-1}TS = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}. \quad (2.114)$$

The first one describes the path around the cusp  $\tau = 0$ , which has width 2. The AD monodromy is conjugate to  $S^{-1}$ , which fixes  $\tau = i$ . The path is then given by the map  $(TS)^{-1} : i \mapsto \tau_{\text{AD}}$ . The matrices (2.114) satisfy  $M_0M_{\text{AD}} = M_\infty$  with  $M_\infty = PT^{-1}$  and  $M_{\text{AD}}^2 = \mathbb{1}$ , such that  $\tau_{\text{AD}}$  is indeed an elliptic fixed point for  $\Gamma_0(2)$  of order 2. They are also related to (2.94) by conjugation with  $\text{diag}(2, 1)$ , which induces the isomorphism between the  $\Gamma_0(2)$  and  $\Gamma^0(2)$  curves.

### 2.6.6 Type II AD mass

On the equal mass  $\mathbf{m} = (m, m, m)$  curve we can also tune the mass to  $m = m_{\text{AD}} = -\frac{1}{64}\Lambda_3$  to find a type II AD theory at  $u_{\text{AD}} = \frac{5}{2^{10}}\Lambda_3^2$ . By fixing the mass to this value we find

$$\frac{u(\tau)}{\Lambda_3^2} = -\frac{f_{3\text{B}}(3\tau) + 7}{2^{12}} \quad (2.115)$$

with  $f_{3\text{B}}$  given in (2.75). At  $u_{\text{AD}}$  two mutually non-local singularities collide, while the other three reside at  $u_0 = -\frac{7}{2^{12}}\Lambda_3^2$ . The physical discriminant is therefore  $\Delta = (u - u_{\text{AD}})^2(u - u_0)^3$ . We know from section 2.4.2 that  $\tau \mapsto f_{3\text{B}}(3\tau)$  is a Hauptmodul for  $\Gamma_0(3)$ , and in fact the fundamental domain is just given by the one for  $\Gamma^0(3)$  as in Fig. 7, with every point divided by 3. It also decomposes into  $\text{SL}(2, \mathbb{Z})$  images of  $\mathcal{F}$ , see Fig. 20.

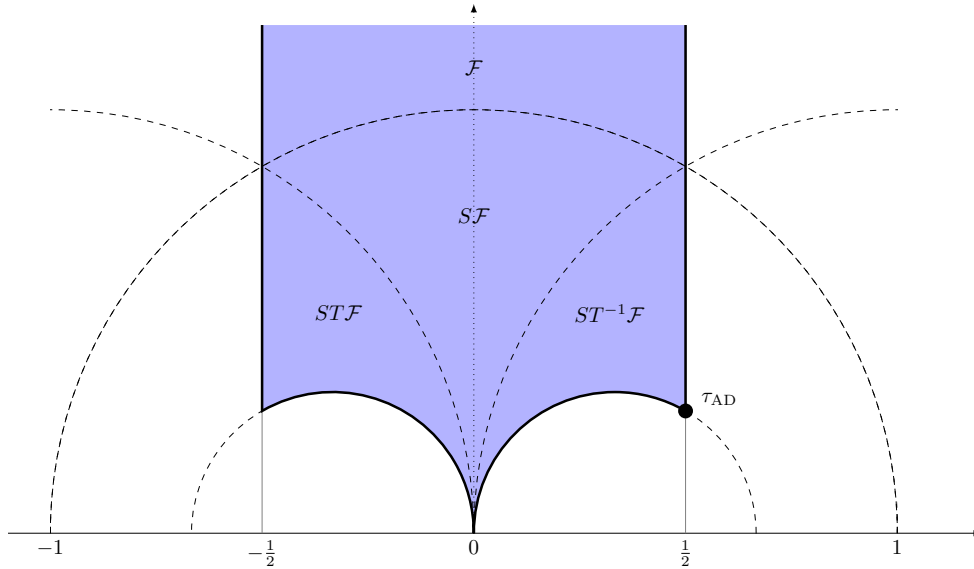


Figure 20: Fundamental domain of  $\Gamma_0(3)$ , the duality group of  $N_f = 3$  with mass  $\mathbf{m} = -\frac{1}{64}(\Lambda_3, \Lambda_3, \Lambda_3)$ . It has index 4 in  $\text{PSL}(2, \mathbb{Z})$  and a type II AD point located at the elliptic fixed point  $\tau_{\text{AD}} = \frac{\omega_{12}}{\sqrt{3}} = \frac{1}{2} + \frac{1}{2\sqrt{3}}i$ . The width of the cusp  $\tau = 0$  is 3.

The AD point  $u(\tau_{\text{AD}}) = u_{\text{AD}}$  translates to  $f_{3\text{B}}(3\tau_{\text{AD}}) = -27$  which has  $\tau_{\text{AD}} = \frac{1}{\sqrt{3}}\omega_{12}$  as a solution (where  $\omega_j = e^{2\pi i/j}$ ). The other singularity satisfies

$f_{3B}(3\tau) = 0$  and therefore  $\tau = 0$ . In terms of the Hauptmodul of  $\Gamma_0(3)$  the  $j$ -invariant of the curve with above given mass  $\mathbf{m}_{AD}$  reads  $j = (f_{3B} + 27)(f_{3B} + 243)^3/f_{3B}^3$ . This proves that the AD singularity is Kodaira type *II* and therefore indeed equivalent to the *II* theory in  $N_f = 1$ , see section 2.4.2. It is interesting that both curves are parametrised by the same Hauptmodul, as the number of singularities on the curves are different.

The monodromies are given by

$$M_0 = ST^3S^{-1} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}, \quad M_{AD} = (TS)^{-1}(ST)^{-1}(TS) = \begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix}, \quad (2.116)$$

which are just (2.82) conjugated by  $\text{diag}(3, 1)$ . They furthermore satisfy  $M_0M_{AD} = M_\infty$  with  $M_\infty = PT^{-1}$ . Since  $M_{AD}^6 = \mathbb{1}$  in  $\text{PSL}(2, \mathbb{Z})$ , the AD point  $\tau_{AD}$  is an elliptic fixed point in  $\Gamma_0(3)$ . Its stabiliser  $M_{AD}$  decomposes into the monodromy  $(ST)^{-1}$  around  $\tau = \omega_3$ , and the path  $(TS)^{-1} : \omega_3 \mapsto \tau_{AD}$ .

## 2.7 Discussion

We have studied the Coulomb branches  $\mathcal{B}_{N_f}$  of  $\text{SU}(2)$   $\mathcal{N} = 2$  Yang–Mills theories with  $N_f \leq 3$  massive hypermultiplets in the fundamental representation. In particular, we have considered the order parameter  $u$  as function of the effective coupling, and derive domains  $\mathcal{F}_{N_f}$  such that  $u : \mathcal{F}_{N_f} \rightarrow \mathcal{B}_{N_f}$  is 1-to-1. We find that generically the function  $u$  has square roots appearing in the expressions for  $u$ , such that  $\mathcal{F}_{N_f}$  is not isomorphic to a domain  $\Gamma \backslash \mathbb{H}$  for a congruence subgroup of  $\text{SL}(2, \mathbb{Z})$ . Nevertheless, exact expressions can be determined, such as for  $N_f = 2$  with 2 equal masses, and  $N_f = 3$  with one non-vanishing mass. For other special values, branch points and cuts can be absent and the fundamental domain is that of a modular curve for a congruence subgroups of  $\text{SL}(2, \mathbb{Z})$ , as also encountered in cases in the literature [45, 100–102].

We described how the order parameters are naturally expressed as roots of certain degree six polynomials with modular functions as coefficients. Many interesting aspects of the theories can be read off from these polynomials:

- The degree of the polynomial tells us that the fundamental domains of the order parameters can be described as six copies of the ordinary  $\text{SL}(2, \mathbb{Z})$  domain.
- For the modular theories this further implies that the duality group needs to be at most index six in  $\text{SL}(2, \mathbb{Z})$ .
- The discriminant of the sextic polynomials includes the branch points as well as the superconformal AD fixed points of the theories.
- We further discussed how one can explicitly construct fundamental regions of order parameters as images of fundamental domains in  $\mathcal{F}_{N_f}$ . The partitioning of the fundamental regions of the order parameters seem to

generalise aspects of the dessins d’enfants [39, 124–127, 158, 163, 164] to the case of non-modular elliptic surfaces.

Physically, the branch points and cuts provide a mechanism for  $\mathcal{F}_{N_f}$  to evolve as function of the mass. This is most apparent in the limits where a hypermultiplet decouples or multiple singularities coincide, where branch cuts appear to “cut” and “glue” regions of  $\mathcal{F}_{N_f}$ . In particular near an AD point, regions with non-local cusps are disconnected from  $\mathcal{F}_{N_f}$ . This makes it manifest that on the  $u$ -plane, not only non-local singularities become coincident, but also branch points, which ceases to be branch points in the limit because also the pre-images in  $\mathcal{F}_{N_f}$  have collided.

We believe that our methods can be adapted to many other rank one theories, such as those of class  $\mathcal{S}$  [21, 113]. The present analyses could perhaps also be used to draw lessons about moduli spaces of other theories, such as  $\mathcal{N} = 2$  SYM with gauge group  $SU(N)$  for  $N > 2$  or Calabi-Yau compactifications in string theory, where in many cases similar structures should arise. Remnants of which could perhaps be seen in [1, 153]. Lastly, we hope our methods find applications in similar geometries such as F-theory [165] and 5d SCFTs [166]. Moreover, our findings benefit the evaluation of the  $u$ -plane integral [71, 73, 81, 82, 84], which we discuss in section 5.

We would further like to mention to explore potential physical consequences of the branch points. It is known that the AD points correspond to critical points of a second order phase transition [157, 167, 168]. It might then be natural to think of the branch cuts in  $\mathcal{F}_{N_f}$ , as in for example Fig. 8, as boundaries over which a first order phase transition takes place. Since branch points and cuts seem to be a generic feature, it would suggest that similar points appear in all theories with these kinds of superconformal fixed points. It would of course be very interesting to study this further and we leave that for future work.

### 3 Four flavours, triality and bimodular forms

In this Section, we study the  $\mathcal{N} = 2$  supersymmetric SYM theory with  $N_f = 4$  fundamental hypermultiplets. Together with the asymptotically free theories investigated in section 2, it completes the study of  $SU(2)$  theories with fundamental hypermultiplets and non-positive one-loop beta functions. This section is based on [3].

#### 3.1 Introduction

The  $\mathcal{N} = 2$  supersymmetric Yang–Mills field theory with gauge group  $SU(2)$  and  $N_f = 4$  fundamental hypermultiplets is distinguished for various reasons [46], including:

- The theory is superconformal up to mass terms for the hypermultiplets, and is a benchmark for four-dimensional SCFT’s with  $\mathcal{N} = 2$  supersymmetry [105, 169–171].
- The theory is a building block for other four-dimensional  $\mathcal{N} = 2$  SCFT’s and the 2d/4d correspondence [21, 172].
- The theory exhibits an intriguing electric-magnetic duality group including triality [46]. This duality group acts on the UV coupling  $\tau_{UV}$  and running coupling constant  $\tau$ , and contains elements which act simultaneously on the two couplings as well as separately.
- The theory is a “parent” theory from which the asymptotically free  $\mathcal{N} = 2$ ,  $SU(2)$  theories with  $N_f \leq 3$  hypermultiplets can be obtained by decoupling one or more hypermultiplets [45, 46].

The focus of the present section is on the third bullet point. We analyse duality groups for the couplings  $\tau_{UV}$  and  $\tau$  of the  $N_f = 4$  theory as function of the masses. To this end, explicit expressions for the order parameter  $u = \langle \text{Tr } \phi^2 \rangle$ , with  $\phi$  the complex scalar of the vector multiplet, are determined as function of both  $\tau_{UV}$  and  $\tau$ . We identify several loci in the space of masses where  $u$  transforms as a modular form for both  $\tau_{UV}$  and  $\tau$ . This extends section 2 on theories with  $N_f \leq 3$  to  $N_f = 4$ . In section 2, we determined fundamental domains for the running coupling  $\tau$  for the asymptotically free theories by analysing in detail the order parameter  $u$  as function of the effective coupling  $\tau \in \mathbb{H}$ . We have demonstrated that for generic masses this function has branch points, with the consequence that the fundamental domain for  $\tau$  is in general not of the form  $\Gamma \backslash \mathbb{H}$  for a congruence subgroup  $\Gamma \subset SL(2, \mathbb{Z})$ . Only for specific values of the masses, such as those giving rise to Argyres-Douglas (AD) points, the order parameter is (weakly) holomorphic as function of  $\tau$ , and the fundamental domain is that of a congruence subgroup. In this section, we find that these features are present as well for the  $N_f = 4$  theory, but with an additional dependence on  $\tau_{UV}$ .

At special modular loci, some properties of the  $N_f = 4$  order parameters are similar to that of the  $\mathcal{N} = 2^*$  SU(2) theory, i.e. the superconformal theory with a single adjoint hypermultiplet. The  $\mathcal{N} = 2^*$  order parameter transforms as a modular form under the group  $\Gamma(2) \times \Gamma(2)$  with the first factor acting on  $\tau$  and the second on  $\tau_{UV}$ , while it also transforms as a modular form under simultaneous  $\text{SL}(2, \mathbb{Z})$  transformations of  $\tau$  and  $\tau_{UV}$  [148, 173, 174]. It was later clarified that  $u(\tau, \tau_{UV})$  is an example of a meromorphic *bimodular* form [85]. Such functions have appeared, although sporadically, in the mathematical literature [175–177].

The SU(2)  $N_f = 4$  theory exhibits a richer structure: It has four mass parameters that give rise to seven singular vacua on the  $u$ -plane. For special choices of the masses, the  $u$ -planes contain any of the three SU(2) Argyres-Douglas superconformal points  $(A_1, A_2)$ ,  $(A_1, A_3)$  and  $(A_1, D_4)$ , while in the massless case there is a non-abelian Coulomb point with a five quaternionic-dimensional Higgs branch [104, 105]. For generic masses, the singularities are roots of a sextic polynomial, for which there is no known expression. The flavour symmetry SO(8) becomes the universal cover Spin(8) in the quantum theory. It has a *triality* group  $\text{Out}(\text{Spin}(8))$  of outer automorphisms, which is isomorphic to the symmetric group  $S_3$  on three letters. The full symmetry group of the  $N_f = 4$  curve is then the semidirect product  $\text{Spin}(8) \rtimes_{\varphi} \text{SL}(2, \mathbb{Z})$ , which is induced by the group homomorphism  $\varphi : \text{SL}(2, \mathbb{Z}) \rightarrow \text{Out}(\text{Spin}(8))$ . As the triality group is of order  $|S_3| = 6$ , the orbits of the group action on mass space  $\mathbb{C}^4$  generally have six elements. However, there are specific mass configurations with enhanced global symmetry that are invariant under subgroups of the triality group, for which the orbits collapse, either to three elements or to a single element.

We study four such configurations in detail, and show that their order parameters, periods and discriminants are bimodular forms for subgroups of  $\text{SL}(2, \mathbb{Z})$ . For the triality invariant case  $(m_1, m_2, m_3, m_4) = (m, m, 0, 0)$  we find that the order parameter is a bimodular form of weight  $(0, 2)$  with  $\Gamma(2)$  acting on both  $\tau$  and  $\tau_{UV}$  individually, while it is also bimodular for  $\text{SL}(2, \mathbb{Z})$  acting on  $\tau$  and  $\tau_{UV}$  simultaneously. If all four hypermultiplets are rather given an equal mass, the triality orbit has three elements. The  $u$ -planes for these three mass configurations are modular curves for the three subgroups of  $\text{SL}(2, \mathbb{Z})$  conjugate to  $\Gamma^0(4)$ . The order parameters, periods and discriminants are permuted by triality, and can thus be organised into vectors to form one-parameter families of vector-valued bimodular forms for  $\text{SL}(2, \mathbb{Z})$ . We further give some examples of exact expressions for order parameters of more complicated theories with two independent mass parameters. These theories then include both AD points and branch cuts.

## 3.2 Four flavours and triality

The one-loop beta function of  $\mathcal{N} = 2$  supersymmetric Yang–Mills theory with  $N_f \leq 4$  hypermultiplets in the fundamental representation is  $\beta_{N_f}(g_{\text{YM}}) = -\frac{g_{\text{YM}}^3}{16\pi^2}(4 - N_f)$ . The gauge coupling  $g_{\text{YM}}$  is combined with the theta angle  $\theta$  in the Lagrangian as  $\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g_{\text{YM}}^2}$ . This complexified gauge coupling can be considered as the expectation value of a background chiral superfield. In the renormalisation scheme where the superpotential remains a holomorphic function of all chiral superfields, the one-loop running coupling at the energy scale  $E$  can be expressed as [10]

$$\tau(E) = \tau_{\text{UV}} - \frac{4 - N_f}{2\pi i} \log \frac{E}{\Lambda_{\text{UV}}}. \quad (3.1)$$

It is one-loop exact in the holomorphic scheme, and thus for  $N_f < 4$  the combination

$$\Lambda_{N_f}^{4-N_f} := \Lambda_{\text{UV}}^{4-N_f} e^{2\pi i \tau_{\text{UV}}} \quad (3.2)$$

of the scale  $\Lambda_{\text{UV}}$  and the coupling  $\tau_{\text{UV}}$  is invariant to all orders in perturbation theory. This complexified dynamical scale  $\Lambda_{N_f}$  sets the overall scale of the theory. For  $N_f = 4$  on the other hand, there is a distinguished dimensionless parameter  $\tau_{\text{UV}}$ , on which the theory depends nontrivially. To shorten the notation, we will also set  $\tau_0 := \tau_{\text{UV}}$  and  $q_0 := e^{2\pi i \tau_0}$  in the following.

### 3.2.1 The curve

The low-energy physics of  $\mathcal{N} = 2$  SYM with  $N_f = 4$  massive hypermultiplets has been determined in [46, 148, 173, 178–180]. Similar to the asymptotically free ( $N_f \leq 3$ ) cases, the physics is encoded in an elliptic curve which depends holomorphically on the Coulomb branch parameter  $u \in \mathcal{B}_4$ . This coordinate  $u$  parametrises the Coulomb branch  $\mathcal{B}_4$  of the  $N_f = 4$  theory. Let us first define the symmetric mass combinations

$$\begin{aligned} \llbracket m_1^k \rrbracket &= \sum_{i=1}^4 m_i^k, & \llbracket m_1^2 m_2^2 \rrbracket &= \sum_{i < j} m_i^2 m_j^2 \\ \llbracket m_1^4 m_2^2 \rrbracket &= \sum_{i \neq j} m_i^4 m_j^2, & \llbracket m_1^2 m_2^2 m_3^2 \rrbracket &= \sum_{i < j < k} m_i^2 m_j^2 m_k^2, \\ \text{Pf}(\mathbf{m}) &= m_1 m_2 m_3 m_4. \end{aligned} \quad (3.3)$$

The  $N_f = 4$  curve for generic masses is then [46]

$$y^2 = W_1 W_2 W_3 + A(W_1 T_1 (e_2 - e_3) + W_2 T_2 (e_3 - e_1) + W_3 T_3 (e_1 - e_2)) - A^2 N, \quad (3.4)$$



where

$$\begin{aligned}
W_i &= x - e_i u - e_i^2 R, \\
A &= (e_1 - e_2)(e_2 - e_3)(e_3 - e_1), \\
R &= \frac{1}{2} \llbracket m_1^2 \rrbracket, \\
T_1 &= \frac{1}{12} \llbracket m_1^2 m_2^2 \rrbracket - \frac{1}{24} \llbracket m_1^4 \rrbracket, \\
T_{2,3} &= \mp \frac{1}{2} \text{Pf}(\mathbf{m}) - \frac{1}{24} \llbracket m_1^2 m_2^2 \rrbracket + \frac{1}{48} \llbracket m_1^4 \rrbracket, \\
N &= \frac{3}{16} \llbracket m_1^2 m_2^2 m_3^2 \rrbracket - \frac{1}{96} \llbracket m_1^4 m_2^2 \rrbracket + \frac{1}{96} \llbracket m_1^6 \rrbracket, \tag{3.5}
\end{aligned}$$

and the half periods

$$e_1 = \frac{1}{3}(\vartheta_3^4 + \vartheta_4^4), \quad e_2 = -\frac{1}{3}(\vartheta_2^4 + \vartheta_3^4), \quad e_3 = \frac{1}{3}(\vartheta_2^4 - \vartheta_4^4) \tag{3.6}$$

are functions of  $\tau_0 = \tau_{\text{UV}}$ , with  $e_1 + e_2 + e_3 = 0$ . The Jacobi theta functions  $\vartheta_i$  are defined in Appendix A. Since the rhs of (3.4) is a cubic polynomial in  $x$ , it is indeed an elliptic curve. We obtain the low energy theory with  $N_f = 3$  flavours by taking the limit  $\tau_0 \rightarrow i\infty$  (or, equivalently,  $q_0 \rightarrow 0$ ) and  $m_4 \rightarrow \infty$  while holding  $\Lambda_3 = 64q_0^{\frac{1}{2}}m_4$  fixed. The order parameters are then related as [46]

$$u_{N_f=4} + \frac{1}{4}e_1 \llbracket m_1^2 \rrbracket \rightarrow u_{N_f=3}. \tag{3.7}$$

See (2.3) for the corresponding curves.

Let us study the singularity structure of the Coulomb branch. For generic masses  $\mathbf{m} = (m_1, m_2, m_3, m_4)$ , there are six distinct strong coupling singularities. By tuning the mass, some of those singularities can collide. If we weight each singularity by the number of massless hypermultiplets at that point, the total weighted number of singularities on the  $u$ -plane is thus always 6. Denote by  $k_l$  the weight of the  $l$ -th singularity, and by  $\mathbf{k}(\mathbf{m}) = (k_1, k_2, \dots)$  the vector of those weights. In Table 1, we list a selection of specifically symmetric mass configurations. One notices that certain a priori unrelated cases have the same weight vector  $\mathbf{k}$  and global symmetries, such as the cases {B, C, D} and {E, F, G}. This will be explained in the next subsection. It is also clear that  $\mathbf{k}(\mathbf{m})$  gives a partition of 6, the total number of singularities on  $\mathcal{B}_4$ .

### 3.2.2 Triality

Let us study the symmetries of the  $N_f = 4$  curve (3.4) with mass  $\mathbf{m} = (m_1, m_2, m_3, m_4)$ . Scale invariance, the  $U(1)_R$  R-symmetry and the  $SL(2, \mathbb{Z})$  symmetry acting on  $\tau_0$  are explicitly broken by the masses. There is a remnant scale invariance on the Coulomb branch, which manifests itself in the  $\mathcal{J}$ -invariant  $\mathcal{B}_4 \times \mathbb{C}^4 \times \mathbb{H} \rightarrow \mathbb{C}$  of the curve being a quasi-homogeneous rational function of degree 0 and type (2, 1, 0),

$$\mathcal{J}(s^2 u, s \mathbf{m}, \tau_0) = \mathcal{J}(u, \mathbf{m}, \tau_0), \quad s \in \mathbb{C}^*. \tag{3.8}$$

Name	$\mathbf{m}$	$\mathbf{k}(\mathbf{m})$	global symmetry
A	$(m, m, 0, 0)$	$(2, 2, 2)$	$SU(2) \times SU(2) \times SU(2) \times U(1)$
B	$(m, m, m, m)$	$(4, 1, 1)$	$SU(4) \times U(1)$
C	$(2m, 0, 0, 0)$	$(4, 1, 1)$	$SU(4) \times U(1)$
D	$(m, m, m, -m)$	$(4, 1, 1)$	$SU(4) \times U(1)$
E	$(m, m, \mu, \mu)$	$(2, 2, 1, 1)$	$SU(2) \times SU(2) \times U(1) \times U(1)$
F	$(m + \mu, m - \mu, 0, 0)$	$(2, 2, 1, 1)$	$SU(2) \times SU(2) \times U(1) \times U(1)$
G	$(m, m, \mu, -\mu)$	$(2, 2, 1, 1)$	$SU(2) \times SU(2) \times U(1) \times U(1)$

Table 1: List of some mass cases with enhanced flavour symmetry in  $N_f = 4$ , with  $\mu \neq m$ . The vector  $\mathbf{k}(\mathbf{m})$  lists the multiplicities of all singularities on the Coulomb branch  $\mathcal{B}_4$  with mass  $\mathbf{m}$ .

The  $N_f = 4$  theory has an  $SO(8)$  flavour symmetry, which becomes the universal double cover  $Spin(8)$  in the quantum theory. In particular, there exists a short exact sequence

$$1 \rightarrow \mathbb{Z}_2 \rightarrow Spin(8) \rightarrow SO(8) \rightarrow 1 \quad (3.9)$$

of Lie groups. The cover  $Spin(8)$  has an order 6 group  $Out(Spin(8))$  of outer automorphisms, which is isomorphic to  $S_3$  [181, 182].<sup>17</sup>

This group of outer automorphisms acts on the  $N_f = 4$  theory as follows. The states with  $(n_m, n_e) = (0, 1)$  are the elementary hypermultiplets, which transform in the fundamental vector representation of  $Spin(8)$ . The magnetic monopole  $(1, 0)$  transforms as one spinor representation, and the dyon  $(1, 1)$  transforms as the conjugate spinor representation [46]. By an accidental isomorphism, these three representations are all 8-dimensional and irreducible, and they are permuted by the outer automorphism group  $Out(Spin(8)) \cong S_3$ . It is generated by

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \mathcal{S} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad (3.10)$$

which act on the column vector  $\mathbf{m} \in \mathcal{M} := \mathbb{C}^4$  from the left [46, 183, 184]. The map  $\mathcal{T}$  exchanges the two spinors keeping the vector fixed, while  $\mathcal{S}$  exchanges the vector with the spinor, keeping the conjugate spinor fixed. This is depicted in Fig. 21.

The generators (3.10) satisfy the algebra

$$\mathcal{T}^2 = \mathcal{S}^2 = (\mathcal{ST})^3 = \mathcal{ST}^2\mathcal{S} = \mathbb{1}, \quad (3.11)$$

<sup>17</sup>For any Lie group  $G$ , there are three associated groups.  $Aut(G)$  is the Lie group consisting of all automorphisms of  $G$  (i.e. group isomorphisms  $G \rightarrow G$ ),  $Inn(G)$  is a normal subgroup of  $Aut(G)$  consisting of inner automorphisms given by  $\alpha_g(h) := ghg^{-1}$  for any  $g \in G$ , and  $Out(G) = Aut(G)/Inn(G)$  is the quotient group. The automorphism group of  $Spin(8)$  is  $Aut(SO(8)) = PSO(8) \rtimes S_3$  [181].

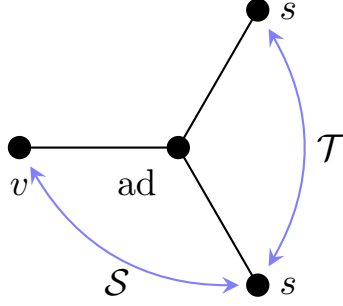


Figure 21: Dynkin diagram of  $\mathfrak{d}_4 = \text{Lie}(\text{Spin}(8))$ . The group  $\mathcal{T} \cong S_3$  of outer isomorphisms acts by permutations on the three conjugacy classes of irreducible representations  $v$ ,  $s$  and  $\bar{s}$  attached to the nodes of the diagram. The 28-dimensional adjoint representation is left invariant by  $\mathcal{T}$ .

which is a presentation of the symmetric group  $S_3$ . Since  $\mathcal{T}^\top \mathcal{T} = \mathcal{S}^\top \mathcal{S} = \mathbb{1}$  but  $\det \mathcal{T} = \det \mathcal{S} = -1$ , the matrices  $\mathcal{T}$  and  $\mathcal{S}$  generate a subgroup

$$\mathcal{T} = \langle \mathcal{T}, \mathcal{S} \rangle \quad (3.12)$$

of the orthogonal group  $O(4, \mathbb{C})$ , isomorphic to  $S_3$ .<sup>18</sup> As a consequence, they leave the inner product  $\llbracket m_1^2 \rrbracket$  (3.3) invariant.

The flavour symmetry mixes with the  $\text{SL}(2, \mathbb{Z})$ -symmetry acting on the UV-coupling  $\tau_0$  in an interesting way. To see this, notice that the reduction  $\mathbb{Z} \rightarrow \mathbb{Z}_2$  modulo 2 induces a homomorphism  $\text{SL}(2, \mathbb{Z}) \rightarrow \text{SL}(2, \mathbb{Z}/2\mathbb{Z})$ . Since  $\text{SL}(2, \mathbb{Z}/2\mathbb{Z}) \cong S_3$  are isomorphic, by transitivity we have a group homomorphism

$$\varphi : \text{SL}(2, \mathbb{Z}) \longrightarrow \text{Out}(\text{Spin}(8)). \quad (3.13)$$

The full symmetry group of the  $N_f = 4$  theory is the semidirect product [46]<sup>19</sup>

$$\mathfrak{T} := \text{Spin}(8) \rtimes_{\varphi} \text{SL}(2, \mathbb{Z}). \quad (3.14)$$

The group  $(\mathfrak{T}, \bullet)$  consists of elements  $(A, \gamma) \in \text{Spin}(8) \times \text{SL}(2, \mathbb{Z})$ , with group operation

$$(A, \gamma) \bullet (\tilde{A}, \tilde{\gamma}) := (A \varphi(\gamma)(\tilde{A}), \gamma \circ \tilde{\gamma}). \quad (3.15)$$

The action of (3.10) is thus accompanied with an action of  $\text{SL}(2, \mathbb{Z})$  on  $\tau$  and  $\tau_0$ . From (3.11) we find that  $\mathcal{T}^2$  and  $\mathcal{S}\mathcal{T}^2\mathcal{S}$  leave any mass configuration invariant. This implies that the theory should also be invariant under the simultaneous action of  $T^2$  and  $ST^2S$  on the two couplings. These two matrices in  $\text{SL}(2, \mathbb{Z})$  generate the principal congruence subgroup  $\Gamma(2)$ . From this it is also clear that

$$\text{SL}(2, \mathbb{Z})/\Gamma(2) = \{I, T, S, TS, ST, TST\} \cong S_3, \quad (3.16)$$

which is another way to see that the group of outer isomorphisms is  $S_3$  [184]. This action is depicted in Fig. 22. The subgroup  $\Gamma(2)$  is the kernel of the

<sup>18</sup>They actually form a subgroup of  $O(4, \mathbb{Q})$ , but act on  $\mathbf{m} \in \mathbb{C}^4$ .

<sup>19</sup>Recall that for two groups  $G$  and  $H$ , a group homomorphism  $\varphi : G \rightarrow \text{Aut}(H)$  defines a semi-direct product  $H \rtimes_{\varphi} G \subset H \times G$  with the multiplication  $(h_1, g_1)(h_2, g_2) := (h_1 \varphi(g_1)(h_2), g_1 g_2)$ . For  $(h, g) \in H \rtimes_{\varphi} G$ , the inverse is found as  $(\varphi(g^{-1})(h^{-1}), g^{-1})$ .

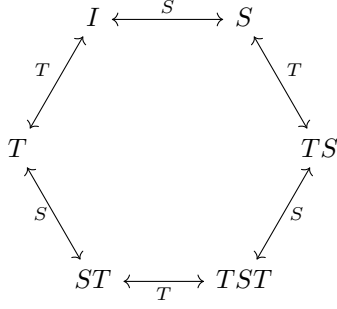


Figure 22: Action of  $SL(2, \mathbb{Z})$  on  $SL(2, \mathbb{Z})/\Gamma(2) \cong S_3$

above group homomorphism  $SL(2, \mathbb{Z}) \rightarrow SL(2, \mathbb{Z}/2\mathbb{Z})$ , such that it is in fact a normal subgroup  $\Gamma(2) \triangleleft SL(2, \mathbb{Z})$ .

The moduli spaces of the cases A–G of Table 1 are related by  $\mathcal{T}$  in the following way. We have that  $\mathbf{m}_A, \mathbf{m}_C, \mathbf{m}_F$  are invariant under  $\mathcal{T}$ . Case A is invariant under both  $\mathcal{T}$  and  $\mathcal{S}$ . The  $\mathcal{S}$ -transformation relates cases B and C, as well as E and F, while leaving cases D and G invariant. We depict the relation among cases B, C and D in Fig. 23. For the cases E, F and G, there is an analogous diagram. An instance of these relations is that the weights of

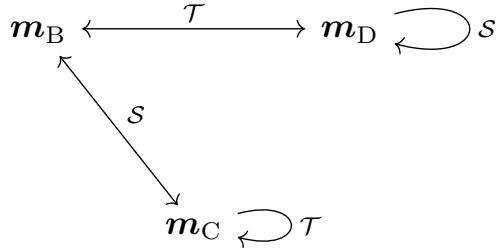


Figure 23: Relation among  $\mathbf{m}_B = (m, m, m, m)$ ,  $\mathbf{m}_C = (2m, 0, 0, 0)$  and  $\mathbf{m}_D = (m, m, m, -m)$ .

the singular structure on the Coulomb branch are invariant under those spaces that are related by triality,

$$\mathbf{k}(\mathcal{T}\mathbf{m}) = \mathbf{k}(\mathbf{m}). \quad (3.17)$$

Using the action of the  $SO(8)$  flavour group, a large range of masses with equivalent duality diagrams can be reached. For example, the mass  $\mathbf{m} = (2m, 0, 0, 0)$  is related to  $\mathbf{m} = (0, 0, 0, 2m)$  by an  $SO(8)$  rotation. The first one is invariant under  $\mathcal{T}$  while the second one is not. The orbit under  $\mathcal{T}$  and  $\mathcal{S}$  for the case  $\mathbf{m} = \mathbf{m}_B = (2m, 0, 0, 0)$  is, as we have just discussed, given by Fig. 23, while that of  $\mathbf{m} = (0, 0, 0, 2m)$  is given in Fig. 24. We see that it is of order six, and includes different relative signs compared to  $\mathbf{m}_A$  and  $\mathbf{m}_D$ . On closer inspection, we note that the mass vectors come in pairs differing by an overall sign, which is an element of  $SO(8)$ . Thus identifying the mass vectors related by  $SO(8)$  in diagram 24, we find that it is equivalent to diagram 23.

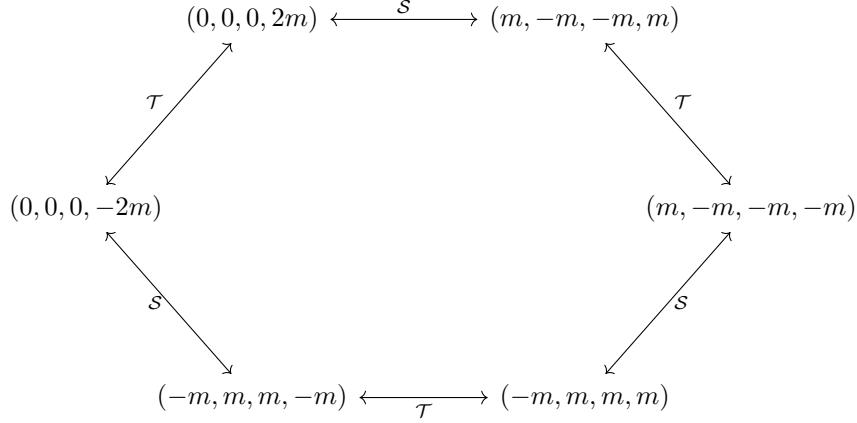


Figure 24: Orbit of the mass vector  $\mathbf{m} = (0, 0, 0, 2m)$  under  $\mathcal{T}$  and  $\mathcal{S}$ .

### 3.2.3 Group action

The action

$$\begin{aligned} \mathcal{T} \times \mathcal{M} &\longrightarrow \mathcal{M} \\ (g, \mathbf{m}) &\longmapsto g \cdot \mathbf{m} \end{aligned} \quad (3.18)$$

of the triality group  $\mathcal{T}$  on mass space  $\mathcal{M}$  can be studied in great detail. It is easy to check that the action is faithful<sup>20</sup>, but neither free<sup>21</sup> nor transitive<sup>22</sup>.

Up to conjugation,  $S_3 \cong \mathcal{T}$  has four subgroups. They are: the trivial group  $\mathbb{Z}_1$ , the symmetric group  $S_2 \cong \mathbb{Z}_2$ , the alternating group  $A_3 \cong \mathbb{Z}_3$ , and  $S_3$  itself. They have order 1, 2, 3, and 6, respectively. All three proper subgroups are abelian. For a given  $\mathbf{m}$ , triality thus not always acts by the full  $S_3$  but rather by a subgroup. For every  $\mathbf{m} \in \mathcal{M}$  we can study the orbit  $\mathcal{T} \cdot \mathbf{m} = \{g \cdot \mathbf{m} \mid g \in \mathcal{T}\}$ . The sets of orbits of  $\mathcal{M}$  then give a partition of  $\mathcal{M}$  under the action (3.18).

First, notice that since  $\mathcal{T}$  is a finite group, all elements have finite order. In particular,  $\mathcal{T}^2 = \mathcal{S}^2 = (\mathcal{T}\mathcal{S}\mathcal{T})^2 = \mathbb{1}$  and  $(\mathcal{S}\mathcal{T})^3 = (\mathcal{T}\mathcal{S})^3 = \mathbb{1}$ . The stabiliser subgroup of a mass  $\mathbf{m} \in \mathcal{M}$  is defined as  $\mathcal{T}_{\mathbf{m}} = \{g \in \mathcal{T} \mid g \cdot \mathbf{m} = \mathbf{m}\}$ . By the orbit-stabiliser theorem

$$|\mathcal{T} \cdot \mathbf{m}| = |\mathcal{T}|/|\mathcal{T}_{\mathbf{m}}|, \quad (3.19)$$

it suffices to study the fixed point equations in order to identify the stabiliser subgroups  $\{\mathbb{Z}_1, S_2, A_3, S_3\}$  with the subgroups of  $\mathcal{T}$ . It is straightforward to identify the fixed point loci

$$\begin{aligned} \mathcal{L}_{\mathcal{T}} &= \{\mathbf{m} \in \mathcal{M} \mid m_4 = 0\}, \\ \mathcal{L}_{\mathcal{S}} &= \{\mathbf{m} \in \mathcal{M} \mid m_1 = m_2 + m_3 + m_4\}, \\ \mathcal{L}_{\mathcal{S}\mathcal{T}\mathcal{S}} &= \{\mathbf{m} \in \mathcal{M} \mid m_1 = m_2 + m_3 - m_4\}, \\ \mathcal{L}_{\mathcal{S}\mathcal{T}} = \mathcal{L}_{\mathcal{T}\mathcal{S}} &= \{\mathbf{m} \in \mathcal{M} \mid m_1 = m_2 + m_3 \text{ and } m_4 = 0\}, \end{aligned} \quad (3.20)$$

<sup>20</sup>For every  $g \neq h \in \mathcal{T}$  there exists an  $\mathbf{m} \in \mathcal{M}$  such that  $g \cdot \mathbf{m} \neq h \cdot \mathbf{m}$ .

<sup>21</sup>A group action is free if it has no fixed points, but  $\mathbf{m} = 0$  is a fixed point for any  $g \in \mathcal{T}$ .

<sup>22</sup>For each pair  $\mathbf{m}, \tilde{\mathbf{m}} \in \mathcal{M}$  there exists  $g \in \mathcal{T}$  such that  $g \cdot \mathbf{m} = \tilde{\mathbf{m}}$ . A counterexample would be  $\mathbf{m} = 0$  and  $\tilde{\mathbf{m}} \neq 0$ .

where  $\mathcal{L}_g = \{\mathbf{m} \in \mathcal{M} \mid g \cdot \mathbf{m} = \mathbf{m}\}$ . For  $\mathbf{m}$  in precisely one of  $\mathcal{L}_\mathcal{T}$ ,  $\mathcal{L}_\mathcal{S}$  or  $\mathcal{L}_{\mathcal{S}\mathcal{T}\mathcal{S}}$ , one finds that  $|\mathcal{T} \cdot \mathbf{m}| = 3$ . From (3.19) it then follows that  $|\mathcal{T}_\mathbf{m}| = 2$ , such that  $\mathcal{T}_\mathbf{m} \cong S_2$ . In fact, since  $\mathcal{T}$ ,  $\mathcal{S}$  and  $\mathcal{S}\mathcal{T}\mathcal{S}$  are all order 2 elements of  $\mathcal{T}$ , the stabiliser groups  $\mathcal{T}_\mathbf{m}$  for  $\mathbf{m}$  in either of the three loci are precisely the three order 2 conjugate subgroups of  $\mathcal{T} \cong S_3$ .<sup>23</sup>

The intersection

$$\mathcal{L}_1 = \mathcal{L}_\mathcal{T} \cap \mathcal{L}_\mathcal{S} = \{\mathbf{m} \in \mathcal{M} \mid m_1 = m_2 + m_3 \text{ and } m_4 = 0\} \quad (3.21)$$

is the locus of triality invariant masses,  $\mathcal{T} \cdot \mathbf{m} = \mathbf{m}$ . Thus, according to (3.19) we have  $|\mathcal{T}_\mathbf{m}| = 6$  for such masses, such that indeed  $\mathcal{T}_\mathbf{m} = \mathcal{T}$ . For the last locus in (3.20), we see immediately that  $\mathcal{L}_{\mathcal{S}\mathcal{T}} = \mathcal{L}_{\mathcal{T}\mathcal{S}} = \mathcal{L}_\mathcal{T} \cap \mathcal{L}_\mathcal{S}$  contains precisely the invariant masses. Therefore, if  $\mathbf{m}$  is kept fixed by either  $\mathcal{T}\mathcal{S}$  or  $\mathcal{S}\mathcal{T}$  then it is also fixed by both  $\mathcal{T}$  and  $\mathcal{S}$  and therefore by all of  $\mathcal{T}$ . Since  $\mathcal{S}\mathcal{T}$  and  $\mathcal{T}\mathcal{S}$  are the only elements of  $\mathcal{T}$  of order 3, there is actually no mass  $\mathbf{m}$  such that  $\mathcal{T} \cdot \mathbf{m}$  has 2 elements, and so there is no stabiliser subgroup isomorphic to  $A_3$ . By case analysis, it is also easy to prove that the set  $\mathcal{T} \cdot \mathbf{m}$  has 1, 3 or 6 elements.

Let us summarise. If  $\mathbf{m} \in \mathcal{L}_1$ , it is invariant under  $\mathcal{T}$ . If  $\mathbf{m}$  is in any of  $\mathcal{L}_\mathcal{T}$ ,  $\mathcal{L}_\mathcal{S}$  or  $\mathcal{L}_{\mathcal{S}\mathcal{T}\mathcal{S}}$ , it could be in the intersection of any two of them. These intersections are however all equal to  $\mathcal{L}_1$ , which is of course because any two elements of  $\{\mathcal{T}, \mathcal{S}, \mathcal{S}\mathcal{T}\mathcal{S}\}$  generate  $\mathcal{T}$ . This is depicted in Fig. 25.

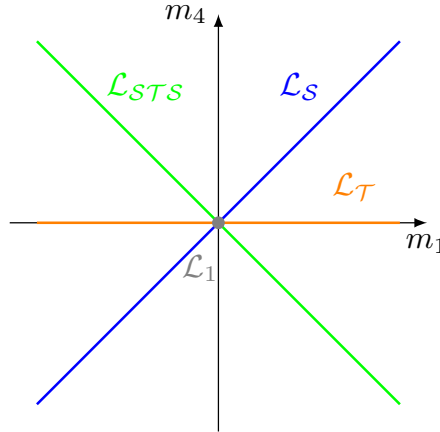


Figure 25: The loci (3.20) with nontrivial stabiliser groups on the subspace  $m_2 = m_3 = 0$  in  $\mathcal{M}$ . They all mutually intersect in the locus  $\mathcal{L}_1$  of triality invariant masses.

If  $\mathbf{m}$  is then an element of

$$\mathcal{L}_3 = \mathcal{L}_\mathcal{T} \cup \mathcal{L}_\mathcal{S} \cup \mathcal{L}_{\mathcal{S}\mathcal{T}\mathcal{S}} \setminus \mathcal{L}_1, \quad (3.22)$$

then the stabiliser group of  $\mathbf{m}$  is isomorphic to  $S_2$ . If  $\mathbf{m}$  does not lie in either  $\mathcal{L}_1$  or  $\mathcal{L}_3$ , then there is no remaining symmetry. It lies in

$$\mathcal{L}_6 = \mathcal{M} \setminus \mathcal{L}_1 \cup \mathcal{L}_3, \quad (3.23)$$

<sup>23</sup>If we represent  $S_3$  in cycle notation of permutations of  $\{1, 2, 3\}$ , the three order 2 conjugate subgroups of  $S_3$  are  $\{(), (1, 2)\}$ ,  $\{(), (1, 3)\}$  and  $\{(), (2, 3)\}$ .

and its stabiliser group is trivial.

### 3.3 Order parameters and bimodular forms

For the  $N_f = 4$  SW theory, there are other curves than the one introduced by Seiberg and Witten [20, 21, 111, 113, 148, 179, 185–187]. In this section, we focus on the modularity of the original SW curve (3.4). We proceed by studying the mass configurations with the largest flavour symmetry groups, A, B, C and D. In all these cases,  $u$  is a weight  $(0, 2)$  bimodular form, which we define below, for a triple of groups related to the duality group of the decoupling theory where the mass of the hypermultiplets is infinitely large, and the stabiliser group of the mass under the triality action. Since case A is triality invariant,  $u$  in that case transforms under the full  $\text{SL}(2, \mathbb{Z})$  group. The other cases B, C and D are permuted by triality, and furnish a vector-valued bimodular form.

The massless case where  $\mathbf{m}_0 = (0, 0, 0, 0)$  is very simple, as  $j(\tau) = \mathcal{J}(u, 0, \tau_0) = j(\tau_0)$  and therefore

$$\tau(u) = \tau_0 \quad (3.24)$$

is constant over the whole Coulomb branch  $\mathcal{B}_4 \ni u$ . In other words, the coupling  $\tau$  is fixed and thus does not run, which is a consequence of the massless  $N_f = 4$  theory being exactly superconformal. There are six singularities, which all sit at the origin  $u = 0$  and form the non-abelian Coulomb point with a five quaternionic-dimensional Higgs branch [105].

#### 3.3.1 Case A

For the mass  $\mathbf{m}_A = (m, m, 0, 0)$ , this allows to express  $u$  as a rational function in Jacobi theta functions of  $\tau_0$  and  $\tau$ . There are in fact six solutions to the correspondence  $\mathcal{J}(u(\tau), \mathbf{m}, \tau_0) = j(\tau)$ . A consistent way of choosing which solution to use, which we will employ throughout, is to take the one that has the right decoupling limit when decoupling the massive hypermultiplets, i.e., the one that decouples to the order parameter of massless  $N_f = 2$ , Eq. (2.86).

In view of the more complicated mass cases, we can further simplify the rather lengthy expression. The dependence on  $\tau$  is in fact only through  $\lambda = \frac{\vartheta_2^4}{\vartheta_3^4}$ . This is not quite true for  $\tau_0$ , for which  $u$  has weight 2 [148]. This weight factor can be extracted by eliminating  $\vartheta_4(\tau_0)$  through the Jacobi identity (A.12) and  $\vartheta_2(\tau_0)$  through the definition of  $\lambda(\tau_0)$ . This gives

$$u_A(\tau, \tau_0) = -\frac{m^2}{3} \vartheta_3(\tau_0)^4 \frac{\lambda(\tau_0)^2 + 2(\lambda(\tau) - 1)\lambda(\tau_0) - \lambda(\tau)}{\lambda(\tau_0) - \lambda(\tau)}. \quad (3.25)$$

The simple mass dependence of  $u_A$  is a consequence of the scaling symmetry (3.8). The second prefactor  $\vartheta_3(\tau_0)^4$  gives the weight 2. The remaining quotient is written in a manifestly invariant fashion. Let us denote by  $\Gamma_\tau$  ( $\Gamma_{\tau_0}$ ) a group acting by linear fractional transformations on  $\tau$  ( $\tau_0$ ). As  $\vartheta_3(\tau_0)^4$  is a modular form of weight 2 and  $\lambda(\tau_0)$  a modular function (of weight 0) for  $\Gamma(2)_{\tau_0}$ , one

can easily see that  $u_A(\tau, \tau_0)$  is a weight 2 modular form for  $\Gamma(2)_{\tau_0}$  for fixed  $\tau$ , and a modular function for  $\Gamma(2)_{\tau}$  for fixed  $\tau_0$ . We thus have that

$$u_A(\gamma_1\tau, \gamma_2\tau_0) = (c_2\tau_0 + d_2)^2 u_A(\tau, \tau_0), \quad \gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \Gamma(2) \quad (3.26)$$

for  $i = 1, 2$ . We call  $u_A$  modular for  $\Gamma(2)_{\tau} \times \Gamma(2)_{\tau_0}$ , where the occurrence of two groups indicates that they act on both variables  $\tau$  and  $\tau_0$  separately.

The mass  $\mathbf{m}_A$  is invariant under the triality group (3.12). As triality acts on  $\tau$  and  $\tau_0$  together, this suggests that  $u_A$  transforms under a simultaneous transformation of  $\mathrm{SL}(2, \mathbb{Z})$ . Indeed, if one acts simultaneously on  $\tau$  and  $\tau_0$  with  $\mathrm{SL}(2, \mathbb{Z})$ , it is easy to check from  $T : \lambda \mapsto \frac{\lambda}{\lambda-1}$  and  $S : \lambda \mapsto 1 - \lambda$  that  $u_A(\tau, \tau_0)$  transforms as

$$u_A(\gamma\tau, \gamma\tau_0) = (c\tau_0 + d)^2 u_A(\tau, \tau_0), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}). \quad (3.27)$$

We call  $u_A$  modular for  $\mathrm{SL}(2, \mathbb{Z})_{(\tau, \tau_0)}$ , where the notation indicates that the single group  $\mathrm{SL}(2, \mathbb{Z})$  acts on both  $\tau$  and  $\tau_0$  simultaneously. The two transformations (3.26) and (3.27) are characteristic properties for functions known as ‘‘bimodular forms’’ [175–177]. For our application to  $N_f = 4$  SQCD, we will adopt the following definition:

**Definition 5** (Bimodular form). *Let  $(\Gamma_1, \Gamma_2; \Gamma)$  be a triple of subgroups of  $\mathrm{SL}(2, \mathbb{R})$  commensurable with  $\mathrm{SL}(2, \mathbb{Z})$ .<sup>24</sup> A two-variable meromorphic function  $F : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{C}$  is called a bimodular form of weight  $(k_1, k_2)$  for the triple  $(\Gamma_1, \Gamma_2; \Gamma)$  if it satisfies both Condition 1 & 2:*

- *Condition 1: For all  $\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \Gamma_i$ ,  $i = 1, 2$ ,  $F$  transforms as*

$$F(\gamma_1\tau_1, \gamma_2\tau_2) = \chi(\gamma_1, \gamma_2) (c_1\tau_1 + d_1)^{k_1} (c_2\tau_2 + d_2)^{k_2} F(\tau_1, \tau_2), \quad (3.28)$$

*for a certain multiplier  $\chi : \Gamma_1 \times \Gamma_2 \rightarrow \mathbb{C}^*$ . We call this the separate transformation of  $F$  under  $(\Gamma_1, \Gamma_2)$ , and denote it by  $(\Gamma_1)_{\tau_1} \times (\Gamma_2)_{\tau_2}$ .*

- *Condition 2: For all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ ,  $F$  transforms as*

$$F(\gamma\tau_1, \gamma\tau_2) = \phi(\gamma) (c\tau_1 + d)^{k_1} (c\tau_2 + d)^{k_2} F(\tau_1, \tau_2), \quad (3.29)$$

*for a multiplier  $\phi : \Gamma \rightarrow \mathbb{C}^*$ . We call this the simultaneous transformation of  $F$  under  $\Gamma$ , and denote it by  $\Gamma_{(\tau_1, \tau_2)}$ .*

Note that condition 2 follows from condition 1 if  $\Gamma$  is the intersection of  $\Gamma_1$  and  $\Gamma_2$ ,  $\Gamma = \Gamma_1 \cap \Gamma_2$  with  $\phi(\gamma) = \chi(\gamma, \gamma)$ ,  $\gamma \in \Gamma$ .

---

<sup>24</sup>A subgroup  $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$  is commensurable with  $\mathrm{SL}(2, \mathbb{Z})$  if  $\Gamma \cap \mathrm{SL}(2, \mathbb{Z})$  has finite index in both  $\mathrm{SL}(2, \mathbb{Z})$  and  $\mathrm{SL}(2, \mathbb{R})$ . This includes in particular all congruence subgroups of  $\mathrm{SL}(2, \mathbb{Z})$ .



This definition contains the main aspects of other definitions of bimodular forms in the literature [85, 175–177].

The definition above for the triple  $(\Gamma_1, \Gamma_2; \Gamma_1 \cap \Gamma_2)$  is equivalent to the definition in [176]. For the triple  $(\Gamma_1, \Gamma_1; \text{SL}(2, \mathbb{Z}))$ , our definition is equivalent to the one of [85]. Finally, for  $k_1 = k_2$  and the triple  $(\Gamma_1, \Gamma_1; \Gamma)$ , our definition is equivalent with [177]. Finally, the definition of Stienstra and Zagier [175], as cited in [176], does require Condition 2 without requiring Condition 1.

From definition 5, we find that  $u_A : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{C}$  in (3.25) is a bimodular form of weight  $(0, 2)$  for the triple

$$(\Gamma(2), \Gamma(2); \text{SL}(2, \mathbb{Z})), \quad (3.30)$$

with trivial multipliers  $\chi$  and  $\phi$ . In fact,  $m \mapsto u_A$  is a 1-parameter family of such bimodular forms.

