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A theoretical model of lateral coherence in isotropic turbulence based on stochastic Fourier wave-number spectrum

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ABSTRACT: With the development of measuring instruments and the study of turbulence theory, the field measurement of the wind field has become increasingly accurate and refined. Thus, these improvements provide us with a new perspective on the old problem, which is how to solve problems through the physical mechanism of objective phenomena rather than relying on phenomenological experience and statistical results. In the present paper, an observation array was built to measure the near-surface strong winds in Southeast China. Based on extensive observation data, the stochastic Fourier spectrum model of fluctuating wind is verified. Then, under the assumption that the flow is stationary and homogeneous, a theoretical model of lateral coherence with a solid physical basis is derived, which is a segmentation function corresponding to the different energy ranges in the turbulence energy spectrum. This model portrays the coherence between the stochastic Fourier spectrum of two spatial points. It is revealed how the evolution of the probabilistic information of fundamental physical quantities affects the spatial coherence of the wind field when the wave number is small or the lateral separation is short. The theoretical results accord with the measurements well.

1. INTRODUCTION

The fluctuating wind speed processes are a typical class of stochastic processes. The power spectral density (PSD) is an important index of the stochastic process, which is one of the prerequisites for the stochastic vibration analysis of structural wind effects (Li & Zhang, 2004). However, the essence of PSD is still the second-order moment statistical characteristic of the stationary process, which is difficult to portray a wealth of probabilistic information on fluctuating wind speed in the atmosphere. To overcome this difficulty, Li (2008) introduced the physical mechanism into the modeling of dynamic excitation and proposed a stochastic function model to express the dynamic excitation, which can effectively overcome the limitations of PSD and make it possible to solve various problems in the study of structural stochastic dynamic systems.

In this study, it is considered that the wind field probabilistic information can be expressed by an abstracted stochastic Fourier function model from the fundamental physical quantities with the help of physical equations. Therefore, combined with the stochastic Fourier function model and the turbulence theory, a stochastic Fourier coherence function model that portrays the coherence degree of wind speed at two points in lateral displacement is established based on the one-dimensional stochastic Fourier spectrum model, and a cross-stochastic Fourier spectrum model is further established.

2. STOCHASTIC FOURIER FUNCTION MODEL

If we regard a set of measured wind speed records $x(t)$ as an ensemble of random samples, the Fourier transform of each random sample is denoted as F'. Then for the corresponding measured record, the relationship between the average energy W and F' can be expressed as following (Li & Zhang, 2007):

$$W = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{T}\left|F'(n)\right|^2 \mathrm{d}n \qquad (1)$$

If we understand Eq. (1) from the perspective of energy decomposition, it can be seen that F' can be regarded as a class of spectral decomposition of the average energy. If a specific physical mechanism is introduced, the Fourier spectral density sample (FSDS) can be defined in combination with Eq. (1) as

$$F_x(\eta,n) = \frac{1}{\sqrt{T}}\int_0^T x(\eta,t)e^{-i2\pi nt}\mathrm{d}t \qquad (2)$$

where η is a fundamental random variable or vector with physical significance that affects the development process of the random excitation.

Further, since the stochastic process $X(t)$ is the statistical ensemble of measured samples $x(t)$, the statistical ensemble of FSDSs which are obtained from all the measured samples after the Fourier transform can be abstracted as a stochastic function, i.e. the stochastic Fourier function

$$F_X(\eta,n) = \frac{1}{\sqrt{T}}\int_0^T X(\eta,t)e^{-i2\pi nt}\mathrm{d}t \qquad (3)$$

It is easy to demonstrate the relationship between the PSD $S_X(n)$ and the stochastic Fourier function $F_X(\eta,n)$ of a stochastic process:

$$S_X(n) = E\left[F_X^2(\eta,n)\right] = \int_0^{\infty} F_X^2(\eta,n)\,p_\eta(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta} \quad (4)$$

where $E(\cdot)$ denotes the expectation operator and $p_\eta(\boldsymbol{\eta})$ is the joint probability density function of the fundamental random variable $\boldsymbol{\eta}$.

