Reliability updating in the presence of distribution parameter uncertainty

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ABSTRACT: Reliability updating (RU) is often used to re-evaluate the failure probability of structural system when new observations are obtained. However, in engineering practice, the specific distribution parameters of input variables are generally unavailable due to the lack of the data. To this end, two reliability updating models with distribution parameter uncertainty (DPU) are firstly constructed in this paper based on the theory of nested reliability approach (NRA) and augmented reliability approach (ARA). The constructed reliability updating models can accurately measure the effect of the DPU on the posterior failure probability. Then, two reliability updating algorithms based on subset simulation (SS), named as SS-NRA-RU and SS-ARA-RU, are developed to solve the corresponding RU model, which can significantly improve the computational efficiency of RU of rare event. Finally, three examples prove the efficiency and accuracy of the proposed reliability updating algorithms.

In engineering practice, physical systems can be replaced by the mathematical models for uncertainty analysis, where the established mathematical model requires a proper description of the underlying system inputs and parameters[1]. However, there are always inevitable deviations between the physical system and its mathematical model [2].To calibrate the mathematical model, observation information can be properly incorporated into the process of modeling and simulation [3]. In probabilistic theory, Bayesian model updating can provide a unified computational framework for this purpose [4], and this theory is also extended to update the failure probability of the structural system, which is termed as reliability updating.

Up to now, a lot of reliability updating methods have been developed, wherein a quite straightforward approach is to obtain the posterior samples by Markov Chain Monte Carlo (MCMC), and then estimate the posterior failure probability based on these posterior samples [2]. To avoid the time-consuming problem of numerical simulation methods, Ref. [5] developed an analytical method for estimating the posterior distribution based on Laplace method. However, this approach cannot guarantee the computational accuracy when the input variables are non-Gaussian form. In recent years, Bayesian network has been widely used in engineering practice [6][7]. Nevertheless, it still has many limitations, for example, the conditional probability table is difficult to calculate, and its operation is hard to non-experts [8].

Currently, one of the most efficient methods is to perform reliability updating with structural reliability methods. However, this method is effective only when all observation information is of inequality type. To solve this problem, Straub [9] redefines the likelihood function by introducing an auxiliary variable P, based on which the equality information expressed by a likelihood function can be described as inequality information, and an inequality limit state function can be constructed. Since this approach is applicable for most cases, some reliability methods are then combined with it to deal with the rare events. For example, Ref. [10] explored the combination of this approach with FORM, line sampling (LS) and subset simulation (SS).

The above methods are carried out under the condition that the prior distribution types and parameters are deterministic. However, in engineering practice, both aleatory and epistemic uncertainties are existing for the input variables due to the data scarcity [11]. For this case, the imprecise probabilistic models [12] are often used to characterize the aleatory and epistemic uncertainties, which includes evidence theory [13], probability-box (p-box) [14], fuzzy probability [15] and so on. Currently, the hybrid uncertainties in the input variables have been also considered in Bayesian analysis. However, in the community of reliability updating, hybrid uncertainties are rarely considered.

This paper develops a reliability updating method in the presence of hybrid uncertainties, where the epistemic uncertainty is considered as distribution parameter uncertainty (DPU). To estimate the posterior failure probability in the presence of DPU, two reliability updating models are firstly constructed based on the theory of nested reliability approach (NRA) and augmented reliability approach (ARA) [16][17]. For the updating model based on NRA, it is relatively stable due to the double-layer framework. But repeat reliability analysis in the inner layer will lead to low computational efficiency. Therefore, ARA is used to develop the alternative updating model, where the double-layer framework is transformed into the single-layer one by incorporating the distribution parameters into the input variable space, thereby avoiding the repeated reliability analysis. Considering the computational efficiency, SS method is embedded into the NRA and ARA, and two efficient algorithms, namely SS-NRA-RU and SS-ARA-RU, are proposed for solving the established reliability updating models in this paper.