The function (3.25) can be easily expanded in either  $q = e^{2\pi i\tau}$  or  $q_0 = e^{2\pi i\tau_0}$ . When expanding  $u_A$  around  $q_0 = 0$ , every coefficient is a modular function for  $\Gamma(2)_\tau$ . If we denote the vector space of holomorphic modular forms of weight  $k$  for  $\Gamma \subseteq \text{SL}(2, \mathbb{Z})$  by  $\mathcal{M}_k(\Gamma)$ , then  $u_A \in \mathcal{M}_0(\Gamma(2))[[q_0^{\frac{1}{4}}]]$ . Conversely, we have that  $u_A \in \mathcal{M}_2(\Gamma(2))[[q_0^{\frac{1}{4}}]]$ .

Recall that  $\Gamma(2)$  is a genus zero congruence subgroup. As such, its Hauptmodul  $\lambda$  is the single transcendental generator of the function field of  $\Gamma(2) \backslash \mathbb{H}^*$ . Since  $u_A$  is modular in  $\tau$  as well as  $\tau_0$  for  $\Gamma(2)$  and no larger subgroup of  $\text{SL}(2, \mathbb{Z})$ , the transcendence of  $\lambda$  then implies that (3.25) cannot be simplified further.

The Coulomb branch  $\mathcal{B}_4$  for the mass  $\mathbf{m} = \mathbf{m}_A$  has six singularities that come in three pairs of two. By expanding  $\lambda(\tau)$  around the cusps, one easily finds

$$\begin{aligned} u_A\left(\frac{1}{2}, \tau_0\right) &= -\frac{m^2}{3} \vartheta_3(\tau_0)^4 (\lambda(\tau_0) - 2), \\ u_A(0, \tau_0) &= -\frac{m^2}{3} \vartheta_3(\tau_0)^4 (\lambda(\tau_0) + 1), \\ u_A(1, \tau_0) &= -\frac{m^2}{3} \vartheta_3(\tau_0)^4 (-2\lambda(\tau_0) + 1). \end{aligned} \quad (3.31)$$

Notice that the singularities are holomorphic modular forms of weight 2 for  $\Gamma(2)_{\tau_0}$ , and are permuted by elements of  $(\text{SL}(2, \mathbb{Z})/\Gamma(2))_{\tau_0}$ . The reason for  $u_A(\frac{1}{2}, \tau_0) = u_A(i\infty, \tau_0)$  is explained in section 3.4.2. Since  $\lambda$  is a Hauptmodul for  $\Gamma(2)$ , for given  $\tau_0 \in \Gamma(2) \backslash \mathbb{H}$  there is exactly one  $\tau \in \Gamma(2) \backslash \mathbb{H}$  where  $u$  has a pole. It is where  $\tau$  approaches  $\tau_0$ ,  $u_A(\tau_0, \tau_0) = \infty$ .

We can furthermore compute the period  $\frac{da}{du}$ . Actually,  $\frac{da}{du}$  is not invariant under the monodromy around  $\infty$ , but multiplied by  $-1$ . Instead of  $\frac{da}{du}$ , we may consider  $\left(\frac{da}{du}\right)^2$ , which is monodromy invariant [2, 146]. In the pure ( $N_f = 0$ )  $\text{SU}(2)$  case, it is a modular form of weight 2 for  $\Gamma^0(4)$ . The weight is the same in  $N_f = 4$ , however it also transforms well under fractional linear transformations of  $\tau_0$ . More specifically, we find that

$$\left(\frac{da}{du}\right)_A^2(\tau, \tau_0) = \frac{1}{8m^2} \frac{\vartheta_3(\tau)^4}{\vartheta_3(\tau_0)^8} \frac{\lambda(\tau) - \lambda(\tau_0)}{\lambda(\tau_0)(\lambda(\tau_0) - 1)}. \quad (3.32)$$

The normalisation may be checked from the fact that  $\frac{da}{du} \sim \frac{1}{\sqrt{8u}}$  for  $u \rightarrow \infty$ , which due to (3.25) corresponds to  $\tau \rightarrow \tau_0$ . Since  $\vartheta_3^4$  is a modular form of weight 2 for  $\Gamma(2)$ , it follows that  $\left(\frac{da}{du}\right)_A^2$  satisfies condition 1 of definition 5 with weight  $(2, -4)$  for  $\Gamma(2)_\tau \times \Gamma(2)_{\tau_0}$ . As  $\mathbf{m}_A$  is left invariant by triality,  $\left(\frac{da}{du}\right)_A^2$  must also be modular for  $\text{SL}(2, \mathbb{Z})$ . Indeed, one easily finds that  $\left(\frac{da}{du}\right)_A^2$  also satisfies condition 2, such that it is a bimodular form of weight  $(2, -4)$  for the triple  $(\Gamma(2), \Gamma(2); \text{SL}(2, \mathbb{Z}))$ .

We can also compute the physical discriminant, which for the case A reads

$$\Delta_A = (u - u_A(\frac{1}{2}, \tau_0))^2 (u - u_A(0, \tau_0))^2 (u - u_A(1, \tau_0))^2. \quad (3.33)$$

Since the singularities (3.31) themselves are modular forms for  $\tau_0$ , it is again a bimodular form. One easily computes

$$\Delta_A(\tau, \tau_0) = m^{12} \vartheta_3(\tau_0)^{24} \frac{\lambda(\tau)^2 (\lambda(\tau) - 1)^2 \lambda(\tau_0)^4 (\lambda(\tau_0) - 1)^4}{(\lambda(\tau) - \lambda(\tau_0))^6}. \quad (3.34)$$

As  $\vartheta_3^{24}$  is a modular form of weight 12 for  $\Gamma(2)$ , this shows that  $\Delta_A$  has modular weight  $(0, 12)$  under  $\Gamma(2)_\tau \times \Gamma(2)_{\tau_0}$ . With the same reasoning as above, we find that  $\Delta_A$  is a bimodular form of weight  $(0, 12)$  for the same triple (3.30).

The SW curve for the  $\mathcal{N} = 2^*$  theory with mass  $m$  is identical to the one of  $N_f = 4$  (3.4) with  $\frac{1}{2} \mathbf{m}_A = (\frac{m}{2}, \frac{m}{2}, 0, 0)$  [46]. A reason for this is that both theories have three singularities each with monodromy being conjugate to  $T^2$ . This was in fact the ansatz of Seiberg and Witten to determine the curve with generic masses. This allows to easily find the order parameter for  $\mathcal{N} = 2^*$ ,

$$u_{\mathcal{N}=2^*}(\tau, \tau_0) = \frac{1}{4} u_A(\tau, \tau_0). \quad (3.35)$$

In particular, it is a bimodular form of weight  $(0, 2)$  for  $(\Gamma(2), \Gamma(2); \text{SL}(2, \mathbb{Z}))$ . The derivative  $\frac{da}{du}$  also only receives an overall normalisation from  $N_f = 4$ . In  $\mathcal{N} = 2^*$  the singularities each have degeneracy 1 and not 2 as in  $N_f = 4$  case A. Therefore, we have that  $\Delta_{\mathcal{N}=2^*} = \sqrt{\Delta_A}$ , which is a polynomial of degree 3 in  $u$  and a bimodular form of weight  $(0, 6)$  [85].

The expressions for  $u_{\mathcal{N}=2^*}$  in the literature [85, 148, 173, 188, 189] are related to  $u_A(\tau, \tau_0)$  by a transformation in (3.16), which corresponds to the choice of a different solution of the sextic equation associated with the  $\mathcal{N} = 2^*$  theory [2]. The different choices can be absorbed in the double scaling limit. The counting of the number of poles of  $u_A$  is immediate from our expression (3.25) (see comment on the transcendence of  $\lambda$  in section 3.3.1).

### 3.3.2 Case B

The equal mass case  $\mathbf{m}_B = (m, m, m, m)$  can be treated with the same technique as in the previous subsection. Since  $N_f = 4$  with four equal masses flows to  $N_f = 0$  for  $m \rightarrow \infty$ , we can express the  $\tau$  dependence through the

Hauptmodul  $f := \frac{\vartheta_2^4 + \vartheta_3^4}{\vartheta_2^2 \vartheta_3^2}$  of  $\Gamma^0(4)$ . In fact, the  $N_f = 0$  order parameter (1.35) is just  $\frac{u}{\Lambda_0^2} = -\frac{1}{2}f$ . The order parameter  $u_B$  reads

$$u_B(\tau, \tau_0) = -\frac{m^2}{3} \vartheta_2(\tau_0)^2 \vartheta_3(\tau_0)^2 \frac{2f(\tau_0)^2 + f(\tau)f(\tau_0) - 12}{f(\tau_0) - f(\tau)}, \quad (3.36)$$

which thus does not involve  $\vartheta_4$ . Since  $\vartheta_2(\tau_0)^2 \vartheta_3(\tau_0)^2$  is a holomorphic modular form of weight 2 for  $\Gamma^0(4)_{\tau_0}$ , we find that  $u_B(\tau, \tau_0)$  has bimodular weight  $(0, 2)$  for the separate transformations under  $\Gamma^0(4)_\tau \times \Gamma^0(4)_{\tau_0}$ .

As  $\mathcal{T} : \mathbf{m}_B \mapsto \mathbf{m}_D$ , there is no simultaneous action of  $T$  on  $\tau$  and  $\tau_0$  leaving  $u_B$  invariant. Also, since  $\mathcal{S} : \mathbf{m}_B \mapsto \mathbf{m}_C$ ,  $S$  does not leave  $u_B$  invariant. However, a subgroup  $\mathcal{T}_{\mathbf{m}_B}$  of  $\mathcal{T}$  leaves  $\mathbf{m}_B$  invariant: Out of the six elements of  $\mathcal{T}$ ,  $\mathbf{m}_B$  is left invariant by  $\mathbb{1}$  and  $\mathcal{TST}$ . As the action of  $\mathcal{T}$  is combined in (3.14) with a simultaneous action on  $\tau$  and  $\tau_0$ , we find that  $u_B$  is expected to be invariant under a simultaneous transformation of  $TST \in \text{SL}(2, \mathbb{Z})$ . However, due to the algebra (3.11), the same holds for  $T^2$ . These two matrices generate the congruence subgroup  $\Gamma^0(2)$  of  $\text{SL}(2, \mathbb{Z})$ . It is straightforward to check from the explicit expression (3.36) that  $u_B$  transforms with weight  $(0, 2)$  under a simultaneous transformation on  $\tau$  and  $\tau_0$  of  $\Gamma^0(2)_{(\tau, \tau_0)}$ . This proves that  $u_B$  is an example of a bimodular form of weight  $(0, 2)$  for the triple

$$(\Gamma^0(4), \Gamma^0(4); \Gamma^0(2)). \quad (3.37)$$

As classified in section 3.2.3, the stabiliser subgroup  $\mathcal{T}_{\mathbf{m}_B} = \{\mathbb{1}, \mathcal{TST}\}$  for the mass  $\mathbf{m}_B$  is isomorphic to the group  $S_2 \cong \mathbb{Z}_2$  of order 2. This agrees with the fact that  $\text{SL}(2, \mathbb{Z})/\Gamma^0(2) \cong S_2$ .

The singularities are

$$\begin{aligned} u_B(1, \tau_0) &= -\frac{m^2}{3} \vartheta_2(\tau_0)^2 \vartheta_3(\tau_0)^2 (-f(\tau_0)), \\ u_B(0, \tau_0) &= -\frac{m^2}{3} \vartheta_2(\tau_0)^2 \vartheta_3(\tau_0)^2 (2f(\tau_0) + 6), \\ u_B(2, \tau_0) &= -\frac{m^2}{3} \vartheta_2(\tau_0)^2 \vartheta_3(\tau_0)^2 (2f(\tau_0) - 6), \end{aligned} \quad (3.38)$$

which again are holomorphic modular forms of weight 2 for  $\Gamma^0(4)_{\tau_0}$ . Due to the duality group  $\Gamma^0(4)_\tau$ , we have that  $u_B(1, \tau_0) = u_B(i\infty, \tau_0)$ . This singularity has degeneracy 4, and flows to  $\infty$  for  $m \rightarrow \infty$ . The singularity in the interior is  $u_B(\tau_0, \tau_0) = \infty$ . One can also check that the singularities (3.38) never collide: The conditions  $u_B(1, \tau_0) = u_B(0, \tau_0)$  or  $u_B(1, \tau_0) = u_B(2, \tau_0)$  are equivalent to  $f(\tau_0) = \pm 2$ , whose only solutions are the two cusps  $\tau_0^+ = 0$  and  $\tau_0^- = 2$  of  $\Gamma(2)$ . Since the SW curve is singular for those values of  $\tau_0$ , the singularities do not merge for any finite masses.

Similarly as before, one finds

$$\left(\frac{da}{du}\right)_B^2(\tau, \tau_0) = \frac{1}{8m^2} \frac{\vartheta_2(\tau)^2 \vartheta_3(\tau)^2}{\vartheta_4(\tau_0)^8} (f(\tau) - f(\tau_0)). \quad (3.39)$$

Since  $f$  is a Hauptmodul,  $\vartheta_2^2\vartheta_3^2$  a modular form of weight 2 and  $\vartheta_4^8$  a modular form of weight 4 for  $\Gamma^0(4)$ , it follows that  $\left(\frac{da}{du}\right)_B^2$  is a bimodular form of weight  $(2, -4)$  for the triple (3.37). Finally, the physical discriminant reads

$$\Delta_B(\tau, \tau_0) = m^{12}\vartheta_4(\tau_0)^{24}\frac{(f(\tau)^2 - 4)(f(\tau_0)^2 - 4)^2}{(f(\tau) - f(\tau_0))^6}, \quad (3.40)$$

which is a bimodular form of weight  $(0, 12)$  for the triple (3.37).

### 3.3.3 Case C

Let us study the case where only one hypermultiplet is massive, that is,  $\mathbf{m}_C = (2m, 0, 0, 0)$ . The particular normalisation of  $\mathbf{m}_C$  is chosen such that the diagram 23 holds without any prefactors. Since in the limit  $m \rightarrow \infty$  we get massless  $N_f = 3$ , we can express the  $\tau$  dependence through the Hauptmodul  $\tilde{f} = \frac{\vartheta_3^2\vartheta_4^2}{(\vartheta_3^2 - \vartheta_4^2)^2}$  of  $\Gamma_0(4)$ . The order parameter of the massless  $N_f = 3$  theory reads  $\frac{u}{\Lambda_3^2} = -\frac{1}{64}\tilde{f}$  (2.105), and the functions  $f$  and  $\tilde{f}$  are related by  $f(4\tau) = 16\tilde{f}(\tau) + 2$ . One finds for the order parameter  $u_C$ ,

$$u_C(\tau, \tau_0) = -\frac{m^2}{3}\vartheta_3(\tau_0)^2\vartheta_4(\tau_0)^2\frac{2\tilde{f}(\tau_0)^2 + (10\tilde{f}(\tau) + 1)\tilde{f}(\tau_0) + 2\tilde{f}(\tau)}{\tilde{f}(\tau_0)(\tilde{f}(\tau_0) - \tilde{f}(\tau))}, \quad (3.41)$$

which is independent of  $\vartheta_2(\tau_0)$ . Again, the factor  $\vartheta_3(\tau_0)^2\vartheta_4(\tau_0)^2$  is a modular form of weight 2 for  $\Gamma_0(4)_{\tau_0}$ , and the quotient is a meromorphic modular function of  $\Gamma_0(4)$  for both  $\tau$  and  $\tau_0$ . Thus  $u_C$  satisfies condition 1 of definition 5 with weight  $(0, 2)$  for  $\Gamma_0(4)_\tau \times \Gamma_0(4)_{\tau_0}$ .

Since  $\mathcal{T} : \mathbf{m}_C \mapsto \mathbf{m}_C$ , there is a simultaneous  $T$ -duality

$$\mathcal{T} : u_C(\tau + 1, \tau_0 + 1) = u_C(\tau, \tau_0), \quad (3.42)$$

which is straightforward to check from (3.41). As  $\mathcal{S} : \mathbf{m}_C \mapsto \mathbf{m}_B$ , this exchanges the order parameters

$$u_C(-\frac{1}{\tau}, -\frac{1}{\tau_0}) = \tau_0^2 u_B(\tau, \tau_0), \quad (3.43)$$

which we can also explicitly check. We can again study the stabiliser subgroup of  $\mathcal{T}_{\mathbf{m}_C}$  of  $\mathcal{S}$ . It is the group generated by  $\mathcal{T}$  and  $\mathcal{S}\mathcal{T}^2\mathcal{S}$ , such that  $u_C$  is expected to transform simultaneously under  $T$  and  $\mathcal{S}\mathcal{T}^2\mathcal{S}$ . These two matrices generate the congruence subgroup  $\Gamma_0(2)$  of  $\text{SL}(2, \mathbb{Z})$ , which is conjugate to  $\Gamma^0(2)$ . Thus we find that  $u_C$  is a bimodular form of weight  $(0, 2)$  for the triple

$$(\Gamma_0(4), \Gamma_0(4); \Gamma_0(2)). \quad (3.44)$$

Lastly, we can also study

$$\left(\frac{da}{du}\right)_C^2(\tau, \tau_0) = \frac{1}{8m^2}\frac{\vartheta_3(\tau)^2\vartheta_4(\tau)^2}{\vartheta_2(\tau_0)^8}\frac{\tilde{f}(\tau) - \tilde{f}(\tau_0)}{\tilde{f}(\tau)\tilde{f}(\tau_0)}. \quad (3.45)$$

It is straightforward to check that  $\left(\frac{da}{du}\right)_C^2$  is a bimodular form of weight  $(2, -4)$  for the triple (3.44). For the discriminant  $\Delta_C$  there exists a similar expression to (3.40), and it is a bimodular form of weight  $(0, 12)$  for (3.44).

### 3.3.4 Case D

Let us finally also study the case  $\mathbf{m}_D = (m, m, m, -m)$ . It is related to cases B and C as in Fig. 23. We have that  $\mathbf{m}_D \in \mathcal{L}_S$ , while  $\mathbf{m}_B \in \mathcal{L}_{S\mathcal{T}S}$  and  $\mathbf{m}_C \in \mathcal{L}_\mathcal{T}$ . From the SW curve one easily finds

$$u_D(\tau, \tau_0) = -\frac{m^2}{3} i \vartheta_2(\tau_0)^2 \vartheta_4(\tau_0)^2 \frac{2\hat{f}(\tau_0)^2 + \hat{f}(\tau)\hat{f}(\tau_0) - 12}{\hat{f}(\tau_0) - \hat{f}(\tau)}, \quad (3.46)$$

where

$$\hat{f}(\tau) = f(\tau + 1) = i \frac{\vartheta_2(\tau)^4 - \vartheta_4(\tau)^4}{\vartheta_2(\tau)^2 \vartheta_4(\tau)^4}. \quad (3.47)$$

Since  $f$  is a Hauptmodul for  $\Gamma^0(4)$ ,  $\hat{f}$  is a Hauptmodul for a subgroup of  $\mathrm{SL}(2, \mathbb{Z})$  conjugate to  $\Gamma^0(4)$ ,

$$\widetilde{\Gamma^0(4)} = T \Gamma^0(4) T^{-1} = \langle T^4, ST^2 \rangle. \quad (3.48)$$

A fundamental domain for  $\widetilde{\Gamma^0(4)}$  is given by

$$\widetilde{\Gamma^0(4)} \backslash \mathbb{H} = \mathcal{F} \cup T\mathcal{F} \cup T^2\mathcal{F} \cup T^3\mathcal{F} \cup TST\mathcal{F} \cup T^3ST\mathcal{F}, \quad (3.49)$$

with  $\mathcal{F} = \mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H}$ . It is straightforward to check that  $u_D(\tau, \tau_0)$  transforms with weight  $(0, 2)$  under  $\widetilde{\Gamma^0(4)}_\tau \times \widetilde{\Gamma^0(4)}_{\tau_0}$ .

The subgroup  $\mathcal{T}_{\mathbf{m}_D} \subset \mathcal{T}$  leaving invariant  $\mathbf{m}_D$  is generated by  $\mathcal{S}$  and  $\mathcal{T}^2$ . The two corresponding  $\mathrm{SL}(2, \mathbb{Z})$  transformations  $S$  and  $T^2$  generate the theta group  $\widetilde{\Gamma^0(2)} := \Gamma_\theta$  (A.2), which is a congruence subgroup of  $\mathrm{SL}(2, \mathbb{Z})$  with index 3, conjugate to  $\Gamma_0(2)$  and  $\Gamma^0(2)$ . Thus we find that  $u_D$  is a bimodular form of weight  $(0, 2)$  for the triple

$$(\widetilde{\Gamma^0(4)}, \widetilde{\Gamma^0(4)}; \widetilde{\Gamma^0(2)}). \quad (3.50)$$

The three groups  $\{\Gamma_0(2), \Gamma^0(2), \widetilde{\Gamma^0(2)}\} \ni \Gamma$  are in fact the three groups  $\mathrm{SL}(2, \mathbb{Z}) \supset \Gamma \supset \Gamma(2)$  with index 3 and 2 cusps, and they correspond to the three conjugate order 2 subgroups of  $S_3$ .

## 3.4 Vector-valued bimodular forms

The analysis of the A, B, C and D theories may suggest that the order parameter  $u_{\mathbf{m}}$  for a generic mass  $\mathbf{m}$  transforms with weight  $(0, 2)$  under  $G_\tau \times G_{\tau_0}$  for some subgroup  $G \subseteq \mathrm{SL}(2, \mathbb{Z})$ . This is however not true in general, as for generic masses there are branch points and associated branch cuts, which spoil the modularity [2]. The discussion in section 2 for  $N_f \leq 3$  suggests that for a fixed  $\tau$  or fixed  $\tau_0$ , there is a natural choice of fundamental domain  $\mathcal{F}(\mathbf{m}) \subseteq \mathbb{H}$  for  $u_{\mathbf{m}}$ , such that  $u_{\mathbf{m}} : \mathcal{F}(\mathbf{m}) \rightarrow \mathcal{B}_4$  is one-to-one. For a generic choice of masses, monodromies on the  $u$ -plane give rise to monodromies of  $\mathcal{F}(\mathbf{m})$ , but these do not generate a congruence subgroup of  $\mathrm{SL}(2, \mathbb{Z})$  for a generic mass. For special cases however,  $\mathcal{F}(\mathbf{m})$  is equal to  $\Gamma \backslash \mathbb{H}$  for some subgroup  $\Gamma \subseteq \mathrm{SL}(2, \mathbb{Z})$ ,

such as when  $\mathbf{m}$  is equal to  $\mathbf{m}_A$ ,  $\mathbf{m}_B$ ,  $\mathbf{m}_C$  or  $\mathbf{m}_D$ , for which  $\Gamma$  is  $\Gamma(2)$ ,  $\Gamma^0(4)$ ,  $\Gamma_0(4)$  or  $\widetilde{\Gamma^0(4)}$ . If the mass  $\mathbf{m}$  is such that  $\mathcal{B}_4$  contains a superconformal Argyres-Douglas point,  $\Gamma \subseteq \mathrm{SL}(2, \mathbb{Z})$  can also be a subgroup of index smaller than 6 [2, 3, 129].

### 3.4.1 Generic masses

In the above discussed examples A–D, the duality groups  $\Gamma_1$  of  $\tau$  and  $\Gamma_2$  of  $\tau_0$  are identical. This is not generally true, even if  $u(\tau, \tau_0)$  is modular in  $\tau$  and  $\tau_0$  [3]. However, we can demonstrate that  $\Gamma_1 \subset \Gamma_2$ . A common non-perturbative definition of the UV coupling constant is the low-energy effective coupling  $\tau$  in the limit where the order parameter is large,

$$\tau_0 = \lim_{u \rightarrow \infty} \tau(u). \quad (3.51)$$

Since it is not associated with a singularity, it is neither a cusp nor an elliptic point and therefore an arbitrary interior point in the space of  $\tau \in \mathbb{H}$ . If  $\Gamma_1 \subsetneq \Gamma_2$  is not a proper subgroup, then in general  $\tau_0 \in \Gamma_2 \backslash \mathbb{H}$  is not an element of a choice of fundamental domain  $\Gamma_1 \backslash \mathbb{H}$ . However, there exists a  $\gamma_1 \in \Gamma_1$  with the property that  $\gamma_1 \tau_0 \in \Gamma_1 \backslash \mathbb{H}$ . Since  $u(\tau, \tau_0)$  has weight 0 in  $\tau$ , we notice that

$$u(\gamma_1 \tau_0, \tau_0) = u(\tau_0, \tau_0) = \infty, \quad (3.52)$$

which is the weak coupling region in  $\mathcal{B}_4$ . If  $\Gamma_2 \subsetneq \Gamma_1$  however, then there exist two points  $\tau_0 \neq \tilde{\tau}_0$  in the fundamental  $\Gamma_1 \backslash \mathbb{H}$ , which are not related by any element  $\gamma_1 \in \Gamma_1$ . Then  $u(\tau, \tau_0)$  and  $u(\tau, \tilde{\tau}_0)$  are two distinct points in  $\mathcal{B}_4$ . This contradicts the fact that the  $N_f = 4$  Coulomb branch  $\mathcal{B}_4$  only contains one such singularity. This shows that indeed  $\Gamma_1 \subseteq \Gamma_2$ .

The weight  $(0, 2)$  of  $u$  can be explained as follows. Monodromies on the  $u$ -plane act on the low-energy effective coupling  $\tau$  and by definition leave  $u$  invariant. Thus  $u(\tau, \tau_0)$  is required to have weight 0 in  $\tau$ . For  $\tau_0$ , recall that the order parameter relates to the prepotential  $F$  of the theory by a logarithmic derivative with respect to the instanton counting parameter [110, 130–132, 190, 191]

$$u = 4\pi i q_0 \frac{\partial F}{\partial q_0} = 2 \frac{\partial F}{\partial \tau_0}. \quad (3.53)$$

As the prepotential  $F$  has weight 0 in  $\tau_0$ , this shows that  $u(\tau, \tau_0)$ , has weight 2 in  $\tau_0$ .

The other possible modular transformations are those involving the masses, which is the action of the triality group  $\mathrm{Spin}(8) \rtimes_{\varphi} \mathrm{SL}(2, \mathbb{Z})$ . From the above analysis, we expect that for generic mass  $\mathbf{m}$  the order parameter  $u_{\mathbf{m}}$  transforms as

$$\begin{aligned} \mathcal{T} : \quad u_{\mathbf{m}}(\tau + 1, \tau_0 + 1) &= u_{\mathcal{T}\mathbf{m}}(\tau, \tau_0), \\ \mathcal{S} : \quad u_{\mathbf{m}}(-\frac{1}{\tau}, -\frac{1}{\tau_0}) &= \tau_0^2 u_{\mathcal{S}\mathbf{m}}(\tau, \tau_0). \end{aligned} \quad (3.54)$$

Due to the branch points and cuts for generic masses, these transformations are again very subtle to perform. From (3.11) and in particular  $\mathcal{T}^2 = \mathbb{1}$ , (3.54)

implies

$$\mathcal{T}^2: \quad u_{\mathbf{m}}(\tau + 2, \tau_0 + 2) = u_{\mathbf{m}}(\tau, \tau_0). \quad (3.55)$$

We can check explicitly that it is true for example for case B as in (3.36), which is not  $\mathcal{T}$ -invariant.

As discussed in section 3.2.3, the group action  $\mathcal{T} \times \mathcal{M} \rightarrow \mathcal{M}$  partitions the mass space  $\mathcal{M} \ni \mathbf{m}$  into three regions  $\mathcal{L}_1$ ,  $\mathcal{L}_3$  and  $\mathcal{L}_6$ , where the orbits  $\mathcal{T} \cdot \mathbf{m}$  have length 1, 3 and 6. The stabiliser subgroups of  $\mathbf{m}$  are then subgroups of  $S_3$  of order 6, 2 and 1, i.e. isomorphic to  $S_3$ ,  $S_2$  or  $S_1 = \{e\}$ . The homomorphism  $\varphi$  (3.13) between  $\text{SL}(2, \mathbb{Z})$  and  $\mathcal{T} = \text{Out}(\text{Spin}(8))$  then dictates the subgroup

$$\varphi^{-1}[\mathcal{T}_{\mathbf{m}}] \quad (3.56)$$

under which  $u_{\mathbf{m}}$  is simultaneously invariant. The preimage of the stabiliser subgroup under  $\varphi$  thus constitutes the third component  $\Gamma$  of the triple  $(\Gamma_1, \Gamma_2; \Gamma)$  in definition 5.

### The case $\mathbf{m} \in \mathcal{L}_1$

When  $\mathbf{m} \in \mathcal{L}_1$ , then the stabiliser group of  $\mathbf{m}$  has six elements and the orbit  $\mathcal{T} \cdot \mathbf{m}$  consists of  $\mathbf{m}$  only. Then there is only one function in (3.54), and  $u_{\mathbf{m}}$  transforms with weight  $(0, 2)$  under  $\text{SL}(2, \mathbb{Z})_{(\tau, \tau_0)}$ , as in condition 2 of definition 5. An example is  $u_{\text{A}}$  as given in (3.25), and the transformation is checked in (3.27).

### The case $\mathbf{m} \in \mathcal{L}_3$

The case  $\mathbf{m} \in \mathcal{L}_3$  is most interesting, as it is not trivial ( $\mathbf{m} \in \mathcal{L}_1$ ) and not generic ( $\mathbf{m} \in \mathcal{L}_6$ ). Namely, when the orbit  $\mathcal{T} \cdot \mathbf{m}$  contains three elements, the stabiliser group is isomorphic to the symmetric group  $S_2$  with two elements. Then the three functions associated with the three elements of the orbit  $\mathcal{T} \cdot \mathbf{m}$  form a vector that transforms under  $\text{SL}(2, \mathbb{Z})$ . An example for this are the functions  $u_{\text{B}}, u_{\text{C}}, u_{\text{D}}$  found in Sections 3.3.2–3.3.4. As is clear from Fig. 23, they are related to each other by triality. If we organise  $\mathbf{u}_3 = (u_{\text{B}}, u_{\text{C}}, u_{\text{D}})^{\text{T}}$ , using (3.36), (3.41) (3.46) one can prove that

$$\begin{aligned} \mathbf{u}_3(\tau + 1, \tau_0 + 1) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \mathbf{u}_3(\tau, \tau_0), \\ \mathbf{u}_3(-1/\tau, -1/\tau_0) &= \tau_0^2 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{u}_3(\tau, \tau_0). \end{aligned} \quad (3.57)$$

As the matrices are in  $\text{GL}(3, \mathbb{C})$ , there exists a 3-dimensional representation  $\text{SL}(2, \mathbb{Z}) \rightarrow \text{GL}(3, \mathbb{C})$ . This shows that  $\mathbf{u}_3(\tau, \tau_0)$  furnishes a vector-valued

bimodular form of weight  $(0, 2)$  for  $SL(2, \mathbb{Z})$ , agreeing with the following definition:<sup>25</sup>

**Definition 6** (Vector-valued bimodular form). *Let*

$$\mathbf{F} = \begin{pmatrix} F_1 \\ \vdots \\ F_p \end{pmatrix} : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{C}^p \quad (3.58)$$

be a  $p$ -tuple of two-variable meromorphic functions,  $p \in \mathbb{N}$ . Then  $\mathbf{F}$  is called a vector-valued bimodular form of weight  $(k_1, k_2)$  for  $\Gamma \subset SL(2, \mathbb{Z})$ , if

- each component  $F_j$  is a bimodular form of weight  $(k_1, k_2)$  for some triple  $(\Gamma_1^j, \Gamma_2^j; \Gamma^j)$ , as in definition 5, and
- there exists a  $p$ -dimensional complex representation  $\rho : \Gamma \rightarrow GL(p, \mathbb{C})$  such that

$$\mathbf{F}(\gamma\tau_1, \gamma\tau_2) = (c\tau_1 + d)^{k_1} (c\tau_2 + d)^{k_2} \rho(\gamma) \mathbf{F}(\tau_1, \tau_2) \quad (3.59)$$

for all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$  and all  $\tau_1, \tau_2 \in \mathbb{H}$ .

Since  $\mathbf{u}_3$  is parametrised by the mass  $m \in \mathbb{C}$ ,  $m \mapsto \mathbf{u}_3(m, \tau, \tau_0)$  is in fact a 1-parameter family of vector-valued bimodular forms of weight  $(0, 2)$  for  $SL(2, \mathbb{Z})$ . The triality action of  $SL(2, \mathbb{Z})$  permutes the triples  $(\Gamma_1^j, \Gamma_2^j; \Gamma^j)$  in an interesting way. The action of the  $SL(2, \mathbb{Z})$  generators on  $u$  is given by (3.57). As  $\Gamma_1^j = \Gamma_2^j$  for the cases B, C, D, both  $\Gamma_1^j$  and  $\Gamma_2^j$  are conjugated by the corresponding element of  $SL(2, \mathbb{Z})$ . An instance of this is the group  $\widetilde{\Gamma^0(4)}$  (3.48), which is the set of elements of  $\Gamma^0(4)$  conjugated by  $T$ . Similarly, we have that  $\Gamma^0(4)$  is conjugate to  $\Gamma_0(4)$  by conjugation with  $S$ . The same is true for the three groups  $\Gamma^0(2)$ ,  $\Gamma_0(2)$  and  $\widetilde{\Gamma^0(2)}$  that the cases B, C, D simultaneously transform under, these three conjugate subgroups are permuted under  $SL(2, \mathbb{Z})$  just as  $\Gamma^0(4)$ ,  $\Gamma_0(4)$  and  $\widetilde{\Gamma^0(4)}$  are.

### The case $\mathbf{m} \in \mathcal{L}_6$

The remaining case is that  $\mathbf{m} \in \mathcal{L}_6$ , where  $\mathcal{T} \cdot \mathbf{m}$  has six elements. Then we can organise  $\mathbf{u}_6 = (u_{\mathbf{m}}, u_{\mathcal{T}\mathbf{m}}, u_{\mathcal{S}\mathbf{m}}, u_{\mathcal{T}\mathcal{S}\mathbf{m}}, u_{\mathcal{S}\mathcal{T}\mathbf{m}}, u_{\mathcal{T}\mathcal{S}\mathcal{T}\mathbf{m}})^\top$ , which is a collection of six pairwise distinct functions. By studying the action of  $\mathcal{T}$  and  $\mathcal{S}$  on the

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<sup>25</sup>It is customary to define vector-valued modular forms for  $SL(2, \mathbb{Z})$ , however vector-valued modular forms for proper subgroups  $\Gamma$  of  $SL(2, \mathbb{Z})$  are familiar in rational CFTs [192–194] and so we leave our definition more generic.



vector  $(\mathbf{m}, \mathcal{T}\mathbf{m}, \mathcal{S}\mathbf{m}, \mathcal{T}\mathcal{S}\mathbf{m}, \mathcal{S}\mathcal{T}\mathbf{m}, \mathcal{T}\mathcal{S}\mathcal{T}\mathbf{m})^\top$ , we find the transformations

$$\mathbf{u}_6(\tau + 1, \tau_0 + 1) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \mathbf{u}_6(\tau, \tau_0), \quad (3.60)$$

$$\mathbf{u}_6(-1/\tau, -1/\tau_0) = \tau_0^2 \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \mathbf{u}_6(\tau, \tau_0).$$

The vector  $\mathbf{u}_6$  is not a vector-valued bimodular form for  $\mathrm{SL}(2, \mathbb{Z})$ , because the components of  $\mathbf{u}_6$  do not transform as modular forms under the separate action of  $\Gamma_{1,2}^j$ ,  $j = 1, \dots, 6$  due to the branch cuts, as discussed for the  $N_f \leq 3$  theories in section 2. The simultaneous action of  $\mathrm{SL}(2, \mathbb{Z})$  on  $\tau$  and  $\tau_0$  is not obstructed by the branch cuts of  $u(\tau, \tau_0)$  [3].

The matrices in (3.57) and (3.60) are not only in  $\mathrm{GL}(n, \mathbb{C})$ , but they are in fact permutation matrices: Because triality permutes the respective moduli spaces, the order parameters are merely permuted and there are no phases. Thus we have that

$$\mathbf{u}(\gamma\tau, \gamma\tau_0) = (c\tau_0 + d)^2 P_{\pi(\gamma)} \mathbf{u}(\tau, \tau_0), \quad (3.61)$$

where  $P_{\pi(\gamma)}$  is the permutation matrix for the permutation  $\pi(\gamma) \in \mathcal{S}_{|\mathcal{F} \cdot \mathbf{m}|}$ , which can be found from the action of  $\mathcal{T}$  on  $\mathbf{m}$ .

For the period  $(\frac{da}{du})^2$ , there are similar results. For instance, one can check that

$$\left( \left( \frac{da}{du} \right)_B^2, \left( \frac{da}{du} \right)_C^2, \left( \frac{da}{du} \right)_D^2 \right)^\top \quad (3.62)$$

is a vector-valued bimodular form of weight  $(2, -4)$  for  $\mathrm{SL}(2, \mathbb{Z})$ . As  $u$  has weight  $(0, 2)$ , it is not obvious how the discriminant  $\Delta$  transforms since it is a polynomial in  $u$ . However, because triality acts on the 6 singularities as well, in general  $\mathbf{\Delta}$  is a vector-valued bimodular form of weight  $(0, 12)$  for  $\mathrm{SL}(2, \mathbb{Z})$ . This can be checked explicitly for the cases B, C, D, where  $\mathbf{\Delta}_3 = (\Delta_B, \Delta_C, \Delta_D)^\top$  is a 1-parameter family of vector-valued bimodular forms of weight  $(0, 12)$  for  $\mathrm{SL}(2, \mathbb{Z})$ .

### 3.4.2 Fundamental domains

In the asymptotically free theories we argued that the  $u$ -planes can be identified with fundamental domains  $\mathcal{F}_{N_f}(\mathbf{m})$ . For this we make the correspondence that

the number of singularities gives the number of rational cusps, the number of BPS states becoming massless at each singularity gives the width of each cusp, and the width at  $i\infty$  is given by  $4 - N_f$ . Then, the sum of all cusps is RG invariant. By following the RG flow from  $N_f = 3$  to  $N_f < 3$  we find that gradually a singularity at strong coupling (a rational cusp) moves to infinity and is identified with the weak coupling region ( $i\infty$ ). Reversing this argument implies that for  $N_f = 4$  there should be six rational cusps and the width at infinity vanishes. This is consistent with the fact [46] that  $u = \infty$  does not correspond to a cusp of the curve anymore. Rather, it lies in the interior of  $\mathbb{H} \ni \tau_0$ .

It is found in the above subsections that depending on the mass configuration, the fundamental domain for an order parameter is related to the one of the underlying theory where all massive hypermultiplets are decoupled. We can depict those domains in an equivalent way that is more suitable to our description. For this, one chooses an equivalent fundamental domain with the property that the width at  $i\infty$  is zero and the number of rational cusps is equal to the number of singularities, with according width. For instance, in case A where  $\mathbf{m} = (m, m, 0, 0)$  the duality group is  $\Gamma(2)$ , whose cusps in the decoupling limit (with the same duality group) we choose as  $\{i\infty, 0, 1\}$ . In  $N_f = 4$  it is more suitable to represent  $i\infty$  by a rational number. For this we can use that  $\Gamma(2) \ni ST^{-2}S : i\infty \mapsto \frac{1}{2}$ , being a preferable representative of the third cusp. As it necessarily also has width 2, both  $\mathcal{F}$  and  $T\mathcal{F}$  can be mapped to the region  $\tau = \frac{1}{2}$ . This is depicted in Fig. 26. The decoupling to massless  $N_f = 2$  is illustrated in Fig. 27. The domains in this case are exactly equivalent, Fig. 26 merely allows to extend the  $N_f \leq 3$  description of the cusps to  $N_f = 4$ .

We stress that the decoupling limit for  $N_f = 4$ , with the order parameter a bimodular form as (3.25), is quite different from the asymptotically free theories with  $N_f \leq 3$ . In the latter theories,  $u(\tau)$  is not holomorphic and modular except for special points in mass space (complex co-dimension  $N_f$ ) [2]. The decoupling of a hypermultiplet is in these theories accompanied by a branch point moving to infinity. In this way, a singularity merges with the weak coupling cusp. For cases A, B, C and D in  $N_f = 4$  on the other hand, there is no branch point for any value of the mass  $m$ , and in particular there is also none for  $m \rightarrow \infty$ .

### 3.5 Discussion

In this section, we have studied in detail the Coulomb branch of the superconformal  $N_f = 4$  theory with gauge group  $SU(2)$ , which has remained of great interest throughout the years [21, 46, 50, 73, 77, 105, 111, 113, 148, 173, 178–180, 183–187, 195, 196]. For the mass configurations with the largest flavour symmetry group, such as when one, two and four hypermultiplets have an equal mass, we show that the Coulomb branch is parametrised by a function  $u(\tau, \tau_0)$

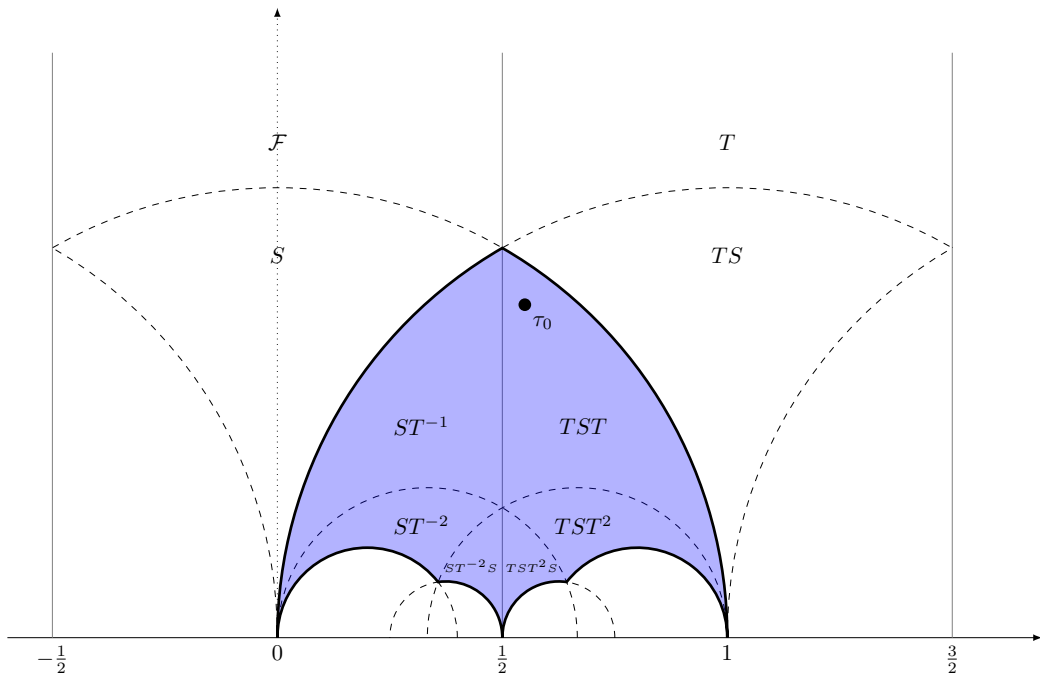


Figure 26: Fundamental domain of  $N_f = 4$  case A with  $\mathbf{m} = (m, m, 0, 0)$ .

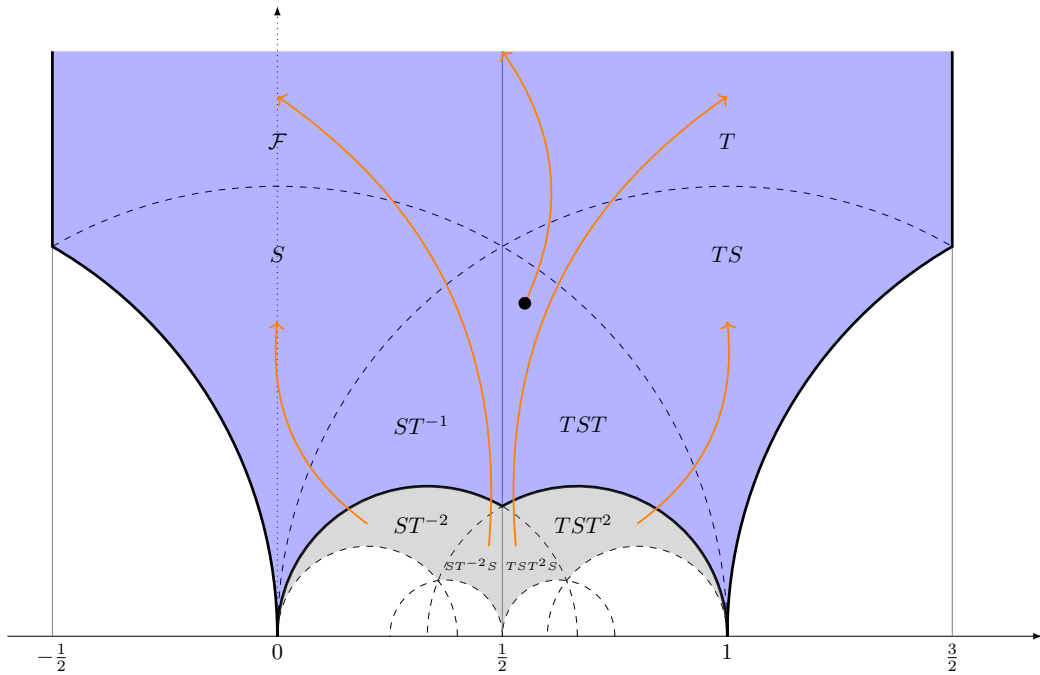


Figure 27: Decoupling the two massive hypermultiplets in  $N_f = 4$  case A gives the domain of massless  $N_f = 2$  (blue). Two of the differing regions (gray) are the regions near  $\tau = \frac{1}{2}$ , which are mapped (orange) to  $i\infty$ . The remaining two are merely mapped to  $\Gamma(2)$  equivalent regions near the same cusp such that the resulting domain is connected. Alternatively, the fundamental domain in this figure is the image of  $ST^{-1}STS$  acting on the domain in Figure 26.

that is not only invariant under separate modular transformations of  $\tau$  and  $\tau_0$ , but also exhibits invariance under a simultaneous transformation under  $\tau$  and  $\tau_0$ . By restricting to the stabiliser subgroup of a given mass under the triality action, such order parameters constitute nontrivial examples of bimodular forms (see (3.25) for example). Furthermore, the moduli spaces are permuted under triality, and the order parameter, periods, discriminants etc. furnish vector-valued bimodular forms, which we also introduce (see definition 6).

The analysis of other mass configurations can be done using the techniques established in section 2. As more complicated mass configurations  $\mathbf{m}$  inevitably introduce branch points and cuts, in general  $u_{\mathbf{m}}$  is not a bimodular form. A simultaneous transformation of  $\tau$  and  $\tau_0$  is yet to be expected by triality, while the separate transformations are induced by monodromies and as such do not in general lie in  $\text{SL}(2, \mathbb{Z})$  [46]. However, even in such cases the action of the monodromy group of the  $u$ -plane can be understood as paths in the fundamental domain for  $\tau$ . See section 4.3.3 for a discussion of these aspects for gauge group  $\text{SU}(3)$ .

Our results allow to study the topologically twisted theory on a four-manifold  $X$  [52, 61, 72–74], where the the path integral can be expressed as an integral over the fundamental domain for the effective coupling  $\tau$ . In fact, a closed expression for the order parameter is enough to define the integrand. The modularity for  $\tau$  allows to show that the integral measure is well-defined. The triality action then gives the S-duality orbit of the  $N_f = 4$  theory on  $X$  [197].

It would also be interesting to apply our results to other theories with an IR moduli space of vacua as well as a non-trivial conformal manifold. Such theories may include subsectors with triality symmetry, such as F-theory [198], quiver gauge theories [21], the AGT correspondence [172], little string theory [199] and string/string/string triality [200].

## 4 Elliptic loci of SU(3) vacua

In this section, we investigate the modularity of the pure SU(3) SW theory. While in the SU(2) theories of Sections 2 and 3 the obstruction to modularity is due to the hypermultiplets, we show here that for rank two the modularity is broken in specific cases even without hypermultiplets. This section is mainly based on [1], and section 4.6 is based on unpublished work.

### 4.1 Introduction

Many observables in supersymmetric theories can be determined non-perturbatively in terms of hypergeometric, modular or other special functions. The best understood example is  $\mathcal{N} = 2$  supersymmetric Yang–Mills theory with gauge group SU(2) [45, 46], which we introduced in detail in section 1.5. Its space of vacua is parametrised by the vacuum expectation value (vev)  $u = \frac{1}{2}\langle\text{Tr}\phi^2\rangle$ , where  $\phi$  is the complex scalar in the  $\mathcal{N} = 2$  vector multiplet. The renormalisation group flow generates a quantum scale  $\Lambda$ , at which the gauge coupling becomes strong. In the weak-coupling region  $|u| \gg \Lambda^2$ , the semi-classical BPS spectrum consists of massive monopoles and dyons. The theory can be solved non-perturbatively in terms of the Seiberg–Witten (SW) curve [45]. This solution demonstrates that the effective abelian gauge theory breaks down at two special points,  $u = \pm\Lambda^2$ . The electric-magnetic duality group is generated by the monodromies around these singular points. It is a subgroup of  $\text{SL}(2, \mathbb{Z})$ , which acts by linear fractional transformations on the effective coupling constant  $\tau$ . With the SW solution, various physical quantities can be exactly determined as functions of  $\tau$  using modular functions [73, 74, 101, 103, 201].

Similar non-perturbative solutions have been developed for gauge theories with matter multiplets [46] and theories with other gauge groups [94–97, 202–204]. In pure Yang–Mills theory with compact gauge group  $G$ , the Coulomb branch has complex dimension  $r = \text{rank}(G)$ . Classically, the moduli space is parametrised by the vevs  $u_{I+1} \sim \langle\text{Tr}\phi^{I+1}\rangle$ ,  $I = 1, \dots, r$ . The  $r(r+1)/2$  couplings  $\tau_{IJ}$  are determined by the  $r$  order parameters  $u_I$ . The electric-magnetic duality group is a subgroup of  $\text{Sp}(2r, \mathbb{Z})$ , generated by monodromies around singular loci. While this also demonstrates a link to modularity, the connection has remained more elusive, and the connection is best established for the superconformal theories [178, 205–208].

One complication for asymptotically free theories is that the structure of the singular loci is in general quite intricate. This section focuses on the asymptotically free SU(3) theory without hypermultiplets, whose singular loci have a rich structure [104, 105, 128, 209, 210]. There are six singular (complex) lines which intersect in five points. A particularly interesting phenomenon occurs at two of these five vacua, namely those where three mutually non-local dyons become massless, such that there is no duality frame in which all of these

states only carry electric charge. This indicates that the system is in a critical phase, which led to the discovery of new superconformal theories [104, 105, 128].

Another complication for  $SU(N > 2)$  is that the number of couplings exceeds the dimension of the Coulomb branch. The observables are therefore defined on a subspace of the genus  $N - 1$  Siegel upper half-space  $\mathbb{H}_{N-1}$ . For the  $SU(3)$  theory, the Coulomb branch is parametrised by two order parameters which determine three coupling constants,  $\tau_{11}$ ,  $\tau_{12}$  and  $\tau_{22}$ . The curve and the SW differential for pure  $SU(N)$  gauge theory have first been proposed in [94]. As a first step to explore the modularity of the  $SU(3)$  theory, we relate the hyperelliptic Seiberg-Witten curve to the Rosenhain form, which is an algebraic expression in terms of Siegel theta series. To exactly match the Rosenhain curve and Seiberg-Witten curve, we use the fact that the complexified masses  $a_I$  and  $a_{D,I} = \frac{\partial \mathcal{F}}{\partial a_I}$  are solutions of second order partial differential equations of Picard-Fuchs (PF) type. The solutions to such equations can be expressed in terms of the generalised hypergeometric function  $F_4$  of Appell [96]. The Siegel theta series and their modular transformations can provide insights for the analytic continuation and monodromies of the solution in terms of  $F_4$ .

The Rosenhain curve allows us to characterise the  $SU(3)$  Coulomb branch, parametrised by the two Casimirs  $u = u_2$  and  $v = u_3$ , as the zero-locus of three equations inside a five-dimensional space. The structure of these equations simplifies on one-dimensional loci of the Coulomb branch. We study two of these loci in detail, namely  $\mathcal{E}_u$  where  $v = 0$  and  $\mathcal{E}_v$  where  $u = 0$ . On each of these loci, the equations reduce to two algebraic relations of Siegel theta functions, relating the couplings  $\tau_{IJ}$  to a single independent one. Interestingly, each of these loci in the space of genus two curves also parametrises a family of (genus 1) elliptic curves. Both loci interpolate between a weak-coupling regime with large order parameters and a strong-coupling regime where  $u/\Lambda^2$  and  $v/\Lambda^3$  are  $\mathcal{O}(1)$ . Locus  $\mathcal{E}_u$  contains three cusps where mutually local dyons become massless, while locus  $\mathcal{E}_v$  contains two special points where mutually non-local dyons becomes massless. The latter are the superconformal Argyres-Douglas points.

Since an elliptic locus parametrises a family of elliptic curves, there must be a coupling  $\tau$  valued in a fundamental domain (or modular curve) for a discrete group in the upper half-plane  $\mathbb{H}$ . We derive the generators of the discrete subgroup from the monodromies of the  $SU(3)$  theory. We provide two solutions for the locus  $\mathcal{E}_u$ . The coupling for the first solution is  $\tau_- = \tau_{11} - \tau_{12}$ , while  $\tau_{22} = \tau_{11}$ . The order parameter  $u$  equals a modular form  $u_-$  for the congruence subgroup  $\Gamma^0(9) \subset SL(2, \mathbb{Z})$  (4.49),

$$u = u_-(\tau_-). \quad (4.1)$$

The cusps of the fundamental domain of  $\Gamma^0(9)$  map exactly to the singular points on this locus. The coupling for the second solution is  $\tau_+ = \tau_{11} + \tau_{12}$ . In terms of this coupling, Equation (4.59) expresses  $u$  as

$$u = u_+(\tau_+), \quad (4.2)$$

where  $u_+$  is expressed in terms of roots of modular forms, while it is not a modular function for a congruence subgroup of  $\mathrm{SL}(2, \mathbb{Z})$ . We call it a *sextic modular function* since it is a solution to a sextic equation. The inverses of the identities (4.1) and (4.2) provide all order  $u$ -expansions for  $\tau_{11} = \tau_{22}$  and  $\tau_{12}$  on this locus. The function  $u_+$  appeared earlier as the solutions for the order parameter on the Coulomb branch of the  $\mathcal{N} = 2$ ,  $\mathrm{SU}(2)$  theory with one massless hypermultiplet [101]. While this Coulomb branch and  $\mathcal{E}_u$  are isomorphic as four punctured spheres, it is striking that the solutions of the order parameters are identical.