The random sample $x(t)$ contains fundamental physical information. And the Fourier transform of the random sample $x(t)$ provides the FSDS. Then a theoretical abstraction of the ensemble of FDSSs provides the random Fourier function $F_X(\eta,n)$. Conversely, the inverse Fourier transform of a sample ensemble of $F_X(\eta,n)$ can also provide the corresponding random process samples $x(t)$. Both types of description methods have equivalence. More importantly, the probabilistic information of the fundamental physical quantities is transferred through the process, and a probabilistic relationship is established between the random process samples and the random Fourier function.

The stochastic Fourier function model in this paper is associated with physically significant fundamental random variables. And this reflective approach not only constitutes an inherent connection but also provides the possibility of modeling by using a set of measured samples, which provides a theoretical basis for a more comprehensive way to reflect the probabilistic information of fluctuating wind fields.

3. STOCHASTIC MODEL OF FLUCTUATING WIND FIELDS

3.1. One-dimensional stochastic Fourier function

Richardson proposed the "turbulence cascades" model which assumes turbulence is composed of vortices in different scales. Based on the turbulent motion and energy transport, Kolmogorov further divides the turbulent energy spectrum into three subranges: the energy-bearing range, the inertial subrange and the dissipative range, and assumes that the turbulent energy spectrum obeys different power laws in different subranges.

First, both theories and experiments show that the energy spectrum in the inertial subrange obeys the "-5/3" power law. An expression can be derived by the magnitude analysis (Kolmogorov, 1941):

$$E_1(k_1) = \alpha_1\varepsilon^{2/3}k_1^{-5/3} \qquad (5)$$

where E_1 is the one-dimensional turbulence energy spectrum, k_1 is the longitudinal wave number, α_1 is the one-dimensional Kolmogorov constant, and ε is the dissipation rate.

Then, in the energy-bearing range, it is more complicated due to the influence of terrain friction. However, several wind tunnel tests and field measurements have shown that the turbulence energy spectrum in the energy-bearing range obeys the "-1" power law (Tchen & Panchev, 1998; Kader & Yaglom, 1984). Based on experiments, Katul and Chu (1998) suggested the transition between the two ranges in the energy spectrum is so short that it can be approximated as a continuous but not smooth point.

Based on scientific inference, Li and Yan (2011) suggested that there is a distinct demarcation point between the energy-bearing and inertial subrange in the energy spectrum, which is named the demarcation wave number. The demarcation wave number can be identified by calculating the FSDS on a period of wind speed record and restoring the energy distribution of vortices in different subranges. Therefore, Li and Yan proposed a bilinear-model for the amplitude of the stochastic Fourier wave number spectrum of longitudinal fluctuating wind speed, by considering the relationship between PSD of the inertial subrange and the stochastic Fourier function as well as the continuity condition:

$$\left|F_1(\eta,k_1)\right| = \begin{cases} \sqrt{\alpha_1}\dfrac{u_*}{(\kappa z k_c)^{1/3}}k_1^{-1/2} & ,k_1 < k_c \\ \sqrt{\alpha_1}\dfrac{u_*}{(\kappa z)^{1/3}}k_1^{-5/6} & ,k_1 \geq k_c \end{cases} \tag{6}$$

where subscript 1 denotes the one-dimensional state; u_* is the shear wave velocity; κ is the von Karman constant; z is the height; and k_c is the demarcation wave number.

In the physical background, the boundary point between the two ranges can be characterized by the demarcation wavelength l_c ($l_c = 1/k_c$). It means, vortices with scales larger than l_c correspond to the energy-bearing range, where they derive energy from the main flow; and vortices with scales smaller correspond to the inertial subrange, where they derive energy from the larger vortices and deliver it to the smaller vortices in the dissipative range (Figure 1.).