1. RELIABILITY UPDATING MODELS IN THE PRESENCE OF DPU

1.1. Reliability updating without DPU

In reliability updating, based on the observation information Z, the likelihood function $L(\boldsymbol{x}|Z)$ can be defined by:

$$L(\boldsymbol{x}|Z) \propto \Pr(Z|\boldsymbol{X}=\boldsymbol{x}) \tag{1}$$

If Z contains m independent observations $\boldsymbol{s}=\{s_1,s_2,\cdots,s_m\}$, and the observation error is additive, the likelihood function constructed based on the i-th observation s_i is expressed by:

$$L_i(\boldsymbol{x}|s_i) = f_{\varepsilon_i}(s_i - g_i(\boldsymbol{x})) \tag{2}$$

where $g_i(\boldsymbol{x})$ is the model output corresponding to s_i, $\varepsilon_i = s_i - g_i(\boldsymbol{x})$ and f_{ε_i} is the PDF of ε_i. Thus, the likelihood function constructed based on the observation information Z is as follows:

$$L(\boldsymbol{x}|Z) = \prod_{i=1}^{m} L_i(\boldsymbol{x}|s_i) \tag{3}$$

To transform the equality information into the inequality type, the likelihood function $L(\boldsymbol{x}|Z)$ can be equivalently expressed as [9]:

$$L(\boldsymbol{x}|Z) = \frac{1}{c}\Pr\{P - cL(\boldsymbol{x}|Z) \le 0\} \tag{4}$$

where P is the introduced standard uniform variable, c is a constant that ensures $cL(\boldsymbol{x}|Z) \le 1$, and c is generally taken as $c=1/\max(L(\boldsymbol{x}|Z))$. Based on Eq. (4), the following inequality limit state function can be constructed:

$$h(\boldsymbol{x},p) = p - c \cdot L(\boldsymbol{x}|Z) \tag{5}$$

The acceptable domain Ω_Z is then defined by the inequality limit state function as follows:

$$\Omega_Z = \{p \le c \cdot L(\boldsymbol{x}|Z)\} \tag{6}$$

For the likelihood function defined in Eq. (4), it can be re-expressed based on Ω_Z as:

$$L(\boldsymbol{x}|Z) = \frac{1}{c}\int_{\boldsymbol{x},p\in\Omega_Z} f_P(p)\,\mathrm{d}p \tag{7}$$

where $f_P(p)$ is the PDF of variable P.

Since $L(\boldsymbol{x}|Z) \propto \Pr(Z|\boldsymbol{X}=\boldsymbol{x})$, $\Pr(Z|\boldsymbol{X}=\boldsymbol{x})$ can be expressed as follows based on Eq. (7):

$$\Pr(Z|\boldsymbol{X}=\boldsymbol{x}) = \frac{\alpha}{c}\int_{\boldsymbol{x},p\in\Omega_Z} f_P(p)\,\mathrm{d}p \tag{8}$$

where α is a proportional constant. Then, the expression of $\Pr(Z)$ can be obtained by:

$$\begin{aligned}\Pr(Z) &= \int_X \Pr(Z|\boldsymbol{X}=\boldsymbol{x}) f_X(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &= \frac{\alpha}{c}\int_{\boldsymbol{x},p\in\Omega_Z} f_P(p) f_X(\boldsymbol{x})\,\mathrm{d}p\mathrm{d}\boldsymbol{x}\end{aligned} \tag{9}$$

Similarly, let event F represent system failure, and the expression of $\Pr(F\cap Z)$ is:

$$\begin{aligned}\Pr(F\cap Z) &= \int_X \Pr(F|\boldsymbol{X}=\boldsymbol{x})\Pr(Z|\boldsymbol{X}=\boldsymbol{x}) f_X(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &= \frac{\alpha}{c}\int_{\boldsymbol{x},p\in[\Omega_F\cap\Omega_Z]} f_P(p) f_X(\boldsymbol{x})\,\mathrm{d}p\mathrm{d}\boldsymbol{x}\end{aligned} \tag{10}$$

Thus, $\Pr(F|Z)$ can be expressed by:

$$\Pr(F|Z) = \frac{\Pr(F\cap Z)}{\Pr(Z)} = \frac{\int_{\boldsymbol{x},p_q\in[\Omega_F\cap\Omega_Z]} f_P(p) f_X(\boldsymbol{x})\,\mathrm{d}p\mathrm{d}\boldsymbol{x}}{\int_{\boldsymbol{x},p_q\in\Omega_Z} f_P(p) f_X(\boldsymbol{x})\,\mathrm{d}p\mathrm{d}\boldsymbol{x}} \tag{11}$$