We find another intriguing structure for the second locus  $\mathcal{E}_v$  where  $u = 0$ . We are able to demonstrate for this locus that  $v$  is left invariant by the action of the principal congruence subgroup  $\Gamma(6) \subset \mathrm{SL}(2, \mathbb{Z})$ . The fundamental domain  $\Gamma(6) \backslash \mathbb{H}$  has 12 cusps, where  $v$  diverges. Surprisingly, this appears to imply the existence of strongly coupled vacua in the region where  $v$  is large, which is unexpected since large  $v$  is known to correspond to weak coupling. The paradox is resolved by realizing that  $v$  is invariant under a transformation which is not contained in  $\mathrm{SL}(2, \mathbb{Z})$ , namely a *Fricke involution*  $\tau \mapsto -1/n\tau$  for integer  $n \geq 2$ . This transformation maps the putative cusps to  $i\infty$ . The result is that  $v$  is a modular function for a discrete subgroup  $\Gamma_v \subset \mathrm{SL}(2, \mathbb{R})$  of Atkin-Lehner type, and we show that the non-trivial monodromies on this locus do generate this group.

We demonstrate furthermore that the elliptic curves underlying the two loci  $\mathcal{E}_u$  and  $\mathcal{E}_v$  are related to the genus two curve in a precise way. For a genus two curve  $\Sigma_2$ , a holomorphic map  $\varphi : \Sigma_2 \rightarrow \Sigma_1$  to an elliptic curve  $\Sigma_1$  may exist. Such maps were studied in the classic works by Legendre and Jacobi, and more recently in [211, 212]. The existence of the map  $\varphi$  depends on the complex structure moduli  $\tau_{IJ}$ . The family of such curves spans a complex co-dimension one locus  $\mathcal{L}_2$  in the complex three-dimensional space of genus two curves. At the elliptic loci of the Coulomb branch of the  $\mathrm{SU}(3)$  theory mentioned above,  $\mathcal{L}_2$  intersects the  $\mathrm{SU}(3)$  Coulomb branch, such that for any point on the elliptic loci, there is a degree two map from the genus two curve to an elliptic curve, or in other words the genus two curve is a double cover of the elliptic curve. Besides  $\mathcal{E}_u$  and  $\mathcal{E}_v$ ,  $\mathcal{L}_2$  also includes a third elliptic locus,  $\mathcal{E}_3$  (4.91), which does not contain any of the singular points of the Coulomb branch.

Our work motivates a similar analysis for  $\mathrm{SU}(N)$  gauge theories, whose Coulomb branch parametrises a curve of genus  $N - 1$ . The order parameters  $u_I$ ,  $I = 2, \dots, N$ , are expected to be given by higher genus modular functions of the coupling matrix  $\tau_{IJ}$ . They should furthermore be invariant under a subgroup of  $\mathrm{Sp}(2r, \mathbb{Z})$  generated by the monodromies. The existence of maps to elliptic or lower genus curves is however more subtle for such theories [213, 214].

## 4.2 The SU(3) Coulomb branch

We study in this section the SU(3) Coulomb branch. We first recall the Seiberg-Witten geometry in section 4.2.1 following [94, 96, 215]. Section 4.2.2 reviews the Picard-Fuchs solution for the complexified masses and couplings. Section 4.2.3 uses those results to write the curve in Rosenhain form.

### 4.2.1 Seiberg-Witten geometry

The vector multiplet scalar  $\phi$  can be gauge rotated into the Cartan subalgebra of SU(3). Then,  $\phi$  can be expanded in terms of the two Cartan generators  $H_I$ ,  $I = 1, 2$ , as

$$\phi = a_1 H_1 + a_2 H_2. \quad (4.3)$$

Non-vanishing vevs of  $\phi$  break the gauge group in general to  $U(1)^2$ . The central charges of the gauge bosons are then given by

$$\begin{aligned} Z_1 &= 2a_1 - a_2, \\ Z_2 &= 2a_2 - a_1, \\ Z_3 &= a_1 + a_2. \end{aligned} \quad (4.4)$$

We denote electric-magnetic charges under  $U(1)^2$  as  $\gamma = (m_1, m_2, n_1, n_2)$ , where  $m_i$  are the magnetic and  $n_i$  the electric charges respectively, and the period vector as  $\pi = (a_{D,1}, a_{D,2}, a_1, a_2)^T$ . The central charge for a generic  $\gamma$  is then given by  $Z_\gamma = \gamma \cdot \pi$ , where  $\cdot$  is the standard scalar product.

The Coulomb branch is parametrised by vevs of Casimirs of  $\phi$ ,  $u_I \sim \langle \text{Tr} \phi^I \rangle$ ,  $I = 2, 3$ . Gauge invariant combinations for SU(3) are

$$\begin{aligned} u &= u_2 = \frac{1}{2} \langle \text{Tr}(\phi^2) \rangle_{\mathbb{R}^4} = a_1^2 + a_2^2 - a_1 a_2, \\ v &= u_3 = \frac{1}{3} \langle \text{Tr}(\phi^3) \rangle_{\mathbb{R}^4} = a_1 a_2 (a_1 - a_2). \end{aligned} \quad (4.5)$$

These relations can be rewritten in terms of two cubic equations for  $a_1$  and  $a_2$  as

$$\begin{aligned} a_1^3 - u a_1 - v &= 0, \\ a_2^3 - u a_2 + v &= 0. \end{aligned} \quad (4.6)$$

There is a spontaneously broken global  $\mathbb{Z}_6$  symmetry acting on  $u$  and  $v$  by  $u \mapsto \alpha u$  and  $v \mapsto -v$ , with  $\alpha = e^{2\pi i/3}$ . Classically, the discriminant is the determinant  $\Delta_{\text{classical}}$  of the matrix  $B_{IJ} = \frac{\partial u_{I+1}}{\partial a_J}$ . It reads

$$\Delta_{\text{classical}} = \det B_{IJ} = (a_1 - 2a_2)(2a_1 - a_2)(a_1 + a_2), \quad (4.7)$$

and vanishes when one of the gauge bosons (4.4) becomes massless.

Let us denote the space parametrised by  $u$  and  $v$  by  $\mathcal{U}$ . We parametrise points on this space by  $(\underline{u}, v) \in \mathcal{U}$ , where  $\underline{u}$  is the normalised parameter,



$\underline{u} = \sqrt[3]{\frac{4}{27}} u$ . The moduli space  $\mathcal{U}$  parametrises a complex two-dimensional family of hyperelliptic curves of genus two [94, 215],

$$y^2 = (x^3 - ux - v)^2 - \Lambda^6, \quad (4.8)$$

which has discriminant

$$\Delta_\Lambda = \Lambda^{18}(4u^3 - 27(v + \Lambda^3)^2)(4u^3 - 27(v - \Lambda^3)^2). \quad (4.9)$$

This can be viewed as a product of the discriminants of two elliptic curves whose  $v$  parameters are separated by  $2\Lambda^3$ . Note that the  $\mathbb{Z}_6$  global symmetry leaves the discriminant invariant. It vanishes if and only if  $\underline{u}^3 = (v \pm \Lambda^3)^2$ . We will frequently use units where the dynamical scale  $\Lambda = 1$  and we note that it can always be restored from dimensional analysis.

If we restrict to  $\text{Im } v = 0$ , the zero locus of the discriminant describes six singular curves which intersect in the following points. On the  $v = 0$  plane, there are four singularities, namely  $\underline{u} \in \{\infty, 1, \alpha, \alpha^2\}$ . On the other hand for  $\underline{u} = 0$ , there are two singularities at  $v = \pm 1$ . These are the Argyres-Douglas points, where mutually non-local BPS states become massless and the theory becomes superconformal [104]. Figure 28 sketches the singular lines on the subset of  $\mathcal{U}$  where  $\text{Im } v = 0$ . The singular lines represent regions in  $\mathcal{U}$  where the effective action of the pure  $\mathcal{N} = 2$  theory becomes singular, and they are associated with vacua where hypermultiplets become massless.

Similarly to the  $\text{SU}(2)$  case, the periods transform under monodromies which generate the duality group of the theory. The classical part of the monodromy group is given by the Weyl group of the  $\text{SU}(3)$  root lattice, which acts as reflections on lines perpendicular to the positive roots. The perturbative quantum correction comes from the one-loop effective action. It contributes to the prepotential as

$$\mathcal{F}_{1\text{-loop}} = \frac{i}{2\pi} \sum_{\alpha} Z_{\alpha}^2 \log Z_{\alpha}, \quad (4.10)$$

where the sum runs over all positive roots  $\alpha_1, \alpha_2$  and  $\alpha_3 = \alpha_1 + \alpha_2$ . Here,  $Z_{\alpha}$  are the central charges (4.4) of the gauge bosons.

### 4.2.2 Picard-Fuchs solution

One way to find the non-perturbative solution is to notice that the periods satisfy second order partial differential equations of Picard-Fuchs (PF) type, whose solution space is spanned by the generalised hypergeometric function  $F_4$  of Appell [96]. We review some aspects of the PF solution in the following, and more details can be found in [1]. We study two interesting regions, one where  $u$  is large and  $v$  small, and the other one where  $v$  is large and  $u$  is small.

In the limit of large  $u$  and small  $v$ , reference [96] determines the  $a_I$  and  $a_{D,I}$  non-perturbatively in terms of the fourth Appell hypergeometric function

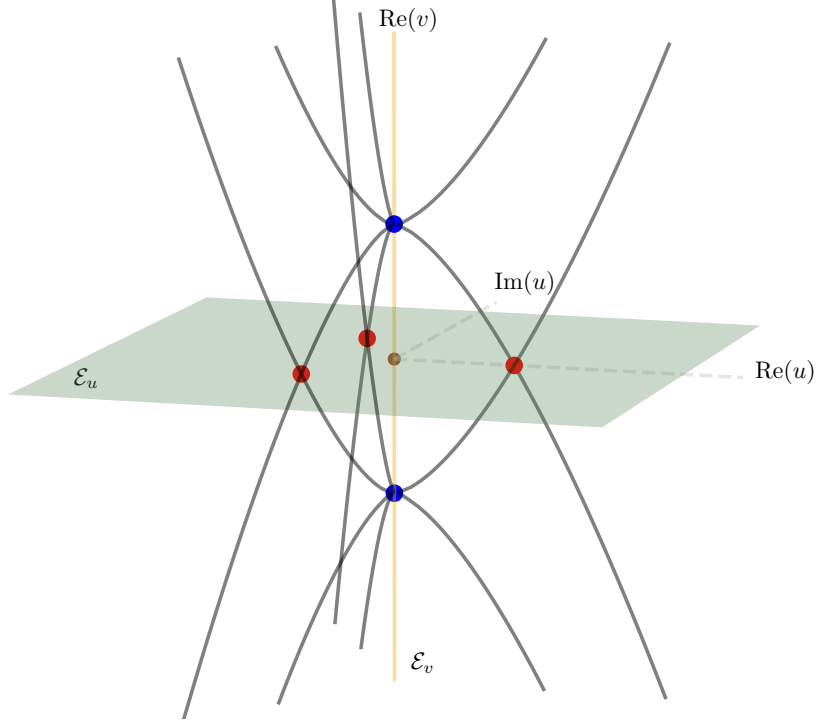


Figure 28: Singular lines  $\Delta(u, v) = 0$  in the  $SU(3)$  moduli space with  $\text{Im } v = 0$ , associated to massless dyons [96]. The red dots represent the strong coupling points  $(\underline{u}, v) = (1, 0)$ ,  $(\alpha, 0)$  and  $(\alpha^2, 0)$  on the  $v = 0$  plane  $\mathcal{E}_u$ , where two singular lines intersect. The blue dots represent the AD points  $(\underline{u}, v) = (0, 1)$  and  $(0, -1)$  respectively, where three singular lines intersect. They lie on  $\mathcal{E}_v$ , which is represented by the  $\text{Re } v$  axis here. The two loci  $\mathcal{E}_u$  and  $\mathcal{E}_v$  intersect in the origin  $(u, v) = (0, 0)$  (brown).

$F_4(a, b, c, d; x, y)$ . For  $\sqrt{|x|} + \sqrt{|y|} < 1$ , this function is given by

$$F_4(a, b, c, d; x, y) = \sum_{m, n \geq 0} \frac{(a)_{m+n} (b)_{m+n}}{m! n! (c)_m (d)_n} x^m y^n, \quad (4.11)$$

where  $(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)}$  is the Pochhammer symbol. We will also need expansions of  $F_4$  for large  $y$ , which can be achieved by replacing the sum over  $n$  by the hypergeometric series  ${}_2F_1$ ,

$$F_4(a, b, c, d; x, y) = \sum_{m \geq 0} \frac{(a)_m (b)_m}{m! (c)_m} {}_2F_1(a+m, b+m, d; y) x^m. \quad (4.12)$$

While analytic continuations are known for  ${}_2F_1$ , they are not well established for  $F_4$ .

In order to match the Picard-Fuchs solutions with the periods, we need to expand the periods around the classical solutions in (4.6). We therefore need to find the roots of these two cubics.

The general formula for the roots of a depressed cubic equation,  $ax^3 + bx + c = 0$ , is given by

$$\xi_k = -\frac{1}{3a} \left( \alpha^k C + \frac{\Delta_0}{\alpha^k C} \right), \quad k \in \{0, 1, 2\}, \quad (4.13)$$

where  $\alpha = e^{2\pi i/3}$ ,  $C^3 = \frac{\Delta_1 \pm \sqrt{\Delta_1^2 - 4\Delta_0}}{2}$ ,  $\Delta_0 = -3ab$  and  $\Delta_1 = 27a^2c$  [216]. The choice of sign in front of the square root in  $C$  is arbitrary, in the sense that it only corresponds to a permutation of the roots.

It is however important to fix the ambiguities in taking the square and cubic root. We fix the ambiguity in the square root by the following choice for the branch of the logarithm: For any complex number  $z \in \mathbb{C}^*$ , we set  $\log(z) = \log|z| + i\text{Arg}(z)$  with  $-\pi < \text{Arg}(z) \leq \pi$ . The ambiguity in the cubic root of a complex number  $z$  is fixed by demanding that the real part of  $\sqrt[3]{z}$  has the largest absolute value among the three solutions to  $\rho^3 = z$ . Thus  $\sqrt[3]{1} = 1$  and  $\sqrt[3]{-1} = -1$ . Two of the cube roots of  $i$  and  $-i$  have equal real parts. We fix the remaining ambiguity by setting  $\sqrt[3]{i} = e^{\pi i/6} = \frac{\sqrt{3}}{2} + \frac{i}{2}$  and  $\sqrt[3]{-i} = e^{-\pi i/6} = \frac{\sqrt{3}}{2} - \frac{i}{2}$ .

To list the roots of our two equations, we define

$$s_{\pm}(a, b) = \sqrt[3]{\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}}. \quad (4.14)$$

Using Eq. (4.13), we then find that the roots of (4.6) for  $a_1$  are given by

$$\begin{aligned} \xi_1(u, v) &= s_+(u, v) + s_-(u, v), \\ \xi_2(u, v) &= \alpha s_+(u, v) + \alpha^2 s_-(u, v), \\ \xi_3(u, v) &= \alpha^2 s_+(u, v) + \alpha s_-(u, v), \end{aligned} \quad (4.15)$$

and the roots for  $a_2$  by  $-\xi_j(u, v)$ . This gives the  $3 \times 3 = 9$  solutions to the equations in (4.6). However, (4.5) is supposed to have only  $2 \times 3 = 6$  solutions. Let us determine the 6 solutions in one of the regimes of interest for SU(3) Yang–Mills theory: we assume  $u$  is large and close to the positive axis:  $u = \lambda - i\epsilon\lambda$  with  $\lambda$  real and very large and  $0 < \epsilon \ll 1$ . Note that in this regime

$$s_{\pm}(u, v) = \sqrt[3]{\frac{v}{2} \pm i\sqrt{\frac{u^3}{27} - \frac{v^2}{4}}}. \quad (4.16)$$

Furthermore,  $s_+(u, v) s_-(u, v) = u/3$  and

$$s_-(u, -v) = e^{-\pi i/3} s_+(u, v) = -\alpha s_+(u, v) \quad (4.17)$$

hold. For  $v = 0$ , we have  $s_+(u, 0) = e^{\pi i/6} \sqrt{u/3}$  and  $s_-(u, 0) = e^{-\pi i/6} \sqrt{u/3}$ , and thus

$$\begin{aligned} \xi_1(u, 0) &= \sqrt{u}, \\ \xi_2(u, 0) &= -\sqrt{u}, \\ \xi_3(u, 0) &= 0. \end{aligned} \quad (4.18)$$

This demonstrates that the solutions to (4.5) for  $(a_1, a_2)$  are given by

$$(\xi_1, -\xi_2), (\xi_1, -\xi_3), (\xi_2, -\xi_1), (\xi_2, -\xi_3), (\xi_3, -\xi_1), (\xi_3, -\xi_2). \quad (4.19)$$

The non-perturbative effective action is characterised by the holomorphic prepotential  $\mathcal{F}$ , which allows to define the dual periods  $a_{D,I} = \frac{\partial \mathcal{F}}{\partial a_I}$ . Both periods  $a_I$  and  $a_{D,I}$  are given by linear combinations of Appell functions. The large  $u$  expansion reads [96]

$$\begin{aligned} a_{D,1}(u, v) &= -\frac{i}{2\pi} \left( \sqrt{u} + \frac{3v}{2u} \right) \log \left( \frac{27}{4u^3} \right) - \frac{1}{\pi} \left( \frac{i}{2} + 2\alpha_1 \right) \sqrt{u} + \dots, \\ a_1(u, v) &= \sqrt{u} + \frac{1}{2} \frac{v}{u} + \dots, \end{aligned} \quad (4.20)$$

with  $a_{D,2}(u, v) = a_{D,1}(u, -v)$ ,  $a_2(u, v) = a_1(u, -v)$  and  $\alpha_1 \in \mathbb{C}$  a constant. The coupling constants  $\tau_{IJ} = \frac{\partial a_{D,I}}{\partial a_J}$  are determined using the chain rule,

$$\tau_{11}(u, v) = \tau_{22}(u, -v) = \frac{i}{\pi} \log(8u^3) + \frac{9iv}{2\pi} u^{-3/2} - \left( \frac{129i}{32\pi} + \frac{63iv^2}{8\pi} \right) u^{-3} + \dots, \quad (4.21)$$

The off-diagonal  $\tau_{12}$  is given by the series

$$\tau_{12}(u, v) = -\frac{\tau_{11}(u, v) + \tau_{22}(u, v)}{4} - \frac{1}{2\pi i} \log(8) + \frac{1}{2\pi i} \frac{27}{4} f(u, v), \quad (4.22)$$

where

$$f(u, v) = \frac{(1 - 4v^2)}{8} u^{-3} + \left( \frac{453}{1024} - 3v^2 - \frac{31}{16} v^4 \right) u^{-6} + \dots \quad (4.23)$$

Similarly, we find that the large  $v$  expansion of the coupling matrix reads ( $\omega = e^{\pi i/6}$ )

$$\tau_{11} \sim \frac{i}{\pi} \log(108v^2) - 1 + \frac{\omega}{\pi} uv^{-2/3} + \frac{\omega^5}{6\pi} u^2 v^{-4/3} - \left( \frac{11i}{18\pi} + \frac{4i}{27\pi} u^3 \right) v^{-2} + \dots, \quad (4.24)$$

and  $\tau_{12}$  and  $\tau_{22}$  are given by similar series. At  $u = 0$  we have  $\tau_{11} = \tau_{22} + 1$  and  $\tau_{12} = -\frac{\tau_{11}}{2} + 1$ .

### 4.2.3 Seiberg-Witten curve in Rosenhain form

In this section, we will relate the SU(3) Seiberg-Witten curve to the curve in Rosenhain form, which is a degree 5 equation. Every genus two hyperelliptic curve can be brought to the Rosenhain form [217]

$$y^2 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3). \quad (4.25)$$

The three roots  $\lambda_i$  of the polynomial are also referred to as *Rosenhain invariants*. These invariants are complementary to the Igusa invariants [218, 219].

By a lemma of Picard, the Rosenhain invariants can be expressed in terms of even theta constants as

$$\lambda_1 = \frac{\Theta_1^2 \Theta_3^2}{\Theta_2^2 \Theta_4^2}, \quad \lambda_2 = \frac{\Theta_3^2 \Theta_8^2}{\Theta_4^2 \Theta_{10}^2}, \quad \lambda_3 = \frac{\Theta_1^2 \Theta_8^2}{\Theta_2^2 \Theta_{10}^2}. \quad (4.26)$$

The functions  $\Theta_j$  are instances of genus two Siegel modular forms,

$$\Theta \begin{bmatrix} a \\ b \end{bmatrix} (\Omega) = \sum_{k \in \mathbb{Z}^2} \exp(\pi i(k+a)^\top \Omega (k+a) + 2\pi i(k+a)^\top b), \quad (4.27)$$

where the entries of the column vectors  $a$  and  $b$  take values in the set  $\{0, \frac{1}{2}\}$ . The argument  $\Omega$  is a  $2 \times 2$ -matrix

$$\Omega = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}, \quad (4.28)$$

valued in the Siegel upper half-plane  $\mathbb{H}_2$ . We refer to Appendix A.2 for a precise definition and references. The moduli space of genus two curves  $\mathcal{M}_2$  is complex three-dimensional. Since the SW order parameters  $u$  and  $v$  are two complex parameters, the SU(3) Coulomb branch maps out a complex two-dimensional space  $\mathcal{U} \subset \mathcal{M}_2$  in the moduli space of genus two curves. In other words,  $\mathcal{U}$  is a divisor of  $\mathcal{M}_2$ .

To relate the Rosenhain curve (4.25) to the Seiberg-Witten curve (4.8), note that a degree 5 polynomial as in (4.25) can be obtained by a linear fractional transformation of a degree 6 hyperelliptic equation  $y^2 = \prod_{j=1}^6 (x - r_j)$ , which maps three of the roots to  $\infty$ , 0 and 1. Linear fractional maps leave cross-ratios invariant, which is a convenient way to relate the  $\lambda_j$  to  $u$  and  $v$ . Let us define the cross-ratio of four points  $z_i \in \mathbb{CP}^1$  as

$$C(z_1, z_2, z_3, z_j) = \frac{(z_1 - z_3)(z_2 - z_j)}{(z_1 - z_j)(z_2 - z_3)}, \quad (4.29)$$

such that  $C(\{\infty, 0, 1, \lambda_j\}) = \lambda_j$ .

Note that we have 120 different possibilities to map three roots among the  $\{r_j\}$  to 0, 1,  $\infty$ , and another 3! possibilities to identify the three cross-ratios in the hyperelliptic setting with the  $\lambda_j$ . By studying the large  $u$  expansions of these for non-zero  $v$ , one can easily identify which cross-ratios, in terms of the  $r_i$ , correspond to which  $\lambda_j$ . To this end, let  $\alpha = e^{2\pi i/3}$  as before. The roots of the rhs of (4.8) are then given by (with  $\Lambda = 1$ )

$$\begin{aligned} r_1 &= s_+(u, v+1) + s_-(u, v+1), & r_4 &= s_+(u, v-1) + s_-(u, v-1), \\ r_2 &= \alpha s_+(u, v+1) + \alpha^2 s_-(u, v+1), & r_5 &= \alpha s_+(u, v-1) + \alpha^2 s_-(u, v-1), \\ r_3 &= \alpha^2 s_+(u, v+1) + \alpha s_-(u, v+1), & r_6 &= \alpha^2 s_+(u, v-1) + \alpha s_-(u, v-1), \end{aligned} \quad (4.30)$$

where

$$s_{\pm}(u, v) = \sqrt[3]{\frac{v}{2} \pm \sqrt{\frac{v^2}{4} - \frac{u^3}{27}}}. \quad (4.31)$$

To simplify notation, let us set  $s_{\pm\pm} := s_{\pm}(u, v \pm 1)$ . The large  $u$ , small  $v$

expansions for the roots are

$$\begin{aligned}
r_1 &= \sqrt{u} + \frac{1+v}{2u} + \dots, & r_4 &= \sqrt{u} - \frac{1-v}{2u} + \dots, \\
r_2 &= -\sqrt{u} + \frac{1+v}{2u} + \dots, & r_5 &= -\sqrt{u} - \frac{1-v}{2u} + \dots, \\
r_3 &= -\frac{1+v}{u} + \dots, & r_6 &= \frac{1-v}{u} + \dots.
\end{aligned} \tag{4.32}$$

Plugging the weak-coupling expansions (4.21) into the Rosenhain invariants gives the leading behaviour for the  $\lambda_j$ . From this we can see that each invariant  $\lambda_j$  approaches 1 in the large  $u$  limit.

We continue by determining which of the 720 possible sets of cross-ratios matches with the theta constants. We have to determine which roots correspond to the first three points  $z_i$ ,  $i = 1, 2, 3$ , in the cross-ratio (4.29). Since the three theta constants approach 1 in the large  $u$  limit, we should take for  $\{z_1, z_2\}$  in (4.29) the roots which vanish in this limit, thus  $\{r_3, r_6\}$ . Together with the choice of  $z_2$ , this reduces to 8 possible triplets. From a further comparison between the Rosenhain invariants and the cross-ratios, we determine that  $z_1 = r_6$ ,  $z_2 = r_3$  and  $z_3 = r_2$ . With  $C_j := C(r_6, r_3, r_2, r_j)$  for  $j = 1, 4$  and 5, we arrive at

$$\lambda_1 = C_5, \quad \lambda_2 = C_1, \quad \lambda_3 = C_4. \tag{4.33}$$

These are three equations for five unknowns, namely  $\tau_{11}$ ,  $\tau_{12}$ ,  $\tau_{22}$ ,  $u$  and  $v$ . To make it more manifest that the right hand side depends on only two variables, let us express the cross-ratios  $C_j$  in terms of  $s_{\pm\pm}$ ,

$$\begin{aligned}
C_1 &= \alpha^2 \frac{[\alpha s_{+-} + s_{--} - s_{++} - \alpha s_{-+}][s_{++} - \alpha s_{-+}]}{[\alpha^2 s_{+-} + \alpha s_{--} - s_{++} - s_{-+}][s_{-+} - s_{++}]}, \\
C_4 &= -\frac{[\alpha s_{+-} + s_{--} - s_{++} - \alpha s_{-+}][\alpha^2 s_{++} + \alpha s_{-+} - s_{+-} - s_{--}]}{3[s_{+-} - \alpha s_{--}][s_{-+} - s_{++}]}, \\
C_5 &= -\alpha^2 \frac{[\alpha s_{+-} + s_{--} - s_{++} - \alpha s_{-+}][\alpha s_{++} + s_{-+} - s_{+-} - \alpha s_{--}]}{3[s_{--} - s_{+-}][s_{-+} - s_{++}]}.
\end{aligned} \tag{4.34}$$

Note that these expressions are true on the full moduli space. For  $u \neq 0$ , we can define

$$X = \frac{s_{++}}{\sqrt{u/3}}, \quad Y = \frac{s_{+-}}{\sqrt{u/3}}, \tag{4.35}$$

such that  $X^{-1} = s_{-+}/\sqrt{u/3}$  and  $Y^{-1} = s_{--}/\sqrt{u/3}$ , since  $s_{+\pm}s_{-\pm} = u/3$ . The cross-ratios can then be expressed as

$$\begin{aligned}
C_1 &= -\alpha^2 \frac{X(X - \alpha Y)(X - Y^{-1})(X - \alpha X^{-1})}{(X^2 - 1)(X - \alpha^2 Y)(X - \alpha Y^{-1})}, \\
C_4 &= -\frac{1}{3}\alpha^2 \frac{(X - \alpha Y)^2(X - Y^{-1})(X - \alpha Y^{-1})}{X(X^2 - 1)(Y - \alpha Y^{-1})}, \\
C_5 &= \frac{1}{3} \frac{(X - \alpha Y)(X - Y^{-1})^2(X - \alpha^2 Y)}{X(X^2 - 1)(Y - Y^{-1})}.
\end{aligned} \tag{4.36}$$

We thus see that the Coulomb branch can be identified with the zero-locus of the three equations (4.36) inside the space  $(\lambda_1, \lambda_2, \lambda_3, X, Y)$ . One may in principle eliminate  $X$  and  $Y$  to arrive at a single equation in terms of the  $\lambda_j$ . In the following two sections, we will restrict to the two one-dimensional sub-loci  $\mathcal{E}_u$  and  $\mathcal{E}_v$  of the solution space of (4.33), where  $v = 0$  and  $u = 0$  respectively.

### 4.3 Locus $\mathcal{E}_u$ : $v = 0$

In this section we analyse the locus  $v = 0$ . We will demonstrate that the order parameter  $u$  can be expressed in terms of classical modular forms on this locus. In fact, we will arrive at two distinct expressions depending on a choice of effective coupling. In Section 4.5, we will discuss these aspects from the geometric point of view.

#### 4.3.1 Algebraic relations

On the locus  $v = 0$  we have that  $\tau_{11}(u, 0) = \tau_{22}(u, 0)$  and  $\tau_{12}(u, 0)$  is given by (4.22). Let us analyse these coupling constants, now from the perspective of Section 4.2.3. For  $u$  large and positive,  $s_{+\pm}$  has a large magnitude and phase  $e^{\pi i/6}$ . Similarly, the phase of  $s_{-\pm}$  is approximately given by  $e^{-\pi i/6}$ . This means that

$$s_{--} = -\alpha s_{++}, \quad s_{+-} = -\alpha^2 s_{-+}, \quad X = -\alpha^2 Y^{-1}. \quad (4.37)$$

Using this and (4.35), we find that (4.36) now turns into

$$\begin{aligned} C_1 &= -\frac{(X + X^{-1})(X - \alpha X^{-1})}{(X - X^{-1})(X + \alpha X^{-1})}, \\ C_4 &= -\frac{1(X + X^{-1})^2}{3(X - X^{-1})^2}, \\ C_5 &= +\frac{1(X + X^{-1})(X + \alpha X^{-1})}{3(X - \alpha X^{-1})(X - X^{-1})}. \end{aligned} \quad (4.38)$$

Since the rhs of (4.38) depends only on one variable  $X$ , the cross-ratios  $C_j$  satisfy two algebraic equations, which can be determined by solving the equations for  $X^2$ . One finds

$$\begin{aligned} C_1 C_5 - C_4 &= 0, \\ (3C_4 - C_1)^2 - C_4(C_1 + 1)^2 &= 0. \end{aligned} \quad (4.39)$$

Using (4.33) and (4.26), the cross-ratios are identified with quotients of Siegel theta functions (see Appendix A.2), and the above equations take the form

$$\begin{aligned} 0 &= \Theta_3^4 - \Theta_4^4, \\ 0 &= \Theta_1^2 \Theta_2^2 \Theta_8^4 \Theta_3^4 - \Theta_2^4 \Theta_8^2 \Theta_{10}^2 \Theta_3^4 + 8 \Theta_1^2 \Theta_2^2 \Theta_4^2 \Theta_8^2 \Theta_{10}^2 \Theta_3^2 + \Theta_1^2 \Theta_2^2 \Theta_4^4 \Theta_{10}^4 - 9 \Theta_1^4 \Theta_4^4 \Theta_8^2 \Theta_{10}^2. \end{aligned} \quad (4.40)$$

The two systems of equations above are equivalent given that none of the  $\lambda_j$  vanish or are infinite, which is an assumption of Picard's lemma (4.26). We

can use the second relation of (4.38) to solve for  $u$ ,

$$u^3 = \frac{\sqrt{27}}{2} \frac{(3C_4 + 1)^3}{\sqrt{C_4}(C_4 - 1)}, \quad (4.41)$$

and in terms of theta constants this gives

$$u^3 = \frac{\sqrt{27}}{2} \frac{(3\Theta_1^2\Theta_8^2 + \Theta_2^2\Theta_{10}^2)^3}{\Theta_1\Theta_2^3\Theta_8\Theta_{10}^3(\Theta_1^2\Theta_8^2 - \Theta_2^2\Theta_{10}^2)}. \quad (4.42)$$

This can be viewed as a generalisation of the rank 1 result (1.35), in the sense that we can write the parameter  $u$  as a rational function of theta series. It follows naively that  $u$  transforms as a weight 0 function under a subgroup of  $\mathrm{Sp}(4, \mathbb{Z})$ .

### 4.3.2 A modular expression for $u$

The solutions to the algebraic relations (4.40) are not unique due to the periodicity in the  $\tau_{IJ}$ . The first equation implies  $\tau_{11} - \tau_{22} = 2k$  with  $k \in \mathbb{Z}$ , but we know from (4.21) that  $k = 0$ . From (4.22) we can make a power series expansion for  $\tau_{12}$  in terms of  $p = e^{2\pi i\tau_{11}}$ . One finds

$$\tau_{12} = -\frac{1}{2}\tau_{11} - \frac{1}{2\pi i} \log(8) + \frac{1}{2\pi i} \frac{27}{4} h(p), \quad (4.43)$$

with

$$h(p) = p^{\frac{1}{2}} - \frac{63}{16}p + \frac{1447}{64}p^{\frac{3}{2}} - \frac{307679}{2048}p^2 + \mathcal{O}(p^{\frac{5}{2}}), \quad (4.44)$$

by satisfying the second relation in (4.40) order by order. Substitution of (4.43) in (4.41) gives the following  $p$ -expansion for  $u$ ,

$$u = \frac{1}{2}p^{-\frac{1}{6}} + \frac{43}{8}p^{\frac{1}{3}} - \frac{2923}{128}p^{\frac{5}{6}} + \frac{1713}{16}p^{\frac{4}{3}} + \mathcal{O}(p^{\frac{11}{6}}). \quad (4.45)$$

One can verify agreement with the Picard-Fuchs approach by substituting this expansion in Eq. (4.21). As this series is only an expansion for small  $p$ , it is not very elucidating. To arrive at a closed expression, we aim to express  $u$  as a function of a ‘‘coupling constant’’ which transforms well under the duality transformations. This is not the case for  $\tau_{11}$ .

However when  $\tau_{11} = \tau_{22}$ , the inversion  $\mathcal{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Sp}(4, \mathbb{Z})$  acts naturally on the linear combinations  $\tau_{\pm} = \tau_{11} \pm \tau_{12}$ , which are in one-to-one correspondence with  $\tau_{11}$  and  $\tau_{12}$ . From (A.24), we deduce for the action of  $\mathcal{S}$  on  $\tau_{\pm}$

$$\mathcal{S} : \tau_{11} \pm \tau_{12} \mapsto -\frac{1}{\tau_{11} \pm \tau_{12}}. \quad (4.46)$$

That is to say, it reduces to the ordinary  $S$ -transformation  $\tau_{\pm} \mapsto -1/\tau_{\pm}$ . Moreover,  $\tau_{\pm} \in \mathbb{H}$  for both  $\pm$ . To see this note that since  $\mathrm{Im}(\Omega)$  is positive definite, we have that  $y_{11} > 0$  and  $y_{11}y_{22} - y_{12}^2 > 0$ , where  $y_{IJ} = \mathrm{Im}(\tau_{IJ})$ . Whenever  $y_{11} = y_{22}$ , the latter inequality implies that  $y_{11}^2 > y_{12}^2$ . Since  $y_{11} > 0$ ,



it implies  $y_{11} > y_{12}$  and  $y_{11} > -y_{12}$  simultaneously. From this we learn that  $y_{11} - y_{12}$  and  $y_{11} + y_{12}$  are both positive and therefore  $\tau_{\pm} := \tau_{11} \pm \tau_{12} \in \mathbb{H}$ .

We will proceed by considering  $\tau_- =: \tau$ , leaving the discussion on  $\tau_+$  for section 4.3.3. To determine  $u$  as function of  $\tau$ , one can first find the series expansion for  $\tau$  in terms of  $p$ , invert and substitute  $p(\tau)$  in (4.45). Alternatively, one can revert to the Picard-Fuchs solution, by inverting the series (4.21) for  $v = 0$ ,

$$q = e^{2\pi i(\tau_{11}(u) - \tau_{12}(u))} = U^3 + 45U^4 + 1512U^5 + 45672U^6 + \dots, \quad U = \frac{1}{4u^3}. \quad (4.47)$$

Either method gives us the following series for  $u$ ,

$$\sqrt[3]{4}u = q^{-\frac{1}{9}} + 5q^{\frac{2}{9}} - 7q^{\frac{5}{9}} + 3q^{\frac{8}{9}} + 15q^{\frac{11}{9}} - 32q^{\frac{14}{9}} + \mathcal{O}(q^{\frac{17}{9}}). \quad (4.48)$$

This expansion is also known as the McKay-Thompson series of class 9B for the Monster group [36, 56–58]. Thus similarly to the  $u$  for rank 1 (1.35), we find a McKay-Thompson series. We then have

$$u = u_-(\tau) = \sqrt[3]{\frac{27}{4}} \frac{b_{3,0}\left(\frac{\tau}{3}\right)}{b_{3,1}\left(\frac{\tau}{3}\right)}, \quad (4.49)$$

where  $b_{3,j}$  are theta series for the  $A_2$  root lattice,

$$b_{3,j}(\tau) = \sum_{k_1, k_2 \in \mathbb{Z} + \frac{j}{3}} q^{k_1^2 + k_2^2 + k_1 k_2}, \quad j \in \{-1, 0, 1\}. \quad (4.50)$$

The theta series  $b_{3,j}$  transform under the generators of  $\mathrm{SL}(2, \mathbb{Z})$  as ( $\alpha = e^{2\pi i/3}$ )

$$\begin{aligned} S: \quad b_{3,j}\left(-\frac{1}{\tau}\right) &= -\frac{i\tau}{\sqrt{3}} \sum_{l \pmod{3}} \alpha^{2jl} b_{3,l}(\tau), \\ T: \quad b_{3,j}(\tau + 1) &= \alpha^{j^2} b_{3,j}(\tau). \end{aligned} \quad (4.51)$$

The solution  $u_-$  can also be expressed in terms of the Dedekind  $\eta$ -function (A.18) as

$$u_-(\tau) = \sqrt[3]{\frac{27}{4}} \left( 1 + \frac{1}{3} \frac{\eta\left(\frac{\tau}{9}\right)^3}{\eta(\tau)^3} \right). \quad (4.52)$$

Using Theorem 1 in Appendix A.1, one finds that  $u_-(9\tau)$  is a modular function for the congruence subgroup  $\Gamma_0(9)$  (also defined in Appendix A.1). This implies that  $u$  is a modular function for  $\Gamma^0(9)$ , which is generated by the matrices  $T^9$ ,  $STS$  and  $(T^3S)T(T^3S)^{-1}$ . In fact, it is easy to see from (4.51) that  $u_-(\tau - 3) = \alpha u_-(\tau)$  for all  $\tau \in \mathbb{H}$ . Furthermore,  $u$  rotates as well under  $TST^{-2}$ ,  $u_-\left(\frac{\tau-3}{\tau-2}\right) = \alpha u_-(\tau)$ . The two elements  $T^3$  and  $TST^{-2}$  generate  $\Gamma^0(3)$  and  $u$  can therefore be interpreted as a modular function for  $\Gamma^0(3)$  with multipliers  $\alpha^k$ .

Let us analyse the strong coupling singularities  $u^3 = \frac{27}{4}$  for  $v = 0$  in terms of the variable  $\tau$ . We will demonstrate that these correspond to  $\tau \rightarrow 0, 3$  and  $-3$ . Using (4.51), one finds that the expansion around 0 takes the form

$$\begin{aligned} \sqrt[3]{\frac{4}{27}} u_{-,D}(\tau_D) &= \frac{b_{3,0}(3\tau_D) + 2b_{3,1}(3\tau_D)}{b_{3,0}(3\tau_D) - b_{3,1}(3\tau_D)} \\ &= 1 + 9q_D + 27q_D^2 + 81q_D^3 + 198q_D^4 + \mathcal{O}(q_D^5), \end{aligned} \quad (4.53)$$

with  $\tau_D = -1/\tau$ ,  $q_D = e^{2\pi i\tau_D}$  and  $u_{-,D}(\tau_D) := u_-(-1/\tau_D)$ . In the same notation we can invert the series to find

$$q_D = \chi - 3\chi^2 + 9\chi^3 - 22\chi^4 + 21\chi^5 + 207\chi^6 + \mathcal{O}(\chi^7), \quad (4.54)$$

where  $\chi := (\sqrt[3]{4/27}u - 1)/9$ . It follows that  $q_D \rightarrow 0$  for  $\sqrt[3]{4/27}u \rightarrow 1$  or  $\chi \rightarrow 0$ . This can be directly confirmed by analytically continuing the Picard-Fuchs expansion around  $u = \sqrt[3]{27/4}$ .

The expansion around  $\pm 3$  can then be obtained from the one around 0 by shifting the argument  $\tau_{D,\pm} = -\frac{1}{\tau} \pm 3$ , and one finds using the  $T$ -transformation (4.51) that

$$u_{-,D}(\tau_{D,\pm}) = \alpha^{\mp 1} \sqrt[3]{\frac{27}{4}} \frac{b_{3,0}(3\tau_D) + 2b_{3,1}(3\tau_D)}{b_{3,0}(3\tau_D) - b_{3,1}(3\tau_D)} \quad (4.55)$$

The expansions around the points 3 and  $-3$  differ from the one around 0 only by the phases  $\alpha^{-1} = \alpha^2$  and  $\alpha$ . Together with (4.53), this proves that indeed  $\tau \rightarrow \{0, -3, 3\}$  corresponds to the three singularities  $u \rightarrow \{1, \alpha, \alpha^2\}$ . Due to the  $T^9$ -invariance of the solution (4.49), there is an ambiguity in identifying the  $\tau$ -parameter with  $\tau + 9\mathbb{Z}$ . These  $\mathbb{Z}_2$  points are studied in detail in [215, 220]. They correspond to the 3 vacua of the  $\mathcal{N} = 1$  theory after deforming the  $\mathcal{N} = 2$  theory by relevant or marginal terms.

The modular analysis is completely analogous to the  $SU(2)$  theory, as reviewed in section 1.5: The cusps of  $\Gamma^0(9)$  are  $\{0, -3, 3, i\infty\}$ , which is exactly where  $u$  assumes the  $\mathbb{Z}_2$  vacua and the semi-classical limit. The fundamental domain of  $\Gamma^0(9)$  is given in Figure 29 and is the union of 12 images of the  $SL(2, \mathbb{Z})$  fundamental domain  $\mathcal{F} = SL(2, \mathbb{Z}) \backslash \mathbb{H}$ ,

$$\Gamma^0(9) \backslash \mathbb{H} = \bigcup_{\ell=-4}^4 T^\ell \mathcal{F} \cup S\mathcal{F} \cup T^3 S\mathcal{F} \cup T^{-3} S\mathcal{F}. \quad (4.56)$$

Using (4.52), we can find the exact coupling at the origin of the moduli space. We have that  $u(\tau_0) = 0$  for the  $\Gamma^0(9)$  orbit of

$$\tau_0 = \sqrt{3}\omega = \frac{3}{2} + \frac{\sqrt{3}}{2}i, \quad (4.57)$$

with  $\omega = e^{\pi i/6}$ . The point  $\tau_0$  lies on the boundary of the fundamental domain, on the point where the boundary arcs from different cusps meet. The elements  $(STS)^k \in \Gamma^0(9)$  map  $\tau_0 \mapsto \tau_0 - 3k$  for integer  $k$ , which identifies the ‘‘corners’’

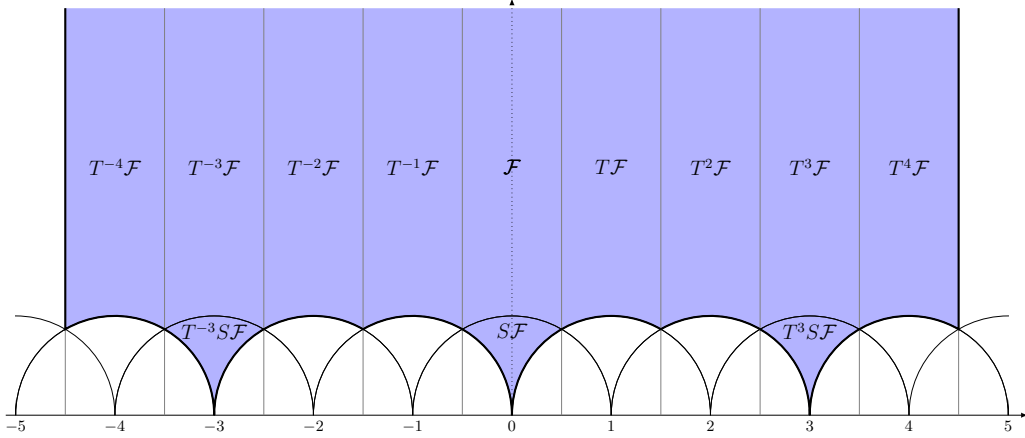


Figure 29: Fundamental domain  $\Gamma^0(9)\backslash\mathbb{H}$  of the congruence subgroup  $\Gamma^0(9)$ . It consists of 12 images of the key-hole fundamental domain  $\mathcal{F}$ .

in Figure 29. This is compatible with the global  $\mathbb{Z}_3$  symmetry, which also acts by  $T^{-3}$  and leaves the origin invariant. It is in complete analogy to the  $SU(2)$  picture: We find the nice picture that the cusps of  $\Gamma^0(9)\backslash\mathbb{H}$  are in one-to-one correspondence with the singularities  $u^3 = \frac{27}{4}$  and  $u = \infty$  and the origin is the symmetric point where the boundary arcs meet.

### 4.3.3 $u$ as a sextic modular function

While we chose in the above the modular parameter  $\tau_- = \tau_{11} - \tau_{12}$ , Equation (4.46) shows that we could equally well consider  $\tau_+ = \tau_{11} + \tau_{12}$ . We will consider the variable  $\tau := \tau_+$  in this subsection. We can determine the first terms in the  $q$ -expansion of  $u$ , which results in

$$u = u_+(\tau) = \frac{1}{4} \left( q^{-1/3} + 104 q^{2/3} - 7396 q^{5/3} + \mathcal{O}(q^{8/3}) \right). \quad (4.58)$$

This series can be recognised as the  $q$ -expansion of

$$u_+(\tau) = \sqrt[3]{\frac{27}{2}} \frac{E_4(\tau)^{1/2}}{(E_4(\tau)^{3/2} - E_6(\tau))^{1/3}}, \quad (4.59)$$

where  $E_4$  and  $E_6$  are the Eisenstein series (1.7). We will derive this explicitly in section 4.5. The function  $u_+$  is a root of the sextic equation

$$X^6 - \frac{j(\tau)}{64} X^3 + \frac{27j(\tau)}{256} = 0, \quad (4.60)$$

where  $j$  is the  $j$ -invariant (A.9). Since the coefficients of this sextic equation are modular functions for  $SL(2, \mathbb{Z})$ , we call  $u_+$  a *sextic modular function* (see also Appendix A.5). Due to the fractional powers in (4.59),  $u_+$  is not a modular function for  $SL(2, \mathbb{Z})$ . In fact,  $E_4^{1/2}$  and  $u_+$  are not invariant under *any* subgroup of  $SL(2, \mathbb{Z})$ . One way to see this is that  $E_4$  has a simple zero for  $\tau = \alpha$ , such that the square root introduces a branch cut. While the family

of sextic modular functions thus includes functions which are not modular for  $\text{SL}(2, \mathbb{Z})$ , this family also includes functions which are modular for an index 6 congruence subgroup of  $\text{SL}(2, \mathbb{Z})$ . The order parameter for  $\text{SU}(2)$  (1.35) is an example of the latter. One can thus view the family of sextic modular functions as an extension of the family of modular functions for index 6 congruence subgroups.

Interestingly,  $u_+$  is up to an overall factor the same function as the order parameter of the massless  $N_f = 1$  theory with gauge group  $\text{SU}(2)$  [101, 103, 221], see (2.66). This aspect distinguishes massless  $N_f = 1$  from  $N_f = 0, 2, 3$ , since for the latter theories the order parameters are modular functions for congruence subgroups isomorphic to  $\Gamma^0(4)$  [101]. On the other hand, it is known since the time of Fricke and Klein that similar fractional powers of modular forms as in  $u_+$  do appear in the context of Picard-Fuchs equations and hypergeometric functions [222, 223].

As mentioned before, the fractional powers in (4.59) are incompatible with any subgroup of  $\text{SL}(2, \mathbb{Z})$ . Nevertheless, if we choose a basepoint, we can show that  $u_+$  is invariant under transformations of  $\tau$ , which combine to a closed trajectory with starting and endpoint equal to the base point. We choose the base point  $\tau_b$  with  $\text{Re}(\tau_b) = 0$  and  $\text{Im}(\tau_b) \gg 1$ . First, using the modular transformation of  $E_4$  and  $E_6$ , we find for the expansion of  $\tau$  near 0,

$$\tau \rightarrow 0 : \quad u_+(\tau) = u_{+,D}(-1/\tau), \quad (4.61)$$

with

$$\begin{aligned} u_{+,D}(\tau_D) &= \sqrt[3]{\frac{27}{2}} \frac{E_4(\tau_D)^{1/2}}{(E_4(\tau_D)^{3/2} + E_6(\tau_D))^{1/3}} \\ &= \sqrt[3]{\frac{27}{4}} (1 + 144 q_D - 3456 q_D^2 + 596160 q_D^3 + \dots). \end{aligned} \quad (4.62)$$

The  $S$ -transform  $u_{+,D}$  is also a solution to (4.60) and thus also a sextic modular function. From Eq. (4.58) we see that  $u_+$  is invariant under  $\tau \mapsto \tau + 3$  at weak coupling,  $\text{Im}(\tau) \gg 1$ . Let us introduce  $T_w$  for the translation at weak coupling. Moreover at strong coupling,  $0 < \text{Im}(\tau) \ll 1$ ,  $u_+$  is invariant under  $\tau_D = -1/\tau \mapsto \tau_D + 1$ . Let us introduce  $T_s$  for the translation at strong coupling. We can get the monodromies around the other cusps,  $\tau = \pm 1$  from conjugation with  $T_w$ . We then find that  $u_+$  is left invariant by

$$T_w^{3n}, \quad (T_w^\ell S) T_s (T_w^\ell S)^{-1}, \quad \ell, n \in \mathbb{Z}, \quad (4.63)$$

where  $S$  is the usual inversion  $\tau \mapsto -1/\tau$ , mapping  $\tau$  from weak to strong coupling. These transformations are sketched in Figure 30 for  $n = 1$  and  $\ell = 0, \pm 1$ .

We denote the invariance group of  $u_+$  by  $\Gamma_{u_+}$ . It is generated by the elements in (4.63) with  $n = 1$ , and  $\ell = 0, 1$ . From the invariance under (4.63), one derives that a fundamental domain is given by

$$\bigcup_{\ell=-1}^1 T^\ell \mathcal{F} \cup T^\ell S \mathcal{F}. \quad (4.64)$$

It consists of six copies of  $\mathcal{F}$ , which is directly related to  $u_+$  being a sextic modular function. This fundamental domain is the grey area in Figure 30. The domain is clearly topologically equivalent to the fundamental domain in Figure 29. The expansions of  $u_+$  and  $u_{+,D}$  demonstrate that  $u_+(i\infty) = \infty$ ,

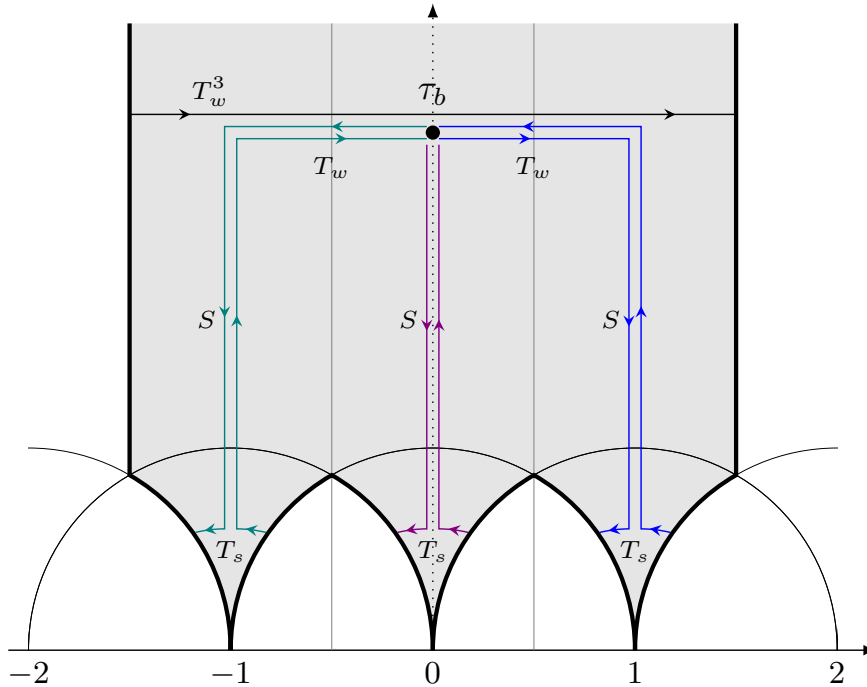


Figure 30: Fundamental domain for  $u_+$ . The vertical lines at  $\tau = \pm 3/2$  are identified, as well as each pair of the two arcs meeting at a cusp  $-1, 0$  or  $1$ . The point  $\tau_b$  is the base point for the monodromies, which are compositions of  $T_w$ ,  $T_s$  and  $S$ .  $T_w$  is a shift  $\tau \mapsto \tau + 1$  at weak coupling,  $T_s$  circles around a strong coupling cusp, and  $S$  maps  $\tau$  from weak to strong coupling.

