

# Finite dimensional surrogates for extreme events

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**ABSTRACT:** Numerical solutions of stochastic problems require the representation of the random functions in their definitions by finite dimensional (FD) models, i.e., deterministic functions of time and finite sets of random variables. It is common to represent the coefficients of these FD surrogates by polynomial chaos (PC) models. We propose a novel model, referred to as the polynomial chaos translation (PCT) model, which matches exactly the marginal distributions of the FD coefficients and approximately their dependence. PC- and PCT- based FD models are constructed for the solution of differential equation with non-Gaussian input and the wind pressure time series recorded at the boundary layer wind tunnel facility at the University of Florida. The PCT-based models capture the joint distributions of the FD coefficients and the extremes of target stochastic processes accurately while PC-based FD models do not have this capability.

## 1. INTRODUCTION

The solution of a broad range of problems in science and engineering involves extremes of random processes  $X(t)$  over finite times intervals, e.g., extreme climate events and design responses of dynamical systems subjected to random loads, Grigoriu. (2020); Easterling et al. (2000); Grigoriu and Samorodnitsky. (2015). Yet, most practical methods for calculating the distribution of the extreme random variable  $X_\tau = \sup_{0 \leq t \leq \tau} |X(t)|$  are based on the mean rate at which the real-valued process  $X(t)$  or its absolute value crosses with positive slope specified levels, Leadbetter et al. (1983), (Chap. 7), which is available analytically for mean square differentiable Gaussian processes  $X(t)$  and memoryless transformation of these processes, referred to as translation processes, Gioffré et al. (2000). If  $X(t)$  does not have these properties, the distribution of the extreme random variable  $X_\tau$  can be approximated from crossing of time series

$(X(t_0), X(t_1), \dots, X(t_n))$  defined by the values of  $X(t)$  at a finite set  $0 = t_0 < t_1 < \dots < t_n = \tau$  of times in  $[0, \tau]$ , Naess and Gaidai. (2008). The accuracy of this approximation depends on the time step and the properties of the samples of  $X(t)$ . For example, the approximation fails if the samples of  $X(t)$  are not differentiable, e.g., the Brownian motion process.

It is proposed to approximate the distribution of  $X_\tau = \sup_{0 \leq t \leq \tau} |X(t)|$  by that of  $X_{d,\tau} = \sup_{0 \leq t \leq \tau} |X_d(t)|$ , where  $X_d(t)$  is a finite dimensional (FD) model of  $X(t)$ , i.e., a deterministic function of time and  $d < \infty$  random variables which has the following two properties. First, the distributions of  $X_\tau$  and  $X_{d,\tau}$  are similar for a sufficiently large stochastic dimension  $d$ . Accordingly, the distribution of  $X_\tau$  can be estimated from samples of  $X_d(t)$ . Second, samples of  $X_d(t)$  can be generated by standard Monte Carlo algorithms. In contrast, samples of  $X(t)$  cannot be generated since, generally, stochas-

tic processes have infinite stochastic dimensions as uncountable families of random variables indexed by time.

## 2. FINITE DIMENSIONAL MODELS

Let  $X(t)$  be a real-valued, zero-mean stochastic process on a bounded time interval  $[0, \tau]$  with correlation function  $c(s, t) = E[X(s)X(t)]$ . Denote by  $\{\lambda_k\}$  and  $\{\varphi_k(t)\}$ ,  $k = 1, 2, \dots$ , the eigenvalues and the eigenfunctions of the correlation function of  $X(t)$ . It is assumed that  $c(s, t)$  is continuous so that its eigenfunctions are real-valued continuous function on  $[0, \tau]$ , Kadota. (1967).

The family of FD models of  $X(t)$  has the form

$$X_d(t) = \sum_{k=1}^d Z_k \varphi_k(t), \quad 0 \leq t \leq \tau, \quad (1)$$

where the random coefficients  $\{Z_k = \int_0^\tau X(t) \varphi_k(t) dt\}$  are the projections of  $X(t)$  on the basis functions  $\{\varphi_k(t)\}$ . Simple calculations show that  $E[Z_k] = 0$  and  $E[Z_k Z_l] = \lambda_k \delta_{kl}$ , so that the zero-mean random variables  $\{Z_k\}$  are uncorrelated. They are independent if  $X(t)$  is Gaussian.