If t observation events are considered, namely $Z=\{Z_1\cap\cdots\cap Z_t\}$, t auxiliary variables and t inequality limit state functions are obtained finally. Denote the introduced t auxiliary variables as $\boldsymbol{P}=[P_1,P_2,\cdots P_t]$, then the posterior failure probability is expressed by:

$$\begin{aligned}\Pr(F|Z_1\cap\cdots\cap Z_t) &= \frac{\Pr(F\cap Z_1\cap\cdots\cap Z_t)}{\Pr(Z_1\cap\cdots\cap Z_t)} \\ &= \frac{\int_{\boldsymbol{x},\boldsymbol{p}\in[\Omega_F\cap\Omega_{Z_1}\cap\cdots\cap\Omega_{Z_t}]} f_{\boldsymbol{P}}(\boldsymbol{p}) f_X(\boldsymbol{x})\,\mathrm{d}\boldsymbol{p}\mathrm{d}\boldsymbol{x}}{\int_{\boldsymbol{x},\boldsymbol{p}\in[\Omega_{Z_1}\cap\cdots\cap\Omega_{Z_t}]} f_{\boldsymbol{P}}(\boldsymbol{p}) f_X(\boldsymbol{x})\,\mathrm{d}\boldsymbol{p}\mathrm{d}\boldsymbol{x}}\end{aligned} \tag{12}$$

where $f_{\boldsymbol{P}}(\boldsymbol{p})$ is the joint PDF of variables $\boldsymbol{P}$, and $f_{\boldsymbol{P}}(\boldsymbol{p}) = \prod_{q=1}^{t} f_{P_q}(p_q)$.

1.2. Reliability updating with DPU

1.2.1. The principles of NRA and ARA

In this section, the basic principles of NRA and ARA are firstly introduced. Denote n-dimensional input variables as $\boldsymbol{X}=(X_1,\cdots,X_n)$, and their distribution parameters with uncertainty as $\boldsymbol{\Theta}=(\Theta_1,\cdots,\Theta_{n_\Theta})$, where the PDF of variables $\boldsymbol{\Theta}$ is $f_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) = \prod_{j=1}^{n_\Theta} f_{\Theta_j}(\theta_j)$. Thus, the failure probability conditional on the realization $\boldsymbol{\theta}$ of variables $\boldsymbol{\Theta}$ can be expressed by:

$$\begin{aligned}P_f(\boldsymbol{\theta}) &= \int_{g(X)\le 0|\boldsymbol{\Theta}=\boldsymbol{\theta}} f_{X|\boldsymbol{\Theta}}(\boldsymbol{x}|\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{x} = \int_X I_F(\boldsymbol{x}) f_{X|\boldsymbol{\Theta}}(\boldsymbol{x}|\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{x} \\ &= \mathbb{E}_X\left(I_F(\boldsymbol{X})|\boldsymbol{\theta}\right)\end{aligned} \tag{13}$$

where $g(\boldsymbol{X})$ is the performance function, $f_{X|\boldsymbol{\Theta}}(\boldsymbol{x}|\boldsymbol{\theta})$ is the conditional PDF of $\boldsymbol{X}$, $I_F(\cdot)$ is the failure indicator function which is defined as $I_F(\boldsymbol{x}) = \begin{cases}1 & g(\boldsymbol{x})\le 0\\ 0 & g(\boldsymbol{x})>0\end{cases}$, and $\mathbb{E}_X(\cdot)$ is the expectation operator of variables $\boldsymbol{X}$.

By integrating $P_f(\boldsymbol{\theta})$ in the space of $\boldsymbol{\Theta}$, the failure probability can be obtained as follows:

$$\tilde{P}_f = \int_{\boldsymbol{\Theta}} P_f(\boldsymbol{\theta}) f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta} = \mathbb{E}_{\boldsymbol{\Theta}}\left(P_f(\boldsymbol{\Theta})\right) \tag{14}$$

where $\mathbb{E}_{\boldsymbol{\Theta}}(\cdot)$ is the expectation operator of $\boldsymbol{\Theta}$.

According to Eq. (14), the computational expression of ARA can be defined as follows:

$$\begin{aligned}\tilde{P}_f &= \int_{\boldsymbol{\Theta}}\left(\int_X I_F(\boldsymbol{x}) f_{X|\boldsymbol{\Theta}}(\boldsymbol{x}|\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{x}\right) f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta} \\ &= \int_{X,\boldsymbol{\Theta}} I_F(\boldsymbol{x}) f_{X,\boldsymbol{\Theta}}(\boldsymbol{x},\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\theta} \\ &= \mathbb{E}_{X,\boldsymbol{\Theta}}\left(I_F(\boldsymbol{X})\right)\end{aligned} \tag{15}$$

where $f_{X,\boldsymbol{\Theta}}(\boldsymbol{x},\boldsymbol{\theta})$ is the joint PDF of variables $(\boldsymbol{X},\boldsymbol{\Theta})$, and $\mathbb{E}_{X,\boldsymbol{\Theta}}(\cdot)$ is the expectation operator of variables $(\boldsymbol{X},\boldsymbol{\Theta})$.