$u_+(0) = \sqrt[3]{\frac{27}{4}}$  and  $u_+(\pm 1) = \alpha^\mp \sqrt[3]{\frac{27}{4}}$ . We will derive  $u_+$  from the SW geometry in section 4.5.

Because  $u_+$  is not a weakly holomorphic modular form, but involves fractional powers of modular forms, it is problematic to identify the transformations (4.63) with elements of  $\text{SL}(2, \mathbb{Z})$ . One way to see that this identification is problematic is that the composition of  $S$ ,  $T_w$  and  $T_s$  does not satisfy the relation  $(ST)^3 = -\mathbb{1}$ , if we identify  $T_w = T_s = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . To further

study this aspect, let us list the  $\mathrm{SL}(2, \mathbb{Z})$  matrices corresponding to (4.63),

$$\begin{aligned} T^3 &= \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \\ STS^{-1} &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \\ (TS)T(TS)^{-1} &= \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \\ (T^{-1}S)T(T^{-1}S)^{-1} &= \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{4.65}$$

These matrices fix each of the cusps  $\{\infty, 0, 1, -1\}$ . On the other hand,  $u_+$  is not invariant under the modular action of the matrices on  $\tau$ ,  $\tau \mapsto (a\tau + b)/(c\tau + d)$  except for  $T^{3n}$ . For example,  $STS^{-1}$  would map  $\tau = i\infty$  to  $-1$ . The values of  $u_+$  are however different for these two arguments:  $u_+(i\infty) = \infty$  and  $u_+(-1) = \alpha \sqrt[3]{\frac{27}{4}}$ . Furthermore, the matrices (4.65) generate the full modular group  $\mathrm{SL}(2, \mathbb{Z})$ .

The origin  $u_+(\tau_0) = 0$  of the moduli space is again given by the points where the boundary arcs meet: At  $\tau_0 = \alpha$  we have that  $E_4$  vanishes but  $E_6$  does not. From (4.59) it is then clear that  $\tau_0 + \mathbb{Z}$  are indeed the zeros of  $u_+$ . This is also compatible with the  $\mathbb{Z}_3$  global symmetry, which according to (4.58) acts as  $T^{-1}$  and leaves the origin invariant.

#### 4.4 Locus $\mathcal{E}_v$ : $u = 0$

We will now consider the second elliptic locus, namely where  $u = 0$ . By doing a similar analysis as in section 4.3 but now for large  $v$ , we find that the correct matching between the cross-ratios and the Rosenhain invariants for this limit is

$$\lambda_1 = C_5, \quad \lambda_2 = C_4, \quad \lambda_3 = C_1. \tag{4.66}$$

Note that the only difference from before is that the rôles of  $\lambda_2$  and  $\lambda_3$  have been interchanged. One could perform a change of symplectic basis to have the same matching as (4.33). This can be done by acting on the periods with the matrix  $\mathcal{T}_\theta = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix} \in \mathrm{Sp}(4, \mathbb{Z})$  with  $\theta = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}$ .<sup>26</sup> This would however also change the Rosenhain form, and we therefore prefer to continue with the identification in (4.66).

We will proceed by deriving the relations satisfied by the couplings  $\tau_{IJ}$  on the locus  $u = 0$ .

##### 4.4.1 Algebraic relations

To determine the algebraic relations among the theta constants, we assume that  $v$  is real, large and positive. In this limit we find that  $s_{+\pm} = \sqrt[3]{v \pm 1}$  and

<sup>26</sup>Note that there is an ambiguity in the choice of  $\mathcal{T}_\theta$ . The  $\lambda_j$  are invariant under a subgroup of  $\mathrm{Sp}(4, \mathbb{Z})$ . Multiplying  $\mathcal{T}_\theta$  with an element of this group thus gives the same result.

$s_{-\pm} = 0$ . The cross-ratios (4.34) simplify, and one finds

$$\begin{aligned} C_1 &= -\alpha^2 \frac{s_{++} - \alpha s_{+-}}{s_{++} - \alpha^2 s_{+-}}, \\ C_4 &= -\frac{\alpha^2 (s_{++} - \alpha s_{+-})^2}{3 s_{++} s_{+-}}, \\ C_5 &= +\frac{1 (s_{++} - \alpha s_{+-}) (s_{++} - \alpha^2 s_{+-})}{3 s_{++} s_{+-}}. \end{aligned} \quad (4.67)$$

From this we find two algebraic relations between the cross-ratios, namely

$$\begin{aligned} C_1 C_5 - C_4 &= 0, \\ C_5^2 + C_4^2 - C_5 C_4 - C_4 &= 0. \end{aligned} \quad (4.68)$$

Writing these in terms of the theta constants, we have

$$\begin{aligned} 0 &= \Theta_1^4 - \Theta_2^4, \\ 0 &= \Theta_2^4 \Theta_3^2 \Theta_8^4 + \Theta_1^4 \Theta_3^2 \Theta_{10}^4 - \Theta_1^2 \Theta_2^2 \Theta_3^2 \Theta_8^2 \Theta_{10}^2 - \Theta_2^4 \Theta_4^2 \Theta_8^2 \Theta_{10}^2. \end{aligned} \quad (4.69)$$

#### 4.4.2 Modular expression for $v$

Our next aim is to determine a modular expression for  $v$  on this elliptic locus. The first relation in (4.69) implies  $\tau_{11} = \tau_{22} + 2\mathbb{Z} + 1$ , while the second one implies  $\tau_{12} = \pm \frac{1}{2} \tau_{11} + \mathbb{Z}$ . We claim that these are all the solutions. As in the case  $v = 0$ , the PF solution (4.24) fixes these relations,

$$\tau_{11} = \tau_{22} + 1, \quad \tau_{12} = -\frac{\tau_{11}}{2} + 1. \quad (4.70)$$

In contrast to the locus  $\mathcal{E}_u$ , these linear relations between the  $\tau_{11}$ ,  $\tau_{22}$  and  $\tau_{12}$  are exact on  $\mathcal{E}_v$ . Using the first equation in (4.67), we can solve for  $v$ ,

$$v = -\frac{i}{\sqrt{27}} \frac{(C_1 - 2)(C_1 + 1)(2C_1 - 1)}{C_1(C_1 - 1)}. \quad (4.71)$$

This can again be written as a rational function of Siegel theta functions,

$$v = -\frac{i}{\sqrt{27}} \frac{(\Theta_8^2 - 2\Theta_{10}^2)(\Theta_8^2 + \Theta_{10}^2)(2\Theta_8^2 - \Theta_{10}^2)}{\Theta_8^2 \Theta_{10}^2 (\Theta_8^2 - \Theta_{10}^2)}. \quad (4.72)$$

As a function of  $\tau_- = \tau_{11} - \tau_{12}$ , one finds ( $q_- = e^{2\pi i \tau_-}$ )

$$v = \frac{i}{2\sqrt{27}} \left( \alpha q_-^{-\frac{1}{6}} - 33 \alpha^2 q_-^{\frac{1}{6}} - 153 q_-^{\frac{1}{2}} - 713 \alpha q_-^{\frac{5}{6}} + \mathcal{O}(q_-^{\frac{7}{6}}) \right). \quad (4.73)$$

The expansion in terms of  $\tau_+ = \tau_{11} + \tau_{12}$  is very similar. One can recognise these series as

$$\begin{aligned} v &= \frac{i}{2\sqrt{27}} m\left(\frac{\tau_+}{2}\right), \\ v &= \frac{i}{2\sqrt{27}} m\left(\frac{\tau_-}{6} + \frac{2}{3}\right), \end{aligned} \quad (4.74)$$

where

$$\begin{aligned} m(\tau) &= \left( \frac{\eta(2\tau)}{\eta(6\tau)} \right)^6 - 27 \left( \frac{\eta(6\tau)}{\eta(2\tau)} \right)^6 \\ &= q^{-1} - 33q - 153q^3 - 713q^5 - 2550q^7 - 7479q^9 + \mathcal{O}(q^{11}). \end{aligned} \quad (4.75)$$

The function  $m$  is known in the literature as the completely replicable function of class 6a [56–58]. The perturbative expansion (4.73) can also be verified from the Picard-Fuchs solution by starting from Eq. (4.24) and setting  $u = 0$ . Then, expand  $q = e^{2\pi i(\tau_{11}(v) - \tau_{12}(v))}$  as a series in  $v$  and invert it to find (4.73).

#### 4.4.3 The $\mathbb{Z}_3$ vacua

Let us study the solution (4.74) near the strong coupling vacua. To this end, we eliminate the phases in (4.73) by substitution of  $\tau := \tau_- + 1$  in (4.74). In the new variable  $\tau$ , the solution reads

$$v = -\frac{i}{2\sqrt{27}} m\left(\frac{\tau}{6}\right). \quad (4.76)$$

It can be shown that the values of  $\tau$  at the Argyres-Douglas (AD) vacua  $v_{\text{AD},1} = 1$  and  $v_{\text{AD},2} = -1$  are ( $\omega = e^{\pi i/6}$ )

$$\begin{aligned} \tau_{\text{AD},1} &= -\frac{3}{2} + \frac{\sqrt{3}i}{2} = \sqrt{3}\omega^5, \\ \tau_{\text{AD},2} &= +\frac{3}{2} + \frac{\sqrt{3}i}{2} = \sqrt{3}\omega, \end{aligned} \quad (4.77)$$

and the origin  $(u, v) = (0, 0)$  is located at  $\tau_0 = \sqrt{3}i$ . This can be rigorously using the properties of  $m$ .

The solutions to  $v = 1$  and  $v = -1$  are not straightforward to obtain. Let us start with the point  $(\underline{u}, v) = (0, -1)$ . In the following, all arguments are those of  $m$ . Due to the prefactor in (4.76),  $v = -1$  is in fact a quadratic equation with zero discriminant and therefore satisfied if and only if

$$\left( \frac{\eta(2\tau)}{\eta(6\tau)} \right)^6 = -\sqrt{27}i. \quad (4.78)$$

A solution to this equation can be found to be

$$\tau_{-1} = \frac{\omega}{2\sqrt{3}} = \frac{1}{4} + \frac{i}{4\sqrt{3}} = \frac{\tau_{\text{AD},2}}{6}, \quad (4.79)$$

with  $\omega = e^{\pi i/6}$  as before and  $\tau_{\text{AD},2}$  the argument of  $v$  in (4.77). The other AD point can be found using the symmetry of  $m$ , and it is given by

$$\tau_{+1} = \frac{\omega^5}{2\sqrt{3}} = -\frac{1}{4} + \frac{i}{4\sqrt{3}} = \frac{\tau_{\text{AD},1}}{6}. \quad (4.80)$$

The zero of  $m$  (and therefore of  $v$ ) is given by

$$\tau_0 = \frac{i}{2\sqrt{3}}. \quad (4.81)$$



Note that all these numbers have the same absolute value  $\frac{1}{2\sqrt{3}}$ .

Let us prove (4.79) first: In order to compute both the numerator and the denominator, we can resort to the  $S$ - and  $T$ -transformations of  $\eta$  as given in A.1,

$$\begin{aligned}\eta(2\tau_{-1}) &\stackrel{S}{=} 3^{\frac{1}{4}}\omega e^{-\frac{\pi i}{12}}\eta\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right) \stackrel{T}{=} 3^{\frac{1}{4}}e^{\frac{\pi i}{12}}\eta(\alpha) \\ \eta(6\tau_{-1}) &= \eta\left(\frac{3}{2} + \frac{\sqrt{3}i}{2}\right) \stackrel{T}{=} e^{\frac{\pi i}{6}}\eta(\alpha).\end{aligned}\tag{4.82}$$

Equation (4.78) follows immediately.

In order to find the point where  $v = +1$ , we can make the observation that  $m(-\frac{1}{\tau}) = -m(\frac{\tau}{12})$ . This implies that under the Fricke involution  $\begin{pmatrix} 0 & -1 \\ 12 & 0 \end{pmatrix}$ , the solution receives a minus sign,

$$m\left(-\frac{1}{12\tau}\right) = -m(\tau).\tag{4.83}$$

Using the  $T$ -transformation of  $\eta$ , one also finds that  $m(\tau \pm \frac{1}{2}) = -m(\tau)$ . We can use either of those maps,  $\tau_{+1} = \tau_{-1} - \frac{1}{2} = -\frac{1}{12\tau_{-1}}$  to obtain (4.80).

We can also study the zeros of  $v$ . Every root of  $m(\tau)$  is given by the equation  $\eta(2\tau)^{12} = 27\eta(6\tau)^{12}$ . A solution to this equation is (4.81), which we can prove: Using the  $S$ -transformation, we find

$$\eta(2\tau_0) = \eta\left(\frac{i}{\sqrt{3}}\right) = 3^{\frac{1}{4}}\eta(\sqrt{3}i) = 3^{\frac{1}{4}}\eta(6\tau_0).\tag{4.84}$$

The result follows immediately. Another proof follows simply from the fact that  $\tau_0$  is the fixed point under (4.83).

The modular group of  $v$  is closely related to the duality group of the SU(3) theory on this locus. It can be shown that  $v$  is a modular form for the principal congruence subgroup  $\Gamma(6)$ , as defined in Appendix A.1. However, the fundamental domain of this group has twelve cusps, and  $v$  diverges at all of them. This suggests that we found strongly coupled vacua in the region of the moduli space where  $v$  is large. But from the discriminant  $\Delta_\Lambda|_{\mathcal{E}_v} = v^2 - 1$  we expect the only singularities to be at  $v \in \{1, -1, \infty\}$ .

To resolve this problem, let us study the function  $m$  in more detail. It is a linear combination of eta quotients, whose modular properties have been studied extensively [224, 225]. Applying Theorem 1 in Appendix A.1, one finds that  $m$  is a modular function for the Hecke congruence subgroup  $\Gamma_0(12)$ . In addition, it satisfies the following non-SL(2,  $\mathbb{Z}$ ) transformations

$$m\left(\tau - \frac{1}{2}\right) = -m(\tau),\tag{4.85a}$$

$$m\left(-\frac{1}{12\tau}\right) = -m(\tau).\tag{4.85b}$$

The transformation (4.85b) is also known as a *Fricke involution*. Translating both equations to the argument of  $v$ , we find that  $v$  picks up a minus sign under both  $T^{-3}$  and  $F = \begin{pmatrix} 0 & -3 \\ 1 & 0 \end{pmatrix}$ . Taking products, we find that  $v$  is properly invariant under  $FT^{-3} = \begin{pmatrix} 0 & -3 \\ 1 & -3 \end{pmatrix}$  and  $T^{-6}$ . Let us normalise the former to

$X = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & -3 \\ 1 & -3 \end{pmatrix}$ , and denote the subgroup of  $\mathrm{PSL}(2, \mathbb{R})$  generated by these two elements as

$$\Gamma_v = \langle X, T^{-6} \rangle. \quad (4.86)$$

This group is a proper subgroup of the modular group  $\Gamma^0(6|2) + 3$  of Atkin-Lehner type, in the notation of [58]. This Atkin-Lehner group extends the ordinary congruence subgroup  $\Gamma^0(\frac{6}{2})$  by elements in  $\mathrm{PSL}(2, \mathbb{R})$ . See Appendix A.1 for the precise definition. If we allow for a non-trivial multiplier system, the modular group associated with  $m$  is  $\Gamma^0(6|2) + 3$  [58]. The latter contains for example  $T^{-3}$ , under which we have shown that  $v$  is anti-invariant. We can write a similar set of matrices as (4.65),

$$M_1 = \begin{pmatrix} -3 & -3 \\ 1 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 3 \\ -1 & 3 \end{pmatrix}, \quad M_\infty = \begin{pmatrix} 1 & -6 \\ 0 & 1 \end{pmatrix} = T^{-6}, \quad (4.87)$$

under which  $v \sim m(\tau/6)$  is invariant. If we consider their normalisation to unit determinant,  $\Pi(M_j) := |\det(M_j)|^{-1/2} M_j$ , they lie in the group  $\Gamma_v$  (4.86), and furthermore satisfy

$$\Pi(M_1)\Pi(M_2) = M_\infty. \quad (4.88)$$

A fundamental domain for  $\Gamma_v$  can be drawn using the algorithm given in [58], and it is shown in Figure 31. The element  $T^6$  contains the domain to  $|\mathrm{Re} \tau| < 3$ .  $X$  identifies the interior of the circle with radius  $\sqrt{3}$  centered at 0, with a region inside the blue domain in Figure 31. Similarly, the interior of the circles centered at  $\pm 3$  is identified with a region of the blue domain. We conclude,

$$\Gamma_v \backslash \mathbb{H} = \{z \in \mathbb{H} \mid |\mathrm{Re} z| < 3\} \setminus \bigcup_{\ell=-1}^1 \overline{D}_{\sqrt{3}}(3\ell). \quad (4.89)$$

where  $\overline{D}_r(c)$  is the closed disc of radius  $r$  and center  $c$ .

The Argyres-Douglas vacua  $v = 1$  and  $v = -1$  correspond to the special points  $\tau_{\mathrm{AD},j}$  (4.77). They are stabilised by  $M_1$  and  $M_2$ , respectively. This makes the AD vacua elliptic points of  $\Gamma_v$ . They are in fact expected to *not* get mapped to cusps of  $v$ , since their coupling matrix lies inside the Siegel upper half-space  $\mathbb{H}_2$  [1]. This is a familiar property of superconformal points [104, 155]. It is different from the  $\mathbb{Z}_2$  points where the coupling matrices are located on the boundary  $\partial\mathbb{H}_2$  and therefore mapped to the real line  $\partial\mathbb{H}_1$ . The origin  $\tau_0 = \sqrt{3}i$  is mapped under  $FT^{-3}$  to  $\tau_0 - 3$ , which is identified with  $\tau_0$  since  $v = 0$  is a fixed point under  $T^{-3} : v \mapsto -v$ . The anti-invariance under  $T^{-3}$  is in fact directly derived from the  $\mathbb{Z}_2$  symmetry  $\rho : v \mapsto e^{\pi i} v$  [1]. The large  $v$  monodromy  $\rho^2$  acts on  $\tau$  as  $T^{-6}$ , under which  $v$  is invariant. The origin of the Fricke involution can therefore be understood from the global structure on the  $u = 0$  plane.

The discussion is similar for the parameter  $\tau_+ = \tau_{11} + \tau_{12}$ . If we introduce here  $\tau = \tau_+ - 1$ ,  $v$  equals  $-\frac{i}{2\sqrt{27}} m(\tau/2)$ , which is again invariant under  $\Gamma(6)$ .

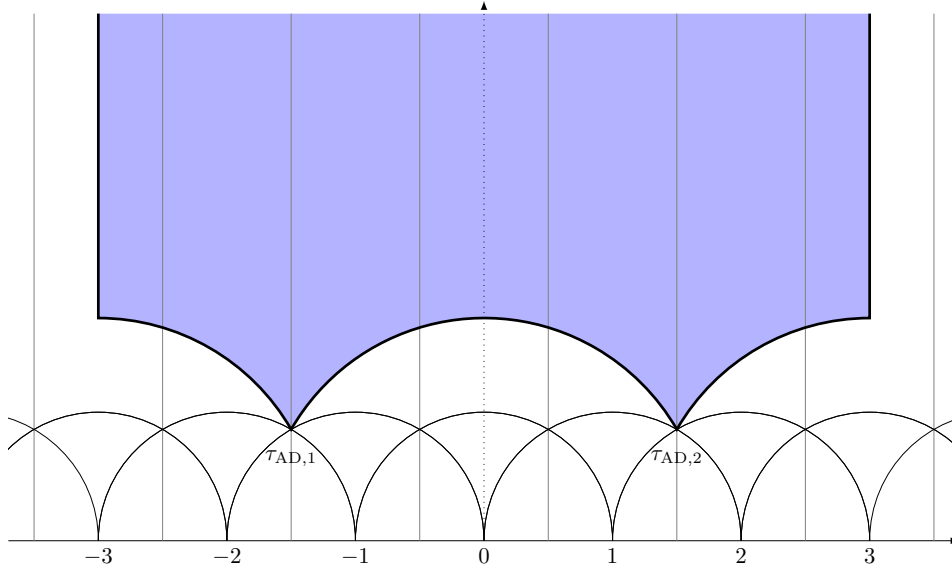


Figure 31: Fundamental domain  $\Gamma_v \backslash \mathbb{H}$  for the group  $\Gamma_v$ . The values of the special points are:  $\tau_{\text{AD},1} = \sqrt{3}\omega^5$  and  $\tau_{\text{AD},2} = \sqrt{3}\omega$ .

It is multiplied by a sign under  $T$  as well as under the Fricke involution  $\tilde{F} = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}$ . This means that it is invariant under  $T^2$  together with the involution  $\tilde{X} := \tilde{F}T^{-1} = \begin{pmatrix} 0 & -1 \\ 3 & -3 \end{pmatrix}$ , which again generate an Atkin-Lehner type group. The fundamental domain of this group equals that in Figure 31, but with all points divided by 3.

## 4.5 Elliptic curves for $\mathcal{E}_u$ and $\mathcal{E}_v$

It is natural to expect that the complexified couplings  $\tau_{\pm}$  for both loci  $\mathcal{E}_u$  and  $\mathcal{E}_v$  have an interpretation as complex structures of elliptic curves. Moreover, these elliptic curves are expected to be related to the geometry of the genus two Seiberg-Witten curve (4.8). We will make these expectations precise in this section.

Recall that the moduli space  $\mathcal{M}_2$  of genus two curves is complex three-dimensional. The moduli space  $\mathcal{M}_2$  contains two-dimensional loci  $\mathcal{L}_2 \subset \mathcal{M}_2$ , for which the genus two curves can be mapped to genus one with a map of degree 2 [226]. The map can be lifted to a map of the Jacobians of the curves. The Jacobian of the genus two curve is a four-torus, while the Jacobian of a genus one curve is a two-torus. For the curves contained in  $\mathcal{L}_2$ , there is a degree two map from the genus two Jacobian to the genus one Jacobian. The Jacobian of a curve in  $\mathcal{L}_2$  factors,  $T^4 \cong T^2 \times T^2$ , which demonstrates that for a generic curve in  $\mathcal{L}_2$ , there are two distinct maps  $\varphi_j : \Sigma_2 \rightarrow \Sigma_{1,j}$ ,  $j = 1, 2$  to two elliptic curves  $\Sigma_{1,j}$ . We will see in this section that these elliptic curves  $\Sigma_{1,j}$  have precisely the complex structures  $\tau_{\pm}$  introduced above.

The locus  $\mathcal{L}_2$  of genus 2 fields with elliptic subfields of degree 2 reads

[211, 212, 227]

$$\begin{aligned}
& 8748J_{10}J_2^4J_6^2 + 507384000J_{10}^2J_4^2J_2 - 19245600J_{10}^2J_4J_2^3 - 592272J_{10}J_4^4J_2^2 + 77436J_{10}J_4^3J_2^4 \\
& - 81J_2^3J_6^4 - 3499200J_{10}J_2J_6^3 + 4743360J_{10}J_4^3J_2J_6 - 870912J_{10}J_4^2J_2^3J_6 + 3090960J_{10}J_4J_2^2J_6^2 \\
& - 78J_2^5J_4^5 - 125971200000J_{10}^3 + 384J_4^6J_6 + 41472J_{10}J_4^5 + 159J_4^6J_2^3 - 236196J_{10}^2J_2^5 - 80J_4^7J_2 \\
& - 47952J_2J_4J_6^4 + 104976000J_{10}^2J_2^2J_6 - 1728J_4^5J_2^2J_6 + 6048J_4^4J_2J_6^2 - 9331200J_{10}J_4^2J_6^2 \\
& + 12J_2^6J_4^3J_6 + 29376J_2^2J_4^2J_6^3 - 8910J_2^3J_4^3J_6^2 - 2099520000J_{10}^2J_4J_6 + 31104J_6^5 - 6912J_4^3J_6^3 \\
& - J_2^7J_4^4 - 5832J_{10}J_2^5J_4J_6 - 54J_2^5J_4^2J_6^2 + 108J_2^4J_4J_6^3 + 972J_{10}J_2^6J_4^2 + 1332J_2^4J_4^4J_6 = 0.
\end{aligned} \tag{4.90}$$

It is the zero locus of a weight 30 polynomial in the genus two Igusa invariants  $J_2, J_4, J_6, J_{10}$ . The Igusa invariants of a generic sextic curve can be found in [228], for example.

Additionally, the  $SU(3)$  vacuum moduli space also corresponds to a two-dimensional locus  $\mathcal{U}$  in  $\mathcal{M}_2$ . On  $\mathcal{U}$  the weight 30 polynomial factors in three terms, such that  $\mathcal{U}$  and  $\mathcal{L}_2$  intersect in three one-dimensional loci:

$$\begin{aligned}
\mathcal{E}_1 = \mathcal{E}_u : & \quad v = 0, \\
\mathcal{E}_2 = \mathcal{E}_v : & \quad u = 0, \\
\mathcal{E}_3 : & \quad 784u^9 - 24u^6(297v^2 + 553) - 15u^3(729v^4 + 5454v^2 - 4775) \\
& \quad + 8(27v^2 - 25)^3 = 0.
\end{aligned} \tag{4.91}$$

Not suprisingly, we have seen the first two of these loci before. The latter is a cubic equation in  $v^2$  as well as in  $u^3$ , which does not reduce further. It does not include special points of the  $SU(3)$  theory. For  $v = 0$ , the equation reduces to the points  $u^3 = 8$  and  $u^3 = \frac{125}{28}$  in the  $u$ -plane, and for  $u = 0$  it intersects in  $v^2 = \frac{25}{27}$  on the  $v$ -plane.

The locus  $\mathcal{L}_2$  can also be characterised in terms of Rosenhain invariants of the curve [211, Equation (18)]. By plugging in the cross-ratios we can check that the  $SU(3)$  Seiberg-Witten curve is not in  $\mathcal{L}_2$  for generic  $u, v$ . For  $v = 0$  we rediscover the first algebraic relation (4.39), while for  $u = 0$  we find both relations (4.68). This arises from an additional symmetry of the  $u = 0$  curve, which we will comment on below.

#### 4.5.1 Elliptic curves for locus $\mathcal{E}_u$

In this subsection we will establish two elliptic curves corresponding to the two modular parameters  $\tau_{\pm}$  in section 4.3. The curves described by the locus  $\mathcal{L}_2$  can be written in the form [211]

$$Y^2 = X^6 - s_1X^4 + s_2X^2 - 1, \tag{4.92}$$

with  $s_1$  and  $s_2$  complex coordinates for  $\mathcal{L}_2$ . This family of curves is left invariant by a non-trivial automorphism group, which contains the Klein four-group  $V_4$  [229]. Namely, the curve (4.92) is left invariant by  $(X, Y) \mapsto (-X, Y)$  and  $(X, Y) \mapsto (X, -Y)$ , which generate the dihedral group  $D_4 \cong V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ . We interpret this group as the symmetry group of BPS/anti-BPS spectrum, and

more precisely the central charges of the W-bosons  $Z_j$  (4.4) and their charge conjugates. For  $v = 0$ , Eq. (4.20) shows that  $a_1 = a_2 = a$ , such that  $Z_1 = Z_2 = a$ , and  $Z_3 = 2a$ . One  $\mathbb{Z}_2 \subset D_4$  corresponds to the charge conjugation symmetry, while the other  $\mathbb{Z}_2$  corresponds to the  $a_1 \leftrightarrow a_2$  symmetry on  $\mathcal{E}_u$ . Note that the automorphism group of a generic genus two curve is  $\mathbb{Z}_2$ , which is consistent with the charge conjugation symmetry for arbitrary  $(u, v)$ .

For  $v = 0$ , the Seiberg-Witten curve  $Y^2 = (X^3 - uX)^2 - 1$  is of the form (4.92), with  $s_1 = 2u$  and  $s_2 = u^2$ . The degree two map to an elliptic curve is

$$(x, y) = (X^2, Y), \quad (4.93)$$

which maps the algebraic equation (4.92) to

$$y^2 = x(x - u)^2 - 1. \quad (4.94)$$

We can determine  $u$  in terms of the complex structure  $\tau$  of the curve from the  $j$ -invariant,  $j = 256u^6/(4u^3 - 27)$ . This gives

$$4u(\tau) = q^{-1/3} + 104q^{2/3} - 7396q^{5/3} + \mathcal{O}(q^{8/3}). \quad (4.95)$$

We immediately recognise this function as the function  $u_+$  (4.58), which was obtained from the Picard-Fuchs solution for the modular parameter  $\tau_+ = \tau_{11} + \tau_{12}$ . The curve (4.94) is exactly the Seiberg-Witten curve for the  $SU(2)$  theory with one massless hypermultiplet in the fundamental representation and scales related by  $\Lambda_{SU(2)} = 2\Lambda_{SU(3)}$  [46], which clarifies the observation in section 4.3.3.

The curve that corresponds to  $\tau_- = \tau_{11} - \tau_{12}$  can be constructed as follows. On the curve (4.92), the transformation  $(X, Y) \mapsto (\frac{1}{X}, \frac{iY}{X^3})$  interchanges  $s_1$  and  $s_2$ . Interchanging those coefficients,  $s_1 = u^2$  and  $s_2 = 2u$ , and setting again  $(x, y) = (X^2, Y)$ , we obtain

$$y^2 = x(x^2 - u^2x + 2u) - 1. \quad (4.96)$$

One finds  $j = 256u^3(u^3 - 6)^3/(4u^3 - 27)$ , which reproduces the solution  $u_-$  for the  $\Gamma^0(9)$  curve (4.49). Note that the equation for  $j$  shows that  $u_-$  is the root of a degree 12 polynomial, which matches with the number of copies of  $\mathcal{F}$  in Figure 29. Another way to obtain this curve is to set  $x = X^2$  and  $y = XY$ , from which one gets a quartic curve with the same  $j$ -invariant.

We have thus demonstrated that the two natural choices  $\tau_{\pm}$  of the modular parameter indeed correspond to the complex structures of two elliptic curves covering the hyperelliptic curve. The physical  $u$  is given in terms of two different functions  $u_{\pm} : \mathbb{H} \rightarrow \mathbb{C}$  with arguments  $\tau_{\pm}$ .

#### 4.5.2 Elliptic curves for locus $\mathcal{E}_v$

The Seiberg-Witten curve  $Y^2 = (X^3 - v)^2 - 1$  for  $u = 0$  is not in form (4.92) for a curve of  $\mathcal{L}_2$ . However, the discussion around (4.91) suggests that it can be written in this form. We can achieve this by comparing the invariants of

the  $u = 0$  hyperelliptic curve and (4.92), and solving for  $s_1, s_2$ . Just as two elliptic curves are isomorphic if and only if their  $j$ -functions are equal, genus two curves are isomorphic if and only if their absolute invariants

$$x_1 = 144 \frac{J_4}{J_2^2}, \quad x_2 = -1728 \frac{J_2 J_4 - 3J_6}{J_2^3}, \quad x_3 = 486 \frac{J_{10}}{J_2^5}, \quad (4.97)$$

are equal [212, 218, 219]. On  $\mathcal{L}_2$ , there are only two independent invariants. For the curve (4.92), we find [212]

$$\begin{aligned} x_1 &= \frac{9 \cdot 405 + (a - 126)a + 12b}{4(15 + a)^2}, \\ x_2 &= \frac{27(a^3 + 729a^2 + 4131a - 36(39 + a)b - 3645)}{8(15 + a)^3}, \\ x_3 &= \frac{243(27 - a(18 + a) + 4b)^2}{8192(15 + a)^5}, \end{aligned} \quad (4.98)$$

where  $a = s_1 s_2$ ,  $b = s_1^3 + s_2^3$  are the dihedral invariants. Comparing these absolute invariants with those of the  $SU(3)$  curve for  $u = 0$ , we arrive at

$$s_1 s_2 = 9(25 - 24v^2), \quad s_1^3 + s_2^3 = 54(216v^4 - 340v^2 + 125). \quad (4.99)$$

To solve the two equations in (4.99), let us denote

$$\mathcal{Q}^\pm(v) = 27(216v^4 - 340v^2 + 125 \pm 8v(27v^2 - 25)\sqrt{v^2 - 1}). \quad (4.100)$$

Then, one of the six solutions is given by

$$s_1^\pm = \sqrt[3]{\mathcal{Q}^\mp(v)}, \quad s_2^\pm = 9 \frac{25 - 24v^2}{\sqrt[3]{\mathcal{Q}^\mp(v)}}. \quad (4.101)$$

In order to get an elliptic curve, we again take the map  $(x, y) = (X^2, Y)$ . This gives us the two curves

$$y^2 = x^3 - s_1^\pm x^2 + s_2^\pm x - 1 \quad (4.102)$$

with  $j$ -function

$$j^\pm = -432 \left( 1458v^6 - 2673v^4 + 1340v^2 - 125 \mp 2v(729v^4 - 972v^2 + 275)\sqrt{v^2 - 1} \right) \quad (4.103)$$

and discriminant  $\Delta = v^2 - 1$ . By inverting (4.103), the resulting function  $v$  matches precisely with (4.74) in section 4.4.2. Note that  $j^\pm$  vanish at the AD points  $v = \pm 1$  and the curve (4.101) becomes a cusp  $y^2 = x^3$ . This implies that the AD points are elliptic fixed points and are in the  $SL(2, \mathbb{Z})$  orbit of  $\alpha$ , which is easy to check from (4.77): We have that  $\tau_{AD,1} = \alpha - 1$  and  $\tau_{AD,2} = \alpha + 2$ . See also Figure 31. They do however not fall into the (classical) Kodaira classification of singular fibers, since the Weierstraß invariants of (4.101) are not polynomials in  $v$  and their order of vanishing is half-integer rather than integer.

In general, the  $j$ -invariants of the two elliptic curves for (4.92) are the two solutions of [212]

$$j^2 + 2^8 \frac{2a^3 - 54a^2 + 9ab - b^2 + 27b}{a^2 + 18a - 4b - 27} j + 2^{16} \frac{(a^2 + 9a - 3b)^3}{(a^2 + 18a - 4b - 27)^2} = 0. \quad (4.104)$$

Since the equations for  $s_1$  and  $s_2$  always have solutions, one elliptic curve is found by substituting  $(x, y) = (X^2, Y)$ , such that it becomes

$$y^2 = x^3 - s_1 x^2 + s_2 x - 1. \quad (4.105)$$

The other elliptic curve is found by relating  $(x, y) = (X^2, XY)$ , such that

$$y^2 = x(x^3 - x_1 x^2 + s_2 x - 1). \quad (4.106)$$

Returning to the curve for  $\mathcal{E}_v$ , we notice that  $Y^2 = X^6 - 2vX^3 + v^2 - 1$  for  $u = 0$  has enhanced symmetry compared to the Klein four-group for (4.92). Since  $v^2 - 1$  is the discriminant, we can divide and rescale  $X$  to find

$$Y^2 = X^6 - \frac{2v}{\sqrt{v^2 - 1}} X^3 + 1. \quad (4.107)$$

It is easy to show that any curve of the form  $Y^2 = X^6 - aX^3 + 1$  is invariant under  $(X, Y) \mapsto (\frac{1}{X}, \frac{Y}{X^3})$  and  $(X, Y) \mapsto (\alpha X, -Y)$ , where again  $\alpha = e^{2\pi i/3}$ . These order 2 and 6 elements generate the dihedral group  $D_{12}$ . Similarly to the enhanced automorphism group for  $\mathcal{E}_u$ , we interpret this group as a symmetry group of the BPS/anti-BPS spectrum. On the locus  $\mathcal{E}_v$ , we find that the periods  $a_1$  and  $a_2$  are related as  $a_2 = -\alpha a_1$ . The central charges  $Z_j$  (4.4) of the W-bosons, together with their charge conjugates, span therefore a regular 6-gon, whose symmetry group is  $D_{12}$ .

Hyperelliptic curves  $C \in \mathcal{L}_2$  with  $\text{Aut}(C) \cong D_{12}$  satisfy an additional constraint, it is given by the zero loci of a weight 12 and a weight 20 polynomial in the Igusa invariants [227, Eq. (24)],

$$\begin{aligned} 0 &= -J_4 J_2^4 + 12J_2^3 J_6 - 52J_4^2 J_2^2 + 80J_4^3 + 960J_2 J_4 J_6 - 3600J_6^2, \\ 0 &= 864J_{10} J_2^5 + 3456000J_{10} J_4^2 J_2 - 43200J_{10} J_4 J_2^3 - 2332800000J_{10}^2 \\ &\quad - J_4^2 J_2^6 - 768J_4^4 J_2^2 + 48J_4^3 J_4^2 + 4096J_4^5. \end{aligned} \quad (4.108)$$

Moreover, the elliptic subcovers of hyperelliptic curves with  $\text{Aut}(C) \cong D_{12}$  are 3-isogenous [211]. We can check explicitly that the  $u = 0$  curve is of this form. Another check on the  $D_{12}$  symmetry is [211]

$$0 = a^2 - 110a - 4b + 1125. \quad (4.109)$$

This explains why the elliptic curves for the two complex structures produce a single modular function (4.74), rather than the two independent functions  $u_{\pm}$  for  $\mathcal{E}_u$ . On  $\mathcal{E}_u$  the first algebraic relation in (4.39) holds and places the curve in  $\mathcal{L}_2$ . On  $\mathcal{E}_v$  both relations (4.68) hold, where the first one projects into  $\mathcal{L}_2$  and the second one gives the augmented  $D_{12}$  symmetry. This is consistent with the argument of Section 4.3.2 that the maps  $\varphi_j$  should exist as long as  $\text{Im}(\tau_{11}) = \text{Im}(\tau_{22})$ , such that it is possible to define  $\tau_{\pm} = \tau_{11} \pm \tau_{12} \in \mathbb{H}$ . The first relations in both (4.39) and (4.68) are equivalent to this condition.

## 4.6 SU(3) theory with matter

With the technology set up in section 4.5, it is in principle straightforward to study other theories characterised by genus two hyperelliptic curves. One such class of theories are the SU(3) theories with  $N_f \leq 6$  hypermultiplets in the fundamental representation.

When  $N_f \leq 5$  and all the masses are equal to  $m$ , the curves are given by [128, 230]

$$Y^2 = C(X)^2 - \Lambda^{2N_c - N_f} (X + m)^{N_f},$$

$$C(X) = X^3 - uX - v + \frac{1}{4} \Lambda^{2N_c - N_f} \sum_{k=0}^{N_f - N_c} X^{N_f - N_c - k} \binom{N_f}{k} m^k. \quad (4.110)$$

These theories are also studied in [8, 178, 205, 231, 232]<sup>27</sup>.

Let us thus study the SU(3) curve for  $\mathcal{N} = 2$  gauge theory with  $N_f \geq 1$  hypermultiplets in the fundamental representation.

$$N_f = 1$$

The locus  $\mathcal{L}_2$  intersects with the  $m = 0$ ,  $N_f = 1$  curve in

$$0 = -6165504u^{12}v^2 - 73809792u^9v^4 - 111484512u^8v^3 + 109220400u^7v^2$$

$$+ 1796349312u^6v^6 - 8196945984u^5v^5 + 9010440000u^4v^4 - 6598371456u^3v^8$$

$$- 4626787500u^3v^3 + 3826375200u^2v^7 + 1261406250u^2v^2 + 2581632u^{11}v$$

$$- 27737500u^6v + 2048u^{15} - 271784u^{10} + 2162500u^5 - 492075000uv^6$$

$$- 175781250uv - 1836660096v^{10} - 59231250v^5 + 9765625, \quad (4.111)$$

and in particular neither  $u = 0$  nor  $v = 0$  are in  $\mathcal{L}_2$ . This is also the case for generic masses, for which the equation because much more complicated.

$$N_f = 2$$

For  $N_f = 2$ ,  $v = 0$  is in fact a singular surface. It intersects with the singular locus  $\Delta_\Lambda = 0$  in the points  $u^2 = 1$ . In fact,  $v = 0$  is a sublocus of  $\mathcal{L}_2$ . However, since it is singular, the usual maps  $\phi_j : \Sigma_2 \rightarrow \Sigma_{1,j}$  are degenerate. Thus the map to the elliptic subcover is ill-defined.

$$N_f = 3$$

For massless  $N_f = 3$  we find that only  $u = 0$  is in  $\mathcal{L}_2$ . The discriminant is

$$\Delta_\Lambda = (4v - 1)^3 (-3456u^3v^2 + 3888u^3v + 256u^6 - 729u^3 + 11664v^4 - 2916v^3), \quad (4.112)$$

and therefore  $u = 0$  is not a singular locus. Rather,  $u = 0$  intersects with  $\Delta_\Lambda = 0$  in the two vacua  $v = 0$  and  $v = \frac{1}{4}$ . We can compute the absolute

<sup>27</sup>The article [205] clashes with both [231] and [230].



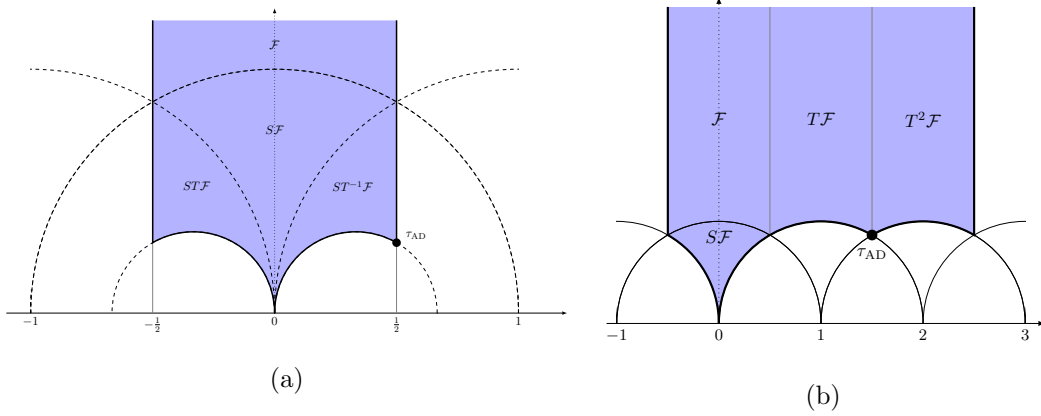


Figure 32: Fundamental domain of  $\Gamma_0(3)$  (Fig. 32a) and  $\Gamma^0(3)$  (Fig. 32b), both the domains are for  $(u = 0) \in \mathcal{L}_2$  for  $N_f = 3$ .

invariants of the  $u = 0$  curve and compare with (4.98). This gives the dihedral invariants

$$a = -\frac{9(48v^2 - 26v + 3)}{2v}, \quad b = \frac{27(6912v^4 - 3968v^3 + 1152v^2 - 248v + 27)}{16v^2}. \quad (4.113)$$

One can then either proceed to compute  $s_1$  and  $s_2$ , which gives fractional powers of  $v$ , or insert (4.113) into (4.104) and compute the  $j$ -invariants of the two elliptic curves. This gives

$$j_1 = -\frac{27(4v - 9)^3(4v - 1)}{64v^3}, \quad j_2 = -\frac{27(4v - 1)(36v - 1)^3}{4v}, \quad (4.114)$$

demonstrating that the underlying elliptic surfaces are rational. From this, one finds

$$v_1 = -\frac{f_{3B}(3\tau)}{108}, \quad v_2 = -\frac{f_{3B}(\tau)}{108}, \quad (4.115)$$

where  $f_{3B}$  is defined in (2.75). This proves that  $v_1$  is a Hauptmodul for  $\Gamma_0(3)$ , while  $v_2$  is a Hauptmodul for  $\Gamma^0(3)$ . The fundamental domains are drawn in Fig. 32. We can see that they are topologically equivalent.

The singular point  $v = 0$  translates to a root of  $f_{3B}$  for both  $v_1$  and  $v_2$ , which holds at the cusp  $\tau = 0$ . For  $v_1$ , we have that  $v_1 = \frac{1}{4}$  if and only if  $f_{3B}(3\tau) = -27$ , and therefore  $\tau = \frac{1}{\sqrt{3}}\omega$ : This is an AD point. For  $v_2$  we get  $f_{3B}(\tau) = -27$ , such that  $\tau = \sqrt{3}\omega$ . It is precisely the same parametrisation as the *II* AD theory in  $SU(2)$  with  $N_f = 1$  and  $m = \frac{3}{4}\Lambda_1$ , where we get a  $\Gamma^0(3)$  curve, or  $N_f = 3$  with  $m = -\frac{1}{64}\Lambda_3$ , where the curve is  $\Gamma_0(3)$  (see section 2.6.6). We can see from (4.114) that both curves are quartic modular functions, which agrees with Fig. 32 as both have index 4 in  $PSL(2, \mathbb{Z})$ .

By solving (4.113) for  $s_1$  and  $s_2$ , we can find the two elliptic curves from (4.105) and (4.106) with  $j$ -invariants (4.114). It allows to check that at the AD point  $u_{AD} = \frac{1}{4}$ , the Kodaira signature is  $\text{ord}(g_2, g_3, \Delta) = (1, 1, 2)$  for both

curves, such that it is a type *II* singularity. This explains the similarity to the *II* AD theories found in rank 1.

The curve for  $u = 0$  reads

$$Y^2 = X^6 - \left(\frac{1}{2} + 2v\right)X^3 + \left(v - \frac{1}{4}\right)^2. \quad (4.116)$$

By a rescaling of  $X$ , we can bring it to the form  $Y^2 = X^6 - fX^3 + 1$ , which is known to have  $D_{12}$  symmetry. This can easily be checked with (4.108) and (4.109). The  $D_{12}$  symmetry implies that the two curves (4.114) are 3-isogenous [211]. The 3-isogeny is related to the fact that there is a determinant 3 Fricke involution that acts on the Hauptmoduln,

$$f_{3B}\left(-\frac{3}{\tau}\right) = \frac{3^6}{f_{3B}(\tau)}. \quad (4.117)$$

It relates the  $j$ -invariants (4.114) as

$$j_1\left(\frac{1}{16v_2}\right) = j_2(v_2). \quad (4.118)$$

$N_f = 4, 5, 6$

For  $N_f = 4$  and  $N_f = 5$ , there are again no simple elliptic loci in the massless case. The  $\mathcal{L}_2$  locus for generic masses becomes increasingly lengthy, compared to (4.111).

The massless  $N_f = 6$  theory is superconformal. It is studied in [178, 180, 205–208, 230, 233, 234]<sup>28</sup>, and it necessarily includes modular forms in the curve itself. It is argued by Minahan and Nemeschansky that for  $m_i = 0$  and  $u = 0$  the curve can be expressed as [178, 205]

$$Y^2 = (f_- X^3 - v)^2 + (f_+^2 - f_-^2)X^6, \quad (4.119)$$

where

$$f_{\pm}(\tau) = \left(\frac{\eta^3(\tau)}{\eta(3\tau)}\right)^3 \pm 27 \left(\frac{\eta^3(3\tau)}{\eta(\tau)}\right)^3. \quad (4.120)$$

The  $q$ -expansions read

$$\begin{aligned} f_+(\tau) &= 1 + 18q + 108q^2 + 234q^3 + 234q^4 + 864q^5 + \mathcal{O}(q^6), \\ f_-(\tau) &= 1 - 36q - 54q^2 - 252q^3 - 468q^4 - 432q^5 + \mathcal{O}(q^6). \end{aligned} \quad (4.121)$$

Both  $f_{\pm}$  are weight 3 modular forms for  $\Gamma_1(3)$ . We find that

$$\begin{aligned} f_+(\tau) &= b_{3,0}(\tau)^3, \\ f_-(\tau) &= b_{3,0}(\tau)^3 - 2b_{3,1}(\tau)^3. \end{aligned} \quad (4.122)$$

This allows to define  $f_1 := \sqrt[3]{f_+} = b_{3,0}$ , which together with  $f_-$  is claimed to generate the ring of modular forms on  $\Gamma_0(3)$ .<sup>29</sup> From (4.119), we can compute

<sup>28</sup>Some of these references contradict each other.

<sup>29</sup>Without this, it is not obvious as in [178, 205] why  $f_1$  should be modular.

the absolute invariants of the curve,

$$\begin{aligned}
x_1 &= \frac{81 (4f_-^2 f_+^2 + 5f_+^4)}{(f_-^2 - 10f_+^2)^2}, \\
x_2 &= -\frac{729 (-26f_-^2 f_+^4 + 4f_-^4 f_+^2 - 5f_+^6)}{(f_-^2 - 10f_+^2)^3}, \\
x_3 &= \frac{729 f_+^4 (f_-^2 - f_+^2)^3}{256 (f_-^2 - 10f_+^2)^5}.
\end{aligned} \tag{4.123}$$

They do not depend on  $v$ . Since  $(x_1, x_2, x_3)$  labels isomorphism classes of hyperelliptic curves, in fact the curve (4.119) does not depend on  $v$  at all. This is reminiscent of the massless  $SU(2)$   $N_f = 4$  curve [46].

There is also a proposed curve for nonzero  $u$ ,

$$Y^2 = (f_- X^3 - f_1 u X - v)^2 + (f_+^2 - f_-^2) X^6. \tag{4.124}$$

Aside from  $u = 0$  and  $v = 0$ , the loci  $0 = 27f_- v^2 - 2f_1^3 u^3$  and

$$\begin{aligned}
0 &= 15f_1^6 f_+^2 u^6 v^4 (512f_-^2 + 325f_+^2) - 96f_1^3 f_- u^3 v^6 (2f_-^2 + 25f_+^2)^2 \\
&\quad - 1160f_1^9 f_- f_+^2 u^9 v^2 + 48f_1^{12} f_+^2 u^{12} + 8v^8 (2f_-^2 + 25f_+^2)^3
\end{aligned} \tag{4.125}$$

are in  $\mathcal{L}_2$ .

### Other gauge groups

We could also run an analysis for other gauge groups. The curves for  $SO(N_c)$  for instance are found in [97, 235, 236], see also [237] for a review. However, for  $SO(N_c)$  the genus of the hyperelliptic curve is  $g = 2l - 1$ , where  $l = \frac{N_c}{2}$  for  $N_c$  even and  $l = \frac{N_c - 1}{2}$  for  $N_c$  odd. So in fact there are no genus 2 curves for  $SO(N_c)$  theories: For  $SO(3)$  of course one finds the same curve  $\Gamma^0(4)$  as for  $SU(2)$ . For  $SO(N_c > 3)$  on the other hand the curve has genus  $g > 2$ .

## 4.7 Discussion

In this section, we have discussed the modular properties of  $\mathcal{N} = 2$  Yang–Mills theory in four dimensions with gauge group  $SU(3)$ . For the pure theory, on the two loci  $\mathcal{E}_u$  and  $\mathcal{E}_v$ , where  $v = 0$  and  $u = 0$  respectively, we express the parameters  $u$  and  $v$  of the moduli space as modular functions for discrete subgroups of  $SL(2, \mathbb{R})$ . See (4.49) and (4.76). To this end, we formulate the genus two  $SU(3)$  SW curve in Rosenhain form in terms of Siegel theta series. The parameters of the theory are then found by relating the Rosenhain form to the PF solution of [96]. We provide an explicit fundamental domain for the effective coupling on the two elliptic loci  $\mathcal{E}_u$  and  $\mathcal{E}_v$ . The relation between cross-ratios of the curve and theta constants suggests that the full moduli space can be parametrised by higher genus modular forms. It would be interesting

to find a general solution to (4.36) by expressing  $u$  and  $v$  as algebraic functions of theta constants.

On  $\mathcal{E}_u$ , we established a nice generalisation of the structure appearing in the  $SU(2)$  case. In rank one, the parameter  $u$  is a weakly holomorphic modular function for the congruence subgroup  $\Gamma^0(4)$ . For  $SU(3)$ , we instead found that on  $\mathcal{E}_u$  the parameter  $u$  is a weakly holomorphic modular function of  $\tau_-$  for the group  $\Gamma^0(9) \subset SL(2, \mathbb{Z})$ . The structure of the moduli space near the special points of this locus also seems to generalise the rank one picture: We find that  $u$  maps the  $\mathbb{Z}_2$  singularities to the cusps of its fundamental domain. Furthermore, the duality group is generated by the nontrivial monodromies on  $\mathcal{E}_u$ . For the other choice of modular parameter  $\tau_+ = \tau_{11} + \tau_{12}$ , we find that  $u$  is not invariant under a congruence subgroup, but is rather a *sextic modular function*, which is the same function as appears for rank 1  $N_f = 1$  SQCD. Nevertheless, we are able to show that the monodromies can be viewed as paths in a new fundamental region, which we propose.

On the other locus  $\mathcal{E}_v$  where  $u = 0$ , we find that  $v$  can be expressed as a modular function for a subgroup  $\Gamma_v \subset SL(2, \mathbb{R})$  of Atkin-Lehner type. The AD points are mapped to the elliptic fixed points of the quotient  $\Gamma_v \backslash \mathbb{H}$ . The group  $\Gamma_v$  includes a Fricke involution, which can be viewed as a manifestation of  $S$ -duality [33, 178, 205]. We derive it from the monodromy group on  $\mathcal{E}_v$ . On the locus  $\mathcal{E}_v$ , the genus two hyperelliptic curve splits into two *elliptic* curves with complex structures  $\tau_{\pm} = \tau_{11} \pm \tau_{12}$ . The appearance of the Fricke involution is a consequence of the two families of elliptic curves being isogenous [223, 238]. Fricke dualities also appear in String theory, where they have been shown to play an important rôle in the web of dualities of CHL models, i.e. orbifolds of heterotic string theory on  $T^6$  or type II on  $K3 \times T^2$  [239, 240]. They are also the natural generalisation of  $S$ -duality in the context of Montonen–Olive duality in  $\mathcal{N} = 4$  super-Yang–Mills theory for non-simply laced gauge groups [241, 242] and the geometric Langlands program [30]. Moreover, Fricke involutions are familiar in topological string theory where they act on higher genus amplitudes, which are described by quasi modular forms. They exchange the large complex structure of the Calabi-Yau threefold with the conifold loci, which gives an analogue of the action of electric-magnetic duality or  $\mathcal{N} = 2$   $S$ -duality in topological string theory [223, 238]. They have also appeared recently in the context of string compactifications and the swampland program [243].