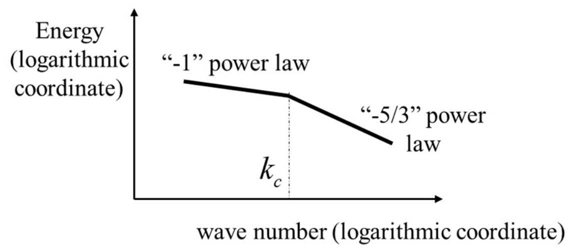


Figure 1. Schematic diagram of the power law

3.2. Stochastic Fourier coherence function

3.2.1. Definition of stochastic Fourier coherence function

In terms of the relationship (4) between the stochastic Fourier function and the PSD, it is useful to infer the relationship between the cross-stochastic Fourier function and the cross-power spectral density (CPSD) as following:

$$S_{XY}(n) = E\left[F_{XY}^2(\eta,n)\right] = \int_0^\infty F_{XY}^2(\eta,n)\,p_\eta(\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta} \tag{7}$$

where S_{XY} is the CPSD and F_{XY} is the cross-stochastic Fourier function. Since atmospheric turbulence can be regarded as homogeneous and isotropic turbulence ideally, the CPSD is completely real. Therefore, the cross-stochastic Fourier function F_{XY} is also real.

Further, let us consider the coherence function of homogeneous and isotropic turbulence. The coherence function is defined as the ratio of the CPSD to the PSD (Welch, 1967):

$$\gamma = \frac{\left|S_{XY}(n)\right|}{S_X(n)} = \frac{E\left[F_{XY}^2(\eta,n)\right]}{E\left[F_X^2(\eta,n)\right]} \tag{8}$$

Obviously, the coherence function is absolutely deterministic. If the coherence of fluctuating wind speed between two points is influenced by the fundamental physical quantities, the coherence function should be random. Based on scientific inference, we can consider the coherence function from the perspective of the sampled ensemble, then the definition of the coherence function can be extended to a random variable. By utilizing the mean relationship and the independence between each random variable, we can derive the expression of $\varGamma$:

$$\varGamma_{XY}=\frac{\left|F_{XY}\left(\eta,n\right)\right|^{2}}{\left|F_{X}\left(\eta,n\right)\right|\cdot\left|F_{Y}\left(\eta,n\right)\right|} \tag{9}$$

Notice that, Eq. (9) can also be applicated to the inhomogeneous turbulence field.

3.2.2. Stochastic Fourier coherence function based on turbulence theory

For a stationary and homogeneous random field, the correlation function at any two points A and B is only associated with the displacement vector difference $\boldsymbol{\xi}=\boldsymbol{x}_A-\boldsymbol{x}_B$ and the time difference $\tau=\tau_A-\tau_B$. For the sake of simplicity, let the mean wind direction be along with the x_1 -axis (denoted as the longitudinal direction), and the line between A and B be along with the x_2 -axis (denoted as the lateral direction) which is perpendicular to the mean wind direction. And the distance between the two points is D . The geometric schematic is shown in Figure 2..

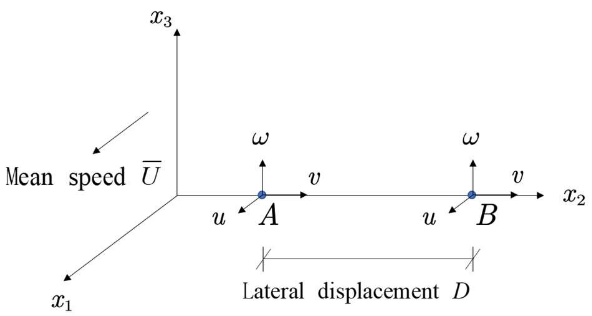


Figure 2. The geometric schematic

For a zero-mean stationary random process in a homogeneous random field, the cross-correlation function of two points is

$$R_{ij}\left(\boldsymbol{\xi},\tau\right)=R_{ij}\left(D\boldsymbol{e}_2,\tau\right) \tag{10}$$

where the first subscript i denotes the velocity component along i direction at point A , the second subscript j denotes the component along j direction at point B , $i,j=1,2,3$; $\boldsymbol{e}_2$ denotes the basis vector along x_2 -axis.