1.2.2. Reliability updating models in the presence of DPU

According to NRA, given a realization $\boldsymbol{\theta}$ of the distribution parameters, $\Pr(F\cap Z|\boldsymbol{\theta})$ and $\Pr(Z|\boldsymbol{\theta})$ can be calculated by the following equations:

$$\begin{aligned}\Pr(F\cap Z|\boldsymbol{\theta}) &= \frac{\alpha}{c}\int_{\boldsymbol{x},p\in[\Omega_F\cap\Omega_Z]} f_P(p) f_{X|\boldsymbol{\Theta}}(\boldsymbol{x}|\boldsymbol{\theta})\,\mathrm{d}p\mathrm{d}\boldsymbol{x} \\ &= \frac{\alpha}{c}\int_{X,P} I_1(\boldsymbol{x},p) f_P(p) f_{X|\boldsymbol{\Theta}}(\boldsymbol{x}|\boldsymbol{\theta})\,\mathrm{d}p\mathrm{d}\boldsymbol{x} \\ &= \frac{\alpha}{c}\mathbb{E}_{X,P}\left(I_1(\boldsymbol{X},P)|\boldsymbol{\theta}\right)\end{aligned} \tag{16}$$

$$\begin{aligned}\Pr(Z|\boldsymbol{\theta}) &= \frac{\alpha}{c}\int_{\boldsymbol{x},p\in\Omega_Z} f_P(p) f_{X|\boldsymbol{\Theta}}(\boldsymbol{x}|\boldsymbol{\theta})\,\mathrm{d}p\mathrm{d}\boldsymbol{x} \\ &= \frac{\alpha}{c}\int_{X,P} I_2(\boldsymbol{x},p) f_P(p) f_{X|\boldsymbol{\Theta}}(\boldsymbol{x}|\boldsymbol{\theta})\,\mathrm{d}p\mathrm{d}\boldsymbol{x} \\ &= \frac{\alpha}{c}\mathbb{E}_{X,P}\left(I_2(\boldsymbol{X},P)|\boldsymbol{\theta}\right)\end{aligned} \tag{17}$$

where $I_1(\boldsymbol{x},p)=\begin{cases}1 & \boldsymbol{x},p\in[\Omega_F\cap\Omega_Z]\\ 0 & \boldsymbol{x},p\notin[\Omega_F\cap\Omega_Z]\end{cases}$ and $I_2(\boldsymbol{x},p)=\begin{cases}1 & \boldsymbol{x},p\in\Omega_Z\\ 0 & \boldsymbol{x},p\notin\Omega_Z\end{cases}$.

By integrating $\Pr(F\cap Z|\boldsymbol{\Theta})$ and $\Pr(Z|\boldsymbol{\Theta})$ in the space of variables $\boldsymbol{\Theta}$, and NRA-based reliability updating model can be obtained by:

$$\Pr(F|Z)=\frac{\int_{\Theta}\Pr(F\cap Z|\boldsymbol{\theta})f_{\Theta}(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta}}{\int_{\Theta}\Pr(Z|\boldsymbol{\theta})f_{\Theta}(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta}}=\frac{\mathbb{E}_{\Theta}\left(\Pr(F\cap Z|\boldsymbol{\Theta})\right)}{\mathbb{E}_{\Theta}\left(\Pr(Z|\boldsymbol{\Theta})\right)} \tag{18}$$

According to Eq. (15), ARA-based reliability updating model can be defined by:

$$\begin{aligned}\Pr(F|Z)&=\frac{\Pr(F\cap Z)}{\Pr(Z)}\\ &=\frac{\int_{X,P,\Theta}I_1(\boldsymbol{x},p)f_P(p)f_{X,\Theta}(\boldsymbol{x},\boldsymbol{\theta})\mathrm{d}p\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\theta}}{\int_{X,P,\Theta}I_2(\boldsymbol{x},p)f_P(p)f_{X,\Theta}(\boldsymbol{x},\boldsymbol{\theta})\mathrm{d}p\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\theta}}\\ &=\frac{\mathbb{E}_{X,P,\Theta}\left(I_1(\boldsymbol{X},P)\right)}{\mathbb{E}_{X,P,\Theta}\left(I_2(\boldsymbol{X},P)\right)}\end{aligned} \tag{19}$$

where $\mathbb{E}_{X,P,\Theta}(\cdot)$ is the expectation operator of joint variables $\boldsymbol{X}$, P and $\boldsymbol{\Theta}$.