It would be interesting to extend this work to other theories, such as those with gauge group  $SU(N)$ , including matter multiplets, theories of class  $\mathcal{S}$  [113], or gravitational couplings to these theories [103, 148]. For theories with  $SU(N > 2)$ , one can for example consider to turn on only the bottom Casimir  $u_2$  and setting  $u_3, \dots, u_N$  to zero. Our analysis naively suggests that it should be parametrised by a modular function for  $\Gamma^0(N^2)$ . The discriminant

of the  $SU(N)$  curve [94]

$$y^2 = \left( x^N - \sum_{j=2}^N u_j x^{N-j} \right)^2 - 1 \quad (4.126)$$

intersects with this locus in  $u_2^N = N^N(N-2)^{2-N}/4$ , confirming that there are  $N$  singularities at strong coupling. However, it is easy to show that  $\Gamma^0(N^2)$  has  $N$  cusps aside from  $i\infty$  if and only if  $N$  is prime. Note that this worked for  $N = 2, 3$ . It is furthermore not obvious how the modular parameter would relate to the coupling matrix, and the map to elliptic subcovers is more subtle in the higher rank case [213].

We would like to finish by mentioning a few potential applications and directions for further research:

- We observe that the functions parametrising the  $SU(2)$  and  $SU(3)$  moduli spaces are all *replicable* [36, 56–58] modular functions. See Appendix A.4 for a definition. The  $SU(2)$  order parameter  $u$  is of class 4C,  $u_-$  of class 9B, and  $v$  of class 6a. It would be interesting to explore whether there is an underlying reason for the functions to have this property.
- This work motivates exploring subloci of Coulomb branches for theories with other gauge groups and including matter multiplets. This could provide a better understanding of the modularity of these theories. Moreover, it would be interesting to understand whether the solution of the theory on a sublocus is equivalent to the solution of another theory, such as we found for  $\mathcal{E}_u$  and the massless  $N_f = 1$ ,  $SU(2)$  theory for example.
- The elliptic loci we consider are somewhat analogous to the special Kähler strata of Coulomb branches being studied in the recent work [244–246]. The latter aims to classify higher rank  $\mathcal{N} = 2$  SCFTs by decomposing the singular locus into a nested series of one-dimensional building blocks. It would be interesting to see if our methods find applications in this programme.
- The last application which we would like to mention, is topological quantum field theory [61]. Evaluation of the path integral or correlation functions for a compact four-manifold  $X$  involves the integration over the Coulomb branch (the so-called  $u$ -plane integral) of the theory [73, 87, 144]. For gauge group  $SU(2)$ , the integral becomes an integral over the modular fundamental domain  $\Gamma^0(4)\backslash\mathbb{H}$  [73, 77, 81, 84]. A better understanding of the modularity of  $SU(N > 2)$  Seiberg-Witten theory could possibly allow further progress in this direction for theories with  $N > 2$ .

## 5 Topological twists of massive SQCD

In this section, we study an infinite family of topological twists of massive  $\mathcal{N} = 2$  supersymmetric QCD on a compact four-manifold, and the formulation of topological correlation functions. This section is based on [5].

### 5.1 Introduction

We consider topologically twisted  $\mathcal{N} = 2$  supersymmetric Yang–Mills theories with additional matter multiplets on a compact four-manifold, which were introduced in [247–252]. After the work by Seiberg and Witten on the full non-perturbative solution [45, 46, 71], these theories have received much attention in physics [65, 66, 73, 74, 82, 85, 88, 89, 98, 99, 188, 253–256] and mathematics [196, 257–267]. For reviews, see for example [52, 53, 268, 269]. More specifically, we consider in this section topological twists of  $\mathcal{N} = 2$  QCD with gauge group  $SU(2)$  and matter multiplets in the fundamental representation of the gauge group. By including background fluxes for the flavour group, we obtain an infinite family of topological theories [255]. The choice of a background flux makes it possible to formulate topologically twists for  $\mathcal{N} = 2$  SQCD for arbitrary 't Hooft fluxes, or first Chern classes of the gauge bundle. This is similar to the topological twist of  $\mathcal{N} = 2^*$   $SU(2)$  gauge theory, which requires a non-vanishing background flux on a non-spin four-manifold [85]. We moreover develop techniques to determine correlation functions for arbitrary values of the masses of the hypermultiplets.

The starting point of our approach is the low-energy effective field theory on the Coulomb branch. This phase of the theory contributes for a compact four-manifold  $X$  with the topological condition that  $b_2^+(X) = 1$  [73]. In this way, the classical Donaldson invariants can be derived starting from the Seiberg–Witten (SW) solution to  $\mathcal{N} = 2$  supersymmetric Yang–Mills theory with gauge group  $SU(2)$ . The Coulomb branch integral (or  $u$ -plane integral) reduces to an integral over zero modes [73], and reads schematically

$$\Phi = \int_{\mathcal{B}} da \wedge d\bar{a} \rho(a) \Psi(a, \bar{a}), \quad (5.1)$$

where  $\mathcal{B}$  is the Coulomb branch with local coordinates  $a$  and  $\bar{a}$ ,  $\rho(a)$  contains the couplings to the background and  $\Psi(a, \bar{a})$  is a sum over fluxes of the unbroken  $U(1)$  gauge group. For simplicity, we have suppressed the dependence on the metric and not included observables here. For the pure  $SU(2)$  theory, the Coulomb branch integral can be formulated and evaluated for arbitrary four-manifolds, without a requirement for Kähler or toric properties.

Recently, progress has been made on evaluating these  $u$ -plane integrals using a change of variables from  $a$  to the running coupling  $\tau$ . As a result, the integration domain becomes a fundamental domain  $\mathcal{F} \subseteq \mathbb{H}$  in the upper half-plane  $\mathbb{H}$  for the running coupling [4, 77–86]. The integral then takes the

form

$$\Phi = \int_{\mathcal{F}} d\tau \wedge d\bar{\tau} \nu(\tau) \Psi(\tau, \bar{\tau}), \quad (5.2)$$

where the measure factor  $\nu(\tau)$  further contains the Jacobian for the change of variables from  $a$  to  $\tau$ . The domain  $\mathcal{F}$  is a modular fundamental domain in previous analyses, corresponding to the duality group  $\Gamma^0(4)$  for the pure  $SU(2)$  theory [73, 81, 84],  $\Gamma(2)$  for the  $\mathcal{N} = 2^*$  theory [85] and similarly  $\Gamma(2)$  and  $\Gamma_0(4)$  for the theories with two and three massless flavours [77].

As mentioned above, we aim to apply this approach to  $\mathcal{N} = 2$  supersymmetric  $SU(2)$  theories with  $N_f \leq 3$  hypermultiplets in the fundamental representation. Topological correlators of these asymptotically free theories have been considered in various papers before, in particular the formulation of the low energy path integral in [73, 74], SW contributions for four-manifolds with  $b_2^+ > 1$  [65, 66, 73, 88], the  $u$ -plane integral for  $\mathbb{P}^2$  [77], and the calculation of the partition function of the AD theory within the  $N_f = 1$  theory [82]. Since no background fluxes are included in these works, the 't Hooft flux necessarily matches the second Stiefel-Whitney class of the four-manifold,  $w_2(E) = w_2(X)$ , since the twisted hypermultiplets are not well-defined otherwise.

Extending to generic 't Hooft fluxes, and application of the above approach (5.2) to fundamental hypermultiplets with generic masses, gives rise to several new aspects. In particular:

1. The fundamental domain of the effective coupling constant becomes more intricate for massive theories, and does for generic masses not correspond to a modular fundamental domain for a subgroup of  $PSL(2, \mathbb{Z})$ . As demonstrated at length in Sections 2 and 3, the domain contains generically a set of branch points, and branch cuts starting from these points. These aspects have to be dealt with appropriately.
2. We couple the hypermultiplets to background fluxes  $\mathbf{k}_j$  for the flavour group to formulate the theories for arbitrary 't Hooft fluxes. This gives rise to additional couplings in (5.1) and (5.2),

$$\prod_{j,k=1}^{N_f} \exp\left(-2\pi i \frac{\partial^2 F}{\partial m_j \partial m_k} B(\mathbf{k}_j, \mathbf{k}_k)\right), \quad (5.3)$$

where  $F$  is the prepotential of the massive theory, and  $B(\cdot, \cdot)$  is the quadratic form associated to the intersection form on the middle homology  $H_2(X, \mathbb{Z})$  of  $X$ . Such couplings were suggested by Shapere and Tachikawa [144], and are also essential for the formulation of the  $\mathcal{N} = 2^*$  Yang–Mills theory on a non-spin four-manifold [85]. Similarly to [85], we also deduce a non-holomorphic coupling to  $\mathbf{k}_j$ . Moreover, for arriving at a single-valued integrand, we fix an ambiguity in the quadratic

terms of the prepotential. These terms have appeared earlier in the literature in the context of singularities of the SW differential and winding numbers [46, 141].

3. Special points on the Coulomb branch give rise to superconformal theories, such as the Argyres-Douglas (AD) theories [104, 105] and the massless  $N_f = 4$  theory [46]. Their topological partition functions and correlators can be found by considering them in certain mass deformations. The case of  $N_f = 1$  is analysed in [269].

The section is organised as follows. In section 5.2, we present the Seiberg-Witten solution of  $SU(2)$   $\mathcal{N} = 2$  SQCD in flat space, focusing on the fundamental domains for the effective coupling, which we illustrate in several interesting examples. In section 5.3, we formulate the topological twist by coupling the hypermultiplets to external fluxes, such that the topological field theory is well-defined for arbitrary 't Hooft flux and non-spin manifolds. The topological low-energy effective theory coupled to  $N_f$  background fluxes is then modelled in section 5.4 as a  $SU(2) \times U(1)^{N_f}$  theory, with the matter fields corresponding to frozen  $U(1)$  factors. This allows to compute the path integral explicitly as an integral over the  $u$ -plane. In section 5.5, we formulate the  $u$ -plane integral as an integral over the fundamental domains. We prove that the single-valuedness under monodromies holds for a specific choice of magnetic winding numbers. Finally, in section 5.6 we demonstrate that such integrals may be evaluated using mock modular forms, and we show that they localise at the cusps, elliptic points and interior singularities of the fundamental domains.

## 5.2 Special geometry and SW theories

In this Section, we review aspects of the non-perturbative solution for the low energy effective theory of  $\mathcal{N} = 2$  SQCD with gauge group  $SU(2)$  and  $0 \leq N_f \leq 3$  fundamental hypermultiplets [45, 46]. See [47] for a review. Throughout, we let  $\Lambda_{N_f}$  denote the scale of the theory with  $N_f$  hypermultiplets having masses  $m_j$ ,  $j = 1, \dots, N_f$ , and  $a$  the mass of the W-boson on the Coulomb branch.

### 5.2.1 Field content

The  $\mathcal{N} = 2$  theories we consider contain a vector multiplet and  $N_f \leq 3$  hypermultiplets. The fields in these multiplets form representations of  $\text{Spin}(4) = SU(2)_+ \times SU(2)_-$  and  $SU(2)_R$ , which we denote by  $(\mathbf{k}, \mathbf{l}, \mathbf{m})$ , with  $\mathbf{k}, \mathbf{l}$  and  $\mathbf{m}$  dimensions of the representations.

The vector multiplet consists of a gauge field  $A_\mu$ , complex scalar field  $\phi$ , and a pair of Weyl fermions  $\Psi_\alpha^I, \bar{\Psi}_\alpha^I$ . This multiplet transforms under the adjoint representation of the gauge group  $G$ . The representation of  $SU(2)_+ \times SU(2)_- \times SU(2)_R$  formed by the bosonic fields is,

$$(\mathbf{2}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1}), \quad (5.4)$$



while the representation for the fermions is

$$(\mathbf{1}, \mathbf{2}, \mathbf{2}) \oplus (\mathbf{2}, \mathbf{1}, \mathbf{2}). \quad (5.5)$$

The hypermultiplet consist of a pair of complex scalar fields,  $q$  and  $\tilde{q}$ , and Weyl fermions,  $\lambda_\alpha, \bar{\lambda}_{\dot{\alpha}}, \chi_\alpha$  and  $\bar{\chi}_{\dot{\alpha}}$ . We fix the gauge group  $G = \text{SU}(2)$ , and let the hypermultiplets transform under the fundamental representation of this group. With the same notation as above, the bosonic fields of this multiplet form the representation,

$$(\mathbf{1}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{2}), \quad (5.6)$$

while the fermions form the representation

$$(\mathbf{2}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1}). \quad (5.7)$$

### 5.2.2 Seiberg-Witten geometry

The Seiberg-Witten geometry underlies the Coulomb branch of  $\mathcal{N} = 2$  gauge theory. The Coulomb branch is the phase of the theory where  $\text{SU}(2)$  is broken to  $\text{U}(1)$  by a vacuum expectation value (vev) of the vector multiplet scalar  $\phi$ . The vev is semi-classically parametrised by a complex parameter (1.21), up to gauge transformations. In particular,  $a \rightarrow -a$  is a gauge transformation. The gauge invariant order parameter is the Coulomb branch expectation value of the theory in  $\mathbb{R}^4$ , (1.22). The non-perturbative effective action of  $\mathcal{N} = 2$  SQCD is characterised by the prepotential  $F(a, \mathbf{m})$ , with  $\mathbf{m}$  the mass vector  $\mathbf{m} = (m_1, \dots, m_{N_f})$ . The semi-classical part of  $F$  reads [141, 143, 191, 270]

$$\begin{aligned} F(a, \mathbf{m}) &= \frac{2i}{\pi} a^2 \log(a/\Lambda_{N_f}) - \frac{1}{2} \sum_{j=1}^{N_f} \left( n_j \frac{m_j}{\sqrt{2}} a + \frac{3}{8} \frac{i}{\pi} m_j^2 \right) \\ &\quad - \frac{i}{4\pi} \sum_{j=1}^{N_f} \left( a + \frac{m_j}{\sqrt{2}} \right)^2 \log\left(\left(a + \frac{m_j}{\sqrt{2}}\right)/\Lambda_{N_f}\right) + \left( a - \frac{m_j}{\sqrt{2}} \right)^2 \log\left(\left(a - \frac{m_j}{\sqrt{2}}\right)/\Lambda_{N_f}\right) \\ &\quad + \dots, \end{aligned} \quad (5.8)$$

where the  $\dots$  indicate further non-perturbative corrections.

The  $n_j \in \mathbb{Z}$  in (5.8) are the *magnetic winding numbers* of the periods  $a_D := \frac{\partial F}{\partial a}$  dual to  $a$  [141, 143, 271]. These numbers seem to be only rarely discussed in the literature beyond these references.<sup>30</sup> Generally, the theory allows for  $N_f$  electric winding numbers for  $a$  and  $N_f$  magnetic winding numbers for  $a_D$ . These appear in the massive  $N_f > 0$  theories since the Seiberg-Witten differentials now have poles with nonzero residues [143]. The choice (5.8) of the prepotential corresponds to fixing the electric winding numbers to be zero, or equivalently fixing the monodromy at infinity to map  $a \rightarrow e^{\pi i} a$ . Compare

<sup>30</sup>Nekrasov's partition function gives a specific choice upon expanding the function  $\gamma_{\hbar}(x; \Lambda)$  in the perturbative part [191, 272].

for example with [143, Eq. (2.17)]. In section 5.5, we will discuss that the single-valuedness of the  $u$ -plane integral requires  $n_j \equiv -1 \pmod{4}$ .

We introduce the period  $a_D$  dual to  $a$ , and the parameters  $m_{D,j}$  dual to  $m_j$  by

$$a_D = \frac{\partial F}{\partial a}, \quad m_{D,j} = \sqrt{2} \frac{\partial F}{\partial m_j}. \quad (5.9)$$

These parameters are further combined into the  $(2 + 2N_f)$ -dimensional vector  $\Pi$ ,

$$\Pi = \begin{pmatrix} a_D \\ a \\ m_{D,1} \\ \frac{m_1}{\sqrt{2}} \\ \vdots \\ m_{D,N_f} \\ \frac{m_{N_f}}{\sqrt{2}} \end{pmatrix}. \quad (5.10)$$

This vector forms a local system over the  $u$ -plane. The elements of the vector form the symplectic form,

$$\omega_{N_f} = da_D \wedge da + \frac{1}{\sqrt{2}} \sum_{j=1}^{N_f} dm_{D,j} \wedge dm_j. \quad (5.11)$$

The effective gauge coupling is related to the prepotential through (1.23). We also introduce the couplings  $v_j$  and  $w_{jk}$  with  $j, k \in 1, \dots, N_f$ .

$$v_j = \sqrt{2} \frac{\partial^2 F}{\partial a \partial m_j}, \quad w_{jk} = 2 \frac{\partial^2 F}{\partial m_j \partial m_k}. \quad (5.12)$$

If we consider  $F$  as a function of  $(a, \frac{1}{\sqrt{2}}m_1, \dots, \frac{1}{\sqrt{2}}m_{N_f})$ , then the dual parameters are encoded in the Jacobian  $\mathbf{J}_F = (a_D, \mathbf{m}_D)$ , while the couplings are the elements of the Hessian  $\mathbf{H}_F = \begin{pmatrix} \tau & \mathbf{v}^T \\ \mathbf{v} & \mathbf{w} \end{pmatrix}$ . The derivative of the prepotential with respect to the scale  $\Lambda_{N_f}$  provides the order parameter  $u$  (1.22) on the Coulomb branch,

$$u = -\frac{4\pi i}{4 - N_f} \Lambda_{N_f} \frac{\partial F}{\partial \Lambda_{N_f}}. \quad (5.13)$$

The weak-coupling limit in our convention is given by  $\tau \rightarrow i\infty$ ,  $a \rightarrow \infty$  and  $u \rightarrow -\infty$ .<sup>31</sup>

The Seiberg-Witten (SW) solution provides a family of elliptic curves parametrised by the order parameter  $u$  and the masses  $m_i$ , whose complex structure corresponds to the running coupling  $\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$ . For the theories of interest in this section, the curves are given by (2.3).

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<sup>31</sup>Note that this differs slightly from some of the previous literature. However, it is shown in section 2 to be the unique limit consistent with the RG flow.

### 5.2.3 Monodromies

This section determines the monodromies around the  $N_f + 2$  monodromies. We define the physical discriminant  $\Delta_{N_f}$  as the monic polynomial  $\Delta_{N_f} = \prod_{j=1}^{N_f+2} (u - u_j)$ , where  $u_j$  for  $j = 1, \dots, N_f + 2$  are the singular points of the effective theory. We let  $j = 1, \dots, N_f$  label the singular points where one of the matter hypermultiplets becomes massless; and  $j = N_f + 1, N_f + 2$  denote the strong coupling singularities where a monopole and a dyon, respectively, becomes massless.

We leave the winding numbers  $n_j$ ,  $j = 1, \dots, N_f$ , for  $a_D$  generic. Starting with the monodromy around infinity,  $a \rightarrow e^{\pi i} a$ , we deduce from the (5.8) that the vector  $\Pi$  transforms as  $\Pi \rightarrow \mathbf{M}_\infty \Pi$ , with  $\mathbf{M}_\infty$  given by

$$\mathbf{M}_\infty = \begin{pmatrix} -1 & 4 - N_f & 0 & -n_1 & \cdots & 0 & -n_{N_f} \\ 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & n_1 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & & \ddots & & \\ 0 & n_{N_f} & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \quad (5.14)$$

The monodromy matrix  $\mathbf{M}_\infty$  is in  $\text{SL}(2+2N_f, \mathbb{Z})$ , while it acts on the couplings by a symplectic transformation, i.e. it preserves the symplectic form (5.11). This can be checked by requiring that any monodromy  $\mathbf{M}_\infty$  satisfies  $\mathbf{M}^T \mathbf{J} \mathbf{M} = \mathbf{J}$ , with

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\oplus N_f+1}. \quad (5.15)$$

The action on the couplings  $\tau$  (1.23),  $v_j$  and  $w_{jk}$  (5.12) is thus

$$\mathbf{M}_\infty : \begin{cases} \tau \rightarrow \tau + N_f - 4, \\ v_j \rightarrow -v_j - n_j, \\ w_{jk} \rightarrow w_{jk} + \delta_{jk}, \end{cases} \quad (5.16)$$

with  $\delta_{jk}$  the Kronecker delta.

If we assume that the mass  $m_j$  is large, we can also deduce the monodromies around  $a = \frac{m_j}{\sqrt{2}}$ ,  $j = 1, \dots, N_f$  from the perturbative prepotential (5.8). For  $a$  encircling  $\frac{m_1}{\sqrt{2}}$  counterclockwise,  $\Pi \rightarrow \mathbf{M}_1 \Pi$ , we find for the monodromy matrix  $\mathbf{M}_1$ ,

$$\mathbf{M}_1 = \begin{pmatrix} 1 & 1 & 0 & -1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & & \ddots & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad (5.17)$$

while the  $\mathbf{M}_j$  for other values of  $j$  are given by permutations. Its action on the couplings is

$$\mathbf{M}_j : \begin{cases} \tau \rightarrow \tau + 1, \\ v_k \rightarrow v_k - \delta_{jk}, \\ w_{kl} \rightarrow w_{kl} + \delta_{kl}\delta_{jl}. \end{cases} \quad (5.18)$$

Besides the monodromies  $\mathbf{M}_\infty$  and  $\mathbf{M}_j$ , there are monodromies  $\mathbf{M}_m$  and  $\mathbf{M}_d$  around the points where a monopole and a dyon becomes massless, respectively. By requiring that the electro-magnetic charges of the massless particles are  $(n_m, n_e) = (1, 0)$  and  $(1, -2)$ , respectively, we can fix the upper left blocks of the monodromies. We fix the remaining entries by assuming that the masses remains invariant,  $m_j \rightarrow m_j$ , and that the other periods only change by a multiple of the vanishing cycle at the corresponding cusp, together with the requirement that

$$\mathbf{M}_\infty = \mathbf{M}_m \mathbf{M}_d \prod_{j=1}^{N_f} \mathbf{M}_j. \quad (5.19)$$

For  $N_f = 1$  and  $n_1 = n$ , this gives for  $\mathbf{M}_m$ ,

$$\mathbf{M}_m = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -(n+1)/2 \\ (n+1)/2 & 0 & 1 & (n+1)^2/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (5.20)$$

This acts on the couplings as

$$\mathbf{M}_m : \begin{cases} \tau \rightarrow \frac{\tau}{-\tau+1}, \\ v \rightarrow \frac{v+(n+1)\tau/2}{-\tau+1}, \\ w \rightarrow w + \frac{(v+(n+1)/2)^2}{-\tau+1}. \end{cases} \quad (5.21)$$

The monodromy  $\mathbf{M}_d$  around the dyon singularity for  $N_f = 1$  is

$$\mathbf{M}_d = \begin{pmatrix} -1 & 4 & 0 & -n-1 \\ -1 & 3 & 0 & -(n+1)/2 \\ (n+1)/2 & -n-1 & 1 & (n+1)^2/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (5.22)$$

This acts on the couplings as

$$\mathbf{M}_d : \begin{cases} \tau \mapsto \frac{-\tau+4}{-\tau+3}, \\ v \mapsto \frac{v+(n+1)\tau-n-1}{-\tau+3}, \\ w \mapsto w + \frac{(v+(n+1)/2)^2}{-\tau+3}. \end{cases} \quad (5.23)$$

We can note that all the above monodromy matrices leave the symplectic form (5.11) invariant and are independent of the masses.

We get similar monodromies for  $N_f = 2, 3$ . The action on the running couplings  $\tau$  are the same for all  $N_f$ , by construction. The transformations of  $v_j$  and  $w_{jk}$  also take the same form for all  $N_f$  and can be summarised as

$$\begin{aligned} \mathbf{M}_m : & \begin{cases} v_j & \rightarrow \frac{v_j + (n_j + 1)\tau/2}{-\tau + 1}, \\ w_{jk} & \rightarrow w_{jk} + \frac{(v_j + (n_j + 1)/2)(v_k + (n_k + 1)/2)}{-\tau + 1}, \end{cases} \\ \mathbf{M}_d : & \begin{cases} v_j & \rightarrow \frac{v_j + (n_j + 1)\tau - n_j - 1}{-\tau + 3}, \\ w_{jk} & \rightarrow w_{jk} + \frac{(v_j + (n_j + 1)/2)(v_k + (n_k + 1)/2)}{-\tau + 3}. \end{cases} \end{aligned} \quad (5.24)$$

### 5.3 The UV theory on a four-manifold

We review various aspects of the formulation of the UV theory on a compact smooth four-manifold.

#### 5.3.1 Aspects of four-manifolds

We let  $X$  be a smooth, compact, oriented Riemannian four-manifold, with Euler number  $\chi = \chi(X)$  and signature  $\sigma = \sigma(X) = b_2^+ - b_2^-$ . The  $u$ -plane integral is non-vanishing only for four-manifolds  $X$  with  $b_2^+ \leq 1$ . In this article, we consider manifolds with  $b_2^+ = 1$ . Such four-manifolds admit a linear complex structure  $\mathcal{J}$  on the tangent space  $TX_p$  at each point  $p$  of  $X$ . The complex structure varies smoothly on  $X$ , such that  $TX$  is a complex bundle. We introduce furthermore the canonical class  $K_X = -c_1(TX)$  of  $X$ , with  $c_1(TX)$  the first Chern class of  $TX$ . For a manifold  $X$  with  $(b_1, b_2^+) = (0, 1)$ , we have that

$$K_X^2 = 8 + \sigma(X). \quad (5.25)$$

The middle cohomology  $H^2(X, \mathbb{Z})$  of  $X$  gives rise to the uni-modular lattice  $L$ . More precisely, we identify  $L$  with the natural embedding of  $H^2(X, \mathbb{Z})$  in  $H^2(X, \mathbb{Z}) \otimes \mathbb{R}$ , which mods out the torsion of  $H^2(X, \mathbb{Z})$ . A characteristic element  $K \in L$  is an element which satisfies  $\mathbf{l}^2 + B(K, \mathbf{l}) \in 2\mathbb{Z}$  for all  $\mathbf{l} \in L$ . The Riemann-Roch theorem demonstrates that the canonical class  $K_X$  of  $X$  is a characteristic element of  $L$ . The Wu formula furthermore shows that any characteristic vector  $K$  of  $L$  is a lift of  $w_2(X)$ .

The quadratic form  $Q$  of the lattice  $L$  for a four-manifold with  $(b_1, b_2^+) = (0, 1)$  can be brought to a simple standard form depending on whether  $Q$  is even or odd [273]. This divides such manifolds into two classes, for which the evaluation of their  $u$ -plane integrals needs to be done separately [84]. The period point  $J \in H^2(X, \mathbb{R})$  is defined as the unique class in the forward light cone of  $H^2(X, \mathbb{R})$  that satisfies  $J = *J$  and  $J^2 = 1$ .

All four-manifolds without torsion and even intersection form admit a Spin structure. More generally, for any oriented four-manifold one can define a  $\text{Spin}^c$ -structure. The group  $\text{Spin}^c(4)$  can be defined as pairs of unitary  $2 \times 2$  matrices with coinciding determinant,

$$\text{Spin}^c(4) = \{(u_1, u_2) \in \text{U}(2) \times \text{U}(2) \mid \det u_1 = \det u_2\}. \quad (5.26)$$

There exists a short exact sequence

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{Spin}^c(4) \longrightarrow \mathrm{SO}(4) \longrightarrow 1. \quad (5.27)$$

A  $\mathrm{Spin}^c$ -structure  $\mathfrak{s}$  on a four-manifold  $X$  is then a reduction of the structure group of the tangent bundle on  $X$ , i.e.  $\mathrm{SO}(4)$ , to the group  $\mathrm{Spin}^c(4)$ . The different  $\mathrm{Spin}^c$ -structures correspond to the inequivalent ways of choosing transition functions of the tangent bundle such that the cocycle condition is satisfied. The  $\mathrm{Spin}^c$ -structure defines two rank two hermitian vector bundles  $W^\pm$ . We let  $c(\mathfrak{s})$  be the first Chern class of the determinant bundles,  $c(\mathfrak{s}) := c_1(\det W^\pm) \in H^2(X, \mathbb{Z})$ .

If  $\mathfrak{s}$  is the canonical  $\mathrm{Spin}^c$  structure associated to an almost complex structure on  $X$ , then  $c(\mathfrak{s})^2 = 2\chi + 3\sigma$ . More generally,

$$c_1(\mathfrak{s})^2 \equiv \sigma \pmod{8}. \quad (5.28)$$

### 5.3.2 Topological twisting with background fluxes

We discuss in this section topological twisting of theories with fundamental hypermultiplets including background fluxes. The discussion is parallel to the case of  $\mathcal{N} = 2^*$  [85], where the hypermultiplet is in the adjoint representation of the gauge group.

We let  $(E \rightarrow X, \nabla)$  be a principal  $\mathrm{SU}(2)/\mathbb{Z}_2 \cong \mathrm{SO}(3)$ -bundle with connection  $\nabla$ . The second Stiefel-Whitney class  $w_2(E) \in H^2(X, \mathbb{Z}_2)$  measures the obstruction to lift  $E$  to an  $\mathrm{SU}(2)$  bundle, which will exist locally but not globally if  $w_2(E) \neq 0$ . We denote a lift of  $w_2(E)$  to the middle cohomology lattice  $L$  by  $\bar{w}_2(E) \in L$ , and define the 't Hooft flux  $\boldsymbol{\mu} = \bar{w}_2(E)/2 \in L/2$ . The instanton number of the principal bundle is defined as  $k = -\frac{1}{4} \int_X p_1(E)$  and satisfies  $k \in -\boldsymbol{\mu}^2 + \mathbb{Z}$ , where  $p_1$  is the first Pontryagin class.

To formulate the theories with  $N_f$  fundamental hypermultiplets on a compact four-manifold, we perform a topological twist. Coupling the four-dimensional  $\mathcal{N} = 2$   $\mathrm{SU}(2)$  theory to background fields means choosing two sets of data:

- A principal  $\mathrm{SU}(2)_R$  R-symmetry bundle, with connection  $\nabla_R$ ,
- and a principal bundle  $\mathcal{L}$  with connection for global symmetries (the flavour symmetries) [85].

The relevant twist for the  $\mathcal{N} = 2$  supersymmetry algebra in four dimensions is the Donaldson-Witten twist. This twist is the local identification of the  $\mathrm{SU}(2)_+$  with the diagonal subgroup of the  $\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_R$  factor of the spin lift of the local spin group  $\mathrm{Spin}(4) \cong \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$  [61]. Alternatively, one can view the fields as sections of a non-trivial R-symmetry bundle, isomorphic to the spin bundle  $S^+$ . Application of this to the representations of the vector multiplet (5.4) and (5.7) gives:

$$\begin{aligned} \text{bosons:} & \quad (\mathbf{2}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}), \\ \text{fermions:} & \quad (\mathbf{2}, \mathbf{2}) \oplus (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}). \end{aligned} \quad (5.29)$$

Thus the bosons remain unchanged, a vector and a complex scalar, while the fermions reorganise to a vector, self-dual two-form and real scalar, which we denote as  $\psi$ ,  $\chi$  and  $\eta$ , respectively. We note that none of these fields are spinors, and can thus be considered on a non-spin four-manifold. The original supersymmetry generators also transform in the representations for the fermions above. Thus the theory contains a scalar fermionic supercharge  $\mathcal{Q} = \epsilon^{AB} \overline{\mathcal{Q}}_{AB}$ , whose cohomology provides the operators in the topological theory [61].

For the fields of a hypermultiplet, (5.6) and (5.7), one finds

$$\begin{aligned} \text{bosons:} & \quad (\mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2}), \\ \text{fermions:} & \quad (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}) \oplus (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}). \end{aligned} \tag{5.30}$$

Thus hypermultiplet bosons become spinors, i.e. sections of the spin bundle  $S^+$ , while the fermions are sections of  $S^+$  and  $S^-$ . Thus the twisted hypermultiplets can in this case only be formulated on four-manifolds which are spin, i.e.  $w_2(X) = 0$  [73, 255].

However, if the hypermultiplets are charged under a gauge field or flux, the product of these bundles with  $S^\pm$  may be a  $\text{Spin}^c$  bundle,  $W^+$  or  $W^-$  [52, 85, 255]. The latter are defined for arbitrary four-manifolds. For example, an almost complex structure on  $X$  determines two canonical  $\text{Spin}^c$  bundles  $W^\pm \simeq S^\pm \otimes K_X^{-1/2}$  with  $K_X$  the canonical class determined by the almost complex structure. Since the hypermultiplets are in the fundamental, two-dimensional representation of  $\text{SU}(2)$ , the topologically twisted hypermultiplets are well-defined on a non-spin four-manifold if  $\boldsymbol{\mu} = -K_X/2$  [73].

Let us state this also in terms of the gauge bundle  $E$ . To this end, we label the two components of the fundamental, two-dimensional representation of  $\text{SU}(2)$  by  $\pm$ . The two components are sections of a line bundle  $\mathcal{L}_E^{\pm 1/2}$  with  $c_1(\mathcal{L}_E) = \bar{w}_2(E)$ . Of course, the square root  $\mathcal{L}_E^{1/2}$  only exists if  $w_2(E) \in 2L$ . On the other hand, the physical requirement is that  $S^+ \otimes \mathcal{L}_E^{1/2}$  is well defined, or  $\bar{w}_2(X) + \bar{w}_2(E) \in 2L$ . Therefore, the obstructions can cancel each other for a suitable choice of  $w_2(E)$ . Thus the topological twisted theory is not well-defined for an arbitrary choice of 't Hooft flux  $\boldsymbol{\mu} := \frac{1}{2}\bar{w}_2(E)$ ; but rather  $\boldsymbol{\mu}$  has to satisfy  $\boldsymbol{\mu} = \frac{1}{2}\bar{w}_2(X) \pmod L$  [73], or

$$\bar{w}_2(X) = \bar{w}_2(E) \pmod{2L}. \tag{5.31}$$

To consider more general 't Hooft fluxes  $\boldsymbol{\mu}$  or equivalently  $w_2(E)$ , we can couple the  $j$ 'th hypermultiplet to a background flux or line bundle  $\mathcal{L}_j$ , with  $\mathcal{L}_j$  possibly different for each  $j$ . We let  $\mathcal{E}_j = \mathcal{L}_E \otimes \mathcal{L}_j$ . Then the requirement that  $S^\pm \otimes \mathcal{E}_j^{\pm 1/2}$  is globally well-defined is that

$$c_1(\mathcal{E}_j) \in \bar{w}_2(X) + 2L, \tag{5.32}$$

which can be satisfied for any  $\bar{w}_2(E)$  for a suitable choice of  $\mathcal{L}_j$ . Thus we can formulate the  $u$ -plane integral for arbitrary  $\bar{w}_2(E)$ , if we require that the

background fluxes satisfy

$$c_1(\mathcal{L}_j) \in \bar{w}_2(X) + \bar{w}_2(E) + 2L, \quad (5.33)$$

for each  $j$ . This is consistent with (5.31) for  $c_1(\mathcal{L}_j) = 0$ .

The Chern classes  $c_1(\mathcal{L}_j)$  can also be seen as the splitting classes of the  $\text{Spin}(2N_f)$  principal bundle  $\mathcal{L}$ . The Chern class of  $\mathcal{L}$  reads

$$c(\mathcal{L}) = \sum_{l=0}^2 c_l(\mathcal{L}) = \prod_{j=1}^{N_f} (1 + c_1(\mathcal{L}_j)). \quad (5.34)$$

The scalar generators of the equivariant cohomology of  $\text{Spin}(2N_f)$  are the masses  $m_j$ , which generate the  $N_f$ -dimensional Cartan subalgebra of  $\text{Spin}(2N_f)$ . The gauge bundle  $E_k$  is also  $\text{Spin}(2N_f)$  equivariant. For generic masses, the flavour group is  $U(1)^{N_f}$ , and is enhanced for special loci of the masses, for example to  $U(N_f)$  for equal masses [46].

The  $\mathcal{Q}$ -fixed equations are the non-Abelian monopole equations with  $N_f$  matter fields in the fundamental representation. For generic gauge group  $G$  and with representation  $R$ , these equations read [254]

$$\begin{aligned} \left(F_{\dot{\alpha}\dot{\beta}}^a\right)^+ + \frac{i}{2} \sum_{j=1}^{N_f} \bar{M}_{(\dot{\alpha}}^j T^a M_{\dot{\beta})}^j &= 0, \\ \not{D}M^j = \sum_{\mu} \sigma^{\mu} D_{\mu} M^j &= 0, \end{aligned} \quad (5.35)$$

where  $T^a$  is a generator of the Lie algebra in the representation  $R$ . Including the sum over matrix elements, we have

$$M_{(\dot{\alpha}}^j T^a M_{\dot{\beta})}^j = \sum_{k,l} (M^j)_{(\dot{\alpha}}^k (T^a)^{kl} (M^j)_{\dot{\beta})}^l. \quad (5.36)$$

We denote the moduli space of solutions to (5.35) by  $\mathcal{M}_{k,\mathcal{L}_j}^{Q,N_f}$ , and leave the dependence on the 't Hooft flux  $\mu$  and the metric  $J$  implicit. For  $N_f = 4$  on  $X = \mathbb{C}\mathbb{P}^2$ , such moduli spaces are studied in [196].

The moduli spaces  $\mathcal{M}_{k,\mathcal{L}_j}^{Q,N_f}$  is non-compact for vanishing masses [65, 265, 274]. This is improved upon turning on masses and localizing with respect to the  $U(1)^{N_f}$  flavour symmetry,  $M_{\alpha}^j \rightarrow e^{i\varphi_j} M_{\alpha}^j$ , which leave invariant the  $\mathcal{Q}$ -fixed equations (5.35). There are two components:

- the instanton component, with  $F^+ = 0$  and  $M^j = 0$ ,  $j = 1, \dots, N_f$ . The moduli space for this component is denoted  $\mathcal{M}_k^i$ . Since the hypermultiplet fields vanish, this component is associated to the Coulomb branch.
- the abelian or monopole component, for which a  $U(1)$  subgroup of the flavour group acts as pure gauge. Here the connection is reducible, and a  $U(1)$  subgroup of the  $SU(2)$  gauge group is preserved. For generic masses, there are  $N_f$  such components, where  $M^{\ell}$  is upper or lower triangular for



some  $\ell$ , and  $M^j = 0$  for all  $j \neq \ell$ . The moduli space of this component is denoted  $\mathcal{M}_k^{a,j}$ ,  $j = 1, \dots, N_f$ . Since some of the hypermultiplet fields are non-vanishing, this component is associated to the Higgs branch [89, 274].

The instanton component  $\mathcal{M}_k^i$  is non-compact due to point-like instantons. This can be cured using the Uhlenbeck compactification or algebraic-geometric compactifications. We assume that the physical path integral chooses a specific compactification, whose details are however not manifest at the level of the low energy effective field theory other than that the compactification must be in agreement with the correlation functions.

### 5.3.3 Correlation functions and moduli spaces

The  $\mathcal{Q}$ -fixed equations (5.35) include a Dirac equation for each hypermultiplet  $j = 1, \dots, N_f$  in the fundamental representation. The corresponding index bundle  $W_k^j$  defines an element of the  $K$ -group of  $\mathcal{M}_k^i$ . Its virtual rank  $\text{rk}(W_k^j)$  is the formal difference of two infinite dimensions. It is given by an index theorem and reads

$$\text{rk}(W_k^j) = -k + \frac{1}{4}(c_1(\mathcal{L}_j)^2 - \sigma) \in \mathbb{Z}, \quad (5.37)$$

where  $c_1(\mathcal{L}_j)$  is the first Chern class of the bundle  $\mathcal{L}_j$ . Note that the rhs is *not* an integer for an arbitrary  $c_1(\mathcal{L}_j) \in H^2(X, \mathbb{Z})$ . To verify that the rhs is integral for the  $c_1(\mathcal{L}_j)$ 's satisfying (5.33), we rewrite  $\text{rk}(W_k^j)$  as

$$\text{rk}(W_k^j) = -(k + \boldsymbol{\mu}^2) - c_1(\mathcal{L}_j) \cdot \boldsymbol{\mu} + \frac{1}{4}((c_1(\mathcal{L}_j) + 2\boldsymbol{\mu})^2 - \sigma). \quad (5.38)$$

Then the first term on the rhs is an integer since  $k \in -\frac{1}{4}w_2(E)^2 + \mathbb{Z}$  for an  $\text{SO}(3)$  bundle. The second term is an integer because  $c_1(\mathcal{L}_j) \cdot \boldsymbol{\mu} = (\bar{w}_2(X) - 2\boldsymbol{\mu}) \cdot \boldsymbol{\mu} \pmod{\mathbb{Z}} \in \mathbb{Z}$ , and the third term is an integer using (5.28) and the fact that  $c_1(\mathcal{L}_j) + 2\boldsymbol{\mu}$  equals the characteristic class of a  $\text{Spin}^c$ -structure  $\mathfrak{s}_j$  by (5.33),

$$c_1(\mathcal{L}_j) + 2\boldsymbol{\mu} = c(\mathfrak{s}_j), \quad (5.39)$$

for each  $j$ .

The mass  $m_j$  is the equivariant parameter of the  $\text{U}(1)$  flavour symmetry associated to the  $j$ 'th hypermultiplet. The equivariant Chern class of  $W_k^j$  reads in terms of the splitting class  $x_l$ ,

$$c(W_k^j) = \prod_{l=0}^{-\text{rk}(W_k^j)} (x_l + m_j) = m_j^{-\text{rk}(W_k^j)} \sum_l \frac{c_l(W_k^j)}{m_j^l}. \quad (5.40)$$

We abbreviate  $c_l(W_k^j)$  to  $c_{l,j}$ , and let  $c(W_k) = \prod_{j=1}^{N_f} c(W_k^j)$ .

The moduli space  $\mathcal{M}_{k,\boldsymbol{\mu},\mathcal{L}_j}^Q$  for  $N_f$  hypermultiplets corresponds to the vanishing locus of the obstructions for the existence of  $N_f$  zero modes of the Dirac operator. As a result, the virtual complex dimension of the moduli

space  $\mathcal{M}_{k,\mu,\mathcal{L}_j}^Q$  is that of the instanton moduli space plus the sum of (typically negative) ranks of the index bundles  $W_k^j$ ,  $\text{vdim}(\mathcal{M}_{k,\mathcal{L}_j}^{Q,N_f}) = \text{vdim}(\mathcal{M}_k^Q)_{N_f=0} + \sum_{j=1}^{N_f} \text{rk}(W_k^j)$  [85, 99, 261, 265]. This gives

$$\text{vdim}(\mathcal{M}_{k,\mathcal{L}_j}^{Q,N_f}) = (4 - N_f)k + \frac{1}{4} \left( -3\chi - (3 + N_f)\sigma + \sum_{j=1}^{N_f} c_1(\mathcal{L}_j)^2 \right). \quad (5.41)$$

## 5.4 The effective theory on a four-manifold

We consider in this section the low energy effective field theory on a four-manifold. We derive the semi-classical action of the theory coupled to background U(1) fields. As in previous cases [72, 73, 85, 144], the final expression takes the form of a Siegel-Narain theta series multiplied by a measure factor.

### 5.4.1 Hypermultiplets and background fields

The effective theory coupled to  $N_f$  background fluxes can be modelled as that of a theory with gauge group  $\text{SU}(2) \times \text{U}(1)^{N_f}$ , where the fields of the U(1) factors have been frozen in a special way [85, 275]. To derive the precise form, we recall the low-energy effective Lagrangian for the  $r$  multiplets  $(\phi^J, \eta^J, \chi^J, \psi^J, F^J)$  of the topologically twisted  $\text{U}(1)^r$  SYM theory [87]. Since the  $u$ -plane integral reduces to an integral over zero-modes [73], it suffices to only include the zero-modes in the Lagrangian. For simply connected four-manifolds, there is no contribution from the one-form fields  $\psi^J$ . The Lagrangian is then given in terms of the prepotential  $F(\{a^J\})$  and its derivatives to the vevs  $\langle \phi^J \rangle = a^J$ , as

$$\begin{aligned} \mathcal{L} = & \frac{i}{16\pi} (\bar{\tau}_{JK} F_+^J \wedge F_+^K + \tau_{JK} F_-^J \wedge F_-^K) - \frac{1}{8\pi} y_{JK} D^J \wedge D^K \\ & + \frac{i\sqrt{2}}{16\pi} \bar{\mathcal{F}}_{JKL} \eta^J \chi^K \wedge (D + F_+)^L, \end{aligned} \quad (5.42)$$

with  $y_{JK} = \text{Im}(\tau_{JK})$ ,  $\tau_{JK} = \partial_J \partial_K F(\{a^J\})$  and  $\mathcal{F}_{JKL} = \partial_J \partial_K \partial_L F(\{a^J\})$ . It is left invariant by the BRST operator  $Q$ , which acts on the zero modes as

$$\begin{aligned} [Q, A^J] &= \psi^J = 0, & [Q, \psi^J] &= 4\sqrt{2} da^J, \\ [Q, a^J] &= 0, & [Q, \bar{a}^J] &= \sqrt{2} i \eta^J, \\ [Q, \eta^J] &= 0, & [Q, \chi^J] &= i(F_+ - D_+)^J, \\ [Q, D^J] &= (d\psi^J)_+ = 0. \end{aligned} \quad (5.43)$$

Using this operator, we can write  $\mathcal{L}$  as the sum of a topological, holomorphic term and a  $Q$ -exact term,

$$\mathcal{L} = \frac{i}{16\pi} \tau_{JK} F^J \wedge F^K + \{Q, W\}, \quad (5.44)$$

with  $W = -\frac{i}{8\pi} y_{JK} \chi^J (F_+ + D)^K$ .

The low-energy theory of SU(2) gauge theory with  $N_f$  hypermultiplets coupled to  $N_f$  background fluxes can then be modelled by the above rank  $r$  description with  $r = N_f + 1$ . We identify  $F(\{a^J\})$  with  $F(a, \mathbf{m})$ . We let the indices  $J, K$  run from 0 to  $N_f$  and identify the index 0 with the unbroken U(1) of the SU(2) gauge group and the indices  $j, k, l = 1, \dots, N_f$  with that of the frozen U(1) $^{N_f}$  factors. We further set  $\phi^0 := \phi$  for any field  $\phi$ . We will proceed by using lower indices for  $j, k, l$ , except where the summation convention is explicitly used, to avoid confusion with powers of the fields.

The masses of the hypermultiplets are the vevs of the frozen scalar fields of the corresponding vector multiplets,  $\frac{m_j}{\sqrt{2}} = \langle \phi_j \rangle = a_j$  [275]. We set  $[F_j] = 4\pi \mathbf{k}_j$  with

$$\mathbf{k}_j = c_1(\mathcal{L}_j)/2 \in L/2. \quad (5.45)$$

To make the BRST variations of the fields from the frozen U(1) factors vanish, we set  $\eta_j = \chi_j = 0$ , as well as  $D^j = F_+^j$ . With these identifications, the Lagrangian becomes

$$\begin{aligned} \mathcal{L} &= \frac{i}{16\pi} \tau_{JK} F^J \wedge F^K + \frac{1}{8\pi} y_{00} F_+ \wedge F_+ - \frac{1}{8\pi} y_{00} D \wedge D \\ &\quad + \frac{i\sqrt{2}}{16\pi} \bar{\mathcal{F}}_{000} \eta \chi \wedge (D + F_+) + \frac{i\sqrt{2}}{8\pi} \bar{\mathcal{F}}_{00j} \eta \chi \wedge F_+^j \\ &\quad + \frac{1}{4\pi} y_{0j} (F_+ - D) \wedge F_+^j. \end{aligned} \quad (5.46)$$

Integrating over  $D$ ,  $\eta$  and  $\chi$  in the standard way [73, 85, 87], we end up with

$$\begin{aligned} &\int dD d\eta d\chi e^{-\int_X \mathcal{L}} \\ &= \frac{\partial}{\partial \bar{a}} \left( i\sqrt{y_{00}} B \left( F + \frac{y_{0j}}{y_{00}} F^j, J \right) \right) e^{-\int_X \mathcal{L}_0}, \end{aligned} \quad (5.47)$$

where

$$\begin{aligned} \mathcal{L}_0 &= \frac{i}{16\pi} \tau_{JK} F^J \wedge F^K + \frac{1}{8\pi} y_{00} F_+ \wedge F_+ + \frac{y_{0j}}{4\pi} F_+ \wedge F_+^j + \frac{1}{8\pi} \frac{y_{0j} y_{0k}}{y_{00}} F_+^j \wedge F_+^k \\ &= \frac{i}{16\pi} (\bar{\tau} F_+ \wedge F_+ + \tau F_- \wedge F_-) + \frac{i}{8\pi} (v_j F_- \wedge F_-^j + \bar{v}_j F_+ \wedge F_+^j) \\ &\quad + \frac{i}{16\pi} w_{jk} F^j \wedge F^k + \frac{y}{8\pi} \text{Im}(v_j) \text{Im}(v_k) F_+^j \wedge F_+^k, \end{aligned} \quad (5.48)$$

and we identified  $\tau := \tau_{00}$ ,  $y = \text{Im}(\tau) = y_{00}$ ,  $v_j := \tau_{0j}$  and  $w_{jk} := \tau_{jk}$ . Thus the coupling  $w_{jk}$  is holomorphic, but the coupling  $v_j$  is *non-holomorphic*. This is similar to the couplings for  $\mathcal{N} = 2^*$  [85].

#### 5.4.2 Sum over fluxes

The path integral includes a sum over fluxes  $\mathbf{k} = [F]/4\pi \in L/2$ . After summing the exponentiated action (5.47) over the fluxes  $\mathbf{k}$  and multiplying by  $\frac{d\bar{a}}{d\bar{\tau}}$ , we

find that this takes the form

$$\sum_{\mathbf{k} \in L + \boldsymbol{\mu}} \int dD d\eta d\chi e^{-\int_X \mathcal{L}} = \left( \prod_{j,k=1}^{N_f} C_{jk}^{B(\mathbf{k}_j, \mathbf{k}_k)} \right) \Psi_{\boldsymbol{\mu}}^J(\tau, \bar{\tau}, \mathbf{z}, \bar{\mathbf{z}}). \quad (5.49)$$

The couplings  $C_{jk}$  are given in terms of  $w_{jk}$  (5.12) by

$$C_{jk} = e^{-\pi i w_{jk}}, \quad (5.50)$$

for  $j, k = 1, \dots, N_f$ . Such couplings were first put forward in [144], and were also crucial in [85].

The term  $\Psi_{\boldsymbol{\mu}}^J$  is an example of a Siegel-Narain theta function. It reads explicitly

$$\begin{aligned} \Psi_{\boldsymbol{\mu}}^J(\tau, \bar{\tau}, \mathbf{z}, \bar{\mathbf{z}}) &= e^{-2\pi y \mathbf{b}_+^2} \sum_{\mathbf{k} \in L + \boldsymbol{\mu}} \partial_{\bar{\tau}} (4\pi i \sqrt{y} B(\mathbf{k} + \mathbf{b}, J)) \\ &\times (-1)^{B(\mathbf{k}, K)} q^{-\mathbf{k}_-^2/2} \bar{q}^{\mathbf{k}_+^2/2} e^{-2\pi i B(\mathbf{z}, \mathbf{k}_-) - 2\pi i B(\bar{\mathbf{z}}, \mathbf{k}_+)}, \end{aligned} \quad (5.51)$$

and discussed in more detail in Appendix A.7. The elliptic variable reads in terms of  $v_j$  and  $\mathbf{k}_j$ ,

$$\mathbf{z} = \sum_{j=1}^{N_f} v_j \mathbf{k}_j, \quad \text{and} \quad \mathbf{b} = \frac{\text{Im}(\mathbf{z})}{y}, \quad (5.52)$$

thus inducing a non-holomorphic dependence on  $v_j$ . Furthermore,  $K$  appearing in the fourth root of unity  $(-1)^{B(\mathbf{k}, K)}$  is a characteristic vector of  $L$ . Note that  $\Psi_{\boldsymbol{\mu}}^J$  changes by the sign  $(-1)^{B(\boldsymbol{\mu}, K - K')}$  upon replacing  $K$  by a different characteristic vector  $K'$  [73, 85, 276].

For  $N_f = 0$ , this phase can be understood as arising from integrating out massive fermionic modes [72]. It also appears naturally in decoupling the adjoint hypermultiplet in the analogous function for  $\mathcal{N} = 2^*$  [85]. For  $N_f > 0$ , the constant part of the couplings  $v_j$  (5.12) effectively contribute to the phase, such that the total phase reads

$$e^{\pi i B(\mathbf{k}, K)} \prod_{j=1}^{N_f} e^{\pi i n_j B(\mathbf{k}_j, \mathbf{k})}, \quad (5.53)$$

with  $n_j$  the magnetic winding numbers. For arbitrary  $n_j \in \mathbb{Z}$ , the phase is an eighth root of unity. It would be interesting to understand this phase from integrating out massive modes.

We deduce from (5.53) that the summand of  $\Psi_{\boldsymbol{\mu}}^J$  changes by a phase

$$e^{\pi i (n'_j - n_j) B(\mathbf{k}_j, \mathbf{k})} \quad (5.54)$$

if the winding numbers  $n_j$  are replaced by  $n'_j$ . Since  $\mathbf{k}_j \in K/2 - \boldsymbol{\mu} \pmod L$  (see (5.33)) and  $\mathbf{k} \in L + \boldsymbol{\mu}$ , this phase is 1 if  $n'_j - n_j = 0 \pmod 4$ . We can

therefore restrict to  $n_j \in \mathbb{Z}_4$ . For specific choices of  $\boldsymbol{\mu}$  and  $\mathbf{k}_j$ , the  $n_j$  can lie in a subgroup of  $\mathbb{Z}_4$ .