If we assume that the mainstream vorticity is much larger than the fluctuating vorticity, and there is no significant variation in a short distance, then we can adopt the Taylor "freezing assumption" to transform the time and space dimensions, which is expressed mathematically as the cross-correlation function of two points being equal to the cross-correlation function by taking the time-lag $\tau=\xi_1/\overline{U}$. The Fourier transform of Eq. (10)gives the CPSD in lateral displacement:

$$S_{ij}\left(De_2,\omega\right)=\frac{1}{2\pi}\int_0^{\infty}R_{ij}\left(De_2-\overline{U}\tau e_1\right)\cdot e^{i\omega\tau}\mathrm{d}\tau \tag{11}$$

According to turbulence theory, the correlation function of two points can also be derived from the energy spectrum tensor $E_{ij}\left(\boldsymbol{k}\right)$ by a three-dimensional Fourier transform in the wave number domain:

$$R_{ij}\left(De_2-\overline{U}\tau e_1\right)=\iiint E_{ij}\left(\boldsymbol{k}\right)\cdot e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{d}\boldsymbol{k} \tag{12}$$

where $\boldsymbol{k}$ is the wave number vector; $E_{ij}\left(\boldsymbol{k}\right)$ is the energy spectrum tensor, which can be represented by the three-dimensional energy spectrum $E\left(k\right)$ (Batchelor, 1953):

$$E_{ij}\left(\boldsymbol{k}\right)=\frac{E\left(k\right)}{4\pi k^2}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right) \tag{13}$$

where $k^2=\left|\boldsymbol{k}\right|^2$, δ_{ij} is the Kronecker notation.

To introduce the stochastic Fourier function model, it is worth defining the three-dimensional stochastic Fourier spectrum $F\left(\eta,k\right)$ similar to Eq. (4) for a transition, i.e.,

$$E\left(k\right)=E\left[F^2\left(\eta,k\right)\right] \tag{14}$$

Considering the relationship between $E\left(k\right)$ and the one-dimensional energy spectrum $E_1\left(k_1\right)$ in turbulence theory. we know that both $E\left(k\right)$ and $E_1\left(k_1\right)$ obey the same power law. Therefore, combining with Eq. (4) and Eq. (14), we can infer that $F(\eta,k)$ and $F_1(\eta,k_1)$ also obey the same power law.

Substituting Eq. (12), (13) and (14) into Eq. (11) and combining the relationship between S_{XY} and F_{XY} in Eq. (7), and considering the mean relationship as a connection, the expression of the cross-stochastic Fourier function corresponding to each statistical sample is established.

For engineering, the effect of longitudinal fluctuating wind speed is considered generally, therefore the situation when $i = j = 1$ is analyzed. Considering the power law of the $|F(\eta,k)|$ discussed above, and utilizing the handbook of mathematics (Abramowitz, 1968) and the polar coordinate, the expression of $|F_{11}|$ is

$$\begin{aligned}&\left|F_{11}(\eta,D,k_1)\right|^2\cdot\overline{U}\\&=\int_0^{\infty}\left|F(\eta,D,k_1)\right|^2\frac{KJ_0(DK)}{2\left(k_1^2+K^2\right)}\mathrm{d}K\\&\quad-k_1^2\int_0^{\infty}\left|F(\eta,D,k_1)\right|^2\frac{KJ_0(DK)}{2\left(k_1^2+K^2\right)^2}\mathrm{d}K\end{aligned} \tag{15}$$

where $J_v(\cdot)$ denotes the first-class Bessel function of order v, $K=\sqrt{k_2^2+k_3^2}$.