2. ALGORITHMS FOR RELIABILITY UPDATING IN THE PRESENCE OF DPU

SS [18] is very popular to reliability updating. Generally, the inequality limit state function in logarithmic form is adopted [18]:

$$\begin{aligned}h'(\boldsymbol{x},p)&=\ln(p)-\ln(c)-\ln\left(L(\boldsymbol{x}|Z)\right)\\ &=\ln(p)+l-\ln\left(L(\boldsymbol{x}|Z)\right)\end{aligned} \tag{20}$$

For SS-NRA-RU algorithm, $\boldsymbol{\Theta}$ takes different realizations $\boldsymbol{\theta}^{(k)}(k=1,\cdots,N_{\Theta})$ in the outer layer, and variables $\boldsymbol{X}$, P are transformed into standard normal variables $\boldsymbol{U}'=\left[\boldsymbol{U},U_p\right]$ with $n+1$ dimensionality, where $\boldsymbol{U}$ can be converted into n-dimensional input variables $\boldsymbol{X}$, and U_P can be converted into variable P by $P=\Phi(U_P)$. Based on this transformation, Eq. (20) can be expressed by:

$$\begin{aligned}h'(\boldsymbol{x},p)&=\ln(p)+l-\ln\left(L(\boldsymbol{x}|Z)\right)\\ &=\ln\left(\Phi(u_P)\right)+l-\ln\left(L\left(\mathrm{T}_1(\boldsymbol{u})|Z\right)\right)\\ &=h_1'(\boldsymbol{u}')\end{aligned} \tag{21}$$

where $\mathrm{T}_1(\cdot)$ represents the transformation which transforms samples $\boldsymbol{u}$ into $\boldsymbol{x}$ based on the given parameter realization $\boldsymbol{\theta}^{(k)}(k=1,\cdots,N_{\Theta})$. The computational steps of SS-NRA-RU algorithm are shown in Algorithm 1:

Algorithm 1： The computational steps of SS-NRA-RU algorithm.

1. Set the value of p_t, where p_t is often taken as 0.1, and determine the sample size N_K of each subset in SS.
2. Generate N_{Θ} samples $\boldsymbol{\theta}^{(k)}(k=1,\cdots,N_{\Theta})$ of variables $\boldsymbol{\Theta}$ according to $f_{\Theta}(\boldsymbol{\theta})$.
3. **For** $k=1:N_{\Theta}$
4. Determine the transformation $\mathrm{T}_1(\cdot)$ based on $\boldsymbol{\theta}^{(k)}$. Generate N_K independent standard normal samples $\boldsymbol{u}'^{(0,j)}=\left[\boldsymbol{u}^{(0,j)},u_p^{(0,j)}\right]$ $(j=1,\cdots,N_K)$ with $n+1$ dimensionality.
5. Let $i=0$ and $b_i^h=\infty$, and calculate $l=\max\left(\ln\left(L\left(\mathrm{T}_1(\boldsymbol{u}^{(0,j)})|Z\right)\right)\right)$.
6. **While** $\left(b_i^h>0\right)$ **do:**
7. Let $i=i+1$.
8. Calculate the values of $h_1'\left(\boldsymbol{u}'^{(i-1,j)}\right)(j=1,\cdots,N_K)$ based on the samples $\boldsymbol{u}'^{(i-1,j)}$, and sort $h_1'\left(\boldsymbol{u}'^{(i-1,j)}\right)$ in ascending order.
9. Determine the value of b_i^h according to the p_t-percentile of the ordered values of $h_1'\left(\boldsymbol{u}'^{(i-1,j)}\right)(j=1,\cdots,N_K)$.
10. Select N_h samples $\boldsymbol{u}'^{(i-1,j)}(j=1,\cdots,N_h)$ satisfying $h_1'\left(\boldsymbol{u}'^{(i-1,j)}\right)\le\max\left(b_i^h,0\right)$ as the seeds of MCMC, and generate N_K-N_h samples, so as to obtain the new samples $\boldsymbol{u}'^{(i,j)}(j=1,\cdots,N_K)$.
11. **If** $b_i^h<0$, let $b_i^h=0$ and $p_i^h=N_h/N_K$.
12. **Else**, let $p_i^h=p_t$.
13. **End**
14. Let $l_{new}=\max\left\{l,\ln\left(L\left(\mathrm{T}_1(\boldsymbol{u}^{(i,j)})|Z\right)\right)\right\}(j=1,\cdots,N_K)$, and update b_i^h into $b_i^h=b_i^h-l+l_{new}$. Let $l=l_{new}$.
15. **End**