The FD models  $X_d(t)$  have two notable properties. First, they are defined on the same probability space as  $X(t)$  so that their samples are paired with those of  $X(t)$ . Second, for given time  $t$ , the random variables  $X_d(t)$  converge in mean square to  $X(t)$  as  $d \rightarrow \infty$  since

$$E[(X_d(t) - X(t))^2] = \sum_{k=d+1}^{\infty} \lambda_k \varphi_k(t)^2 \rightarrow 0,$$

as  $d \rightarrow \infty$  by Mercer's theorem, Mercer. (1909). This convergence implies the converges in probability of  $X_d(t)$  to  $X(t)$  and, therefore, in distribution. This observation and Theorem 18.10 of van der Vaart. (1998) imply that the finite dimensional distributions of  $X_d(t)$  converge to those of  $X(t)$  as  $d \rightarrow \infty$ .

## 3. FD-BASED ESTIMATES OF EXTREMES

Denote by  $F_\tau$  and  $F_{d,\tau}$  the distributions of the extremes  $X_\tau = \sup_{0 \leq t \leq \tau} |X(t)|$  and  $X_{d,\tau} = \sup_{0 \leq t \leq \tau} |X_d(t)|$  of  $X(t)$  and  $X_d(t)$  in the bounded time interval  $[0, \tau]$ . We give conditions under which

$F_{d,\tau}$  converges to  $F_\tau$  as  $d \rightarrow \infty$ . Under these conditions,  $F_\tau$  can be estimated from samples of  $X_d(t)$  provided that  $d$  is sufficiently large. This is essential in applications since samples of  $X_d(t)$  can be generated by standard Monte Carlo algorithms while samples of  $X(t)$  are not available. For simplicity, we assume as in the previous section that  $X(t)$  is real-valued. Extension to vector-valued processes is straightforward.

**Property 1.** If  $X(t)$  has continuous samples and its correlation function  $c(s, t) = E[X(s)X(t)]$  is continuous, the finite dimensional distributions of  $X_d(t)$  converge to those of  $X(t)$  as  $d \rightarrow \infty$  and  $\sum_{k=1}^{\infty} \lambda_k C_k < \infty$  with  $C_k = \sup_{t \in [0, \tau]} \varphi_k(t)^2$ , then the distribution  $F_{d,\tau}$  of  $\sup_{0 \leq t \leq \tau} |X_d(t)|$  converges to the distribution  $F_\tau$  of  $\sup_{0 \leq t \leq \tau} |X(t)|$  as  $d \rightarrow \infty$ .

It can be shown by using the Theorem 8.2 in Billingsley. (1968) that this property holds. The practical implication of this property is that the distribution of the extreme random variable  $\sup_{0 \leq t \leq \tau} |X(t)|$  can be estimated from samples of FD models  $X_d(t)$  of  $X(t)$  provided that  $d$  is sufficiently large. This is essential in applications since the distribution of  $\sup_{0 \leq t \leq \tau} |X(t)|$  is available analytically in special cases of limited practical interest and samples of  $X(t)$  cannot be generated.

**Property 2.** If  $X(t)$  is a Gaussian process with continuous samples and its correlation function  $c(s, t) = E[X(s)X(t)]$  is continuous and if the finite dimensional distributions of  $X_d(t)$  converge to those of  $X(t)$  as  $d \rightarrow \infty$ , then the sequence of random variables  $\sup_{0 \leq t \leq \tau} |X_d(t) - X(t)|$  converges to zero in probability as  $d \rightarrow \infty$ .

The proof is based on arguments similar to those used for the previous property. This means that the "bad" subset

$$\Omega_d(\varepsilon) = \{\omega \in \Omega : \sup_{0 \leq t \leq \tau} |X_d(t) - X(t)| > \varepsilon\}$$

of the sample space  $\Omega$  on which the samples of  $X(t)$  and  $X_d(t)$  differs by more than any  $\varepsilon > 0$  can be made as small as desired by increasing  $d$  since  $P(\Omega_d(\varepsilon)) \rightarrow 0$  as  $d \rightarrow \infty$ . Accordingly, most of the samples of  $X(t)$  can be represented by the samples

of  $X_d(t)$  for a sufficiently large stochastic dimension  $d$ .  $G_1, \dots, G_d$  defined by

#### 4. PC AND PCT MODELS

Our objective is to construct models of the random coefficients  $Z = (Z_1, \dots, Z_d)^T$  of  $X_d(t)$  in (1) from its samples which are accurate in the sense that their joint distributions match the joint distribution of  $Z$ , and efficient, i.e., standard Monte Carlo algorithms can be used to generate samples of these models.