Substituting Eq. (15) into Eq. (9), a distinct expression for the stochastic Fourier coherence function of the longitudinal fluctuating wind speed in the lateral distance is derived.

Since the transition of the "-1" power law and "-5/3" power law in the energy spectrum is considered as a continuous but not smooth point (Katul & Chu, 1998), $F_1(\eta,k_1)$ is a segmentation function and is not derivable at demarcation wave number k_c. Consequently, Γ_{11} derived from $F_1(\eta,k_1)$ is discontinuous, which is not consistent with the real physical situation. Therefore, a dimensionless function $\varphi(B)=\exp(B)$ in exponential form is considered to make the segmentation function continuous:

$$\begin{aligned}&\Gamma_{11}(D,k_1)\\&=\begin{cases}\exp(B)\cdot\dfrac{(Dk_1/2)^{1/2}}{\Gamma(1/2)}\left[2K_{1/2}(Dk_1)-Dk_1K_{1/2}(Dk_1)\right], & k_1<k_c\\ \dfrac{(Dk_1/2)^{5/6}}{\Gamma(5/6)}\left[2K_{5/6}(Dk_1)-Dk_1K_{1/6}(Dk_1)\right] & ,k_1\geq k_c\end{cases}\end{aligned} \tag{16}$$

where

$$B=\frac{k_1}{k_c}\cdot\ln\frac{(Dk_c/2)^{5/6}\left[2K_{5/6}(Dk_c)-Dk_cK_{1/6}(Dk_c)\right]\Gamma(1/2)}{(Dk_c/2)^{1/2}\left[2K_{1/2}(Dk_c)-Dk_cK_{1/2}(Dk_c)\right]\Gamma(5/6)} \tag{17}$$

where $K_v(\cdot)$ denotes the second-class Bessel function of order v, $\Gamma(\cdot)$ denotes the gamma function.

It can be seen that the stochastic Fourier coherence function is derived on the theoretical basis of homogeneous and isotropic turbulence. For the atmospheric boundary layer turbulence, it tends to be more anisotropic due to the friction by buildings and vegetation, etc. When the scale of the structure is much larger than the turbulence integral scale L (i.e. $D \gg L$, where L is along the longitudinal direction), some correction needs to be considered. In this study, we define the modified one-dimensional wave number k_1' to replace the one-dimensional wave number k_1:

$$k_1'=\frac{2\pi}{D}\sqrt{\left(\frac{D}{L}\right)^2+(k_1D)^2} \tag{18}$$

where L is calculated by the formula (ESDU):

$$L(z)=L_{ref}\left(z/z_{ref}\right)^{0.3},10\mathrm{m}\leq z\leq 200\mathrm{m} \tag{19}$$

witth the reference height $z_{ref}=10\mathrm{m}$ and the turbulence integral scale $L_{ref}=100\mathrm{m}$.

4. VERIFICATION

4.1. Statistics of fundamental random variables

In this paper, field measurements of near-surface strong winds in Southeast China were carried out, where an observation array consisting of four towers and twelve ultrasonic anemometers were employed (Figure 3.). The lateral displacement can be 30m, 60m, 90m, 120m, 150m and 180m according to the permutations and combinations.

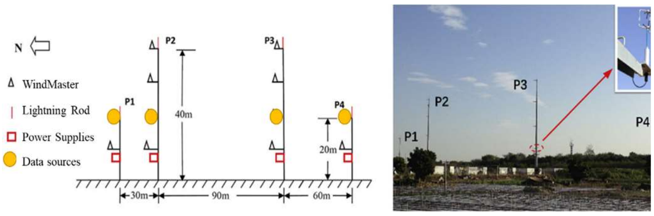
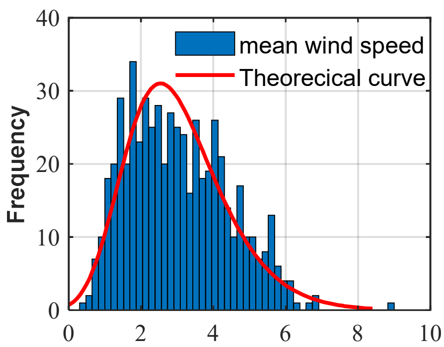