16. Collect the samples generated in the last subset as the posterior samples $\boldsymbol{u}'^{(0,j)}_{post}\left(j=1,\cdots,N_K\right)$, and determine the final inequality limit state function $h_1'(\boldsymbol{u}')$ according to the final *l*. Let $\Pr\left(Z\middle|\boldsymbol{\theta}^{(k)}\right)=\prod_i p_i^h$.
17. **For** $i=0:N_{\max}$ ($N_{\max}$ is the maximum of iterations)
18. Let $i=i+1$.
19. Calculate the values of performance function $g\left(\mathrm{T}_1(\boldsymbol{u}_{post}{}^{(i-1,j)})\right)\left(j=1,\cdots,N_K\right)$, and sort them in ascending order.
20. Determine the value of b_i^g according to the p_t -percentile of the ordered values of $g\left(\mathrm{T}_1(\boldsymbol{u}_{post}{}^{(i-1,j)})\right)\left(j=1,\cdots,N_K\right)$.
21. **If** $b_i^g>0$, select N_g samples satisfying $g\left(\mathrm{T}_1(\boldsymbol{u}_{post}{}^{(i-1,j)})\right)\le b_i^g$ and $h_1'\left(\boldsymbol{u}'^{(i-1,j)}_{post}\right)\le 0$ as the seeds of MCMC, and generate N_K-N_g samples, so as to obtain the new samples $\boldsymbol{u}'^{(i,j)}_{post}\left(j=1,\cdots,N_K\right)$. Let $p_i^g=p_t$.
22. **Else**, let $b_i^g=0$, and determine N_f failure samples that satisfy $g\left(\mathrm{T}_1(\boldsymbol{u}_{post}{}^{(i-1,j)})\right)\le 0$ and $h_1'\left(\boldsymbol{u}'^{(i-1,j)}_{post}\right)\le 0$. Then $p_i^g=N_f/N_K$. **Break**.
23. **End**
24. **End**
25. $\Pr\left(F\cap Z\middle|\boldsymbol{\theta}^{(k)}\right)=\Pr\left(F\middle|Z,\boldsymbol{\theta}^{(k)}\right)\Pr\left(Z\middle|\boldsymbol{\theta}^{(k)}\right)$, and $\Pr\left(F\middle|Z,\boldsymbol{\theta}^{(k)}\right)=\prod_i p_i^g$.
26. **End**
27. Calculate the posterior failure probability based on $\Pr\left(F\cap Z\middle|\boldsymbol{\theta}^{(k)}\right)$ and $\Pr\left(Z\middle|\boldsymbol{\theta}^{(k)}\right)$, $k=1,\cdots,N_\Theta$.

In SS-ARA-RU algorithm, $\boldsymbol{\Theta}$ is also considered as random variables. Denote input variables in the standard normal space as $\boldsymbol{U}''=\left[\boldsymbol{U},U_p,\boldsymbol{U}_\Theta\right]$, and Eq. (20) can be expressed by:

$$\begin{aligned}h'(\boldsymbol{x},p)&=\ln(p)+l-\ln\left(L\left(\boldsymbol{x}\middle|Z\right)\right)\\&=\ln\left(\Phi(u_P)\right)+l-\ln\left(L\left(\mathrm{T}_2(\boldsymbol{u},\boldsymbol{u}_\Theta)\middle|Z\right)\right)\\&=h_2'(\boldsymbol{u}'')\end{aligned}\quad(22)$$

where $\mathrm{T}_2(\cdot)$ represents the transformation which transforms samples $\boldsymbol{u}$ into $\boldsymbol{x}$ based on the corresponding parameter samples $\boldsymbol{u}_\Theta$. The computational steps of SS-ARA-RU algorithm are shown in Algorithm 2:

Algorithm 2： The computational steps of SS-ARA-RU algorithm.