The Rosenblatt transformation Rosenblatt. (1952) shows that the components of  $Z = (Z_1, \dots, Z_d)$  can be related to the components of, e.g., a vector  $G = (G_1, \dots, G_d)$  with independent standard Gaussian variables, by the mapping

$$\begin{aligned} Z_1 &= F_1^{-1} \circ \Phi(G_1) \\ Z_k | Z_{k-1}, \dots, Z_1 &= F_{k|k-1, \dots, 1}^{-1} \circ \Phi(G_k), \quad (2) \end{aligned}$$

where  $F_k$  is the distribution of  $Z_k$ ,  $F_{k|k-1, \dots, 1}$  is the distribution of  $Z_k | Z_{k-1}, \dots, Z_1$ . If the mapping in (2) is available, samples of  $Z$  can be obtained from samples of  $G$ , which can be generated by standard algorithms. Since the conditional distributions in the mapping  $G \mapsto Z$  are available analytically only in special cases, they have to be constructed numerically in most applications. Their construction from the joint distribution of  $Z$  is computationally demanding and the resulting conditional distributions are likely to be unsatisfactory, particularly when dealing with heavy tail distributions. The construction of the conditional distributions  $F_{k|k-1, \dots, 1}$  from data is not feasible when dealing with high dimensional vectors and relatively small data sets.

This section develops approximations of the Rosenblatt transformation for the random coefficients  $(Z_1, \dots, Z_d)$  based on polynomial chaos (PC) and an extension of this representation, referred to as PCT models. These models of  $(Z_1, \dots, Z_d)$  are denoted by  $Z^{PC} = (Z_1^{PC}, \dots, Z_d^{PC})$  and  $Z^{PCT} = (Z_1^{PCT}, \dots, Z_d^{PCT})$ .

The PC models considered here are quadratic forms of independent standard Gaussian variables

$$\begin{aligned} Z_k^{PC} &= E[Z_k] + \sum_{j=1}^d a_{k,j} G_j \\ &+ \sum_{1 \leq j < l \leq d} a_{k,j,l} (G_j G_l - E[G_j G_l]). \quad (3) \end{aligned}$$

The coefficients  $\{a_{k,j}, a_{k,j,l}\}$  in (3) are determined by minimizing the objective function

$$\begin{aligned} e_1(a_{k,j}, a_{k,j,l}) &= g_1 E[ \|Z - Z^{PC}\|_2^2 ] \\ &+ g_2 \max_{1 \leq i_1 < i_2 \leq d} \|h_{i_1, i_2}(\cdot) - h_{i_1, i_2}^{PC}(\cdot | a_{k,j}, a_{k,j,l})\|_2 \\ &+ g_3 ( \|E[ZZ^T] - E[Z^{PC}(Z^{PC})^T]\| ) \quad (4) \end{aligned}$$

where  $h_{i_1, i_2}(\cdot)$  is the histogram of  $(Z_{i_1}, Z_{i_2})$  and  $h_{i_1, i_2}^{PC}(\cdot | a_{k,j}, a_{k,j,l})$  is the histogram of  $(Z_{i_1}^{PC}, Z_{i_2}^{PC})$  for given coefficients  $\{a_{k,j}, a_{k,j,l}\}$ . The Matlab function `histcounts2` is used to construct the two dimensional histograms of  $(Z_{i_1}, Z_{i_2})$  and  $(Z_{i_1}^{PC}, Z_{i_2}^{PC})$ . The error between the two matrices is quantified by the norm  $\|\cdot\|_2$ , i.e., the absolute largest eigenvalue of the matrix. We consider the set of all pairs of components rather than all components to minimize calculations. The weighting coefficients  $g_1, g_2, g_3$  are such that the components  $E[ \|Z - Z^{PC}\|_2^2 ]$ ,  $\max_{1 \leq i_1 < i_2 \leq d} \|h_{i_1, i_2}(\cdot) - h_{i_1, i_2}^{PC}(\cdot | a_{k,j}, a_{k,j,l})\|_2$  and  $\|E[ZZ^T] - E[Z^{PC}(Z^{PC})^T]\|$  contribute equally to the objective function (4). We set  $g_1 = 0$  if  $Z$  and  $Z^{PC}$  are not defined on the same probability space since the mean error  $E[ \|Z - Z^{PC}\|_2^2 ]$  cannot be calculated.