Figure 3. Xiamen observation array

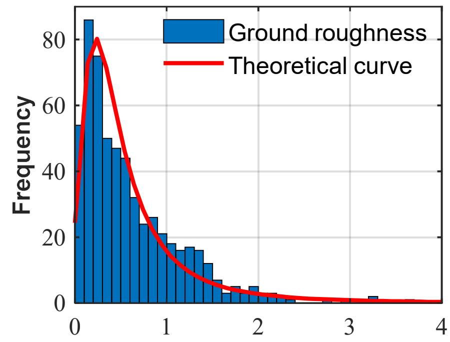
According to the assumptions in the literature (Yan, 2011), the fundamental random variables are taken as the ground roughness z_0 and the 10-minute mean wind speed $\overline{U}_{10}$ at a height of 10 m. z_0 follows a lognormal distribution and $\overline{U}_{10}$ follows an extreme type-I distribution. And 576 sets of 10-minute wind speed records at 20m of the P1, P2, P3 and P4 towers in July 2015 were chosen to verify the coherence model in different lateral directions. The statistical results of the fundamental random variables are shown in Table 1 and Figure 4..

Table 1: Statistics of fundamental random variables

Tower	z_0		U_{10}	
	μ	σ	μ	σ
P1	0.6168	0.5756	3.8403	1.8113
P2	0.6586	0.6668	3.8593	1.8220
P3	1.7312	0.8889	3.1230	1.4628
P4	1.1940	1.2798	3.0304	1.3843



(a) Histogram of $\overline{U}_{10}$ at P4

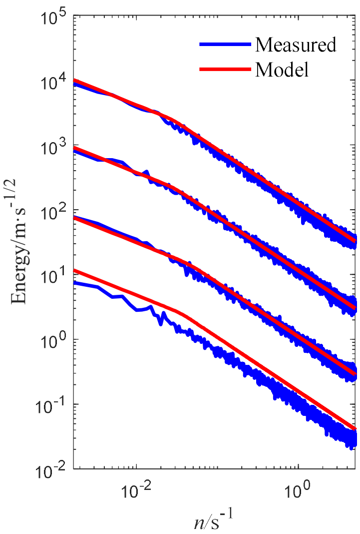


(b) Histogram of z_0 at P1

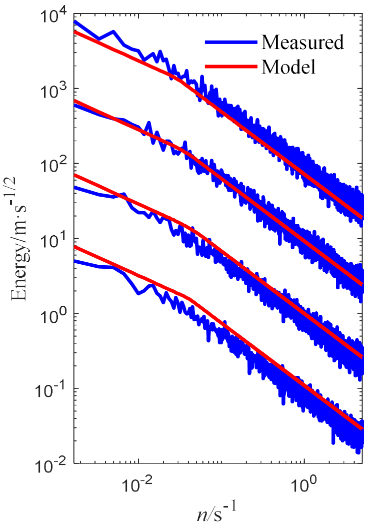
Figure 4. Statistical histograms of fundamental random variables

4.2. Verification of one-dimensional stochastic Fourier function

The FSDSs of the measurements are calculated according to Eq. (2), and the mean and standard deviation spectra are counted, respectively. Then 600 samples of the fundamental random variables are randomly generated according to the corresponding distribution types by the computer. The fundamental random variables are transferred through the physical Eq. (6), and FSDSs of the model can be obtained to calculate several pieces of probabilistic information. In this study, we mainly compare the mean and standard deviation spectra.