1. Set the value of p_t, where p_t is often taken as 0.1, and determine the sample size N_K of each subset in SS.
2. Generate N_K independent standard normal samples $\boldsymbol{u}''^{(0,k)}\left(k=1,\cdots,N_K\right)$ with $n+1+n_\Theta$ dimensionality.
3. Let $i=0$ and $b_i^h=\infty$, and calculate $l=\max\left(\ln\left(L\left(\mathrm{T}_2(\boldsymbol{u}^{(0,k)},\boldsymbol{u}_\Theta{}^{(0,k)})\middle|Z\right)\right)\right)\left(k=1,\cdots,N_K\right)$
4. **While** $\left(b_i^h>0\right)$ **do:**
5. Let $i=i+1$.
6. Calculate the values of $h_2'\left(\boldsymbol{u}''^{(i-1,k)}\right)\left(k=1,\cdots,N_K\right)$ based on the samples $\boldsymbol{u}''^{(i-1,k)}$ $\left(k=1,\cdots,N_K\right)$, and sort them in ascending order.
7. Determine the value of b_i^h according to the p_t -percentile of the ordered values of $h_2'\left(\boldsymbol{u}''^{(i-1,k)}\right)\left(k=1,\cdots,N_K\right)$.
8. Select the N_h samples $\boldsymbol{u}''^{(i-1,k)}(k=1,\cdots,N_h)$ satisfying $h_2'\left(\boldsymbol{u}''^{(i-1,k)}\right)\le\max\left(b_i^h,0\right)$ as the seeds of MCMC, and generate N_K-N_h samples, so as to obtain the new samples $\boldsymbol{u}''^{(i,k)}\left(k=1,\cdots,N_K\right)$.
9. **If** $b_i^h<0$, let $b_i^h=0$ and $p_i^h=N_h/N_K$.
10. **Else**, $p_i^h=p_t$.
11. **End**
12. Let $l_{new}=\max\left\{l,\ln\left(L\left(\mathrm{T}_2(\boldsymbol{u}''^{(i,k)},\boldsymbol{u}''^{(i,k)}_\Theta)\middle|Z\right)\right)\right\}\left(k=1,\cdots,N_K\right)$, and update b_i^h into $b_i^h=b_i^h-l+l_{new}$. Let $l=l_{new}$.
13. **End**
14. Denote the samples generated in the last subset as the posterior samples $\boldsymbol{u}''^{(0,k)}_{post}\left(k=1,\cdots,N_K\right)$, and determine the final inequality limit state function $h_2'(\boldsymbol{u}'')$ according to the final *l*.
15. **For** $i=0:N_{\max}$ ($N_{\max}$ is the maximum of iterations)

16. Let $i = i + 1$.
17. Calculate the values of performance function $g\left(\mathrm{T}_2(\boldsymbol{u}_{post}^{(i-1,k)}, \boldsymbol{u}_{\Theta post}^{(i-1,k)})\right)\left(k = 1, \cdots, N_K\right)$, and sort them in ascending order.
18. Determine the value of b_i^g according to the p_t -percentile of the ordered values of $g\left(\mathrm{T}_2(\boldsymbol{u}_{post}^{(i-1,k)}, \boldsymbol{u}_{\Theta post}^{(i-1,k)})\right)\left(k = 1, \cdots, N_K\right)$.
19. **If** $b_i^g > 0$, select N_g samples satisfying $g\left(\mathrm{T}_2(\boldsymbol{u}_{post}^{(i-1,k)}, \boldsymbol{u}_{\Theta post}^{(i-1,k)})\right) \le b_i^g$ and $h_2'\left(\boldsymbol{u}''^{(i-1,k)}_{post}\right) \le 0$ as the seeds of MCMC, and generate $N_K - N_g$ samples based on these seeds, to obtain the new samples $\boldsymbol{u}''^{(i,k)}_{post}\left(k = 1, \cdots, N_K\right)$. Let $p_i^g = p_t$.
20. **Else**, let $b_i^g = 0$, and determine N_f failure samples satisfying $g\left(\mathrm{T}_2(\boldsymbol{u}_{post}^{(i-1,k)}, \boldsymbol{u}_{\Theta post}^{(i-1,k)})\right) \le 0$ and $h_2'\left(\boldsymbol{u}''^{(i-1,k)}_{post}\right) \le 0$. Then $p_i^g = N_f / N_K$. **Break**.
21. **End**
22. **End**
23. The posterior failure probability can be obtained by $\Pr\left(F|Z\right) = \prod_i p_i^g$.

3. EXAMPLES

In this section, two examples are investigated to test the effectiveness of developed algorithms. For MC-NRA-RU, MC-ARA-RU and SS-ARA-RU algorithms, reliability updating processes are repeated 30 times, and the mean values of the results are provided in tables. Since repeated SS is too time-consuming, SS-NRA-RU and SS-NRA-RU are performed for one time in each example.