The components of the PCT models are defined by

$$Z_k^{PCT} = F_k^{-1} \circ F_k^{PC}(Z_k^{PC}), \quad k = 1, \dots, d, \quad (5)$$

where  $F_k^{PC}$  is the distribution of  $Z_k^{PC}$  for given coefficients  $\{a_{k,j}, a_{k,j,l}\}$ . The coefficients  $\{a_{k,j}, a_{k,j,l}\}$  in (5) are determined by minimizing the objective

function

$$\begin{aligned}
 & e_2(a_{k,j}, a_{k,j,l}) \\
 & = w_1 E[ \|Z - Z^{PCT}\|_2^2 ] \\
 & + w_2 \max_{1 \leq i_1 < i_2 \leq d} \|s_{i_1, i_2}(\cdot) - s_{i_1, i_2}^{PCT}(\cdot | a_{k,j}, a_{k,j,l})\|_2 \\
 & + w_3 \max_{1 \leq i_1 < i_2 \leq d} \|h_{i_1, i_2}(\cdot) - h_{i_1, i_2}^{PCT}(\cdot | a_{k,j}, a_{k,j,l})\|_2,
 \end{aligned} \tag{6}$$

where  $h_{i_1, i_2}(\cdot)$  is as in (4),  $s_{i_1, i_2}(\cdot)$  is the spectral measure of  $(Z_{i_1}, Z_{i_2})$ ,  $s_{i_1, i_2}^{PCT}(\cdot | a_{k,j}, a_{k,j,l})$  and  $h_{i_1, i_2}^{PCT}(\cdot | a_{k,j}, a_{k,j,l})$  are the spectral measure and the histogram of  $(Z_{i_1}^{PCT}, Z_{i_2}^{PCT})$  for given coefficients  $\{a_{k,j}, a_{k,j,l}\}$ . Spectral measures of  $(Z_{i_1}, Z_{i_2})$  are metrics which quantify the likelihood that  $(Z_{i_1}, Z_{i_2})$  are simultaneously large, see (5.3) and (5.4) in Grigoriu. (2019) for definitions and Resnick. (2007), (Chap. 6) for technical details. We sort the samples of the two-dimensional vectors  $(Z_{i_1}, Z_{i_2})$  and  $(Z_{i_1}^{PCT}, Z_{i_2}^{PCT})$  according to their lengths such that the first sample is the furthest to the origin and construct the spectral measures from the top 10% of these samples. The Matlab function `histcounts2` is used to construct the two dimensional histograms and spectral measures of  $(Z_{i_1}, Z_{i_2})$  and  $(Z_{i_1}^{PCT}, Z_{i_2}^{PCT})$ . We consider the set of all pairs of components rather than all components to minimize calculations. The weighting coefficients  $w_1, w_2, w_3$  are such that the components  $E[ \|Z - Z^{PCT}\|_2^2 ]$ ,  $\max_{1 \leq i_1 < i_2 \leq d} \|s_{i_1, i_2}(\cdot) - s_{i_1, i_2}^{PCT}(\cdot | a_{k,j}, a_{k,j,l})\|_2$  and  $\max_{1 \leq i_1 < i_2 \leq d} \|h_{i_1, i_2}(\cdot) - h_{i_1, i_2}^{PCT}(\cdot | a_{k,j}, a_{k,j,l})\|_2$  contribute equally to the objective function (6). We set  $w_1 = 0$  if  $Z$  and  $Z^{PCT}$  are not defined on the same probability space since the mean error  $E[ \|Z - Z^{PCT}\|_2^2 ]$  cannot be calculated. The second and third terms of  $e_2(a_{k,j}, a_{k,j,l})$  quantify differences between the dependence structure of  $Z$  and  $Z^{PCT}$ . The third term is an approximate metric for the differences between the joint distributions of  $Z$  and  $Z^{PCT}$  while the second term measures the differences between the tail dependence of these random vectors.