(a) Mean spectrum



(b) Standard deviation spectrum

Figure 5. Verification of one-dimensional stochastic Fourier function

As shown in Figure 5., the mean and standard deviation spectra which are obtained from the measurements of the four towers at a height of 20 m agree well with the model, indicating that the proposed one-dimensional stochastic Fourier function can reflect the real probability situation precisely. Therefore, the probability distribution characteristics of the target function can be obtained by grasping the probability information of the fundamental physical quantities. For the clarity of the drawing, the curves of the P2, P3 and P4 towers are shifted upward by a factor of 10, 100 and 1000, respectively.

4.3. Verification of cross-stochastic Fourier function

According to the permutation and combination, we can calculate 6 different distances. For the stochastic Fourier coherence function, the mean value is verified. The measured coherence function is calculated according to Eq. (8) and used as the mean value. The computer is used to generate 600 samples of the fundamental random variables randomly according to the corresponding distribution type and the modified one-dimensional wave number is calculated by Eq. (18). Then the stochastic Fourier coherence function samples are calculated by the physical equation (16), and the probabilistic information is obtained statistically. The empirical coherence formula is calculated by using the Davenport decay exponential model (Davenport, 1962).

For the cross-stochastic Fourier function, the mean and standard deviation spectra are mainly verified.

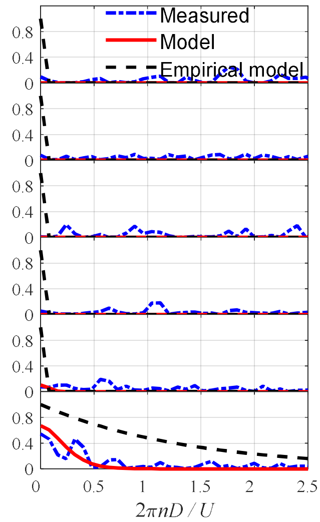
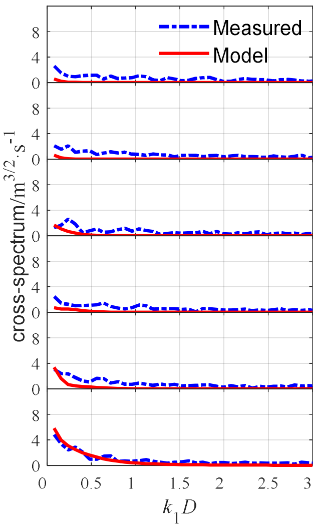
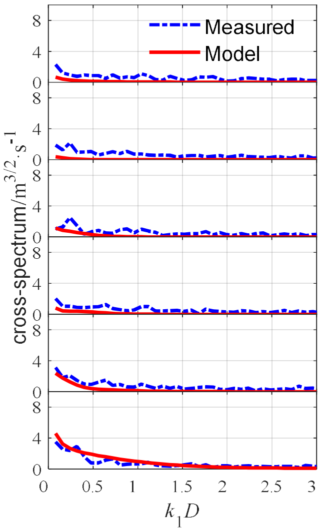


Figure 6. Verification of stochastic Fourier coherence function



(a) Mean spectrum



(b) Standard deviation spectrum

Figure 7. Verification of cross-stochastic Fourier function

From Figure 6. and Figure 7. we can see that both the results of the measurement and the theoretical model are in good agreement, which means the stochastic Fourier function can describe the transformation of the probabilistic information of fundamental random variables.

5. CONCLUSION

This paper investigates the probability evolution in the fluctuating wind field and applies the stochastic Fourier function to reveal the physical mechanism. Some conclusions are as follows:

(1) The stochastic Fourier function can effectively describe the probabilistic information. The probabilistic information of FSDSs can be obtained by the statistics of the fundamental random variables and using the physical equations.

(2) The stochastic Fourier coherence function based on the turbulence theory and stochastic Fourier function can not only reflect the coherence of wind speed in mean value, but also provides more probabilistic information.

(3) The cross-stochastic Fourier function based on the stochastic Fourier coherence function is no longer a second-order moment statistical characteristic compared to CPSD, but a physical model with rich probabilistic information from the fundamental physical quantities.

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