The modular transformations of  $\Psi_\mu^J$  are discussed in Appendix A.7, which are crucial input for single-valuedness of the  $u$ -plane integrand. We will demonstrate in section 5.5.2 that the  $u$ -plane integrand is single-valued if we impose further constraints on the winding numbers  $n_j$ .

Finally, if the theory is considered on a curved background, topological couplings arise in the effective field theory [72]. These terms couple to the Euler characteristic and the signature of the four-manifold  $X$ , respectively denoted  $A$  and  $B$ . These take the form [72, 73],

$$A = \alpha \left( \frac{du}{da} \right)^{1/2}, \quad B = \beta \Delta_{N_f}^{1/8}. \quad (5.55)$$

Here,  $\Delta_{N_f}$  is the physical discriminant incorporating the singularities of the effective theory, while  $\frac{du}{da}$  is the (reciprocal of) the periods of the SW curves as introduced in section 5.2. Both can be determined directly from the SW curve, as described in section 2.3.1. The prefactors  $\alpha$  and  $\beta$  are independent of  $u$ , but can be functions of other moduli such as the masses  $\mathbf{m}$ , the dynamical scale  $\Lambda_{N_f}$  or the UV coupling  $\tau_{UV}$ . However, it turns out that for the theories with fundamental matter they are independent of the masses and only depend on the scale [66, 111]. They satisfy several constraints from holomorphy, RG flow, homogeneity and dimensional analysis, and can in principle be fixed for any Lagrangian theory from a computation in the  $\Omega$ -background [66, 82, 111, 129].

### 5.4.3 Observables and contact terms

The observables in the topologically twisted theories are the point observable or 0-observable  $u$ , as well as  $d$ -observables supported on a  $d$ -dimensional submanifold of  $X$ . The  $d$ -observables are only non-vanishing if the submanifold corresponds to a non-trivial homology class. For  $b_1 = 0$ , the  $d$ -observables with  $d$  odd therefore do not contribute.

To introduce the surface observable, let  $\mathbf{x} \in H_2(X, \mathbb{Q})$ . Then the surface observable reads in terms of the UV fields,

$$I(\mathbf{x}) = \frac{1}{4\pi^2} \int_{\mathbf{x}} \text{Tr} \left[ \psi \wedge \psi - \frac{1}{\sqrt{2}} \phi F \right]. \quad (5.56)$$

In the effective infrared theory, this operator becomes,

$$\tilde{I}(\mathbf{x}) = \frac{i}{\sqrt{2}\pi} \int_{\mathbf{x}} \frac{1}{32} \frac{d^2 u}{da^2} \psi \wedge \psi - \frac{\sqrt{2}}{4} \frac{du}{da} (F_- + D). \quad (5.57)$$

Generating functions of correlation functions are obtained by inserting

$$e^{p u / \Lambda_{N_f}^2 + \tilde{I}(\mathbf{x}) / \Lambda_{N_f}} \quad (5.58)$$

in the path integral. The surface observable leads to a change in the argument of the sum over fluxes (5.51),

$$z \rightarrow z + \frac{\mathbf{x}}{2\pi \Lambda_{N_f}} \frac{du}{da}, \quad \bar{z} \rightarrow \bar{z}. \quad (5.59)$$

and to analytically continue  $\mathbf{b}$  (5.52) to the complex number by setting  $\mathbf{b} = (z - \bar{z})/(2iy)$ .

The inclusion of the surface observable also gives rise to a contact term [70, 73, 277], which in particular ensures that the  $u$ -plane integrand is single-valued. For  $0 \leq N_f \leq 3$ , the contact term is  $\exp(\mathbf{x}^2 G_{N_f})$  with [74, 98, 278]

$$G_{N_f} = -\frac{1}{24 \Lambda_{N_f}^2} E_2 \left( \frac{du}{da} \right)^2 + \frac{1}{3 \Lambda_{N_f}^2} \left( u + \frac{\Lambda_3^2}{64} \delta_{N_f,3} \right), \quad (5.60)$$

while for  $N_f = 4$  it is given by [98, 174]

$$G_{N_f=4} = -\frac{1}{24 \Lambda_4^2} E_2 \left( \frac{du}{da} \right)^2 + \frac{u}{3 \Lambda_4^2} E_2(\tau_{UV}) + \frac{1}{18 \Lambda_4^2} \llbracket m_1^2 \rrbracket E_4(\tau_{UV}). \quad (5.61)$$

A more general scheme to fix the contact terms is proposed in [74]. Contact terms can also be derived from the corresponding Whitham hierarchies [278, 279]. In the presence of surface observables, there are additional mixed contact terms  $\frac{\partial^2 \mathcal{F}}{\partial \tau_0 \partial m}$  for the external fluxes  $\{\mathbf{k}_j\}$  as encountered in [85] for the  $\mathcal{N} = 2^*$  theory.

## 5.5 The $u$ -plane integral

In this section, we set up the  $u$ -plane integral schematically given in (5.2), and demonstrate that it is well-defined on the integration domain for any  $\boldsymbol{\mu}$  with appropriate background fluxes. The case  $\boldsymbol{\mu} = \bar{w}_2(X)/2$  and  $\mathbf{k}_j = 0$  was analysed in [73].

### 5.5.1 Definition of the integrand

As discussed in the previous sections, the  $u$ -plane integral on a closed four-manifold  $X$  with  $(b_1, b_2^+) = (0, 1)$  depends on many parameters. We summarise:

- The scale  $\Lambda_{N_f}$  and masses  $\mathbf{m} = (m_1, \dots, m_{N_f})$  of the theory. See section 5.2.
- The magnetic winding numbers  $n_j$ ,  $j = 1, \dots, N_f$ . See section 5.2.2.
- The four-manifold  $X$ , in particular its signature  $\sigma = \sigma(X)$ , Euler characteristic  $\chi = \chi(X)$ , period point  $J$  and intersection form  $Q$ . See section 5.3.1.
- The 't Hooft flux  $\boldsymbol{\mu}$ , and the external fluxes  $\{\mathbf{k}_j\} = (\mathbf{k}_1, \dots, \mathbf{k}_{N_f})$ . See section 5.3.3.

- The fugacities for the point and surface observables  $p$  and  $\mathbf{x}$ . See section 5.4.3.

In terms of these parameters, the  $u$ -plane path integral reduces to the following finite dimensional integral over  $\mathcal{F}_{N_f}(\mathbf{m})$ ,

$$\begin{aligned} \Phi_{\mu, \{\mathbf{k}_j\}}^J(p, \mathbf{x}, \mathbf{m}, \Lambda_{N_f}) = \\ \mathcal{K}_{N_f} \int_{\mathcal{F}_{N_f}(\mathbf{m})} d\tau \wedge d\bar{\tau} \nu(\tau; \{\mathbf{k}_j\}) \Psi_{\mu}^J(\tau, \bar{\tau}, \mathbf{z}, \bar{\mathbf{z}}) e^{2pu + \mathbf{x}^2 G_{N_f}}. \end{aligned} \quad (5.62)$$

We summarise the different elements on the rhs:

- $\mathcal{K}_{N_f}$  is an overall normalisation factor. For  $N_f = 0$ , it is fixed by matching to known Donaldson invariants. Due to  $\chi + \sigma = 4$ , there is an ambiguity [82]

$$(\mathcal{K}_{N_f}, \alpha, \beta) \sim (\zeta^{-4} \mathcal{K}_{N_f}, \zeta \alpha, \zeta \beta), \quad (5.63)$$

with  $\alpha$  and  $\beta$  the  $u$ -independent prefactors in (5.55).

- The integration domain  $\mathcal{F}_{N_f}(\mathbf{m})$  in (5.62) is crucially the fundamental domain of the effective gauge coupling. As discussed in section 2, this domain requires new aspects compared to integration domains for earlier discussions of  $u$ -plane integrals. The evaluation of integrals over  $\mathcal{F}_{N_f}(\mathbf{m})$  will be discussed in more detail in section 5.6.
- $\nu$  is the “measure factor” [66, 72–74, 85]

$$\nu(\tau; \{\mathbf{k}_j\}) = \frac{da}{d\tau} A^\chi B^\sigma \prod_{i,j=1}^{N_f} C_{ij}^{B(\mathbf{k}_i, \mathbf{k}_j)}. \quad (5.64)$$

It combines the topological couplings (5.55) and the couplings to the background fluxes (5.50) with the Jacobian  $\frac{da}{d\tau}$  of the change of variables from  $a$  to  $\tau$ .

- The function  $\Psi_{\mu}^J$  arises from the sum over U(1) fluxes. It is a Siegel-Narain theta function (5.51) and discussed in detail in section 5.4.2. The elliptic parameter  $\mathbf{z}$  of the Siegel-Narain theta function reads

$$\begin{aligned} \mathbf{z} &= \frac{\mathbf{x}}{2\pi \Lambda_{N_f}} \frac{du}{da} + \sum_{j=1}^{N_f} v_j \mathbf{k}_j, \\ \bar{\mathbf{z}} &= \sum_{j=1}^{N_f} \bar{v}_j \mathbf{k}_j. \end{aligned} \quad (5.65)$$

- Finally,  $G_{N_f}$  is the contact term, discussed in more detail in section 5.4.3.

While the path integral set up in section 5.4 integrates the exponentiated action over the local coordinates  $a$  and  $\bar{a}$ , in (5.62) we have changed variables to  $\tau$  and  $\bar{\tau}$ . This change of variables  $(a, \bar{a}) \rightarrow (\tau, \bar{\tau})$  is valid as long as the Jacobian is nonsingular in the integration region. Since the coordinates  $a$  and  $\bar{a}$  are holomorphic and anti-holomorphic respectively, the Jacobian is diagonal and the functional determinant accordingly reads  $\frac{da}{d\tau} \frac{d\bar{a}}{d\bar{\tau}}$ . We thus need to show that  $\frac{da}{d\tau}$  is not singular away from isolated points in  $\mathcal{F}_{N_f}(\mathbf{m})$ , which in (5.62) we remove implicitly from the integration domain.

Using  $\frac{da}{du} = \frac{da}{du} \frac{du}{d\tau}$ , we can study the singular points in detail. First, it is shown in section 2 that the singularities of  $\frac{du}{d\tau}$  are in one-to-one correspondence with the branch points. In fact, both  $\frac{du}{d\tau} = 0$  and  $\frac{du}{d\tau} = \infty$  are realised as branch points in  $\mathcal{N} = 2$  SQCD. In the following section 5.6, we remove a small circle in  $\mathcal{F}_{N_f}(\mathbf{m})$  around the branch points, and show that they do not give an extra contribution. Furthermore, the solutions to  $\frac{du}{da} = 0$  are shown to be the Argyres-Douglas (AD) points. We exclude them from the integration region, and study their contribution also in section 5.6. Finally, we know that  $\eta^{24} \propto \left(\frac{da}{du}\right)^{12} \Delta_{N_f}$  (2.48), with  $\eta$  the Dedekind eta-function as defined in (A.18). Since  $\eta \neq 0$  and  $\Delta_{N_f}$  does not have poles, we find that  $\frac{da}{du}$  never vanishes. This agrees with the fact that  $\frac{da}{du}$  is the period of a holomorphic differential and therefore is never zero.

We conclude that the functional determinant is singular in  $\mathbb{H}$  precisely at the branch points and AD points, however with the proper exclusion of those as done in the following Section, it is non-singular and the change of variables is well-defined. This furthermore conveniently solves the problem that there is no natural integration region in  $(a, \bar{a})$  space [73].

### 5.5.2 Monodromy transformations of the integrand

We continue by explicitly verifying that the  $u$ -plane integral is single-valued around the singular points of the moduli space. We find that this puts a constraint on the magnetic winding numbers  $n_j$ , in addition to the constraints on the background fluxes  $\mathbf{k}_j$  discussed in section 5.3.3.

#### Monodromy around infinity

Let us determine how the  $u$ -plane integrand transforms under the monodromy around infinity. As a function of the effective coupling  $\tau$ , the measure factor (5.64) is proportional to  $\frac{da}{d\tau} \left(\frac{du}{da}\right)^{\frac{\sigma}{2}} \Delta^{\frac{\sigma}{8}}$  times the product over the couplings  $C_{ij}$ . We take the monodromy at infinity to be oriented as  $u \rightarrow e^{2\pi i} u$  and  $a \rightarrow e^{\pi i} a$ , as in section 5.2.3. Then this path also encircles all singularities  $u_j$ , which are the roots of the physical discriminant,  $\Delta = \prod_{j=1}^{N_f+2} (u - u_j)$ . We thus have that  $\Delta \rightarrow e^{2\pi i(N_f+2)} \Delta$ , and hence

$$\Delta^{\frac{\sigma}{8}} \rightarrow e^{\pi i(N_f+2)\sigma/4} \Delta^{\frac{\sigma}{8}} \quad (5.66)$$



Next, since  $u \rightarrow e^{2\pi i}$  and  $a \rightarrow e^{\pi i} a$  we find  $\frac{du}{da} \rightarrow e^{\pi i} \frac{du}{da}$ , and therefore

$$\left(\frac{du}{da}\right)^{\frac{\chi}{2}} \rightarrow e^{\pi i \chi/2} \left(\frac{du}{da}\right)^{\frac{\chi}{2}}. \quad (5.67)$$

For  $\frac{da}{d\tau}$  we have that  $a \rightarrow e^{\pi i} a$ , while  $d\tau \rightarrow d\tau$ , and thus

$$\frac{da}{d\tau} \rightarrow -\frac{da}{d\tau}. \quad (5.68)$$

From (5.16) we recall that  $w_{ij} \rightarrow w_{ij} + \delta_{ij}$ , such that with the definition (5.50) we find  $C_{ij} \rightarrow e^{-\pi i \delta_{ij}} C_{ij}$ . The couplings  $C_{ij}$  transform in the measure factor as

$$\prod_{i,j=1}^{N_f} C_{ij}^{B(\mathbf{k}_i, \mathbf{k}_j)} \rightarrow e^{-\pi i \sum_j \mathbf{k}_j^2} \prod_{i,j=1}^{N_f} C_{ij}^{B(\mathbf{k}_i, \mathbf{k}_j)}. \quad (5.69)$$

Combining (5.66), (5.67), (5.68), (5.69), and using  $\chi = 4 - \sigma$ , we obtain

$$\nu \rightarrow -e^{\pi i N_f \sigma/4} e^{-\pi i \sum_j \mathbf{k}_j^2} \nu. \quad (5.70)$$

This phase for  $\mathbf{k}_j = 0$  can be checked directly by taking  $q$ -expansions from the SW curves, for generic masses.

From (5.16) we recall that under the monodromy around infinity  $v_j \rightarrow -v_j - n_j$ , and thus  $\mathbf{z} \rightarrow -\mathbf{z} - \sum_{j=1}^{N_f} n_j \mathbf{k}_j$ . Recall from (5.33) that  $c_1(\mathcal{L}_j) \equiv K - 2\boldsymbol{\mu} \pmod{2L}$ . For the sum over fluxes, in [5] we show that

$$\mathbf{M}_\infty : \quad \nu(\tau; \{\mathbf{k}_j\}) \Psi_\mu^J(\tau, \mathbf{z}) \rightarrow e^{2\pi i \boldsymbol{\mu} \sum_j (n_j + 1) \mathbf{k}_j} \nu(\tau; \{\mathbf{k}_j\}) \Psi_\mu^J(\tau, \mathbf{z}), \quad (5.71)$$

and the  $u$ -plane integrand is invariant under  $T^{N_f - 4}$  if and only if  $\boldsymbol{\mu} \sum_j (n_j + 1) \mathbf{k}_j \in \mathbb{Z}$ . Using the fact that  $K$  is a characteristic vector of  $L$ , we find

$$n_j \equiv 1 \pmod{2} \quad (5.72)$$

for all  $j = 1, \dots, N_f$ , which implies the above constraint.

### Monodromy $\mathbf{M}_j$

Let us determine how the integrand transforms under the monodromy  $M_j$  around the mass singularity  $m_j/\sqrt{2}$ . Since the mass singularity corresponds to a singularity  $u_j$  on the  $u$ -plane, we have that  $(u - u_j) \rightarrow e^{2\pi i} (u - u_j)$ . This implies that  $\Delta = (u - u_j) \prod_{i \neq j}^{2+N_f} (u - u_i) \rightarrow e^{2\pi i} \Delta$ , such that  $\Delta^{\frac{\sigma}{8}} \rightarrow e^{\pi i \sigma/4} \Delta^{\frac{\sigma}{8}}$ . The transformation of  $\frac{da}{du}$  can be determined from (2.46): While  $u \rightarrow u_j$ , both  $g_2$  and  $g_3$  remain finite and nonzero (otherwise  $u_j$  would be an AD point). This implies that  $\frac{g_3}{g_2}$  contains no factors of  $(u - u_j)$ , and thus  $\frac{du}{da} \rightarrow \frac{du}{da}$ . Similarly, we have that  $\frac{da}{d\tau} \rightarrow \frac{da}{d\tau}$ . From (5.18) we finally have that  $w_{ik} \rightarrow w_{ik} + \delta_{ij} \delta_{ik}$ . We combine

$$\mathbf{M}_j : \quad \nu \rightarrow e^{\pi i \sigma/4} e^{-\pi i \mathbf{k}_j^2} \nu. \quad (5.73)$$

For the monodromy around the mass singularity  $m_j/\sqrt{2}$ , we find for  $\Psi_\mu^J$  with (A.58)

$$\Psi_\mu^J(\tau + 1, \mathbf{z} - \mathbf{k}_j) = e^{-\pi i \sigma/4 + \pi i \mathbf{k}_j^2} \Psi_\mu^J(\tau, \mathbf{z}). \quad (5.74)$$

The phases thus cancel precisely,

$$\mathbf{M}_j : \quad \nu(\tau + 1) \Psi_\mu^J(\tau + 1, \mathbf{z} - \mathbf{k}_j) = \nu(\tau) \Psi_\mu^J(\tau, \mathbf{z}), \quad (5.75)$$

without any constraints.

### Monodromy $\mathbf{M}_m$

For the monopole singularity in  $N_f = 1$  we find a further constraint. Since  $\Psi_\mu^J$  is required to transform to itself up to an overall factor, we must demand that  $(n + 1)\mathbf{k}_1/2 \in L$ . Therefore for  $\mathbf{k}_1 \in L/2$ , we find the requirement that  $n = -1 \in \mathbb{Z}_4$ . This simplifies the transformations considerably, and we find

$$\Psi_\mu^J(\tau, \mathbf{z}) \rightarrow (-\tau + 1)^{b_2/2} (-\bar{\tau} + 1)^2 e^{-\pi i \sigma/4} e^{\pi i \mathbf{k}_1^2 \frac{v^2}{-\tau+1}} \Psi_\mu^J(\tau, \mathbf{z}). \quad (5.76)$$

The  $\mathbf{k}_j$ -independent part of the measure factor transforms precisely as under  $\mathbf{M}_j$  (see (5.73)), as the same argument holds. However, due to the transformation  $\tau \rightarrow \frac{\tau}{-\tau+1}$ , the measure also picks up its modular weight  $\frac{\sigma}{2} + 1$ . From (5.21) we furthermore find the transformation of  $C_{11}$ , such that

$$\nu(\tau, \mathbf{k}_1) \rightarrow e^{\pi i \sigma/4} e^{-\pi i \mathbf{k}_1^2 \frac{v^2}{-\tau+1}} (-\tau + 1)^{\frac{\sigma}{2} + 1} \nu(\tau, \mathbf{k}_1), \quad (5.77)$$

where we have already used  $n = -1 \in \mathbb{Z}_4$ . If we multiply (5.76) and (5.77) with  $d\tau \wedge d\bar{\tau}$  (which has modular weight  $(-2, -2)$ ), then

$$\mathbf{M}_m : \quad d\tau \wedge d\bar{\tau} \nu(\tau, \mathbf{k}_1) \Psi_\mu^J(\tau, \mathbf{z}) \rightarrow d\tau \wedge d\bar{\tau} \nu(\tau, \mathbf{k}_1) \Psi_\mu^J(\tau, \mathbf{z}), \quad (5.78)$$

where we have used  $\sigma + b_2 = 2$ . Thus, the  $u$ -plane integrand is also invariant under  $\mathbf{M}_m$ .

For  $N_f > 1$  we find the same condition, namely that  $n_j = -1 \pmod{4}$  for all  $j$ .

### Monodromy $\mathbf{M}_d$

Given the relation (5.19), it is not necessary to explicitly check single-valuedness of the integrand under this monodromy, as it is a product of the above monodromies.

To conclude this section, let us stress the constraints for the winding number  $n_j$ , such that the  $u$ -plane integral is invariant under all monodromies in  $N_f \leq 3$ . To this end, we need to satisfy the constraints  $n_j = 1 \pmod{2}$  (5.72) from  $\mathbf{M}_\infty$ , and  $n_j = -1 \pmod{4}$  for  $\mathbf{M}_m$ . Since the latter is the stronger constraint, we require

$$n_j = -1 \pmod{4}, \quad (5.79)$$

for all  $j = 1, \dots, N_f$ .

## 5.6 Integration over fundamental domains

As discussed in Sections 5.2 and 5.5,  $u$ -plane integrals for massive  $\mathcal{N} = 2$  theories with fundamental hypermultiplets include new aspects. This section discusses how to evaluate such integrals (5.62). More concretely, we aim to define and evaluate integrals of the form

$$\mathcal{I}_f = \int_{\mathcal{F}(\mathbf{m})} d\tau \wedge d\bar{\tau} y^{-s} f(\tau, \bar{\tau}), \quad (5.80)$$

with  $s \leq 1$ . The domain  $\mathcal{F}(\mathbf{m})$  is the fundamental domain for the effective coupling constant as discussed in section 2, and  $f$  a non-holomorphic function of weight  $(2-s, 2-s)$  arising from the topologically twisted Yang–Mills theory. For  $\mathcal{F}(\mathbf{m})$  a fundamental domain of a congruence subgroup, such integrals (5.80) have been studied in the context of theta lifts of weakly holomorphic modular forms and harmonic Maass forms [280–282] as well as one-loop amplitudes in string theory [283–285].

We assume that the integrand  $y^{-s} f(\tau, \bar{\tau})$  can be expressed as

$$\partial_{\bar{\tau}} \widehat{h}(\tau, \bar{\tau}) = y^{-s} f(\tau, \bar{\tau}), \quad (5.81)$$

for a suitable function  $\widehat{h}(\tau, \bar{\tau})$  using mock modular forms. This was indeed the case in [81, 83, 85], and will be demonstrated for massive  $\mathcal{N} = 2$  theories with fundamental hypermultiplets. The integral  $\mathcal{I}_f$  then reads

$$\mathcal{I}_f = - \int_{\partial\mathcal{F}(\mathbf{m})} d\tau \widehat{h}(\tau, \bar{\tau}), \quad (5.82)$$

with  $\partial\mathcal{F}(\mathbf{m})$  the boundary of  $\mathcal{F}(\mathbf{m})$ . We will carry this out evaluation in Part II [286].

There are a number of aspects to be addressed in order to evaluate integrals over  $\mathcal{F}(\mathbf{m})$ :

1. Identifications of boundary components of  $\mathcal{F}(\mathbf{m})$  due to monodromies on the  $u$ -plane.
2. Contributions from the cusps, that is  $\tau \rightarrow i\infty$  or  $\tau \rightarrow \gamma(i\infty) \in \mathbb{Q}$  for an element  $\gamma \in \text{PSL}(2, \mathbb{Z})$ .
3. Contributions from a singular point in the interior of  $\mathcal{F}(\mathbf{m})$ .
4. Contributions from an elliptic point  $p \in \mathbb{H}$  of  $\text{PSL}(2, \mathbb{Z})$ .
5. Branch points and branch cuts.

We will discuss these aspects 1.–5. in the following.

### 1. *Identifications*

The modular transformation induced by monodromies identify components

of the boundary of the fundamental domain  $\partial\mathcal{F}(\mathbf{m})$  pairwise. Their contributions to the integral (5.82) vanish, which is, for example, familiar from deriving valence formulas for modular forms [287, Fig. 2]. See Fig. 1 for an example.

## 2. Cusps

Contributions near the cusps require a regularisation [73, 83]. Such regularisations have been developed in the context of string amplitudes [283–285] and analytic number theory [287–289].

Let us first consider the cusp  $\tau \rightarrow i\infty$ . To regularise the divergence, one introduces a cut-off  $\text{Im } \tau = Y \gg 1$ , and takes the limit  $Y \rightarrow \infty$  after evaluation. We require that  $f$  near  $i\infty$  has a Fourier expansion of the form<sup>32</sup>

$$f(\tau, \bar{\tau}) = \sum_{m \gg -\infty, n \geq 0} c(m, n) q^m \bar{q}^n. \quad (5.83)$$

Then the function  $\widehat{h}$  has the form,

$$\widehat{h}(\tau, \bar{\tau}) = h(\tau) + 2^s \int_{-\bar{\tau}}^{i\infty} \frac{f(\tau, -v)}{(-i(v + \tau))^s} dv, \quad (5.84)$$

where  $h(\tau)$  is a weakly holomorphic  $q$ -series, with expansion

$$h(\tau) = \sum_{m \gg -\infty} d(m) q^m. \quad (5.85)$$

The cusp  $\tau \rightarrow i\infty$  then contributes

$$[\mathcal{I}_f]_\infty = w_\infty d(0), \quad (5.86)$$

with  $d(0)$  the constant term of  $h(\tau)$  (5.85), and  $w_\infty$  the width of the cusp  $\mathcal{F}(\mathbf{m})$  at  $i\infty$ . For  $N_f \leq 3$ ,  $w_\infty$  is  $4 - N_f$  (see section 2).

The other cusps can be treated in a similar fashion using modular transformations. We label the  $n_c$  cusps in  $\mathcal{F}(\mathbf{m})$  by  $j = 1, \dots, n_c$ . If the cusp is on the horizontal axis at  $-\frac{d_j}{c_j} \in \mathbb{Q}$  with relative prime  $(c_j, d_j) \in \mathbb{Z}^2$ , we can map the cusp to  $i\infty$  by a modular transformation

$$\gamma_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}. \quad (5.87)$$

We let  $\tau_j = \gamma_j \tau$ . Then the holomorphic part  $h_j(\tau_j)$  of  $(c_j \tau + d_j)^{-2} \widehat{h}(\gamma \tau_j, \gamma \bar{\tau}_j)$  can be expanded for  $\tau$  near  $-\frac{d_j}{c_j}$  as

$$h_j(\tau_j) = \sum d_j(n) q_j^n, \quad q_j = e^{2\pi i \tau_j}. \quad (5.88)$$

As a result, the cusp  $j$  contributes

$$[\mathcal{I}_f]_j = w_j d_j(0). \quad (5.89)$$

<sup>32</sup>Also if  $f$  does not satisfy this requirement, the integral can be regularised as explained in [83, 289]. We do not need this regularisation for the correlators in this section.

### 3. Singular points in the interior of $\mathcal{F}(\mathbf{m})$

The integrand can be singular at a point  $\tau_s$  in the interior of  $\mathcal{F}(\mathbf{m})$ . Such singularities appear typically for deformations of superconformal theories, such as the  $\mathcal{N} = 2^*$  theory and the  $N_f = 4$  theory, where the UV coupling  $\tau_{\text{UV}}$  gives rise to such a singularity [3, 85]. See Fig. 26 for an example. We require that the expansion of  $f$  near such a singularity reads,

$$f(\tau, \bar{\tau}) = \sum_{m \gg -\infty, n \geq 0} c_s(m, n) (\tau - \tau_s)^m (\bar{\tau} - \bar{\tau}_s)^n. \quad (5.90)$$

Then, the anti-derivative  $\widehat{h}(\tau, \bar{\tau})$  has similar expansion,

$$\widehat{h}(\tau, \bar{\tau}) = \sum_{m \gg -\infty, n \geq 0} d_s(m, n) (\tau - \tau_s)^m (\bar{\tau} - \bar{\tau}_s)^n. \quad (5.91)$$

The contour integral for a small contour around  $\tau_s$ ,

$$C_\varepsilon(\tau_s) = \{ \tau = \tau_s + \varepsilon e^{i\varphi}, \varphi \in [0, 2\pi) \}, \quad (5.92)$$

is bounded for such a function. Moreover, in the limit  $\varepsilon \rightarrow 0$ , the contour integral is finite. We define the “residue” of a non-holomorphic function  $g(\tau, \bar{\tau})$

$$\text{nRes}_{\tau=\tau_s} [g(\tau, \bar{\tau})] = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0} \oint_{C_\varepsilon(\tau_s)} g(\tau, \bar{\tau}) d\tau. \quad (5.93)$$

For the expansion (5.90) this evaluates to

$$[\mathcal{I}_f]_s = 2\pi i \text{nRes}_{\tau=\tau_s} [\widehat{h}(\tau, \bar{\tau})] = d_s(-1, 0), \quad (5.94)$$

with  $d_s(-1, 0)$  the coefficient in the expansion (5.91).

### 4. Elliptic points

For  $\mathcal{N} = 2$  QCD, AD points are the elliptic points of the duality group, and lie on the boundary of  $\mathcal{F}(\mathbf{m})$ . See Fig. 7 for an example. The elliptic points are  $\alpha = e^{\pi i/3}$  and  $i$ , and their images under  $\text{PSL}(2, \mathbb{Z})$ . Contour integrals around such points can be regularised using a cut-off  $\varepsilon$ . We assume that the anti-derivative  $\widehat{h}$  has the following expansion near an elliptic point  $\tau_e$ ,

$$\widehat{h}(\tau, \bar{\tau}) = \sum_{m \gg -\infty, n \geq 0} d_e(m, n) (\tau - \tau_e)^m (\bar{\tau} - \bar{\tau}_e)^n. \quad (5.95)$$

As a result, the boundary arc around  $\tau_{\text{AD}}$  in  $\mathbb{H}$  is a fraction of  $2\pi$ , which needs to be properly accounted for. These neighbourhoods have an angle  $\frac{2\pi}{k_e}$ , with  $k_e = 2$  for  $\tau_e = i$ , and  $k_e = 6$  for  $\tau_e = \alpha$  [145]. Furthermore, it is important how many images of  $\mathcal{F}$  in  $\mathcal{F}(\mathbf{m})$  coincide at the elliptic point. We denote this number by  $n_e$ . For  $\mathcal{N} = 2$  SQCD, we found examples with  $n_e = 2$

and 4 for  $\tau_e \sim \alpha$ , while for  $\tau_e \sim i$ ,  $n_e = 1$  [2]. The contribution from an elliptic point is then,

$$[\mathcal{I}_f]_e = 2\pi i \frac{n_e}{k_e} \text{nRes}_{\tau=\tau_e} \left[ \widehat{h}(\tau, \bar{\tau}) \right] = \frac{n_e}{k_e} d_e(-1, 0), \quad (5.96)$$

### 5. Branch points and cuts

Branch points and cuts are a new aspect compared to previous analyses (see for instance Fig. 8 and 15). We will demonstrate that their contribution vanishes for the integrands of interest.

We assume that the integrand  $f$  satisfies

$$\widehat{h}(\tau, \bar{\tau}) = (\tau - \tau_{\text{bp}})^n g(\tau, \bar{\tau}), \quad (5.97)$$

with  $n \in \mathbb{Z}/2$  and  $n \geq -1/2$ ,  $g(\tau, \bar{\tau})$  being a real analytic function near  $\tau_{\text{bp}}$ . This assumption is satisfied for the twisted Yang–Mills theories [286]. To treat this type of singularity, we remove a  $\delta$  neighbourhood and analyse the  $\delta \rightarrow 0$  limit. Let  $C_\delta$  be the contour

$$C_\delta = \{\tau_{\text{bp}} + \delta e^{i\theta} \mid \theta \in (0, 2\pi)\} \quad (5.98)$$

around  $\tau_{\text{bp}}$  with radius  $\delta > 0$ . Therefore, on the contour  $|y^{-s}f|$  is bounded by

$$|\widehat{h}| \leq \delta^n K \quad (5.99)$$

for some  $K > 0$ . The integral around the branch point therefore vanishes in the limit,

$$\begin{aligned} \mathcal{I}_f^{\text{bp}} &= \lim_{\delta \rightarrow 0} \int_{C_\delta} \widehat{h} |d\tau| \leq \lim_{\delta \rightarrow 0} \int_0^{2\pi} \delta^n K \delta d\theta \\ &= \lim_{\delta \rightarrow 0} 2\pi K \delta^{n+1} = 0. \end{aligned} \quad (5.100)$$

The branch points necessarily give rise to branch cuts. For the purpose of integration, we remove a neighbourhood with distance  $r$  from the cut, and take the limit  $r \rightarrow 0$  after determining the integral. Since the value of the integrand is finite near the branch cut, the contribution to the integral vanishes.

### Summary

Combining all the contributions discussed above, we find

$$\mathcal{I}_f = \sum_{j=1}^n w_j d_j(0) + \sum_s d_s(-1, 0) + \sum_e \frac{n_e}{k_e} d_e(-1, 0). \quad (5.101)$$

This formula generalises [73] for the pure  $N_f = 0$  theory on a smooth four-manifold  $X$  that admits a metric of positive scalar curvature, [84, Equation (5.10)] for the pure  $\text{SU}(2)$  theory on generic  $X$ , [85, Equation (4.88)] for the  $\mathcal{N} = 2^*$  theory on  $X$ , and [77] for the massless  $N_f = 2$  and  $N_f = 3$  theories on  $X = \mathbb{CP}^2$ .

## 6 The $u$ -plane integral for non-simply connected manifolds

In this final section, we study the  $u$ -plane integral on compact four-manifolds  $X$  with  $b_1(X) > 0$ , which are non-simply connected. This section is based on [4].

### 6.1 Introduction

Recently, interest in DW theory, and in particular the  $u$ -plane integral  $Z_u$ , was revived due to observations relating the latter for special four-manifolds to the theory of mock theta functions and harmonic Maass forms [77, 78]. For more generic, but simply connected, compact four-manifolds, it was later reformulated in terms of the modular completion of a mock modular form [81, 83, 84]. In this series of papers, the possibility of adding  $\mathcal{Q}$ -exact operators to the action without affecting the correlation functions was studied in detail. In particular, a specific new  $\mathcal{Q}$ -exact operator related to the 2-cycles of the background geometry was added to the action of the low-energy U(1) theory, which makes the connection to mock modular forms apparent and elegant [81, 84]. This technique circumvents the cumbersome method of lattice reduction, and allows to evaluate correlation functions efficiently. Previous results relating  $Z_u$  and mock modular forms are restricted to the case where the low-energy U(1) theory is formulated on simply connected four-manifolds only.

Taking inspiration from [76], we ask the natural question how these recent results carry over to the case when the four-manifold  $X$  has a non-trivial fundamental group and non-zero first Betti number  $b_1(X)$ . When the four-manifold is non-simply connected, the theory is more complicated. This is due to the fact that the manifold now admits more structures, in the form of 1-form fields and 1- and 3-cycles, which are not present in the simply connected case. These cycles give rise to further contact terms in the low-energy U(1) action [74, 76]. As a result, we will consider more general  $\mathcal{Q}$ -exact operators related to these cycles.

We present a natural extension of the recent results [81, 84] to the case of non-simply connected four-manifolds with  $b_2^+ = 1$ . Specifically, we introduce a number of new  $\mathcal{Q}$ -exact operators in the low-energy effective U(1) theory that allow us to express the integrand of the  $u$ -plane integral elegantly as the non-holomorphic completion of a mock modular form. This further allows us to derive a closed-form expression for the  $u$ -plane integral for any such four-manifold and for arbitrary period point  $J$ , as is evident from the result (6.57). This solution depends on  $H_1(X)$ , a fact that is easily seen in the case of product ruled surfaces where it manifests as a genus dependence, while when  $H_1(X)$  is trivial, (6.57) reduces to Equation (4.10) of [81].

DW theory on product ruled surfaces  $X = \mathbb{CP}^1 \times \Sigma_g$ , with  $\Sigma_g$  a genus  $g$  Riemann surface, in the limit of vanishing volume for  $\Sigma_g$  has been argued to

be equivalent to an  $\mathcal{N} = (2, 2)$  2d topological A-model on  $\mathbb{CP}^1$  with target space the moduli space  $\mathcal{M}_{\text{flat}}(\Sigma_g)$  of flat  $SU(2)$  connections on  $\Sigma_g$  [74, 90, 290–294]. Reference [4] presents a concrete derivation of this equivalence, and in turn shows that due to the relation between DW theory and its low-energy  $U(1)$  effective theory as given by Eq. (1.39), a connection between Gromov–Witten (GW) theory (realised physically by the A-model) and mock modular forms (appearing in the low-energy effective action) exists, such that one can compute GW invariants using modular data originating from the 4d theory. The above equivalence is expected to hold in the limit small  $\Sigma_g$ , since the twisted  $\mathcal{N} = 2$  gauge theory is topological and we are thus free to shrink  $\Sigma_g$ . As a consequence, flat  $SU(2)$  connections along the directions tangent to  $\Sigma_g$  are required to prevent the effective 2d action from blowing up when the limit of small  $\Sigma_g$  is taken.

## 6.2 The effective theory for $b_1 > 0$

DW theory is the topologically twisted formulation of the pure  $\mathcal{N} = 2$  supersymmetric Yang–Mills theory with gauge group  $G$  of rank  $r_G = 1$  on a smooth four-manifold  $X$  [61]. In the IR, the theory becomes a  $U(1)$  gauge theory that depends on the complexified effective gauge coupling  $\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2} \in \mathbb{H}$ , where  $\mathbb{H}$  denotes the Poincaré half-plane. DW theory contains a scalar fermionic BRST operator  $\mathcal{Q} := \epsilon^{AB} \overline{\mathcal{Q}}_{AB}$  that obeys  $\mathcal{Q}^2 = 0$ .<sup>33</sup> The field content of the theory is a collection of bosonic and fermionic degree 0, 1 and 2 operator valued differential forms on  $X$ , where the degree of the differential form is equal to the ghost number of the physical operator. In Table 2 we summarise the field content of the DW theory. The BRST transformations on these fields are

Bosons	Fermions	Form degree
$a, \bar{a}$	$\eta$	0
$A$	$\psi$	1
$D$	$\chi$	2

Table 2: Field content of DW theory. The  $a, \bar{a}$  fields originate from the vacuum expectation value of the scalar field of the UV theory. The  $D$  field is an auxiliary field.

given in (5.43), where the label  $J$  can be removed since we are studying a rank 1 theory. The physical observables of the theory belong to the  $\mathcal{Q}$ -cohomology. We are interested in computing the path integral of the theory, the *u-plane integral* or *Coulomb branch integral*, when evaluated on a non-simply connected four-manifold. To this end, let us first introduce some notation.

Let  $b_j := b_j(X) = \dim H^j(X)$  be the Betti numbers of the smooth, closed and oriented four-manifold  $X$  with  $b_2(X) = b_2^+(X) + b_2^-(X)$ , where the first (second) summand corresponds to the number of positive (negative) eigenvalues of the quadratic form  $Q$  of  $X$ . For  $a \in H^i(X)$  and  $b \in H^{4-i}(X)$  we

<sup>33</sup>Across the literature, this operator is often denoted as  $\overline{\mathcal{Q}}$  instead.



define  $B(a, b) = \int_X a \wedge b$ . For  $a \in H^2(X)$  the quadratic form  $Q$  of  $X$  corresponds to  $Q(a) := B(a, a)$ . Furthermore, the signature of  $X$  is defined as  $\sigma(X) = b_2^+(X) - b_2^-(X)$ . Hereafter we consider four-manifolds with  $b_2^+(X) = 1$ . By Poincaré duality, we have that  $b_0 = b_4$ ,  $b_1 = b_3$ . We can assume  $b_1$  to be even, since the correlation function of the theory are known to vanish unless  $1 - b_1 + b_2^+$  is even [73].

The Coulomb branch integral is the path integral of the low-energy U(1) theory with the insertion of the observables arising from the descent formalism as well as contact terms and  $\mathcal{Q}$ -exact operators. It takes the form

$$Z_u(p, \gamma, S, Y) = \int [\mathcal{D}\Phi] \nu(\tau) e^{-\int_X \mathcal{L}' + I(S, Y) + I_{\mathcal{O}} + I_{\mathcal{Q}}}, \quad (6.1)$$

where  $\Phi = \{a, \bar{a}, A, \eta, \psi, \chi, D\}$  is the collection of fields of the theory (as in Table 2).

### 6.2.1 Ingredients of the $u$ -plane integrand

In this section, we give explicit modular expressions for the ingredients of the  $u$ -plane integrand. The integrand transforms under the duality group  $\Gamma^0(4)$ , which is generated by  $T^4$  and  $S^{-1}T^{-1}S$ . Let us introduce the shorthand  $f = (\phi_1, \phi_2)^{(k, l)}$  if a function  $f$  is a non-holomorphic modular form of weight  $(k, l)$  for  $\Gamma^0(4)$  with multipliers, i.e. transforms as

$$\begin{aligned} f(\tau + 4, \bar{\tau} + 4) &= \phi_1 f(\tau, \bar{\tau}), \\ f\left(\frac{\tau}{\tau + 1}, \frac{\bar{\tau}}{\bar{\tau} + 1}\right) &= \phi_2 (\tau + 1)^k (\bar{\tau} + 1)^l f(\tau, \bar{\tau}). \end{aligned} \quad (6.2)$$

It is clear that

$$\begin{aligned} (\phi_1, \phi_2)^{(k_1, l_1)} (\varphi_1, \varphi_2)^{(k_2, l_2)} &= (\phi_1 \varphi_1, \phi_2 \varphi_2)^{(k_1 + k_2, l_1 + l_2)}, \\ \overline{(\phi_1, \phi_2)^{(k, l)}} &= (\bar{\phi}_1, \bar{\phi}_2)^{(l, k)}, \\ \frac{1}{(\phi_1, \phi_2)^{(k, l)}} &= (\bar{\phi}_1, \bar{\phi}_2)^{(-k, -l)}, \end{aligned} \quad (6.3)$$

since  $|\phi_i| = 1$ . The functions<sup>34</sup>

$$\begin{aligned} \frac{d^2 u}{da^2} &= 4 \frac{2E_2 + \vartheta_2^4 + \vartheta_3^4}{3\vartheta_4^8}, & \frac{u}{\Lambda^2} &= \frac{\vartheta_2^4 + \vartheta_3^4}{2\vartheta_2^2 \vartheta_3^2}, & \frac{a}{\Lambda} &= \frac{2E_2 + \vartheta_2^4 + \vartheta_3^4}{6\vartheta_2 \vartheta_3}, \\ \frac{du}{d\tau} &= \frac{\pi \Lambda^2}{4i} \frac{\vartheta_4^8}{\vartheta_2^2 \vartheta_3^2}, & \frac{da}{du} &= \frac{1}{2\Lambda} \vartheta_2 \vartheta_3, & \frac{d\tau}{da} &= \frac{8i}{\pi \Lambda} \frac{\vartheta_2 \vartheta_3}{\vartheta_4^8} \end{aligned} \quad (6.4)$$

transform as

$$\begin{aligned} u &= (1, 1)^{(0, 0)}, & \frac{du}{d\tau} &= (1, 1)^{(2, 0)}, & \frac{da}{du} &= (-1, 1)^{(1, 0)}, \\ \rho &= (-1, 1)^{(-1, 0)}, & \frac{d\tau}{da} &= (-1, 1)^{(-3, 0)}, & y &= (1, 1)^{(-1, -1)} \end{aligned} \quad (6.5)$$

<sup>34</sup>In this section, we choose the traditional convention where  $\frac{u}{\Lambda^2} \rightarrow +\infty$  for  $\tau \rightarrow i\infty$ .

### 6.2.2 Effective Lagrangian

The low-energy U(1) effective Lagrangian  $\mathcal{L}$  of the twisted theory is given in [73, (2.15)]. The  $\mathcal{Q}$ -exact terms as well as the kinetic terms do not contribute since the zero modes are constant in DW theory on a four-manifold  $X$  with  $b_2^+(X) = 1$ . For such manifolds there is a useful fact stating that for any  $\beta_1, \beta_2, \beta_3, \beta_4 \in H^1(X, \mathbb{Z})$ , we have [295]

$$\beta_1 \wedge \beta_2 \wedge \beta_3 \wedge \beta_4 = 0. \quad (6.6)$$

We will make extensive use of this fact below.

Let us define  $\mathcal{L}'$  as the part of the zero-mode low-energy U(1) effective Lagrangian that contributes to the  $u$ -plane integral. It is given by [73]

$$\begin{aligned} \mathcal{L}' = & \pi i \bar{\tau} k_+^2 + \pi i \tau k_-^2 - \frac{y}{8\pi} D \wedge *D + \frac{i\sqrt{2} d\bar{\tau}}{16\pi da} \eta \chi \wedge (F_+ + D) \\ & - \frac{i\sqrt{2} d\tau}{2^7 \pi da} \psi \wedge \psi \wedge (F_- + D), \end{aligned} \quad (6.7)$$

where  $F_{\pm} = 4\pi k_{\pm}$  and for any two-form  $x$  we abbreviate  $B(x, x) = x^2$ . In  $\mathcal{L}'$ , we disregard any summands of  $\mathcal{L}$  containing  $\mathcal{Q}$ -exact terms, exact differential forms and  $\wedge$ -products of four 1-forms. Here and throughout the rest of the section we use units where the dynamical scale  $\Lambda$  of the low-energy effective U(1) theory is equal to one. The gravitational contributions to  $\mathcal{L}'$  are described in the following section.

### 6.2.3 Measure factors

Assuming  $X$  is connected, the (holomorphic) measure factor [52, 73] is<sup>35</sup>

$$\nu(\tau) := - (2^{7/2} \pi)^{\frac{b_1}{2}} \frac{2^{\frac{3\sigma(X)+1}{4}}}{\pi} (u^2 - 1)^{\frac{\sigma(X)}{8}} \left( \frac{da}{du} \right)^{\frac{\sigma(X)}{2} + b_1 - 2}. \quad (6.8)$$

Here we used  $\chi(X) + \sigma(X) = 4 - 2b_1$  to eliminate the Euler character of  $X$ ,  $\chi(X)$ . This expression reduces to Eq. (2.9) in [81] if we take  $b_1 = 0$ . For the simply connected theory one can use the microscopic definition of the theory to determine the effective gravitational couplings (e.g. by considering expansions of the Nekrasov partition function) [53, 111].

The zero modes of the one-forms  $\psi$  live in the tangent space of a  $b_1$ -dimensional torus  $\mathbb{T}^{b_1} = H^1(X, \mathbb{R})/H^1(X, \mathbb{Z}) = H^1(X, \mathcal{O}_X^*)$  which corresponds to isomorphism classes of invertible sheaves (for  $X$  a smooth complex variety, that is, holomorphic line bundles) on  $X$  which are topologically trivial. We can expand  $\psi$  in zero-modes as  $\psi = \sum_{i=1}^{b_1} c_i \beta_i$  with  $\beta_i$  an integral basis of harmonic

<sup>35</sup>Note that this differs from the notation  $\nu$  used in Section 5, in particular (5.64), by the factor  $\frac{da}{d\tau}$ , which we insert later, and we do not consider any external fluxes  $\mathbf{k}_j$ , since we study the  $N_f = 0$  theory in this section.

one-forms, and  $c_i$  Grassmann variables. We then have the measure

$$\prod_{i=1}^{b_1} \frac{dc_i}{\sqrt{y}} = y^{-\frac{b_1}{2}} \prod_{i=1}^{b_1} dc_i. \quad (6.9)$$

The photon partition function will also include an integration over  $b_1$  zero modes of the gauge field corresponding to flat connections [76]. These zero modes span the tangent space of  $\mathbb{T}^{b_1}$ . As a consequence of this, the photon partition function will have an overall factor of  $y^{\frac{1}{2}(b_1-1)}$  [72]. Combining this with the measure factor (6.9) we see that in the end there will only be a factor of  $y^{-1/2}$  surviving.

We can also consider the  $c_j$  in the expansion of  $\psi$  as a basis of one-forms  $\beta_j^\# \in H^1(\mathbb{T}^{b_1}, \mathbb{Z})$ , dual to  $\beta_j$ , such that

$$\psi = \sum_{j=1}^{b_1} \beta_j \otimes \beta_j^\#. \quad (6.10)$$

A useful fact about four-manifolds with  $b_2^+ = 1$  is that the image of the map

$$\wedge : H^1(X, \mathbb{Z}) \otimes H^1(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z}) \quad (6.11)$$

is generated by a single rational cohomology class, which we denote as  $W$  [295].<sup>36</sup> This means that we can write  $\beta_i \wedge \beta_j = a_{ij}W$ ,  $i, j = 1, \dots, b_1$ , where  $a_{ij}$  is an anti-symmetric matrix. This further implies that the two-form on  $\mathbb{T}^{b_1}$  can be written as

$$\Omega = \sum_{i < j} a_{ij} \beta_i^\# \wedge \beta_j^\#, \quad (6.12)$$

where  $\beta_i^\# \in H^1(\mathbb{T}^{b_1}, \mathbb{Z})$ , such that

$$\text{vol}(\mathbb{T}^{b_1}) = \int_{\mathbb{T}^{b_1}} \frac{\Omega^{b_1/2}}{(b_1/2)!}. \quad (6.13)$$

Using the analysis above we can now write  $\psi \wedge \psi = 2(W \otimes \Omega)$  [76]. This is useful when one performs the integral over  $\mathbb{T}^{b_1}$  for the product ruled surfaces, for instance.

## 6.2.4 Observables

$\mathcal{Q}$ -invariant observables can be constructed using the celebrated descent formalism. By starting with the zero-form operator  $\mathcal{O}^{(0)} = 2u$ , we find all  $k$ -form valued observables  $\mathcal{O}^{(k)}$  for  $k = 1, 2, 3, 4$  that are  $\mathcal{Q}$ -invariant modulo exact forms by solving the descent equations

$$d\mathcal{Q}^{(j)} = \{\mathcal{Q}, \mathcal{O}^{(j+1)}\} \quad (6.14)$$

<sup>36</sup>This class is denoted  $\Sigma$  in [76] and  $\Lambda$  in [90]. However, since we want to reserve  $\Sigma$  for the Riemann surfaces studied below and  $\Lambda$  for the dynamical scale of the theory we choose to call the class  $W$ .

inductively. This ensures that for a  $k$ -cycle  $\Sigma^{(k)} \in H_k(X)$  in  $X$ , the integrals  $\int_{\Sigma^{(k)}} \mathcal{O}^{(k)}$  are  $\mathcal{Q}$ -invariant and only depend on  $\Sigma^{(k)}$ . Fortunately, there is a canonical solution to the descent equations: Due to the fact that the translation generator is  $\mathcal{Q}$ -exact, there is the one-form valued *descent operator*  $K$ , which satisfies  $d = \{\mathcal{Q}, K\}$  [73]. This implies that (6.14) can be solved by  $\mathcal{O}^{(j)} = K^j \mathcal{O}^{(0)}$ , where the iterated (anti)-commutators are implicit. The action of the operator  $K$  can be inferred from the BRST transformations (5.43) as [73]

$$\begin{aligned} [K, a] &= \frac{1}{4\sqrt{2}}\psi, & [K, \bar{a}] &= 0, & [K, \psi] &= -2(F_- + D), & [K, A] &= -2i\chi, \\ [K, \eta] &= -\frac{i}{\sqrt{2}}d\bar{a}, & [K, \chi] &= -\frac{3\sqrt{2}i}{4} * d\bar{a}, & [K, D] &= \frac{3i}{4}(2d\chi - *d\eta). \end{aligned} \quad (6.15)$$

Let us study the insertion of all possible observables. For ease of notation, let us denote  $p = \Sigma^{(0)}$  a point class,  $\gamma = \Sigma^{(1)}$  a 1-cycle,  $S = \Sigma^{(2)}$  a 2-cycle and  $Y = \Sigma^{(3)}$  a 3-cycle. The cycles  $\gamma$ ,  $S$  and  $Y$  can be expanded in formal sums as

$$\gamma = \sum_{i=1}^{b_1} \zeta_i \gamma_i, \quad S = \sum_{i=1}^{b_2} \lambda_i S_i, \quad Y = \sum_{i=1}^{b_3} \theta_i Y_i, \quad (6.16)$$

where  $\gamma_i$ ,  $S_i$  and  $Y_i$  are a basis of one-, two- and three-cycles respectively,  $\lambda_i$  are complex numbers, while  $\zeta_i$  and  $\theta_i$  are Grassmann variables. By the common abuse of notation, we use the same notation for the 3-, 2-, and 1-forms Poincaré dual to the cycles, and use the convention

$$\int_{\gamma} \omega_1 = \int_X \omega_1 \wedge \gamma, \quad \int_S \omega_2 = \int_X \omega_2 \wedge S, \quad \int_Y \omega_3 = \int_X \omega_3 \wedge Y. \quad (6.17)$$

The most general  $\mathcal{Q}$ -invariant observable we can add is then

$$I_{\mathcal{O}} = 2pu + a_1 \int_{\gamma} Ku + a_2 \int_S K^2 u + a_3 \int_Y K^3 u, \quad (6.18)$$

where  $a_2 = \frac{i}{\sqrt{2}\pi}$  is fixed from matching with the mathematical literature [73] and

$$\begin{aligned} Ku &= \frac{1}{4\sqrt{2}} \frac{du}{da} \psi, \\ K^2 u &= \frac{1}{32} \frac{d^2 u}{da^2} \psi \wedge \psi - \frac{\sqrt{2}}{4} \frac{du}{da} (F_- + D), \\ K^3 u &= \frac{1}{2^7 \sqrt{2}} \frac{d^3 u}{da^3} \psi \wedge \psi \wedge \psi - \frac{3}{16} \frac{d^2 u}{da^2} \psi \wedge (F_- + D) - \frac{3\sqrt{2}i}{16} \frac{du}{da} (2d\chi - *d\eta). \end{aligned} \quad (6.19)$$

### 6.2.5 Contact terms

The existence of the canonical solution to the descent equations allows to map an observable of the UV theory to the low-energy U(1) effective theory on

the  $u$ -plane. For instance, the operator  $I(S) = \int_S K^2 u$  of the UV theory is mapped to the same observable  $\tilde{I}(S) = \int_S K^2 u$  in the IR. This is not quite true for products  $I(S_1)I(S_2)\dots I(S_n)$  of such operators for distinct Riemann surfaces  $S_i \in H_2(X, \mathbb{Z})$ . At the intersection of the surfaces, contact terms will appear [73, 74]. When mapping a product of surface operators to the IR, the product is corrected by a sum over the intersection points. Due to the  $Q$ -invariance, the inserted operator is holomorphic and the point at which it is inserted is irrelevant.