**Example 1.** Let  $X_1(t), X_2(t)$ ,  $0 \leq t \leq \tau$ , be real-valued processes defined by the differential equa-

tions

$$\begin{aligned}
 \dot{X}_1(t) + \alpha_1 \dot{X}_1(t) + \beta_1 X_1(t) & = \kappa_1 V(t), \\
 \dot{X}_2(t) + \alpha_2 \dot{X}_2(t) + \beta_2 X_2(t) & = \kappa_2 V(t)
 \end{aligned} \tag{7}$$

with the initial conditions  $X_i(0) = 0$  and  $\dot{X}_i(0) = 0$ ,  $i = 1, 2$ , where  $\alpha_i, \beta_i, \kappa_i > 0$ ,  $i = 1, 2$  are constants. The input is the translation process  $V(t) = F^{-1} \circ \Phi(W(t))$ , where  $F$  is the Gamma distribution function with the shape parameter  $\nu$  and scale parameter 1,  $W(t)$  is the stationary solution of  $dW(t) = -\vartheta W(t) dt + \sqrt{2\vartheta} dB(t)$ ,  $\vartheta > 0$ , and  $B$  denotes the standard Brownian motion.

From Grigoriu. (2021), (Chap. 2), the solution of (7) is

$$X_i(t) = \int_0^t \frac{\kappa_i}{\psi_i} e^{-\alpha_i(t-u)/2} \sin(\psi_i(t-u)) V(u) du, \tag{8}$$

where  $\psi_i = (\beta_i - \alpha_i^2/4)^{1/2}$ ,  $i = 1, 2$ .

Our objective is to construct FD models for the vector-valued process  $(X_1(t), X_2(t))$ . Since (7) has to be solved numerically,  $V(t)$  and  $(X_1(t), X_2(t))$  are defined and calculated at a finite set of times, e.g., the equally spaced times  $t_i = i\Delta t$ ,  $i = 1, \dots, n$ , where  $\Delta t = \tau/n$  denotes the integration time step. Denote by  $\eta = (V(t_1), \dots, V(t_n))$  and  $\zeta_i = (X_i(t_1), \dots, X_i(t_n))$ ,  $i = 1, 2$ , the discrete versions of the input  $V(t)$  and of the processes  $X_i(t)$ ,  $i = 1, 2$ . The random vector  $\eta$  admits the representation  $\eta = \sum_{k=1}^n Z_k v_k$ , where  $\{v_k\}$  are the eigenvectors of the covariance matrix  $E[\eta \eta^T]$  and the random coefficients  $\{Z_k\}$  are defined sample by sample by projection, i.e.,  $Z_k(\omega) = \eta^T(\omega) v_k$ ,  $\omega \in \Omega$ . The corresponding FD model is  $\eta_d = \sum_{k=1}^d Z_k v_k$ ,  $d \leq n$ . Since the differential equations (7) are linear, their solutions to  $\eta$  and  $\eta_d$  are linear forms of  $\{Z_k\}$  denoted by  $\zeta_i = \{\zeta_{i,j}\}$  and  $\zeta_{d:i} = \{\zeta_{d:i,j}\}$ ,  $i = 1, 2$ ,  $j = 1, \dots, n$ .

The thin solid lines of the top and bottom panels of Fig. 1 are estimates of  $P(\|\zeta_i\| > x)$  for  $i = 1$  and  $i = 2$  which are obtained directly from data, where  $\|\zeta_i\| = \max_{1 \leq j \leq n} |\zeta_{i,j}|$ . These probabilities are viewed as truth. The other lines of the figure are calculated from samples of  $\zeta_{d:i}$  (heavy solid lines),  $\zeta_{d:i}^{PC}$  (dotted lines) and  $\zeta_{d:i}^{PCT}$  (dashed lines) for the first and second components (top and bottom panels). The heavy solid lines are the closest to the truth.