Such contact terms appear for all cycles in  $X$  that can intersect. They have been classified and the corresponding contact terms have been found in [76, Equations (2.8)-(2.12)],

$$\begin{aligned} I_\cap = & \int_{S \cap S} T + a_{13} \int_{Y \cap \gamma} T + a_{32} \int_{Y \cap S} KT + a_{33} \int_{Y \cap Y} K^2 T \\ & + a_{332} \int_{S \cap Y \cap Y} \frac{\partial^3 \mathcal{F}}{\partial \tau_0^3} + a_{333} \int_{Y \cap Y \cap Y} K \frac{\partial^3 \mathcal{F}}{\partial \tau_0^3} + a_{3333} \int_{Y \cap Y \cap Y \cap Y} \frac{\partial^4 \mathcal{F}}{\partial \tau_0^4}. \end{aligned} \quad (6.20)$$

Here  $\tau_0$  is the deformation parameter of the prepotential, related to the dynamical scale by  $\Lambda^4 = e^{\pi i \tau_0}$ . The coefficient functions can all be expressed as quasi-modular functions on the  $u$ -plane. For instance, the contact term for  $S \cap S$  is

$$T = \frac{u}{2} - \frac{a}{4} \frac{du}{da} = \frac{\vartheta_2^4 + \vartheta_3^4 - E_2}{6\vartheta_2^2 \vartheta_3^2}. \quad (6.21)$$

In terms of the prepotential  $\mathcal{F}$ , it is given by  $T(\tau) = \frac{4}{\pi i} \frac{\partial^2 \mathcal{F}}{\partial \tau_0^2}$  [98]. We can use the action (6.15) to find

$$\begin{aligned} KT &= \frac{1}{4\sqrt{2}} \frac{dT}{da} \psi, \\ K^2 T &= \frac{1}{32} \frac{d^2 T}{da^2} \psi \wedge \psi - \frac{1}{2\sqrt{2}} \frac{dT}{da} (F_- + D). \end{aligned} \quad (6.22)$$

The intersection constants can be obtained from duality invariance [76]. Due to the identity (6.6), the two last terms in (6.20) vanish and we can disregard them. Thus, from (6.20) and (6.22) we see that all terms in  $I_\cap$  except for one are only integrated over  $\psi$  and  $\tau$ , which we do in a later step. The remaining term

$$- \frac{\sqrt{2} a_{33}}{4} \frac{dT}{da} B(F_- + D, Y \wedge Y). \quad (6.23)$$

is to be integrated over  $D$ ,  $\chi$  and  $\eta$ .

### 6.2.6 $Q$ -exact operators

As we will later see, the photon path integral combines with the insertion of the surface observable to a Siegel-Narain theta function  $\Psi_\mu^J(\tau, z)$ . See (6.52) for the definition. This function can be expressed as a total derivative to a non-holomorphic modular completion of an indefinite theta function, as has

been previously shown in the simply connected case [81, 84]. To facilitate the calculation further the authors of those papers add the  $\mathcal{Q}$ -exact operator  $I_S$ .<sup>37</sup> In this section we will generalise this operator insertion to simplify the calculations also in the case of non-simply connected manifolds. This then allows us to evaluate correlation functions efficiently using mock modular forms [81, 83, 84].

Since our computations should be valid for any  $b_1 \geq 0$  and in particular  $b_1 = 0$ , this suggests that it is instructive to add the same  $\mathcal{Q}$ -exact operator [81, (2.11) and (2.12)]

$$\begin{aligned} I_S &= -\frac{1}{4\pi} \int_S \left\{ \mathcal{Q}, \frac{d\bar{u}}{d\bar{a}} \chi \right\} \\ &= -\frac{\sqrt{2}i}{4\pi} \frac{d^2\bar{u}}{d\bar{a}^2} \int_S \eta\chi - \frac{i}{4\pi} \frac{d\bar{u}}{d\bar{a}} \int_S (F_+ - D). \end{aligned} \quad (6.24)$$

The  $u$ -plane integrand (6.1) with  $I_S$  inserted can also in the case where  $b_1 \neq 0$  be written as an anti-holomorphic derivative. However, it does not give the same kind of Siegel-Narain theta function as in the simply-connected case. The reason is that the elliptic argument  $z$  of  $\Psi_\mu^J$  does not couple to  $H_-^2(X)$  symmetrically to how its conjugate  $\bar{z}$  couples to  $H_+^2(X)$ . The insertion of  $I_S$  in the case  $b_1 = 0$  can be viewed as the unique correction to the path integral that symmetrises the couplings to  $H_\pm^2(X)$ . Without such an insertion, the resulting theta functions are not symmetric, see for instance [73, Equation (3.18)].

As we demonstrate below, for  $b_1 \neq 0$  this issue can be cured by introducing additional  $\mathcal{Q}$ -exact operators. More precisely, the new observables and related contact terms require three new  $\mathcal{Q}$ -exact terms. The first two

$$\begin{aligned} I_Y &= -\frac{3i\bar{a}_3}{16} \int_Y \left[ \mathcal{Q}, \frac{d^2\bar{u}}{d\bar{a}^2} \chi \wedge \psi \right] + \frac{\sqrt{2}}{2^7\pi} \int_X \left\{ \mathcal{Q}, \frac{d\bar{\tau}}{d\bar{a}} \chi \wedge \psi \wedge \psi \right\} \\ &= \frac{3\sqrt{2}\bar{a}_3}{2^4} \frac{d^3\bar{u}}{d\bar{a}^3} B(\eta\chi, \psi \wedge Y) + \frac{3\bar{a}_3}{2^4} \frac{d^2\bar{u}}{d\bar{a}^2} B(F_+ - D, \psi \wedge Y) \\ &\quad + \frac{i}{2^6\pi} \frac{d^2\bar{\tau}}{d\bar{a}^2} B(\eta\chi, \psi \wedge \psi) + \frac{\sqrt{2}i}{2^7\pi} \frac{d\bar{\tau}}{d\bar{a}} B(F_+ - D, \psi \wedge \psi) \end{aligned} \quad (6.25)$$

compensate the observables (6.18). From the collection of contact terms (6.20), only the one from the intersection  $Y \cap Y$  gives a term (6.23) that is integrated over  $D$ ,  $\eta$  and  $\chi$ . This term requires the addition of the  $\mathcal{Q}$ -exact operator

$$\begin{aligned} I_{Y \cap Y} &= -\frac{\sqrt{2}i\bar{a}_{33}}{4} \int_{Y \cap Y} \left\{ \mathcal{Q}, \frac{d\bar{T}}{d\bar{a}} \chi \right\} \\ &= \frac{\bar{a}_{33}}{2} \frac{d^2\bar{T}}{d\bar{a}^2} B(\eta\chi, Y \wedge Y) + \frac{\sqrt{2}\bar{a}_{33}}{4} \frac{d\bar{T}}{d\bar{a}} B(F_+ - D, Y \wedge Y). \end{aligned} \quad (6.26)$$

We can note that  $\bar{a}_{33} = -a_{33}$  [76]. The sum of these additional  $\mathcal{Q}$ -exact terms can be compactly written as

$$I_Y + I_{Y \cap Y} = -\sqrt{2}\eta B(\chi, \partial_{\bar{a}}(y\bar{\omega})) - yB(F_+ - D, \bar{\omega}), \quad (6.27)$$

<sup>37</sup>This term is called  $\tilde{I}_+(\mathbf{x})$  in [81, 83]. For ease of notation, we remove the tilde from such expressions since we only discuss operators in the IR.

where we introduced the 2-form

$$\omega := \frac{\sqrt{2}i}{2^7 \pi y} \frac{d\tau}{da} \psi \wedge \psi - \frac{3a_3}{2^4 y} \frac{d^2 u}{da^2} \psi \wedge Y - \frac{\sqrt{2}a_{33}}{4y} \frac{dT}{da} Y \wedge Y. \quad (6.28)$$

This 2-form has the property that  $y\omega$  is holomorphic and thus  $y\bar{\omega}$  is anti-holomorphic. The form of (6.25) is derived in Appendix 6.3, where we furthermore show that its one-point function evaluates to zero, such that it is safe to include it into the path integral, following the analysis in [83, 84]. We furthermore find it useful to follow [81] and introduce the notation

$$\rho = \frac{S}{2\pi} \frac{du}{da}, \quad b = \frac{\text{Im}(\rho)}{y}. \quad (6.29)$$

Anticipating the result as a Siegel-Narain theta function, the elliptic variable will turn out to be  $z = \rho + 2iy\omega$ , which is a 2-form with holomorphic coefficients. In terms of this variable, the sum of all  $\mathcal{Q}$ -exact insertions (6.24), (6.25) and (6.26) combine nicely as

$$\begin{aligned} I(S, Y) &:= I_S + I_Y + I_{Y \cap Y} \\ &= -\frac{i}{2} \left( \sqrt{2} B(\eta\chi, \partial_{\bar{a}} \bar{z}) + B(F_+ - D, \bar{z}) \right). \end{aligned} \quad (6.30)$$

It is clear that this is purely anti-holomorphic. The operator  $I(S, Y)$  is then included into the path integral, as in (6.1).

### 6.3 $\mathcal{Q}$ -exact operators

In this section, we explain the construction of the  $\mathcal{Q}$ -exact operator  $I(S, Y)$  in (6.30), which aids the evaluation of the  $u$ -plane integral using mock modular forms. A constructive approach is to classify all  $\mathcal{Q}$ -exact operators in DW theory, add all of them to the path integral, evaluate the path integral and solve for all coefficient functions that lead to the desired properties. For two reasons, this is fortunately not necessary. First, it is convenient that most such operators do not even alter the  $u$ -plane integrand after integrating out the fermions and the auxiliary field. Second, the path integral can be performed without insertions of any additional operators, or with the insertion of just  $I_S$  as was done in the case that  $b_1(X) = 0$  [81]. Such calculations lead to integrands that do not contain the symmetric Siegel-Narain theta function  $\Psi_\mu^J(\tau, z)$  for any  $z$ , however only a few terms are missing with an educated guess of  $z$ . Only very specific  $\mathcal{Q}$ -exact operators can provide the necessary terms for the new integrands to complete into  $\Psi_\mu^J(\tau, z)$ .

In section 6.3.1, we classify all possible  $\mathcal{Q}$ -exact operators that contribute to the  $u$ -plane *integrand*. In section 6.3.2 we demonstrate how the correct  $\mathcal{Q}$ -exact operators can be selected, for the simplified example where the intersection  $Y \cap Y$  is empty (such that there is no intersection term for  $Y \cap Y$ ).

### 6.3.1 Construction of $\mathcal{Q}$ -exact operators

Let us complete the result of [83] by computing the all  $\mathcal{Q}$ -exact observables on a four-manifold  $X$  with  $\pi_1(X) \neq 0$ . Let us first collect

$$\begin{aligned}
\mathcal{C}_1 &= \{\psi\}, \\
\mathcal{C}_2 &= \{D, F_\pm, \chi, \psi \wedge \psi\}, \\
\mathcal{C}_3 &= \{\psi \wedge D, \psi \wedge \chi, \psi \wedge F_\pm, \psi \wedge \psi \wedge \psi\}, \\
\mathcal{C}_4 &= \{\psi \wedge \psi \wedge D, \psi \wedge \psi \wedge \chi, \psi \wedge \psi \wedge F_\pm, \psi \wedge \psi \wedge \psi \wedge \psi, \\
&\quad D \wedge D, D \wedge F_+, D \wedge \chi, F_\pm \wedge F_\pm, F_+ \wedge \chi\}.
\end{aligned} \tag{6.31}$$

These are all 1...4-forms that can be constructed out of the field content in Table 2. Since any operator must be gauge invariant, we do not use the 1-form  $A$  to construct operators but only  $F = dA$ . Furthermore, some operators are identically zero due to fermion saturation. The sets  $\mathcal{C}_k$  are then generating sets for the spaces of  $k$ -form observables [83],

$$\mathcal{O}_k = \sum_{j=0}^1 \sum_{X \in \mathcal{C}_k} f_{X,j}(a, \bar{a}) \eta^j X. \tag{6.32}$$

Here,  $f_{X,j}(a, \bar{a})$  are real-analytic functions without singularities away from strong and weak coupling. The most generic 0-form observable is  $\mathcal{O}_0 = f_0(a, \bar{a}) + f_1(a, \bar{a})\eta$ . Let us restrict now to the  $\mathcal{Q}$ -exact  $k$ -observables  $[\mathcal{Q}, \mathcal{O}_k]^f$  that survive integration. These do in particular either contain  $\eta\chi$  or neither, since otherwise they would not survive the fermionic integration, and they do not contain any derivative term  $dX$ , as we consider  $b_2^+(X) = 1$  and thus their zero modes vanish. By the notation  $[\mathcal{Q}, \mathcal{O}]$  we furthermore mean either  $\{\mathcal{Q}, \mathcal{O}\}$  or  $[\mathcal{Q}, \mathcal{O}]$ , depending on whether  $\mathcal{O}$  is Grassmann odd or even.

Recall the action (5.43) of the supersymmetry generator  $\mathcal{Q}$ . It follows that  $[\mathcal{Q}, F_\pm] = (d\psi)_\pm$ . The action of  $\mathcal{Q}$  on functions  $f(a, \bar{a})$  is given by

$$[\mathcal{Q}, f(a, \bar{a})] = \partial_{\bar{a}} f(a, \bar{a}) [\mathcal{Q}, \bar{a}] = \sqrt{2}i \partial_{\bar{a}} f(a, \bar{a}) \eta. \tag{6.33}$$

The general  $\mathcal{Q}$ -exact observable  $[\mathcal{Q}, \mathcal{O}_k]$  from (6.32) is very tedious to compute, luckily in  $[\mathcal{Q}, \mathcal{O}_k]^f$  generally not many terms survive. Furthermore, (6.32) has  $2|\mathcal{C}_k|$  terms, however due to (5.43) we have  $[\mathcal{Q}, \eta\mathcal{O}] = -\eta[\mathcal{Q}, \mathcal{O}]$  and for the terms with  $j = 1$  it remains to multiply the  $j = 0$  term by  $-\eta$ . Then only one of  $[\mathcal{Q}, \mathcal{O}]$  and  $-\eta[\mathcal{Q}, \mathcal{O}]$  is Grassmann even in the variables  $\eta$  and  $\chi$ , such that only one of those can contribute to  $[\mathcal{Q}, \mathcal{O}]^f$ . Lastly, if  $\mathcal{O} = \prod_l \tilde{\mathcal{O}}_l$  is a composite operator, its  $\mathcal{Q}$ -action  $[\mathcal{Q}, \mathcal{O}]$  is an alternating sum  $\sim \sum_l [\mathcal{Q}, \tilde{\mathcal{O}}_l] \prod_{k \neq l} \tilde{\mathcal{O}}_k$ , and so if all such summands do not contribute (which can be easily checked) then the whole commutator does not either. With this, it is now slightly less work to extract those summands  $\mathcal{O}_k^f$  of (6.32) that



contribute,  $[\mathcal{Q}, \mathcal{O}_k]^f = [\mathcal{Q}, \mathcal{O}_k^f]$ . They are

$$\begin{aligned}\mathcal{O}_2^f &= f_1\chi, \\ \mathcal{O}_3^f &= f_2\psi \wedge \chi, \\ \mathcal{O}_4^f &= f_3\psi \wedge \psi \wedge \chi + f_4D \wedge \chi + f_5F_+ \wedge \chi,\end{aligned}\tag{6.34}$$

while  $[\mathcal{Q}, \mathcal{O}_0]^f = [\mathcal{Q}, \mathcal{O}_1]^f = 0$ . Their  $\mathcal{Q}$ -commutators give

$$\begin{aligned}\{\mathcal{Q}, \mathcal{O}_2^f\} &= +\sqrt{2}i\partial_{\bar{a}}f_1\eta\chi + if_1(F_+ - D), \\ [\mathcal{Q}, \mathcal{O}_3^f] &= -\sqrt{2}i\partial_{\bar{a}}f_2\eta\chi \wedge \psi - if_2(F_+ - D) \wedge \psi, \\ \{\mathcal{Q}, \mathcal{O}_4^f\} &= +\sqrt{2}i\partial_{\bar{a}}f_3\eta\chi \wedge \psi \wedge \psi + if_3(F_+ - D) \wedge \psi \wedge \psi \\ &\quad + \sqrt{2}i\partial_{\bar{a}}f_4\eta\chi \wedge D + if_4D \wedge (F_+ - D) \\ &\quad + \sqrt{2}i\partial_{\bar{a}}f_5\eta\chi \wedge F_+ + if_5F_+ \wedge (F_+ - D).\end{aligned}\tag{6.35}$$

These are all  $\mathcal{Q}$ -exact operators in DW theory. The following  $\mathcal{Q}$ -exact terms can then be added to the action

$$I_2 = \int_S \{\mathcal{Q}, \mathcal{O}_2^f\}, \quad I_3 = \int_Y [\mathcal{Q}, \mathcal{O}_3^f], \quad I_4 = \int_X \{\mathcal{Q}, \mathcal{O}_4^f\}.\tag{6.36}$$

### 6.3.2 Solution for $I_Y$

By adding only  $I_S$  as suggested in [81, 83, 86], the  $u$ -plane integrand can be written as a total derivative, however it does not complete to a Siegel-Narain theta function. Let us construct the operator  $I_Y$  such that this becomes true. For simplicity, we ignore the contact terms  $I_\cap$ . This is possible since all contact terms other than the  $Y \cap Y$  are integrated over  $\psi$  and  $\tau$  only and therefore do not affect the path integral calculation. For simplicity and only in this section, we take the intersection  $Y \cap Y$  to be empty.

We therefore aim to find the functions  $f_1, \dots, f_5$ . In the case  $\pi_1(X) = 0$ , the total integrand must go back to (6.45). If  $f_4$  and  $f_5$  are nonzero, this is not the case since they alter the integral.<sup>38</sup> We therefore set  $f_4 = f_5 = 0$ . Thus, in the simply connected case, we have  $I_2 = I_S$ , which implies  $f_1 = -\frac{1}{4\pi} \frac{d\bar{u}}{d\bar{a}}$ . We shall therefore consider adding the correction

$$\begin{aligned}I_Y &= -\sqrt{2}i\partial_{\bar{a}}f_2\eta B(\chi, \psi \wedge Y) - if_2B(F_+ - D, \psi \wedge Y) \\ &\quad + \sqrt{2}i\partial_{\bar{a}}f_3\eta B(\chi, \psi \wedge \psi) + if_3B(F_+ - D, \psi \wedge \psi).\end{aligned}\tag{6.37}$$

to the exponential in (6.45). The terms  $\psi \wedge Y$  and  $\psi \wedge \psi$  are precisely the terms that lead to the problems if only  $I_S$  is added. We can organise  $h := f_3\psi \wedge \psi - f_2\psi \wedge Y$ , such that

$$I_Y = \sqrt{2}i\eta B(\chi, \partial_{\bar{a}}h) + iB(F_+ - D, h).\tag{6.38}$$

<sup>38</sup>This is certainly true if  $f_4$  and  $f_5$  can be varied. It is possible in principle that for specific functions  $f_4$  and  $f_5$  the  $\pi_1(X) = 0$  *integral* does not change.

Inserting it into the path integral we find

$$D = \frac{\sqrt{2}i}{4y} \frac{d\bar{\tau}}{d\bar{a}} \eta \chi - 4\pi(b_+ + \omega_+) + \frac{4\pi i}{y} h_+. \quad (6.39)$$

After integrating out  $D$ , this produces new terms

$$4\pi i B(k_+ + b_+, h) + \frac{\sqrt{2}}{4y} \frac{d\bar{\tau}}{d\bar{a}} \eta B(\chi, h) + \sqrt{2}i \eta B(\chi, \partial_{\bar{a}} h) \quad (6.40)$$

to (6.48) (notice that  $\omega \wedge h = h \wedge h = 0$ ). The first term is only integrated over  $\psi$  and  $\tau$ , so it will not play a role immediately. The second and third term yield after the fermionic integration,

$$\frac{d\bar{\tau}}{d\bar{a}} \left( -\frac{\sqrt{2}}{4y} h - \sqrt{2}i \partial_{\bar{\tau}} h \right). \quad (6.41)$$

In view of (6.50) and the above discussion, we can aim this new contribution to give the missing factor

$$\sqrt{y} \frac{d\bar{\tau}}{d\bar{a}} \partial_{\bar{\tau}} \sqrt{2y} \bar{\omega} = \frac{d\bar{\tau}}{d\bar{a}} \left( \frac{\sqrt{2}i}{4} \bar{\omega} + \sqrt{2}y \partial_{\bar{\tau}} \bar{\omega} \right), \quad (6.42)$$

such that the Siegel-Narain theta function has an elliptic variable  $z = \rho + 2iy\omega$  and  $\beta = b + \omega + \bar{\omega}$ .<sup>39</sup> Motivated by the computation

$$\partial_{\bar{\tau}} y = \frac{i}{2}, \quad \partial_{\bar{\tau}} \sqrt{2y} = \frac{\sqrt{2}i}{4\sqrt{y}}, \quad \partial_{\bar{\tau}} \frac{1}{y} = \frac{1}{2iy^2}, \quad \partial_{\bar{\tau}} b = \frac{b - \partial_{\bar{\tau}} \bar{\rho}}{2iy}, \quad \partial_{\bar{\tau}} \omega = \frac{1}{2iy} \omega, \quad (6.43)$$

we make the ansatz  $h = icy\bar{\omega}$ , with  $c \in \mathbb{C}$  some number. From this it follows that  $y\partial_{\bar{\tau}}\bar{\omega} = -\frac{i}{c}\partial_{\bar{\tau}}h - \frac{i}{2}\bar{\omega}$ . Notice that  $h$  is purely anti-holomorphic, while  $\bar{\omega}$  is not. We find that (6.41) equals (6.42) precisely for  $c = 1$ . From this, it is easy to find

$$f_2 = \frac{3ia_3}{16} \frac{d^2\bar{u}}{d\bar{a}^2}, \quad f_3 = \frac{\sqrt{2}}{2^7\pi} \frac{d\bar{\tau}}{d\bar{a}}. \quad (6.44)$$

In the simply connected case, the correction  $I_2 = I_S$  is necessary in order for the surface observable  $\tilde{I}_-(S)$  to combine into a Siegel-Narain function such that the  $u$ -plane integral is a total derivative. In the case  $\pi_1(X) \neq 0$ , an analogous procedure is required for the 3-cycle  $Y$ , which combines to a 2-form as  $\psi \wedge Y$ . In the  $\pi_1(X) \neq 0$  Lagrangian (6.7) there is a new term  $\psi \wedge \psi$  that is integrated over  $\eta$ ,  $\chi$  and  $D$ , such that the  $u$ -plane integral is a total derivative but does not contain a SN theta function. After the insertion of an anti-holomorphic  $\mathcal{Q}$ -exact 4-form operator, the integrand indeed becomes a Siegel-Narain theta function.

<sup>39</sup>Another possibility would be to chose  $h$  to be holomorphic and cancelling the  $\omega$  inside the derivative. This is possible, however, the  $\omega$  dependence does not drop from the SN theta function.

## 6.4 The $u$ -plane integral for $b_1 > 0$

The  $u$ -plane integral (6.1) can be expressed as

$$Z_u(p, \gamma, S, Y) = \int [dad\bar{a}d\eta d\chi dD] \int_{\text{Pic}(X)} d\psi \nu(\tau) \frac{1}{\sqrt{y}} e^{-\int_X \mathcal{L}' + I_{\mathcal{O}} + I_{\Gamma} + I(S, Y)}, \quad (6.45)$$

where  $\int_{\text{Pic}(X)}$  denotes a sum over isomorphism classes of line bundles, equivalent to a sum over  $H^2(X, \mathbb{Z})$ , followed by an integration over  $\mathbb{T}^{b_1}$ . The  $\psi$  zero modes are tangent to  $\text{Pic}(X)$ , so the integral over these modes is understood as the integral of a differential form on  $\text{Pic}(X)$  [73]. At this point let us make a remark. The  $\mathcal{Q}$ -exact operator  $I(S, Y)$  is not strictly required in order to derive our end result (6.57). As a matter of fact, as shown in [84] this operator can be added freely as  $\alpha I(S, Y)$ , with  $\alpha$  any number. In particular, we can have  $\alpha = 0$ . However, the case of  $\alpha = 1$  makes the analysis simpler and more elegant, why we choose to include it.

Let us perform the integrals above in steps, using an economical notation. We integrate first over the auxiliary field  $D$ , and then over the fermionic 0- and 2-forms,  $\eta$  and  $\chi$ .

### 6.4.1 Integration over $D$ , $\eta$ and $\chi$

Using (6.28) and (6.29), we can expand the terms in the exponential of (6.45) that are affected by the integrals over  $D$ ,  $\eta$  and  $\chi$  as (ignoring the remaining terms for now)

$$\begin{aligned} & - \int_X (\mathcal{L}' + a_2 K^2 u + a_3 K^3 u) + I(S, Y) - \frac{\sqrt{2} a_{33}}{4} \frac{dT}{da} B(F_- + D, Y \wedge Y) \\ & = -\pi i \bar{\tau} k_+^2 - \pi i \tau k_-^2 + \frac{y}{8\pi} D^2 - \frac{\sqrt{2} i}{4} \frac{d\bar{\tau}}{d\bar{a}} B(\eta\chi, k_+) - \frac{\sqrt{2} i}{16\pi} \frac{d\bar{\tau}}{d\bar{a}} B(\eta\chi, D) \\ & - \frac{i}{\sqrt{2}} B(\eta\chi, \frac{d\bar{\rho}}{d\bar{a}}) - 2\pi i B(k_-, \rho) - 2\pi i B(k_+, \bar{\rho}) + y B(D, b_+) + \frac{\sqrt{2} i}{2^5} B(\psi \wedge \psi, \frac{d\rho}{d\bar{a}}) \\ & - \sqrt{2} \eta B(\chi, \partial_{\bar{a}}(y\bar{\omega})) + 4\pi y B(k_-, \omega_-) - 4\pi y B(k_+, \bar{\omega}) + y B(D, \omega_+) + y B(D, \bar{\omega}_+). \end{aligned} \quad (6.46)$$

At any point we discard terms that vanish identically, such as 4-fermion terms or any instance of (6.6) such as  $\psi \wedge \psi \wedge \psi \wedge \psi$ ,  $\psi \wedge \psi \wedge \psi \wedge Y$  or  $\omega \wedge \omega$ . The exponential (6.46) is Gaussian in  $D$  with saddle point

$$D = \frac{\sqrt{2} i}{4y} \frac{d\bar{\tau}}{d\bar{a}} \eta\chi - 4\pi(b_+ + \omega_+ + \bar{\omega}_+). \quad (6.47)$$

This can be found by differentiating (6.46) with respect to  $D$  and setting it to zero. Inserting  $D$  in (6.46) gives <sup>40</sup>

$$\begin{aligned}
& + \frac{\sqrt{2}i}{2^5} B(\psi \wedge \psi, \frac{d\rho}{da}) - 2\pi y(b_+ + \omega_+ + \bar{\omega}_+)^2 - \pi i \tau k_+^2 - \pi i \tau k_-^2 \\
& - 2\pi i B(k_-, \rho) - 2\pi i B(k_+, \bar{\rho}) + 4\pi y B(k_-, \omega) - 4\pi y B(k_+, \bar{\omega}) \\
& - \frac{\sqrt{2}i}{4} \frac{d\bar{\tau}}{d\bar{a}} B(\eta\chi, k_+ - b_+ - \omega_+ - \bar{\omega}_+) - \frac{i}{\sqrt{2}} B(\eta\chi, \frac{d\bar{\rho}_+}{d\bar{a}}) - \sqrt{2}\eta B(\chi, \partial_{\bar{a}}(y\bar{\omega})).
\end{aligned} \tag{6.48}$$

The third line are the only terms involving  $\eta$  and  $\chi$ , which we will integrate over next. Before, we can combine those terms in the expression

$$- \frac{\sqrt{2}i}{4} \frac{d\bar{\tau}}{d\bar{a}} B(\eta\chi, k - b - \omega + \bar{\omega} - 4iy\partial_{\bar{\tau}}\bar{\omega} + 2\partial_{\bar{\tau}}\bar{\rho}). \tag{6.49}$$

Integrating over  $\eta$  and  $\chi$ , we can rewrite this in a compact way as a total anti-holomorphic derivative times an overall factor that, as we discuss below, cancels with contributions from the rest of the measure,

$$\frac{\sqrt{2}i}{4} \frac{d\bar{\tau}}{d\bar{a}} B(k - b - \omega + \bar{\omega} - 4iy\partial_{\bar{\tau}}\bar{\omega} + 2\partial_{\bar{\tau}}\bar{\rho}, \underline{J}) = \sqrt{y} \frac{d\bar{\tau}}{d\bar{a}} \partial_{\bar{\tau}} \sqrt{2y} B(k+b+\omega+\bar{\omega}, \underline{J}), \tag{6.50}$$

where  $\partial_{\bar{\tau}}$  acts on everything to its right and  $\underline{J} = J/\sqrt{Q(J)} \in H_+^2(X)$  is the normalised self-dual harmonic form on  $X$ . This result follows directly from the the identities (6.43).

As previously discussed, the photon path integral together with the measure for the zero modes of  $\psi$  contains a sum over all fluxes times a factor of  $1/\sqrt{y}$ , and additionally contributes  $(-1)^{B(k,K)}$ , where  $K$  is the canonical class of  $X$  [72]. The  $1/\sqrt{y}$  factor is thus absorbed by the  $\sqrt{y}$  on the rhs of (6.50).

Using the change of variables  $u : \Gamma^0(4)\backslash\bar{\mathbb{H}} \xrightarrow{\sim} \mathbb{C}\mathbb{P}^1$  provided by (6.4), we can further integrate over  $d\tau \wedge d\bar{\tau}$  rather than over  $da \wedge d\bar{a}$ . This motivates the definition of the transformed measure

$$\tilde{\nu} = \nu \frac{da}{d\tau}, \tag{6.51}$$

such that  $da \wedge d\bar{a} \nu = d\tau \wedge d\bar{\tau} \frac{d\bar{a}}{d\bar{\tau}} \tilde{\nu}$ . The factor  $\frac{d\bar{a}}{d\bar{\tau}}$  cancels with the  $\frac{d\bar{\tau}}{d\bar{a}}$  of (6.50).

## 6.4.2 Siegel-Narain theta function

Let us demonstrate that the  $u$ -plane integrand for  $\pi_1(X) \neq 0$ , as in the simply-connected case [81], evaluates to a Siegel-Narain theta function. To this end,

<sup>40</sup>If we integrate over  $D$  instead of inserting the equations of motion we get an additional factor of  $2\pi i \sqrt{\frac{2}{y}}$  in front of the integral [83]. The result of the integration does not change otherwise since both methods agree for Gaussian integrals up to an overall factor of  $\sqrt{\frac{\pi}{a}}i$ , if we integrate  $e^{aD^2}$  over  $D$ . It is the same factor as in the simply-connected case because the quadratic  $D$ -term is the same. According to [73, p. 68] the  $D$  determinant should be ignored because it cancels in any case with the fermionic determinants.

let us define

$$\begin{aligned} \Psi_\mu^J(\tau, z) &= e^{-2\pi y \beta_+^2} \sum_{k \in L + \mu} \partial_{\bar{\tau}} \left( \sqrt{2y} B(k + \beta, \underline{J}) \right) \\ &\times (-1)^{B(k, K)} q^{-k_-^2/2} \bar{q}^{k_+^2/2} e^{-2\pi i B(z, k_-) - 2\pi i B(\bar{z}, k_+)} \end{aligned} \quad (6.52)$$

with  $q = e^{2\pi i \tau}$  and  $\beta = \frac{\text{Im}z}{y} \in L \otimes \mathbb{R}$ , where  $L = H^2(X, \mathbb{Z})$ .

For the elliptic variable  $z = \rho + 2iy\omega$ , we have  $\beta = b + \omega + \bar{\omega}$  (here, we use that  $y\omega$  is holomorphic). Both variables appear naturally in (6.48) and (6.50). In fact, we can combine everything to find

$$Z_u(p, \gamma, S, Y) = \int_{\Gamma^0(4) \backslash \mathbb{H}} d\tau \wedge d\bar{\tau} \int_{\mathbb{T}^{b_1}} [d\psi] \tilde{\nu} \Psi_\mu^J(\tau, \rho + 2iy\omega) e^{I'_\mathcal{O} + I'_\mathcal{H}}. \quad (6.53)$$

Here,

$$I'_\mathcal{H} = \int_{S \cap S} T + a_{13} \int_{Y \cap \gamma} T + a_{332} \int_{S \cap Y \cap Y} \frac{\partial^3 \mathcal{F}}{\partial \tau_0^3} + \frac{a_{32}}{4\sqrt{2}} \frac{dT}{da} \int_{Y \cap S} \psi \quad (6.54)$$

and

$$I'_\mathcal{O} = 2pu + \frac{\sqrt{2}a_1}{8} \frac{du}{da} \int_\gamma \psi + \frac{\sqrt{2}i}{2^6\pi} \frac{d^2u}{da^2} \int_S \psi \wedge \psi, \quad (6.55)$$

are the (holomorphic) remainders of the collections of 0, ..., 3-form observables and their contact terms that has not yet been integrated over, and we eliminated all terms that do not contribute.

Let us check that (6.53) is indeed true from the computations in section 6.4.1. Aside from the  $\psi \wedge \psi$  term, the exponential of the first two lines in (6.48) immediately combine into the definition (6.52) with said parameters,  $z = \rho + 2iy\omega$  and  $\bar{z} = \bar{\rho} - 2iy\bar{\omega}$ . Everything not exponentiated is given by the  $\bar{\tau}$  derivative term in (6.50), which precisely gives the derivative term in (6.52). This proves (6.53).

The expression (6.53) generalises the result of the  $u$ -plane integral [84, (4.32)] to four-manifolds  $X$  with  $b_1(X) > 0$  by giving a decomposition of the integrand into a holomorphic and metric-independent measure  $\tilde{\nu} e^{I'_\mathcal{O} + I'_\mathcal{H}}$  and a metric-dependent, non-holomorphic component  $\Psi_\mu^J(\tau, z)$ . Therefore, the evaluation techniques of [84] apply. Namely, we can express the integrand of the  $u$ -plane integral as an anti-holomorphic derivative,

$$\frac{d}{d\bar{\tau}} \widehat{\mathcal{H}}_\mu^J(\tau, \bar{\tau}) = \tilde{\nu} \Psi_\mu^J(\tau, z) e^{I'_\mathcal{O} + I'_\mathcal{H}}. \quad (6.56)$$

The holomorphic exponential  $e^{I'_\mathcal{O} + I'_\mathcal{H}}$  does not affect the anti-holomorphic derivative, and thus the extension to  $\pi_1(X) \neq 0$  is simply through the elliptic argument  $z = \rho + 2iy\omega$ .

Once  $\widehat{\mathcal{H}}_\mu^J(\tau, \bar{\tau})$  is found, we can use coset representatives of  $\text{SL}(2, \mathbb{Z})/\Gamma^0(4)$  to map the six images of  $\mathcal{F} = \text{SL}(2, \mathbb{Z}) \backslash \mathbb{H}$  back to  $\mathcal{F}$  (see Fig. 2). The regularisation and renormalisation of such integrals originating from insertions

of  $\mathcal{Q}$ -exact operators has been rigorously established in [83]. This then allows to evaluate the partition function as

$$Z_u(p, \gamma, S, Y) = 4 \mathcal{I}_\mu(\tau)|_{q^0} + \mathcal{I}_\mu(-\frac{1}{\tau})|_{q^0} + \mathcal{I}_\mu\left(\frac{2\tau-1}{\tau}\right)|_{q^0}, \quad (6.57)$$

where by  $|_{q^0}$  we denote the  $q^0$  coefficient of the resulting Fourier expansion, and the  $\tau$ -integrand of (6.53) is given by <sup>41</sup>

$$\mathcal{I}_\mu(\tau) = \int_{\mathbb{T}^{b_1}} [d\psi] \widehat{\mathcal{H}}_\mu^J(\tau, \bar{\tau}). \quad (6.58)$$

The prefactors in (6.57) can be recognised as the widths of the cusps  $i\infty$ , 0 and 1 of the modular curve  $\Gamma^0(4)\backslash\mathbb{H}$ .

To derive a suitable anti-derivative  $\widehat{\mathcal{H}}_\mu^J(\tau, \bar{\tau})$ , it is auxiliary to choose a convenient period point  $J$ . The  $u$ -plane integral for a different choice  $J'$  is then related to the one for  $J$  by a wall-crossing formula, given explicitly in [76]. It is shown in [84] that for convenient choices of  $J$ ,  $\Psi_\mu^J(\tau, z)$  factors into holomorphic and anti-holomorphic terms, and the anti-derivative  $\widehat{\mathcal{H}}_\mu^J$  can be found for both  $L$  even and odd. Furthermore, the  $u$ -plane integral can be evaluated using mock modular forms for point observables  $p \in H_0(X)$  and Appell-Lerch sums for surface observables  $z \in H_2(X)$  [84].

In [83] it is furthermore shown that in the above mentioned renormalisation, any  $\mathcal{Q}$ -exact operator (such as  $I(S, Y)$ ) decouples in DW theory. However, it is clear that the insertion of  $I(S, Y)$  crucially changes the *integrand*, making the Siegel-Narain theta function symmetric. Instead of inserting  $I(S, Y)$ , we can contemplate adding  $\alpha I(S, Y)$  for an arbitrary constant  $\alpha$ . It was noticed in [84] that the Siegel-Narain theta function  $\Psi_{\mu, \alpha}^J$  for  $b_1 = 0$  with the insertion  $\alpha I_S$  remains finite at weak coupling ( $\text{Im}\tau \rightarrow \infty$ ) if and only if  $\alpha = 1$ . This can be seen from the exponential prefactor in (6.52), whose exponent is negative definite if and only if  $\bar{z}$  (which we suppress in the notation) is the complex conjugate of  $z$ .

### 6.4.3 Single-valuedness of the integrand

An essential requirement, for the consistency of the theory, is that the path integral (6.53) is single-valued. For this it is advantageous to first change variables in the  $\psi$ -integral as

$$\psi' = \psi + \frac{12\pi i a_3}{\sqrt{2}} \frac{da}{d\tau} \frac{d^2 u}{da^2} Y. \quad (6.59)$$

This is because the coefficient function of  $\psi \wedge \psi$  in  $y\omega$  is modular, while the  $\psi \wedge Y$  and  $Y \wedge Y$  coefficients of  $y\omega$  are only quasi-modular. Such shifts (6.59)

<sup>41</sup>One could also contemplate switching the order of integration, and integrate over  $\psi$  first. This would however not necessarily result in a similar function to (6.57), and it might not be possible to use the results of [84].

leave the measure of  $\int[d\psi]$  invariant, as  $d\psi = d\psi'$ . Due to the order of integration in (6.53), the change of variables (6.59) is well-defined. Since  $Y$  is also Grassmann-odd,  $\psi$  and  $Y$   $\wedge$ -commute. This gives

$$\omega = \frac{\sqrt{2}i}{2^7\pi y} \frac{d\tau}{da} \psi' \wedge \psi' + \frac{9\sqrt{2}\pi^2 a_3^2}{16y} \frac{du}{da} Y \wedge Y. \quad (6.60)$$

Let us use the notation of section 6.2.1. It is argued in [76] that  $\psi'$  transforms as  $(-1, 1)^{(1,0)}$ . Using (6.5), one then finds that  $y\omega = (-1, 1)^{(-1,0)}$  transforms precisely as  $\rho = (-1, 1)^{(-1,0)}$ , such that  $z = \rho + 2iy\omega = (-1, 1)^{(-1,0)}$  is a modular form and transforms exactly as in the  $\pi_1(X) = 0$  case.

Furthermore, it is auxiliary to define [76, (2.14)]

$$S' = S + 4\pi iy \frac{da}{du} \omega. \quad (6.61)$$

It is well-defined, as  $S' = (1, 1)^{(0,0)}$  is fully invariant. In contrast to (6.59), this is not a change of variables or a redefinition, but rather a substitution to simplify some expressions. For instance, the elliptic variable now reads

$$z = \frac{S' du}{2\pi da}, \quad (6.62)$$

which takes the same form (6.29) as in the simply-connected case.

By incorporating the shift of  $\psi \rightarrow \psi'$  together with (6.61) we find that the contact terms and observables in (6.54) and (6.55) can be written as

$$\begin{aligned} I_{\mathcal{O}+\cap} = & 2pu + S'^2 T + \frac{\sqrt{2}a_1}{8} \frac{du}{da} \int_{\gamma} \psi' - 3\pi^2 a_1 a_3 u \int_{\gamma} Y + \frac{\sqrt{2}}{32} \frac{d\tau}{du} u \int_{S'} \psi' \wedge \psi' \\ & - \frac{3\pi i}{8} a_3 \frac{du}{da} \int_Y S' \wedge \psi' + \frac{3\sqrt{2}}{4} i\pi^3 a_3^2 u \int_{S'} Y \wedge Y. \end{aligned} \quad (6.63)$$

All terms but  $S'^2 T$  are modular functions with trivial multipliers. Due to (6.62), the quasi-modular shift of  $T$  combines precisely with the one of  $\Psi(\tau, z)$ .

### Measure factor

Since  $\Delta \propto \frac{\vartheta_4^8}{\vartheta_2^4 \vartheta_3^4}$ ,  $\frac{da}{d\tau} = \frac{\pi}{8i} \frac{\vartheta_4^8}{\vartheta_2 \vartheta_3}$  and  $\frac{da}{du} = \frac{1}{2} \vartheta_2 \vartheta_3$ , from (6.8) we have that  $\nu \propto \frac{\vartheta_4^\sigma}{(\vartheta_2 \vartheta_3)^{2-b_1}}$  and therefore

$$\tilde{\nu} \propto \frac{\vartheta_4^{8+\sigma}}{(\vartheta_2 \vartheta_3)^{3-b_1}}. \quad (6.64)$$

We find that under the generators of  $\Gamma^0(4)$ ,  $\tilde{\nu} = (-1, e^{-\pi i \sigma/4})^{(2-\frac{b_2}{2}+b_1, 0)}$ . For this we have used that  $\sigma + b_2 = 2$  and that  $b_1$  is even.

We also need to consider the fermion measure. As we have discussed earlier, this comes with an overall factor of  $y^{-\frac{b_1}{2}}$  which gets absorbed by a similar factor coming from the photon partition function. This leaves us with  $\prod_{i=1}^{b_1} dc_i$ , which has weight  $(-b_1, 0)$ , since  $\psi$  has weight  $(1, 0)$  [72]. So after the integration over

$D$ ,  $\eta$  and  $\chi$ , and after changing integration variables from  $da \wedge d\bar{a}$  to  $d\tau \wedge d\bar{\tau}$  the measure of the integral will have weight  $(-2 - b_1, -2)$ , and we thus need the rest of the integrand to have weight  $(2 + b_1, 2)$ . Finally, the transformations of the Siegel-Narain theta function  $\Psi_\mu^J(\tau, z)$  can be found in Appendix A.7.

The integrand of the  $u$ -plane integral (6.53) reads

$$\mathcal{J}_\mu^J = d\tau \wedge d\bar{\tau} \int_{\mathbb{T}^{b_1}} [d\psi] \tilde{\nu} \Psi_\mu^J(\tau, z) e^{I'_\sigma + I'_\eta}. \quad (6.65)$$

Since it is integrated over the fundamental domain of  $\Gamma^0(4)$ , in order to check whether the integral is well-defined  $\mathcal{J}_\mu^J$  must transform as a modular function for  $\Gamma^0(4)$  with no phases. In Table 3 we collect the phases and weights of the individual factors as discussed above. This shows that the integral is indeed well-defined.

object	$d\tau \wedge d\bar{\tau}$	$\int_{\mathbb{T}^{b_1}} [d\psi]$	$\tilde{\nu}$	$\Psi_\mu^J(\tau, z)$	$e^{I'_\sigma + I'_\eta}$	$\mathcal{J}_\mu^J$
weight	$(-2, -2)$	$(-b_1, 0)$	$(2 - \frac{b_2}{2} + b_1, 0)$	$(\frac{b_2}{2}, 2)$	$(0, 0)$	$(0, 0)$
$T^4$	1	1	-1	-1	1	1
$S^{-1}T^{-1}S$	1	1	$e^{-\frac{\pi i \sigma}{4}}$	$e^{\frac{\pi i \sigma}{4}} e^{-\frac{\pi i z^2}{\tau+1}}$	$e^{\frac{\pi i z^2}{\tau+1}}$	1

Table 3: Modular weights and phases of the  $u$ -plane integrand (6.65) under  $\Gamma^0(4)$  transformations. This proves that  $\mathcal{J}_\mu^J(\gamma\tau) = \mathcal{J}_\mu^J(\tau)$  for any  $\gamma \in \Gamma^0(4)$ .



## 7 Discussion and conclusion

In this thesis, we have studied in detail the modularity of a class of  $\mathcal{N} = 2$  supersymmetric field theories. Being a manifestation of duality, modularity gives a precise description of duality and thus provides nontrivial insight into the dynamics. For the asymptotically free  $SU(2)$  theories with fundamental matter, we have found that the modularity is mildly obstructed by branch points, which give rise to the superconformal Argyres-Douglas points. In the superconformal  $N_f = 4$  theory, for specific masses the order parameters are enhanced to bimodular forms, as they are modular in two parameters. For more general mass configurations however, branch points will also appear.

When the gauge group is enlarged to  $SU(3)$ , we rather found that there exist elliptic loci which allow a modular parametrisation. With the inclusion of fundamental matter, such elliptic loci become more complicated, but in certain examples can be found explicitly.

These results vastly generalise earlier work by Seiberg and Witten [45, 46], Nahm [101], Malmendier and Ono [77], Klemm, Lerche, Theisen [96] and others on the modularity of the Coulomb branch for  $\mathcal{N} = 2$  supersymmetric Yang-Mills theory and  $\mathcal{N} = 2$  supersymmetric QCD. They open up several new directions for further research, as will be discussed in the following.

The fundamental domains obtained in this thesis are used as integration domains for topological correlation functions, when the theory is formulated on a four-manifold. We found that by coupling the theory to background fluxes, the constraint on the gauge bundle can be lifted. This allows to study a larger class of topological correlation functions. We investigated the definition, analysis and evaluation of such  $u$ -plane integrals for  $SU(2)$   $\mathcal{N} = 2$  supersymmetric QCD with  $N_f \leq 4$  arbitrary masses on a compact four-manifold. The change of variables from the  $u$ -plane to fundamental domains for the effective coupling allows to express the integrands as generalisations of mock modular forms.

Another application of the fundamental domains is the study of BPS spectra. Many four-dimensional  $\mathcal{N} = 2$  theories have the property that their spectrum of BPS states can be encoded in a quiver, the *BPS quiver* [296, 297]. Recently, it has been proposed that BPS quivers can be obtained from the fundamental domain of a particular 4d  $\mathcal{N} = 2$  theory through a simple map [1, 129, 298]. This is due to the fact that fundamental domains encode the monodromy matrices, whose eigenvectors are the charge vectors of BPS states. It would be interesting to expand these ideas to non-modular configurations.

As mentioned in section 4.7, it would be interesting to extend our work on  $SU(3)$  to higher rank gauge groups, such as  $SU(N)$ . While the elliptic loci are more subtle in this case, an approach similar to the one for the rank one theories could be worth exploring. In rank one, we match the  $\mathcal{J}$ -invariant of the elliptic curve with the modular  $j$ -invariant. When  $\mathcal{J}$  is a rational function of the order parameter, then this relation can be equivalently expressed as a polynomial equation (such as in Sections 2 and 3). For rank two, genus

two curves are isomorphic if and only if their absolute invariants  $x_1$ ,  $x_2$  and  $x_3$  (as defined in (4.97)) coincide. The absolute invariants themselves are rational functions of Siegel modular forms of weight 0, which are meromorphic on the genus two Siegel upper-half plane  $\mathbb{H}_2 \ni \Omega$ . Thus from  $x_i = x_i(\Omega)$ ,  $i = 1, 2, 3$  we find three equations for the two variables  $u$  and  $v$ . For pure  $SU(3)$  for example, these equations are necessarily equivalent to the system (4.36). By multiplying with the denominators of  $x_i(\Omega)$ , we obtain three two-variable polynomial equations over the field of meromorphic Siegel modular forms of weight 0. Such a reformulation however likely runs into the same problem as the map to Rosenhain form, essentially since it does not circumvent the Schottky problem [299, 300]. It would be interesting to find a general solution to this problem.

It would furthermore be very interesting and challenging to extend our techniques and results to other  $\mathcal{N} = 2$  supersymmetric theories. Many  $\mathcal{N} = 2$  field theories in four dimensions can be obtained from geometric engineering [20, 21] in Type IIA string theory on local Calabi-Yau threefold singularities [22, 23]. They are also closely related to five-dimensional theories compactified on a circle, as a consequence of the Type-IIA/M-theory duality [301, 302]. Arguably the simplest such five-dimensional theories are the rank one theories with  $E_n$  flavour symmetry [129, 166, 298, 303]. Their circle compactification gives a class of four-dimensional  $\mathcal{N} = 2$  supersymmetric theories, which are related by mass deformations, starting from  $E_8$  to  $E_n$  with  $n < 8$ . The  $E_8$  theory itself can be obtained from the  $E$ -string theory [304].

Another motivation to study  $\mathcal{N} = 2$  theories in four dimensions is to explore indirectly the non-trivial six-dimensional  $\mathcal{N} = (2, 0)$  superconformal theories [305–307]. It can be realised as the worldvolume theory of a stack of  $N$  parallel M5-branes [308]. By a twisted compactification of the 6d superconformal theory on a Riemann surface, the so-called class  $\mathcal{S}$  of 4d  $\mathcal{N} = 2$  supersymmetric theories are obtained [20, 21, 112, 113]. The class  $\mathcal{S}$  theories include the rank 1 SCFTs [155, 309, 310], general non-Lagrangian theories [64, 82, 109] and the above mentioned circle compactifications of 5d SCFTs. This six-dimensional construction gives a geometric perspective on the S-duality of Seiberg-Witten theory. It furthermore gives rise to the AGT correspondence, which is a conjectured equality of many class  $\mathcal{S}$  observables with 2d CFT correlation functions [172, 311, 312].

It would be interesting to generalise our results on the topologically twisted theories and study the Coulomb branch integrals of other  $\mathcal{N} = 2$  theories, such as those of class  $\mathcal{S}$ . While the topological twist of Donaldson-Witten theory can be implemented in any such theory, the formulation and evaluation of correlation functions as integrals over fundamental domains has been achieved to date only for a selection of  $\mathcal{N} = 2$  theories and four-manifolds [73, 74, 77–80, 84, 85]. It would further be fruitful to study the relations to 2d  $\mathcal{N} = (0, 2)$  theories [63], vertex operator algebras [65], topological modular forms [62], and investigate the possibility to establish new four-manifold invariants [63, 85].

In view of the arguments given in section 1.1 on the complexity for the dynamics of supersymmetric quantum field theory with decreasing  $\mathcal{N}$ , it would be interesting to explore whether our results can be used to learn about the dynamics of theories with less supersymmetry, for instance  $\mathcal{N} = 1$ . Such supersymmetry breaking has already been studied in the 1990s [45,149,150,251,313,314], with revived interest due to relations to integrable systems [121,315] as well as line defects and 't Hooft anomalies [276,316].

# A Automorphic forms and elliptic curves

In this Appendix, we collect some properties of modular and automorphic forms, as well as elliptic curves. For further reading see for example [59, 145, 154, 217, 224, 225, 287, 317–321].

## A.1 Elliptic modular forms

We make use of modular forms for the congruence subgroups  $\Gamma_0(n)$  and  $\Gamma^0(n)$  of  $\mathrm{SL}(2, \mathbb{Z})$ . They are defined as

$$\begin{aligned}\Gamma_0(n) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \mid c \equiv 0 \pmod{n} \right\}, \\ \Gamma^0(n) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \mid b \equiv 0 \pmod{n} \right\},\end{aligned}\tag{A.1}$$

and are related by conjugation with the matrix  $\mathrm{diag}(n, 1)$ . We furthermore define the *principal congruence subgroup*  $\Gamma(n)$  as the subgroup of  $\mathrm{SL}(2, \mathbb{Z}) \ni A$  with  $A \equiv \mathbb{1} \pmod{n}$ . A subgroup  $\Gamma$  of  $\mathrm{SL}(2, \mathbb{Z})$  is called a congruence subgroup if there exists an integer  $n \in \mathbb{N}$  such that it contains  $\Gamma(n)$ . The smallest such  $n$  is then called the *level* of  $\Gamma$ .