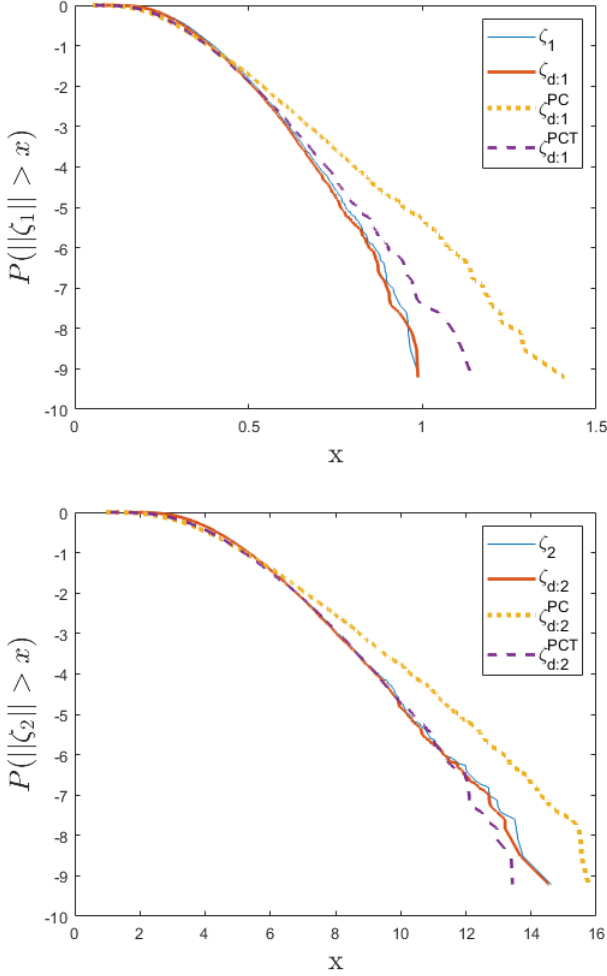


Figure 1: Estimates of the target probability  $P(\|\zeta_i\| > x)$  based on data (thin solid line), FD model (heavy solid line), finite dimensional PC model (dotted line) and finite dimensional PCT model in logarithmic scale (dashed line) for  $\zeta_i = (X_i(t_1), \dots, X_i(t_n))$ ,  $i = 1, 2$  (top and bottom panels).

The next best model is  $\zeta_{d:i}^{PCT}$  while  $\zeta_{d:i}^{PC}$  differs significantly from the truth. We prefer  $\zeta_{d:i}^{PCT}$  to  $\zeta_{d:i}$  since the set of samples of  $\zeta_{d:i}$  is defined by the available data so that it cannot be extended. In contrast, samples of any size can be generated from  $\zeta_{d:i}^{PCT}$  since its probability law is known.

## 5. FD MODEL FOR WIND FORCES

FD models are constructed for the vector-valued wind pressure time series  $X(t) = (X_1(t), \dots, X_m(t))$  recorded in the wind tunnel of the University of Florida and these models are used to estimate distributions of this time series.

Wind tunnel records are used to estimate the correlation functions of the vector-valued process  $X(t)$ , find the eigenfunctions of these functions and calculate the samples of the random coefficients  $\{Z_{i,k}\}$  of the FD models of the components of  $X(t)$  by projection as discussed in Sect. 2.

The joint distribution of the random vector whose components are the random variables  $\{Z_{i,k}\}$  is obtained by translating polynomial chaos representation such that they match exactly the target marginal distribution, see (5). These models can be used to generate samples of the random coefficients  $\{Z_{i,k}\}$  which are used to find the corresponding samples of FD models of  $X(t)$ .

Let  $\mathcal{X}_i = (X_i(t_1), \dots, X_i(t_n))^T$ ,  $i = 1, \dots, m$ , denote the components of the  $m$ -dimensional time series describing the wind model. The functional form of the FD models under consideration is in Sect. 2. The models, denoted by  $\mathcal{X}_{d:i}^{IND}$ ,  $\mathcal{X}_{d:i}^{PC}$  and  $\mathcal{X}_{d:i}^{PCT}$ , are elements of the space spanned by the same vectors  $\{v_{i,k}\}$ , but their coefficients differ. The random coefficients of  $\mathcal{X}_{d:i}^{PC}$  are given by the PC model. The random coefficients of  $\mathcal{X}_{d:i}^{IND}$  and  $\mathcal{X}_{d:i}^{PCT}$  have the same marginal distributions but they are independent for  $\mathcal{X}_{d:i}^{IND}$  and dependent given by the PCT model for  $\mathcal{X}_{d:i}^{PCT}$ .