We furthermore make use of the theta group [322]

$$\Gamma_\theta := \langle T^2, S \rangle \subseteq \mathrm{SL}(2, \mathbb{Z}).\tag{A.2}$$

It is a congruence subgroup of  $\mathrm{SL}(2, \mathbb{Z})$ , as [321, 323]<sup>42</sup>

$$\Gamma_\theta = \{A \in \mathrm{SL}(2, \mathbb{Z}) \mid A \equiv \mathbb{1} \text{ or } S \pmod{2}\}.\tag{A.3}$$

### Eisenstein series

We let  $\tau \in \mathbb{H}$  and define  $q = e^{2\pi i\tau}$ . Then the Eisenstein series  $E_k : \mathbb{H} \rightarrow \mathbb{C}$  for even  $k \geq 2$  are defined as the  $q$ -series

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,\tag{A.4}$$

with  $\sigma_k(n) = \sum_{d|n} d^k$  the divisor sum. For  $k \geq 4$  even,  $E_k$  is a modular form of weight  $k$  for  $\mathrm{SL}(2, \mathbb{Z})$ . On the other hand  $E_2$  is a quasi-modular form, which means that the  $\mathrm{SL}(2, \mathbb{Z})$  transformation of  $E_2$  includes a shift in addition to the weight,

$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) - \frac{6i}{\pi} c(c\tau + d).\tag{A.5}$$

From the  $S$ -transformation, we find that

$$E_4(e^{\pi i/3}) = 0, \quad E_6(i) = 0,\tag{A.6}$$

<sup>42</sup>It can also be written as the group of matrices  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with  $a + b + c + d \equiv 0 \pmod{2}$ , or  $ab \equiv cd \equiv 0 \pmod{2}$ .

and the zeros are unique in  $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H}$  according to the valence formula for modular forms on  $\mathrm{SL}(2, \mathbb{Z})$ . Any modular form for  $\mathrm{SL}(2, \mathbb{Z})$  can be related to the Jacobi theta functions (1.14) by

$$E_4 = \frac{1}{2}(\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8), \quad E_6 = \frac{1}{2}(\vartheta_2^4 + \vartheta_3^4)(\vartheta_3^4 + \vartheta_4^4)(\vartheta_4^4 - \vartheta_2^4). \quad (\text{A.7})$$

All quasi-modular forms for  $\mathrm{SL}(2, \mathbb{Z})$  can be expressed as polynomials in  $E_2$ ,  $E_4$  and  $E_6$ . The derivatives of the Eisenstein series are quasi-modular,

$$E_2' = \frac{2\pi i}{12}(E_2^2 - E_4), \quad E_4' = \frac{2\pi i}{3}(E_2 E_4 - E_6), \quad E_6' = \frac{2\pi i}{2}(E_2 E_6 - E_4^2). \quad (\text{A.8})$$

These equations give the differential ring structure of quasi-modular forms on  $\mathrm{PSL}(2, \mathbb{Z})$ . With our normalisation (1.7) the  $j$ -invariant can be written as

$$j = 1728 \frac{E_4^3}{E_4^3 - E_6^2} = 256 \frac{(\vartheta_3^8 - \vartheta_3^4 \vartheta_4^4 + \vartheta_4^8)^3}{\vartheta_2^8 \vartheta_3^8 \vartheta_4^8}. \quad (\text{A.9})$$

### Theta functions

The Jacobi theta functions  $\vartheta_j : \mathbb{H} \rightarrow \mathbb{C}$ ,  $j = 2, 3, 4$ , are defined as

$$\vartheta_2(\tau) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} q^{r^2/2}, \quad \vartheta_3(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2/2}, \quad \vartheta_4(\tau) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2/2}, \quad (\text{A.10})$$

with  $q = e^{2\pi i \tau}$ . These functions transform under  $T, S \in \mathrm{SL}(2, \mathbb{Z})$  as

$$\begin{aligned} S : \quad \vartheta_2(-1/\tau) &= \sqrt{-i\tau} \vartheta_4(\tau), & \vartheta_3(-1/\tau) &= \sqrt{-i\tau} \vartheta_3(\tau), & \vartheta_4(-1/\tau) &= \sqrt{-i\tau} \vartheta_2(\tau) \\ T : \quad \vartheta_2(\tau + 1) &= e^{\frac{\pi i}{4}} \vartheta_2(\tau), & \vartheta_3(\tau + 1) &= \vartheta_4(\tau), & \vartheta_4(\tau + 1) &= \vartheta_3(\tau). \end{aligned} \quad (\text{A.11})$$

They furthermore satisfy the Jacobi abstruse identity

$$\vartheta_2^4 + \vartheta_4^4 = \vartheta_3^4. \quad (\text{A.12})$$

Derivatives of modular functions are described by Ramanujan's differential operator. It increases the holomorphic weight by 2 and it can be explicitly constructed using the theory of Hecke operators. For the derivatives of the Jacobi theta functions, one finds

$$\begin{aligned} D\vartheta_2^4 &= \frac{1}{6}\vartheta_2^4 (E_2 + \vartheta_3^4 + \vartheta_4^4), \\ D\vartheta_3^4 &= \frac{1}{6}\vartheta_3^4 (E_2 + \vartheta_2^4 - \vartheta_4^4), \\ D\vartheta_4^4 &= \frac{1}{6}\vartheta_4^4 (E_2 - \vartheta_2^4 - \vartheta_3^4), \end{aligned} \quad (\text{A.13})$$

where  $D := \frac{1}{2\pi i} \frac{d}{d\tau} = q \frac{d}{dq}$  and  $E_2$  is the quasi-modular Eisenstein series (1.7) of weight 2, transforming as (A.5).

The modular lambda function is defined as  $\lambda = \frac{\vartheta_2^4}{\vartheta_3^4}$ , and is a Hauptmodul for  $\Gamma(2)$ .

Another class of theta series is provided by the one of the  $A_2$  root lattice,  $b_{3,j} : \mathbb{H} \rightarrow \mathbb{C}$ ,

$$b_{3,j}(\tau) = \sum_{k_1, k_2 \in \mathbb{Z} + \frac{i}{3}} q^{k_1^2 + k_2^2 + k_1 k_2}, \quad j \in \{-1, 0, 1\}. \quad (\text{A.14})$$

It is clear that  $b_{3,-1} = b_{3,1}$ . The transformation properties under  $\text{SL}(2, \mathbb{Z})$  are

$$\begin{aligned} S : \quad b_{3,j} \left( -\frac{1}{\tau} \right) &= -\frac{i\tau}{\sqrt{3}} \sum_{l \pmod{3}} \omega_3^{2jl} b_{3,l}(\tau), \\ T : \quad b_{3,j}(\tau + 1) &= \omega_3^{j^2} b_{3,j}(\tau), \end{aligned} \quad (\text{A.15})$$

with  $\omega_3 = e^{2\pi i/3}$ . The  $b_{3,j}$  series can be expressed through the Dedekind eta function (A.19) as

$$b_{3,0}(\tau) = \frac{\eta(\frac{\tau}{3})^3 + 3\eta(3\tau)^3}{\eta(\tau)}, \quad b_{3,1}(\tau) = 3 \frac{\eta(3\tau)^3}{\eta(\tau)}. \quad (\text{A.16})$$

A relation to the Jacobi theta functions is given by

$$b_{3,0}(\tau) = \vartheta_3(2\tau)\vartheta_3(6\tau) + \vartheta_2(2\tau)\vartheta_2(6\tau). \quad (\text{A.17})$$

### Dedekind eta function

The Dedekind eta function  $\eta : \mathbb{H} \rightarrow \mathbb{C}$  is defined as the infinite product

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi i\tau}. \quad (\text{A.18})$$

It transforms under the generators of  $\text{SL}(2, \mathbb{Z})$  as

$$\begin{aligned} S : \quad \eta(-1/\tau) &= \sqrt{-i\tau} \eta(\tau), \\ T : \quad \eta(\tau + 1) &= e^{\frac{\pi i}{12}} \eta(\tau), \end{aligned} \quad (\text{A.19})$$

and relates to the Jacobi theta series as  $\eta^3 = \frac{1}{2}\vartheta_2\vartheta_3\vartheta_4$ . The derivative of  $\eta$  is given by  $\eta' = \frac{\pi i}{12}\eta E_2$ .

Quotients of  $\eta$ -functions are frequently used to generate bases for the spaces of modular functions for congruence subgroups of  $\text{SL}(2, \mathbb{Z})$ . We use the following

**Theorem 1** ([224, 225]). *Let  $f(\tau) = \prod_{\delta|N} \eta(\delta\tau)^{r_\delta}$  be an  $\eta$ -quotient with  $k = \frac{1}{2} \sum_{\delta|N} r_\delta \in \mathbb{Z}$  and  $\sum_{\delta|N} \delta r_\delta \equiv \sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}$ . Then,  $f$  is a weakly holomorphic modular form for  $\Gamma_0(N)$  with weight  $k$ .*

### Atkin-Lehner involutions

The modular groups of  $n|h$ -type are defined in the following way [58]. Consider matrices of the form

$$\begin{pmatrix} ae & b/h \\ cn & de \end{pmatrix} \quad (\text{A.20})$$

with determinant  $e$ , where  $a, b, c, d, e, h, n \in \mathbb{Z}$ , and  $h$  is the largest integer for which  $h^2|N$  and  $h|24$  with  $n = N/h$ . These matrices are also referred to as *Atkin-Lehner involutions*.

In the case that  $n$  is a positive integer and  $h|n$ , we define  $\Gamma_0(n|h)$  as the set of above matrices with  $e = 1$ . For any positive integer  $e$  which satisfies  $e|n/h$  and  $(e, n/eh) = 1$  ( $e$  is called an *exact divisor* of  $n/h$ ), one can include also matrices of the above form with  $e > 1$ , forming a group denoted by  $\Gamma_0(n|h) + e$ . In fact, this construction works for any choice  $\{e_1, e_2, \dots\}$  of exact divisors of  $n/h$ , resulting in the group  $\Gamma_0(n|h) + e_1, e_2, \dots$ . If  $h = 1$ , the  $|h$  is omitted in the notation, and in case that all the possible  $e_i$  are included, the group is simply denoted by  $\Gamma_0(n|h) +$ .

In the  $\Gamma^0$  convention the notation simplifies, since  $\Gamma^0(n|h) = \Gamma^0(\frac{n}{h})$ . This can be checked by conjugating (A.20) with  $\text{diag}(n, 1)$ . The extension by non-unity determinant matrices follows by analogy.

## Fundamental domains

A key concept of the theory of modular forms is the *fundamental domain*. A fundamental domain for a group  $\Gamma \subset \text{SL}(2, \mathbb{R})$  is an open subset  $\mathcal{F} \subset \mathbb{H}$  with the property that no two distinct points of  $\mathcal{F}$  are equivalent under the action of  $\Gamma$  and every point in  $\mathbb{H}$  is mapped to some point in the closure of  $\mathcal{F}$  by the action of an element in  $\Gamma$ . The quotient  $\Gamma \backslash \mathbb{H}$  can be compactified by adding finitely many points called *cusps*. Cusps are  $\Gamma$ -equivalence classes of  $\mathbb{Q} \cup \{i\infty\}$ . Special points in the fundamental domain are the *elliptic fixed points*, which are points in  $\mathbb{H}$  that have a non-trivial  $\Gamma$ -stabiliser. There, the quotient  $\Gamma \backslash \mathbb{H}$  becomes singular. Elliptic points can always be mapped to the boundary of the fundamental domain. They furthermore contribute non-trivially to the order of vanishing, which determines the dimension of the spaces of modular forms for fixed weight.

## A.2 Siegel modular forms

Ordinary modular forms are constructed by the action of an  $\text{SL}(2, \mathbb{Z})$  Möbius transformation on the upper half-plane  $\mathbb{H}$ . Siegel modular forms [287, 319, 324] generalise this notion by introducing an action of  $\text{Sp}(2g, \mathbb{Z})$  on the so-called Siegel upper half-plane  $\mathbb{H}_g$ , which works for any *genus*  $g \in \mathbb{N}$ .

Define the Siegel modular group of genus  $g$  as

$$\text{Sp}(2g, \mathbb{Z}) = \{M \in \text{Mat}(2g; \mathbb{Z}) \mid M^T J M = J\} \quad \text{with } J = \begin{pmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{pmatrix}. \quad (\text{A.21})$$

The group  $\text{Sp}(4, \mathbb{Z})$  can be generated [287] by the elements  $J$  and  $T = \begin{pmatrix} \mathbb{1}_g & s \\ 0 & \mathbb{1}_g \end{pmatrix}$  with  $s = s^T$ . The Siegel upper half-plane

$$\mathbb{H}_g = \{\Omega \in \text{Mat}(g; \mathbb{C}) \mid \Omega^T = \Omega, \text{Im}\Omega > 0\} \quad (\text{A.22})$$

consists of complex symmetric  $g \times g$  matrices whose (componentwise) imaginary part is positive definite. This generalises the ordinary upper half-plane  $\mathbb{H} = \mathbb{H}_1$ . For example, for  $g = 2$  this means that

$$\Omega = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}, \quad \text{Im}\tau_{11} > 0, \quad \text{Im}\tau_{11}\text{Im}\tau_{22} - (\text{Im}\tau_{12})^2 > 0. \quad (\text{A.23})$$

An element  $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z})$  acts on the Siegel upper half-plane by

$$\Omega \mapsto \gamma(\Omega) = (A\Omega + B)(C\Omega + D)^{-1}. \quad (\text{A.24})$$

A (classical) Siegel modular form of weight  $k$  and genus  $g$  is then a holomorphic function  $f : \mathbb{H}_g \rightarrow \mathbb{C}$  satisfying

$$f(\gamma(\Omega)) = \det(C\Omega + D)^k f(\Omega) \quad \forall \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z}), \quad (\text{A.25})$$

where for  $g = 1$  holomorphicity at  $i\infty$  is required in addition.

Theta series provide an explicit class of classical Siegel modular forms. For  $a, b \in \mathbb{Q}^2$  and  $\Omega \in \mathbb{H}_2$ , define

$$\Theta \begin{bmatrix} a \\ b \end{bmatrix} (\Omega) = \sum_{k \in \mathbb{Z}^2} \exp(\pi i(k+a)^\top \Omega (k+a) + 2\pi i(k+a)^\top b). \quad (\text{A.26})$$

We are especially interested in the case where the entries of these column vectors take values in the set  $\{0, \frac{1}{2}\}$ . The corresponding theta functions are usually referred to as the theta characteristics. We call  $\gamma = \begin{bmatrix} a \\ b \end{bmatrix}$  an even (odd) characteristic if  $4a^\top b$  is even (odd). In the case of genus two there are ten even theta constants [318],

$$\begin{aligned} \Theta_1 &= \Theta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Theta_2 = \Theta \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \Theta_3 = \Theta \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}, \quad \Theta_4 = \Theta \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad \Theta_5 = \Theta \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}, \\ \Theta_6 &= \Theta \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad \Theta_7 = \Theta \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}, \quad \Theta_8 = \Theta \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}, \quad \Theta_9 = \Theta \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \quad \Theta_{10} = \Theta \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}. \end{aligned} \quad (\text{A.27})$$

All even theta constants can be related through algebraic identities to four *fundamental* ones,  $\Theta_1, \Theta_2, \Theta_3, \Theta_4$  [318].

The above theta functions are weight  $\frac{1}{2}$  Siegel modular forms for a subgroup of  $\text{Sp}(4, \mathbb{Z})$ . Their transformation properties under the Siegel modular group can be found in [319].

### A.3 Fundamental domains

Here we present an algorithm to draw fundamental domains for discrete groups  $G \subset \text{SL}(2, \mathbb{R})$  acting on  $\mathbb{H}$  by  $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ , based on [58]. Let us assume that  $G$  contains  $T^s$  for some  $s \in \mathbb{Q}$ , and define  $\mathcal{O}(T^s) = \{\tau \in \mathbb{H} \mid \text{Re}\tau < \frac{s}{2}\}$ . For all other elements  $X$  where  $c \neq 0$ , define

$$\mathcal{O}(X) = \mathcal{O}\left(\frac{a}{c}, \frac{\sqrt{\det X}}{c}\right) \quad (\text{A.28})$$



where  $\mathcal{O}(c, r) = \{\tau \in \mathbb{H} \mid |\tau - c| > r\}$  is the exterior of a half-circle in  $\mathbb{H}$  around  $c$  with radius  $r > 0$ . A fundamental domain for  $G$  is then given by

$$\mathcal{R} = \bigcap_{X \in G \setminus 1} \mathcal{O}(X). \quad (\text{A.29})$$

If a set of generators of  $G$  is known, (A.29) is an intersection of finitely many regions: for products of generators the circles becomes smaller. In practise it is enough to run (A.29) over the generators and their inverses. We denote by

$$\mathcal{C}_r(c) = \partial\mathcal{O}(c, r) = \{\tau \in \mathbb{H} \mid |\tau - c| = r\} \quad (\text{A.30})$$

the half-circles around  $c$  with radius  $r$ . For any  $X \in G$  they are given analogously to (A.28). Then by (A.29), the boundary of  $\mathcal{R}$  which does not have constant real part (generated by  $\partial\mathcal{O}(T^s)$ ) are piecewise arcs of circles  $\mathcal{C}_r(c)$ .

A simple Mathematica code to draw fundamental domains from a set of generators is the following:

```

1  g1 = MatrixPower[T, 4]; g2 = S.T.S; g3 = g1; g4 = g2;
2  w = 4; (*width at infinity*)
3  left = -1/2; (*left boundary*)
4  p = 2; (*depth of algorithm*)
5  prod = 4; (*number of generators*)
6  T = {{1,1}, {0,1}}; S = {{0,-1}, {1,0}};
7  Flatten[Table[MatrixPower[g1, k1].MatrixPower[g2, k2].MatrixPower[g3, k3].
8      MatrixPower[g4, k4], {k1,-p,p}, {k2,-p,p}, {k3,-p,p}, {k4,-p,p}], prod-1];
9  list = DeleteCases[%, x_ /; x[[2, 1]] == 0];
10 t = DeleteDuplicates[Flatten[{list, Inverse /@ list}, 1]];
11 circ[x_?MatrixQ] := {x[[1, 1]]/x[[2, 1]], 1/Abs[x[[2, 1]]]};
12 CR = Thread[circ[t]];
13 lines = ContourPlot[Product[(x - (left+k)), {k, 0, w}] == 0, {x, left-0.01,
14     left+w+0.01}, {y, 0, w}, ContourStyle -> {Orange}, ImageSize -> Large];
15 Table[pl[k]=ContourPlot[(x-CR[[k,1]])^2+y^2==CR[[k,2]]^2,{x,left,left+w},{y,0,w},
16     Axes->True, ImageSize->Large, AspectRatio->Automatic], {k,1, Length[CR]};
17 Show[Table[pl[k], {k, 1, Length[CR]}], lines]

```

For standard subgroups, fundamental domains can be drawn using Verrill's Java applet [325] or the FareySymbol class in Sage [326].

## Modular curves

A subgroup  $\Gamma$  of  $\text{SL}(2, \mathbb{Z})$  is a congruence subgroup if  $\Gamma \supset \Gamma(N)$  for some  $N \in \mathbb{N}$ , which is called the *level* of  $\Gamma$ . The (projective) *index* of a congruence subgroup  $\Gamma$  is defined as

$$\text{ind } \Gamma = [\text{PSL}(2, \mathbb{Z}) : \Gamma], \quad (\text{A.31})$$

and it is finite for all  $N$ . By  $\text{SL}(2, \mathbb{Z})$  we strictly mean  $\text{PSL}(2, \mathbb{Z})$  in the following. In fact, one can prove [145]

$$\text{ind } \Gamma(N) = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right), \quad \text{ind } \Gamma^0(N) = N \prod_{p|N} \left(1 + \frac{1}{p}\right), \quad (\text{A.32})$$

where the sum is over all prime divisors of  $N$ . It can also be computed in the following way. The volume of the curve  $\Gamma \backslash \mathbb{H}$  is defined as

$$\text{vol}(\Gamma \backslash \mathbb{H}) = \int_{\Gamma \backslash \mathbb{H}} d\mu, \quad (\text{A.33})$$

where  $d\mu = y^{-2} dx dy$  is the hyperbolic metric on  $\mathbb{H}$ , with  $\tau = x + \text{Im}y$ . Since  $\text{vol}(\text{SL}(2, \mathbb{Z}) \backslash \mathbb{H}) = \frac{\pi}{3}$  can easily be computed, the index of any  $\Gamma \subseteq \text{SL}(2, \mathbb{Z})$  is then given by

$$\text{ind } \Gamma = \frac{3}{\pi} \text{vol}(\Gamma \backslash \mathbb{H}). \quad (\text{A.34})$$

Let  $\Gamma$  be a congruence subgroup of  $\text{SL}(2, \mathbb{Z})$ . Cusps of  $\Gamma$  are  $\Gamma$ -equivalence classes of  $\mathbb{Q} \cup \{\infty\}$ . Adjoining coordinate charts to the cusps and compactifying gives the modular curve  $X(\Gamma) := \Gamma \backslash (\mathbb{H} \cup \mathbb{Q} \cup \{i\infty\})$ . The isotropy (stabiliser) group of  $\infty$  in  $\text{SL}(2, \mathbb{Z})$  is the abelian group of translations,

$$\text{SL}(2, \mathbb{Z})_{\infty} = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z} \right\}. \quad (\text{A.35})$$

For each cusp  $s \in \mathbb{Q} \cup \{i\infty\}$  some  $\delta_s \in \text{SL}(2, \mathbb{Z})$  maps  $s \mapsto \infty$ . The *width* of  $s$  is defined as

$$h_{\Gamma}(s) = \left| \text{SL}(2, \mathbb{Z})_{\infty} / (\delta_s \Gamma \delta_s^{-1})_{\infty} \right|. \quad (\text{A.36})$$

It can be proven that this definition is independent of  $\delta_s$ . For a fixed group  $\Gamma$  it can be viewed as a well-defined function  $\mathbb{Q} \cup \{i\infty\} \rightarrow \mathbb{N}_0$ . It is straightforward to show that the sum over the widths of all inequivalent cusps  $\mathcal{C}$  is equal to the index [327]

$$\sum_{s \in \mathcal{C}_{\Gamma}} h_{\Gamma}(s) = \text{ind } \Gamma. \quad (\text{A.37})$$

The width of 0 (which is the level) is the lcm of all widths, and the width of  $\infty$  is the gcd of all widths.

Other invariants of modular curves are the elliptic fixed points. A point  $\tau \in \mathbb{H}$  is an *elliptic point* for  $\Gamma$  if its isotropy (stabiliser) group is nontrivial. The *period* of  $\tau$  is defined as the order of the isotropy group. It can be shown that any congruence subgroup of  $\text{SL}(2, \mathbb{Z})$  has only finitely many elliptic points, and the period for any point  $\tau \in \mathbb{H}$  is 1, 2 or 3.

### Riemann-Hurwitz formula

Let  $f : X \rightarrow Y$  be a nonconstant holomorphic map between compact Riemann surfaces  $X, Y$ . It has a degree  $n \in \mathbb{N}$ , such that  $|f^{-1}(y)| = n$  for all but finitely many  $y \in Y$ . More precisely, for each point  $x \in X$  let  $e_x \in \mathbb{N}$  be the ramification degree of  $f$  at  $x$ , i.e. the multiplicity with which  $f$  takes 0 to 0 as a map in local coordinates, making  $f$  an  $e_x$ -to-1 map around  $x$ . Then there exists a positive integer  $n$  such that

$$\sum_{x \in f^{-1}(y)} e_x = n \quad (\text{A.38})$$

for all  $y \in Y$ . If  $g_X$  and  $g_Y$  are the genera of  $X$  and  $Y$ , the Riemann-Hurwitz formula

$$2g_X - 2 = n(2g_Y - 2) + \sum_{x \in X} (e_x - 1) \quad (\text{A.39})$$

states that the Euler characteristic of  $X$  is that of  $Y$  multiplied by the degree  $n$  of the cover, corrected by contributions from the ramification points. It is obvious that  $g_X \geq g_Y$ , otherwise  $f$  is not holomorphic.

This allows to compute the genus of a modular curve  $X(\Gamma)$  for any congruence subgroup  $\Gamma \subseteq \text{SL}(2, \mathbb{Z})$ . For this, let  $X = X(\Gamma)$  and  $Y = X(1)$ . Let  $y_2 = \text{SL}(2, \mathbb{Z}) \cdot i$ ,  $y_3 = \text{SL}(2, \mathbb{Z}) \cdot e^{\frac{\pi i}{3}}$  and  $\varepsilon_2$  and  $\varepsilon_3$  be the number of elliptic fixed points of period 2 and 3 for  $X(\Gamma)$ , and finally  $y_\infty = \text{SL}(2, \mathbb{Z}) \cdot \infty$  and  $\varepsilon_\infty$  be the number of cusps  $X(\Gamma)$ . Then  $e_x - 1$  is only nonzero when  $x \in f^{-1}(y_h)$  for  $h = 2, 3, \infty$ . Since  $g_{X(1)} = 0$ , it follows from (A.39) that

$$g = 1 + \frac{n}{12} - \frac{\varepsilon_2}{4} - \frac{\varepsilon_3}{3} - \frac{\varepsilon_\infty}{2}, \quad (\text{A.40})$$

where  $g = g_{X(\Gamma)}$  and  $n = \text{ind } \Gamma$ . See, for example, [145] for a more detailed derivation of this formula.

Conjugacy classes of subgroups of  $\text{SL}(2, \mathbb{Z})$  can be classified by the data  $(n, \varepsilon_\infty, \varepsilon_2, \varepsilon_3)$  together with the set of widths of the  $\varepsilon_\infty$  cusps. For instance, the subgroups of  $\text{SL}(2, \mathbb{Z})$  of index  $n = 6$  have been completely classified [154, 328]. There are precisely 22 subgroups that fall into 8 conjugacy classes, and they are listed in Table 4. It has been shown in [329] that every subgroup of  $\text{SL}(2, \mathbb{Z})$  with index  $n \leq 6$  is a congruence subgroup. Examples of *noncongruence* subgroups of  $\text{SL}(2, \mathbb{Z})$  of index 7 have been constructed already in the 19th century by Fricke. Hauptmoduln for low index subgroups of  $\text{SL}(2, \mathbb{Z})$  are studied in more detail in [115, 330].

## A.4 McKay-Thompson series

For discrete genus 0 subgroups of  $\text{PSL}(2, \mathbb{R})$ , Hauptmoduln corresponding to 174 Atkin-Lehner groups are known [331]. Conway and Norton [36] conjectured a natural  $\mathbb{Z}$ -graded representation  $V_n$  of the Fischer-Griess monster group  $\mathbb{M}$ . For every element  $g \in \mathbb{M}$ , we can collect the characters in a  $q$ -series

$$T_g(\tau) = \sum_{n=-1}^{\infty} \chi_{V_n}(g) q^n, \quad (\text{A.41})$$

which are called the *McKay-Thompson series* (see also [240, 332–336]). For the identity element  $g = 1 \in \mathbb{M}$ , the character counts the dimension  $\chi_{V_n}(1) = \dim V_n$ ,

$$T_1(\tau) = \sum_{n=-1}^{\infty} \dim V_n q^n = J(\tau), \quad (\text{A.42})$$

which famously gives the Klein  $j$ -function  $J = j - 744$ . Since the character of a representation is a class function, there are as many distinct McKay-Thompson

$(\varepsilon_\infty, \varepsilon_2, \varepsilon_3)$	cusps	Hauptmodul	conj
(1, 0, 0)	6	$g = 1$	1
(1, 0, 3)	6	$j = -27x^3(x^3 + 16)$	2
(1, 4, 0)	6	$j = 27(x^2 + 4)^3$	3
(2, 2, 0)	3 + 3	$j = \frac{x^3(x+12)^3}{(x+9)^3}$	3
(2, 2, 0)	4 + 2	$j = \frac{x^3(x+8)^3}{(x+4)^2}$	3
(2, 2, 0)	5 + 1	$j = \frac{(x^2+10x+5)^3}{x}$	6
(3, 0, 0)	2 + 2 + 2	$j = \frac{(x^2+192)^3}{(x^2-64)^2}$	1
(3, 0, 0)	4 + 1 + 1	$j = \frac{(x^2+48)^3}{x^2+64}$	3

Table 4: Classification of conjugacy classes of index 6 subgroups of  $\mathrm{SL}(2, \mathbb{Z})$  [154]. The first column gives the number of cusps and elliptic fixed points of order 2 and 3. According to the Riemann-Hurwitz formula (A.40) all but the (1, 0, 0) one induce genus 0 modular curves, whose Hauptmodul  $x$  is expressed in terms of the  $j$ -invariant. In fact the group with signature (1, 0, 0) is the only subgroup of  $\mathrm{SL}(2, \mathbb{Z})$  of nonzero genus and index  $\leq 7$ . The second column gives the indices of the  $\varepsilon_\infty$  number of cusps, and the last column counts the number of conjugate groups with the same data. From the expression of  $j = \frac{p(x)}{q(x)}$  as a rational function in  $x$  we learn that  $\deg p = n$  gives the index and the difference  $\deg p - \deg q = h(\infty)$  gives the width of  $\infty$ . The roots of  $q(x) = 0$  then give the cusps in  $\mathbb{Q}$  with widths provided by their corresponding multiplicity.

series as there are conjugacy classes of the Monster, namely 194. If  $g$  is not the identity element, the series (A.41) are not modular for  $\mathrm{PSL}(2, \mathbb{Z})$  but rather for a discrete subgroup  $\Gamma_g$  of  $\mathrm{PSL}(2, \mathbb{R})$ . In fact, the  $\Gamma_g$  are all genus 0 groups and  $T_g$  is a Hauptmodul on  $\Gamma_g \backslash \mathbb{H}$  [280]. It turns out that all but 3 of the 174 Atkin-Lehner groups of genus zero correspond to McKay-Thompson series. The series (A.41) are all examples of *replicable* functions [56–58, 337–339]

Let  $f$  be a  $q$ -series

$$f(q) = \frac{1}{q} + \sum_{n=1}^{\infty} a_n q^n \quad (\text{A.43})$$

with vanishing constant term. For every  $n \in \mathbb{N}_0$  there exists a unique monic polynomial  $F_n$  such that

$$F_n(f(q)) = \frac{1}{q^n} + \mathcal{O}(q) \quad (\text{A.44})$$

as  $q \rightarrow 0$ . The  $F_n$  are called *Faber polynomials*. For any  $f$  as in (A.43), they can be defined as  $F_n(z) = \det(z\mathbb{1} - A_n)$ , where

$$A_n = \begin{pmatrix} a_0 & 1 & & & & \\ 2a_1 & a_0 & 1 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ (n-2)a_{n-3} & a_{n-4} & a_{n-5} & \cdots & 1 & \\ (n-1)a_{n-2} & a_{n-3} & a_{n-4} & \cdots & a_0 & 1 \\ na_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \end{pmatrix}. \quad (\text{A.45})$$

This allows to write

$$F_n(f(q)) = \frac{1}{q^n} + n \sum_{m=1}^{\infty} h_{m,n} q^m. \quad (\text{A.46})$$

The  $q$ -series  $f$  is then said to be replicable if  $h_{m,n} = h_{r,s}$  whenever  $mn = rs$  and  $\gcd(m, n) = \gcd(r, s)$ . This gives a relatively sparse list of equivalences between the entries of the infinite matrix  $(h_{m,n})_{m,n \in \mathbb{N}}$ . In practise one can truncate the matrix and compare  $h_{m,n}$  and  $h_{r,s}$  for pairs with the given constraints. They can be worked out before. For instance, if all series  $F_n(f(q))$  are computed up to  $q^{14}$ , there are 8 constraints

$$\begin{aligned} h_{1,6} = h_{2,3}, \quad h_{1,10} = h_{2,5}, \quad h_{1,12} = h_{3,4}, \quad h_{1,14} = h_{2,7}, \\ h_{2,12} = h_{4,6}, \quad h_{3,10} = h_{5,6}, \quad h_{3,14} = h_{6,7}, \quad h_{5,14} = h_{7,10}. \end{aligned} \quad (\text{A.47})$$

## A.5 Algebraic modular functions

In this Appendix, we discuss aspects of roots of polynomials over the field of modular functions for  $\text{PSL}(2, \mathbb{Z})$ , which describe order parameters found throughout this thesis. In the following, by  $\Gamma$  we mean  $\text{PSL}(2, \mathbb{Z})$ , unless specified otherwise. We begin with a

**Lemma 1** (Proposition 18 of [59]). *Let  $f \in M_0(\Gamma')$  be a modular form of weight 0 for a congruence subgroup  $\Gamma' \subset \Gamma$ . Then  $f$  is constant.*

*Proof.* Let  $a = f(\tau_0)$  for some fixed  $\tau_0 \in \mathbb{H}$ . Let  $\Gamma = \bigcup_j \alpha_j \Gamma'$ , where the disjoint union is over  $j = 1, \dots, \text{ind } \Gamma' = [\Gamma : \Gamma']$ . Consider the function

$$g : \mathbb{H} \rightarrow \mathbb{C}, \quad \tau \mapsto \prod_{j=1}^{\text{ind } \Gamma'} (f(\alpha_j^{-1} \tau) - a), \quad (\text{A.48})$$

where  $\alpha_j^{-1} \tau$  is the usual action of  $\Gamma$  on  $\mathbb{H}$ . One finds that  $g(\gamma^{-1} \tau) = \prod_j (f((\gamma \alpha_j)^{-1} \tau) - a)$  for  $\gamma \in \Gamma$ . This however only permutes the cosets, such that the terms in the product (A.48) are merely rearranged. Therefore,  $g(\gamma \tau) = g(\tau)$  and  $g$  is a holomorphic modular form for  $\Gamma$  and thus constant. As  $g(\tau_0) = 0$ , we have  $g = 0$  identically. This means that one of the factors  $f(\alpha_j^{-1} \tau) - a$  in (A.48) is zero, such that  $f(\tau) = a$  for all  $\tau \in \mathbb{H}$ .  $\square$

**Lemma 2** (Exercise III §3 7(a) of [59]). *Let  $f$  be a modular function for a congruence subgroup  $\Gamma' \subset \Gamma$ . Then  $f$  satisfies a polynomial of degree  $\text{ind } \Gamma' = [\Gamma : \Gamma']$  over the field  $\mathbb{C}(\Gamma)$  of modular functions on  $\Gamma$ .*

*Proof.* We can recycle most parts of the proof of Lemma 1. Let  $X \in \mathbb{C}$  be a constant, and let  $f$  be a modular function for  $\Gamma' \subset \Gamma$ , which can be written as  $\Gamma = \bigcup_j \alpha_j \Gamma'$ . The function

$$\tilde{g} : \mathbb{H} \rightarrow \mathbb{C}, \quad \tau \mapsto \prod_{j=1}^{\text{ind } \Gamma'} (f(\alpha_j^{-1} \tau) - X) \quad (\text{A.49})$$

is meromorphic on  $\mathbb{H}$  and admits a Fourier series, since  $f$  does. For the same reason as before, one finds that  $\tilde{g}(\gamma\tau) = \tilde{g}(\tau)$  for all  $\gamma \in \Gamma$  and  $\tau \in \mathbb{H}$ . Therefore,  $\tilde{g}$  is a modular function for  $\Gamma$ . The coefficients of the polynomial  $\tilde{g}$  in  $X$  are the elementary symmetric polynomials in  $\{f \circ \alpha_1^{-1}, \dots, f \circ \alpha_{\text{ind}\Gamma}^{-1}\}$ . Since  $X$  is arbitrary, the coefficients are also modular functions for  $\Gamma$ . Therefore,

$$P(X) = \prod_{j=1}^{\text{ind}\Gamma'} (f \circ \alpha_j^{-1} - X) \in \mathbb{C}(\Gamma)[X] \quad (\text{A.50})$$

is a polynomial of degree  $\text{ind}\Gamma'$  in  $X$  over  $\mathbb{C}(\Gamma)$  with  $f$  as a solution, as demanded.  $\square$

It is clear from the above proof that some properties of the coefficients of the modular polynomial are inherited from those of  $f$ . For instance, instead of  $\Gamma = \text{PSL}(2, \mathbb{Z})$  we can take  $\Gamma = \text{PSL}(2, \mathbb{R})$  and  $\Gamma'$  to be an arithmetic (i.e. finite index) Fuchsian group. Another possibility is to study modular polynomials for weakly holomorphic functions  $f$  on  $\text{PSL}(2, \mathbb{Z})$ , for which the coefficients of  $P$  are polynomials in  $j$ . This motivates the following

**Definition 7** (Algebraic modular form). *Let  $\Gamma'$  be an arithmetic Fuchsian group, i.e. a discrete finite index subgroup of  $\text{PSL}(2, \mathbb{R})$ . An algebraic modular form of degree  $d$  for  $\Gamma'$  is a root of a polynomial of degree  $d$  over the field of automorphic functions on  $\Gamma'$ . The degree is taken to be minimal in the obvious sense.*

According to Lemma 2, every congruence subgroup  $\Gamma'$  of  $\text{PSL}(2, \mathbb{Z})$  is an algebraic modular form for  $\text{PSL}(2, \mathbb{Z})$  of degree  $\text{ind}\Gamma'$ . This is because congruence subgroups of  $\text{PSL}(2, \mathbb{Z})$  have finite index. In some cases, algebraic modular forms form Galois extensions of the field over which the polynomial is defined:

**Lemma 3** (Section 2.1 in [118]). *Let  $\Gamma$  be a Fuchsian group of the first kind, and let  $\Gamma'$  be a subgroup of  $\Gamma$  of finite index. Identify  $A_0(\Gamma)$  (resp.  $A_0(\Gamma')$ ) with the field of all meromorphic functions on  $\Gamma \backslash \mathbb{H}$  (resp.  $\Gamma' \backslash \mathbb{H}$ ). Then,  $\Gamma' \backslash \mathbb{H}$  is a covering of  $\Gamma \backslash \mathbb{H}$  of degree  $[\Gamma : \Gamma']$ . If  $\Gamma'$  is a normal subgroup of  $\Gamma$ , one can consider the automorphisms of  $\Gamma' \backslash \mathbb{H}$ , or equivalently of  $A_0(\Gamma')$ . Then  $A_0(\Gamma')$  is a Galois extension of  $A_0(\Gamma)$ , and  $\text{Gal}(A_0(\Gamma')/A_0(\Gamma))$  is isomorphic to  $\Gamma/\Gamma'$ .*

The principal congruence subgroup  $\Gamma(N)$  is normal in  $\text{PSL}(2, \mathbb{Z})$ , however neither  $\Gamma_0(N)$  nor  $\Gamma_1(N)$  are for  $N \geq 2$ . The extension  $\mathbb{C}(X(N))/\mathbb{C}(X(1))$  of the modular curve  $\Gamma(N)$  is Galois with Galois group  $\text{SL}(2, \mathbb{Z}/N\mathbb{Z})\{\pm 1\}$ . Modular function field extensions are further studied in [340] and [145, Section 7.5].

### Application to the SW curves

The above allows us to find a general algebraic description of the  $N_f = 0, 1, 2, 3, 4$  SW curves. From section 2 it is clear that  $g_2$  and  $g_3$  are polynomials in  $u$  of degree 2 and 3. In  $N_f \leq 3$ , the coefficients of  $u$  contain the

masses  $\mathbf{m}$  and the dynamical scale  $\Lambda_{N_f}$ . In  $N_f = 4$ , they contain the masses  $\mathbf{m}$  and the UV-coupling  $\tau_{\text{UV}}$ . We can consider them to be fixed complex numbers.

Consider now the polynomial  $P(X)$  (2.13) of degree 6 in  $X$  with coefficients in the field  $\mathbb{C}(\Gamma)$  of modular functions on  $\Gamma \backslash \mathbb{H}$ . Any solution to

$$P(X) = 0 \tag{A.51}$$

is called an order parameter for the SW curve. It is important to note that the polynomial (2.13) is not an invariant of the SW curve. However, since it is a relative invariant, its roots (A.51) are absolute invariants.

### Examples

It is instructive to study some examples. For the pure  $\text{SU}(2)$  case, up to  $\mathbb{C}$ -proportionality one finds<sup>43</sup>

$$P(X) = (-1728 + j) + (6912 - j)X^2 - 9216X^4 + 4096X^6. \tag{A.52}$$

Such a relation also gives a recurrence relation on the coefficients of the Fourier expansion of any solution  $u = X$ . The fact that the coefficients of  $P(X)$  can be normalised to be in  $\mathbb{Z}$  does however not imply an integer  $q$ -expansion of  $u$  (it can for instance also be done for some  $\mathbf{m} \in \mathbb{Q}^{N_f}$  in  $N_f > 0$ ).

Another example is  $N_f = 2$  with  $\mathbf{m} = (m, m)$  and  $m = m_{\text{AD}} = \frac{1}{2}$ . The polynomial factors over  $\mathbb{C}(\Gamma)$  as

$$P(X) = (X - \frac{3}{8})^3 ((1728 - 5j) + 8(1728 - j)X + 36864X^2 + 32768X^3). \tag{A.53}$$

Three solutions are  $X = \frac{3}{8}$ , which is a constant function in  $\mathbb{C}(\Gamma)$ . If we assume that any solution to above equation has index 6, then Lemma 2 gives a contradiction since if  $f \circ \alpha_j^{-1}$  is constant for some coset representative  $\alpha_j$ , then since  $\alpha_j$  is a bijection on  $\mathbb{H}$  also  $f$  is constant. But  $j \in \mathbb{C}(\Gamma)$  is transcendental and therefore does not satisfy a polynomial equation over  $\mathbb{Q}$  such as (A.53). This is true in general, any linear factor of  $P(X)$  over  $\mathbb{C}$  needs to be divided from  $P(X)$ . Such constant solutions signal the appearance of an AD point. The remaining degree 3 polynomial in (A.53) does not factor over  $\mathbb{C}(\Gamma)$ . However, it does factor over  $\mathbb{C}(\Gamma^0(2))$ , which is genus 0 and generated by  $\mathbb{C}(\Gamma^0(2)) \ni h : \tau \mapsto f_{2\text{B}}(\frac{\tau}{2})$  (as demonstrated in section 2.5.3). In order to see this, let us use (2.92) to write  $j = \frac{(h+16)^3}{h}$ . Then one solution to the degree equation in (A.53) is  $X = -(h+40)/64$ , which is precisely what is found in (2.89). The other solutions are related by  $\text{SL}(2, \mathbb{Z})$  images of  $h$ .

One also finds the following

**Fact 1.** For  $N_f = 0, 1, 2$ , the sum over all 6 solutions to (2.13) is zero.

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<sup>43</sup>We set  $\Lambda_{N_f} = 1$  in this subsection.

*Proof.* The relative invariants  $g_2$  and  $g_3$  are polynomials of degree 2 and 3 in  $u$ . For  $N_f = 0, 1, 2$ , by direct inspection one finds that the coefficients of  $u$  in  $g_2$  and the coefficient of  $u^2$  in  $g_3$  are zero. Therefore, the coefficient of  $u^5$  in (2.13) is zero. By Vieta's theorem, the sum over the roots of a polynomial  $\sum_{k=0}^n a_k x^k$  is  $-\frac{a_{n-1}}{a_n}$ . Since the  $u^6$  coefficient is never zero, the claim follows.  $\square$

This does not hold for any mass in  $N_f = 3, 4$ . Field extensions have been studied in the context of SW theory in [121–123].

## Geometric description

The above algebraic construction has a geometric description in terms of Riemann surfaces. We again first lay out the mathematical structure and then apply it to the family of SW curves.

Let  $Y$  be a Riemann surface, i.e. a connected complex one-dimensional manifold. Furthermore, let

$$T = y^n + a_1 y^{n-1} + \dots + a_n \in K(Y)[y] \quad (\text{A.54})$$

be a polynomial in  $y$  of degree  $n$  over the field  $K(Y)$  of meromorphic functions  $Y \rightarrow \mathbb{P}^1(\mathbb{C})$ . In case that in the factorisation of  $T$  every irreducible factor occurs with multiplicity one, the discriminant  $D$  of  $T$  is nonzero. This does however not need to be the true. Denote by  $O$  the discrete subset of  $Y$  containing all poles of the coefficients  $a_i \in K(Y)$  and all zeros of the discriminant  $D$ . Then, for every point  $x_0 \in Y \setminus O$  the polynomial  $T_{x_0} = y^n + a_1(x_0)y^{n-1} + \dots + a_n(x_0)$  has exactly  $n$  distinct roots over  $\mathbb{C}$ . We have the

**Theorem 2** (Thm. 2.2.12 in [119]). *The Riemann surface of the equation  $T = 0$  is an  $n$ -fold ramified covering  $\pi : M \rightarrow Y$  together with a meromorphic function  $y : M \rightarrow \mathbb{P}^1(\mathbb{C})$ , such that for every point  $x_0 \in Y \setminus O$  the set of roots of the polynomial  $T_{x_0}$  coincides with the set of values of the function  $y$  on the preimage  $\pi^{-1}(x_0)$  of the point  $x_0$  under the projection  $\pi$ .*

The set  $\tilde{O}$  of critical values of the ramified covering  $\pi : M \rightarrow Y$  associated with the Riemann surface of the equation  $T = 0$  is strictly contained in  $O$ . It is called the *ramification set* of the equation  $T = 0$ . In the field of germs of meromorphic functions at a point  $a \in Y \setminus \tilde{O}$ , the equation  $T = 0$  has only roots of multiplicity 1, and their number is equal to the degree of the polynomial  $T$ . Each of the meromorphic germs at  $a$  satisfying the equation  $T = 0$  corresponds to a point over  $a$  in the Riemann surface of the equation.

## A.6 Kodaira classification

Let us consider an elliptic curve in Weierstraß form,

$$y^2 = 4x^3 - g_2x - g_3, \quad (\text{A.55})$$



$F_v$	$\tau$	$\text{ord}(g_2)$	$\text{ord}(g_3)$	$\text{ord}(\Delta)$	$\mathbb{M}$	$\mathfrak{g}$
$I_k$	$i\infty$	0	0	$k$	$T^k$	$\mathfrak{su}(k)$
$I_k^*$	$i\infty$	2	3	$k+6$	$PT^k$	$\mathfrak{so}(2k+8)$
$I_0^*$	$\tau_0$	$\geq 2$	$\geq 3$	6	$P$	$\mathfrak{so}(8)$
$II$	$e^{\frac{2\pi i}{3}}$	$\geq 1$	1	2	$(ST)^{-1}$	-
$II^*$	$e^{\frac{2\pi i}{3}}$	$\geq 4$	5	10	$(ST)$	$\mathfrak{e}_8$
$III$	$i$	1	$\geq 2$	3	$S^{-1}$	$\mathfrak{su}(2)$
$III^*$	$i$	3	$\geq 5$	9	$S$	$\mathfrak{e}_7$
$IV$	$e^{\frac{2\pi i}{3}}$	$\geq 2$	2	4	$(ST)^{-2}$	$\mathfrak{su}(3)$
$IV^*$	$e^{\frac{2\pi i}{3}}$	$\geq 3$	4	8	$(ST)^2$	$\mathfrak{e}_6$

Table 5: Kodaira classification of singular fibers based on the orders of vanishing of  $g_2$ ,  $g_3$  and  $\Delta$  in the Weierstraß model. The value of the complex structure is denoted by  $\tau$ , while  $\mathbb{M}$  is the monodromy and  $\mathfrak{g}$  the associated flavour algebra.

where  $g_2$  and  $g_3$  are functions of some parameter  $u$ . This is the case for all SW curves (2.3), where  $g_2$  and  $g_3$  are polynomials of degree 2 and 3 in  $u$ . The discriminant of the Weierstraß curve is  $\Delta = g_2^3 - 27g_3^2$ . The study of the elliptic curves on the discriminant divisor  $\Sigma = \{u \mid \Delta(u) = 0\}$  is the famous Kodaira classification. At a generic point in  $\Sigma$ ,  $g_2$  and  $g_3$  do not vanish simultaneously. If we denote by  $\text{ord } f$  the order of vanishing (order of zero) of a polynomial or power series  $f$  at a given point, then on a generic point in  $\Sigma$  we have  $\text{ord}(g_2, g_3, \Delta) = (0, 0, 1)$ . The various combinations of orders of vanishing of the invariants  $g_2$  and  $g_3$  are classified [41, 42], see for a review [341]. At special points in  $\Sigma$  both  $g_2 = g_3 = 0$ . From (A.55) we see that the curve becomes cuspidal,  $y^2 = x^3$ . However the shape of the curve depends on the orders of vanishing of the respective quantities. The full classification is given in Table 5. The singular fibres of the Kodaira classification are all realised as curves of physical theories [245, 246].

## A.7 Siegel-Narain theta function

Let  $L$  be an  $n$ -dimensional uni-modular lattice with signature  $(1, n-1)$ . For the application to the  $u$ -plane integral,  $n = b_2(X)$ . Let  $K$  be a characteristic vector of  $L$ . Its defining property is  $\mathbf{l}^2 = \mathbf{l} \cdot K \pmod{2}$  for every  $\mathbf{l} \in L$ . Furthermore, we have that  $\boldsymbol{\mu} \in L/2$ .

We consider the Siegel-Narain theta function  $\Psi_{\boldsymbol{\mu}}^J : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$  defined in the main text in (5.51). We repeat it here for convenience,

$$\begin{aligned} \Psi_{\boldsymbol{\mu}}^J(\tau, \bar{\tau}, \mathbf{z}, \bar{\mathbf{z}}) &= e^{-2\pi y \mathbf{b}_+^2} \sum_{\mathbf{k} \in L + \boldsymbol{\mu}} \partial_{\bar{\tau}} (4\pi i \sqrt{y} B(\mathbf{k} + \mathbf{b}, \underline{J})) \\ &\times (-1)^{B(\mathbf{k}, K)} q^{-\mathbf{k}_-^2/2} \bar{q}^{\mathbf{k}_+^2/2} e^{-2\pi i B(\mathbf{z}, \mathbf{k}_-) - 2\pi i B(\bar{\mathbf{z}}, \mathbf{k}_+)}, \end{aligned} \quad (\text{A.56})$$

where  $\mathbf{b} = \text{Im}(\mathbf{z})/y$ . The transformations under the generators  $S$  and  $T$  of  $\text{PSL}(2, \mathbb{Z})$  are most easily determined if we shift  $\boldsymbol{\mu} \rightarrow \boldsymbol{\mu} + K/2$ . One finds

[81, 84]

$$\begin{aligned}
S : \quad & \Psi_{\mu+K/2}^J(-1/\tau, -1/\bar{\tau}, \mathbf{z}/\tau, \bar{\mathbf{z}}/\bar{\tau}) = -i(-i\tau)^{n/2}(i\bar{\tau})^2 \\
& \quad \times e^{-\pi i \mathbf{z}^2/\tau + \pi i K^2/2} (-1)^{B(\mu, K)} \Psi_{K/2}^J(\tau, \bar{\tau}, \mathbf{z} - \boldsymbol{\mu}, \bar{\mathbf{z}} - \boldsymbol{\mu}), \\
T : \quad & \Psi_{\mu+K/2}^J(\tau + 1, \bar{\tau} + 1, \mathbf{z}, \bar{\mathbf{z}}) = \\
& \quad e^{\pi i(\mu^2 - K^2/4)} \Psi_{\mu+K/2}^J(\tau, \bar{\tau}, \mathbf{z} + \boldsymbol{\mu}, \bar{\mathbf{z}} + \boldsymbol{\mu}).
\end{aligned} \tag{A.57}$$

Using these transformations, one finds for the periodicity in  $\tau$ ,

$$\Psi_{\mu}^J(\tau + 1, \bar{\tau} + 1, \mathbf{z}, \bar{\mathbf{z}}) = e^{\pi i(\mu^2 - B(\mu, K))} \Psi_{\mu}^J(\tau, \bar{\tau}, \mathbf{z} + \boldsymbol{\mu} - K/2, \bar{\mathbf{z}} + \boldsymbol{\mu} - K/2) \tag{A.58}$$

and for  $S^{-1}T^{-k}S = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ ,

$$\Psi_{\mu}^J\left(\frac{\tau}{k\tau+1}, \frac{\bar{\tau}}{k\bar{\tau}+1}, \frac{\mathbf{z}}{k\tau+1}, \frac{\bar{\mathbf{z}}}{k\bar{\tau}+1}\right) = (k\tau + 1)^{\frac{n}{2}} (k\bar{\tau} + 1)^2 e^{-\frac{\pi i k \mathbf{z}^2}{k\tau+1}} e^{\frac{\pi i}{4} k K^2} \Psi_{\mu}^J(\tau, \bar{\tau}, \mathbf{z}, \bar{\mathbf{z}}). \tag{A.59}$$

We furthermore list the following transformations for  $\mathbf{z}$ :

- For the reflection  $\mathbf{z} \rightarrow -\mathbf{z}$ ,

$$\Psi_{\mu}^J(\tau, \bar{\tau}, -\mathbf{z}, -\bar{\mathbf{z}}) = -e^{2\pi i B(\mu, K)} \Psi_{\mu}^J(\tau, \bar{\tau}, \mathbf{z}, \bar{\mathbf{z}}). \tag{A.60}$$

- For shifting  $\mathbf{z} \rightarrow \mathbf{z} + \boldsymbol{\nu}$  with  $\boldsymbol{\nu} \in L$ ,

$$\Psi_{\mu}^J(\tau, \bar{\tau}, \mathbf{z} + \boldsymbol{\nu}, \bar{\mathbf{z}} + \boldsymbol{\nu}) = e^{-2\pi i B(\boldsymbol{\nu}, \mu)} \Psi_{\mu}^J(\tau, \bar{\tau}, \mathbf{z}, \bar{\mathbf{z}}). \tag{A.61}$$

- For shifting  $\mathbf{z} \rightarrow \mathbf{z} + \boldsymbol{\nu}\tau$  with  $\boldsymbol{\nu} \in L \otimes \mathbb{R}$ ,

$$\Psi_{\mu}^J(\tau, \mathbf{z} + \boldsymbol{\nu}\tau) = e^{2\pi i B(\mathbf{z}, \boldsymbol{\nu})} q^{\boldsymbol{\nu}^2/2} (-1)^{-B(\boldsymbol{\nu}, K)} \Psi_{\mu+\boldsymbol{\nu}}^J(\tau, \bar{\tau}, \mathbf{z}, \bar{\mathbf{z}}). \tag{A.62}$$

We can restrict to  $\boldsymbol{\nu} \in L/2$ , if the characteristic  $\boldsymbol{\mu} + \boldsymbol{\nu}$  is required to be in  $L/2$ .

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