The plots of Fig. 2 are estimates of the distribution of the fifth component of the vector-valued time series  $\mathcal{X}_i$ ,  $i = 1, \dots, m$ . The thin solid lines are the probabilities  $P(\|\mathcal{X}_5\| > x)$ , which are obtained directly from data, where  $\|\mathcal{X}_5\| = \max_{1 \leq j \leq N} |X_5(t_j)|$ . These probabilities are viewed as truth. The other lines of the figure are calculated from samples of  $\mathcal{X}_{d:5}$  (heavy solid lines),  $\mathcal{X}_{d:5}^{IND}$  (dash-dotted lines),  $\mathcal{X}_{d:5}^{PC}$  (dotted lines) and  $\mathcal{X}_{d:5}^{PCT}$  (dashed lines). The heavy solid lines are the closest to the truth. The next best model is  $\mathcal{X}_{d:5}^{PCT}$  while  $\mathcal{X}_{d:5}^{PC}$  differs significantly from the truth. The estimates are unsatisfactory for  $\mathcal{X}_{d:5}^{IND}$ , an expected result since the resulting FD wind model is approximately Gaussian.

## 6. CONCLUSIONS

Finite dimensional (FD) models, i.e., deterministic functions of time and finite sets of random variables, have been constructed for a test case and a wind pressure time series recorded at the UFBLWT

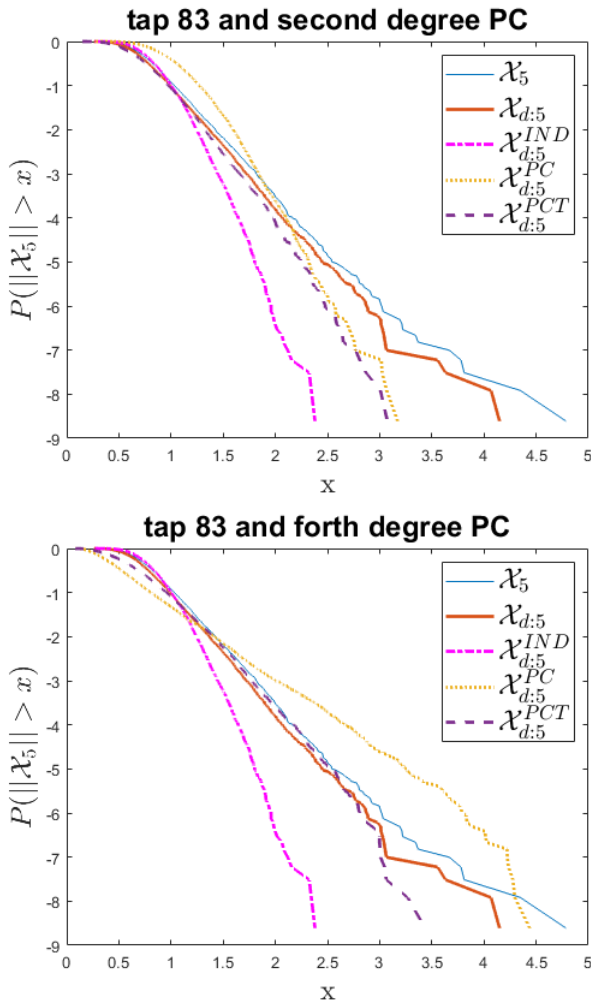


Figure 2: Estimates of the target probability  $P(\|\mathcal{X}_5\| > x)$  based on data (thin solid line), FD model (heavy solid line), finite dimensional independent model (dash-dotted line), finite dimensional PC models (dotted line) and finite dimensional PCT model in logarithmic scale (dashed line) with second and fourth degree polynomial chaos (top and bottom panels).

facility in Gainesville by using polynomial chaos (PC) and polynomial chaos translation (PCT) models to represent their random coefficients. The components of PCT models are obtained from those of PC models by translation, so that they match exactly the target marginal distributions irrespective of the coefficients in their definition. The optimal values of the PCT coefficients minimize the discrepancy between the PCT and target joint properties, which are quantified by joint distributions and spectral measures.

The FD models with PCT random coefficients

are superior to those with PC coefficients in the sense that the distributions of extremes of PCT-based FD models are similar to those of target time series while PC-based FD models exhibit notable errors.

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