3.1. Cantilever beam model

This example studies a cantilever beam model [19]. As shown in Fig. 1, when the cantilever beam bears two independent loads *X* and *Y*, the displacement of the end of the cantilever beam can be expressed by the following equation:

$$g = D_0 - D(X, Y, w, t)$$

where $D(X, Y, w, t) = \dfrac{4L^3}{Ewt}\sqrt{\left(\dfrac{X}{w^2}\right)^2 + \left(\dfrac{Y}{t^2}\right)^2}$, $D_0 = 2.18$, $E = 2.9 \times 10^7$ is the elastic modulus, *L*, *w* and *t* are the length, width and thickness of the cantilever beam, respectively. The random variables in this structure are *X*, *Y*, *w* and *t*. The obtained output observations of the cantilever beam are 0.11 and 0.15, respectively, and the likelihood function constructed based on the observations is expressed as follows:

$$L\left(\boldsymbol{x}|Z\right) = \exp\left(-\frac{(g-0.11)^2 + (g-0.13)^2}{2\sigma_\varepsilon^2}\right)$$

where $\sigma_\varepsilon = 0.05$.

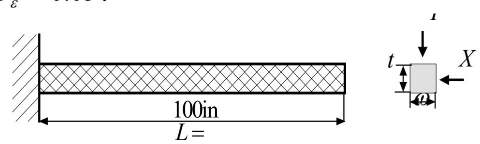

Fig. 1 Diagram of the cantilever beam structure.

Table 1. *Prior distribution of the input variables.*

Variables	*Distribution*	*Mean*	*Standard*
X / N	*Lognormal*	μ_1	*100*
Y / N	*Lognormal*	μ_2	*100*
E / Pa	*Lognormal*	μ_3	*2.5×10⁶*

Table 2. *Distribution of the prior distribution parameters.*

Variables	*Distribution*	*Mean*	*Standard*
μ_1	*Normal*	*500*	*50*
μ_2	*Normal*	*1200*	*120*
μ_3	*Normal*	2.5×10^7	2.5×10^6

Table 3. *The results of reliability updating.*

	MC-NRA-RU	*MC-ARA-RU*	*SS-NRA-RU*	*SS-ARA-RU*
$\Pr(Z)$	*0.0974*	*0.0974*	*0.0968*	*0.0970*
$\Pr(F\cap Z$	2.935×10^{-5}	2.972×10^{-5}	2.904×10^{-5}	3.113×10^{-5}
$\Pr(F\|Z)$	3.015×10^{-4}	3.050×10^{-4}	2.999×10^{-4}	3.210×10^{-4}
N	5×10^8	1×10^7	7×10^7	7×10^4

In this example, the reliability updating problem of a cantilever beam model is studied. Table 3 shows that four methods have quite close solutions, although SS-ARA-RU method has the largest error. In terms of the computational cost, it is obvious that ARA-based methods require less computational cost than NRA-based methods. In this example, SS takes 1×10^4 samples per subset, and aBUS requires three subsets to obtain the posterior samples while another four subsets are required to calculate the posterior failure probability. Therefore, it indicates that SS-ARA-

RU method requires the least cost, which proves that under the approximate calculation accuracy, SS-ARA-RU has the highest efficiency.

3.2. A roof truss model

A roof truss model is studied in this example [20]. As shown in Fig.2, the top chords and compression bars of the roof truss are made of steel reinforced concrete, while the bottom chords and tension bars are made of steel. Assume the uniformly distributed load q that can be transformed into the nodal load $P = ql/4$ is applied on the roof truss structure, where l is the length of the truss. The perpendicular deflection of the truss peak node C is derived as:

$$\Delta_C = \frac{ql^2}{2}\left(\frac{3.81}{A_C E_C} + \frac{1.13}{A_S E_S}\right)$$

where A_C and A_S are the cross sectional areas of the steel reinforced concrete and the steel bars respectively, and E_C , E_S are the corresponding elastic modulus. The performance functions of three failure modes are given by:

$$g_1 = 0.03 - \frac{ql^2}{2}\left(\frac{3.81}{A_C E_C} + \frac{1.13}{A_S E_S}\right)$$

$$g_2 = f_C A_C - 1.185ql$$

$$g_3 = f_S A_S - 0.75ql$$

Likelihood functions are expressed as follows:

$$L_1\left(q, l, A_S, A_C \mid \boldsymbol{y}\right) = \exp\left(-\frac{\left((g_1 - 0.0168)^2 + (g_1 - 0.017)^2\right)}{2\sigma_{\varepsilon_1}^2}\right)$$

$$L_2\left(q, l, A_S, A_C \mid \boldsymbol{y}\right) = \exp\left(-\frac{\left((g_2 - 3.1\times10^5)^2 + (g_3 - 2.1\times10^4)^2\right)}{2\sigma_{\varepsilon_2}^2}\right)$$

where $\sigma_{\varepsilon_1} = 6\times10^{-4}$ and $\sigma_{\varepsilon_2} = 1\times10^4$.

Table 4. *Prior distribution of the input variables.*

Variables	*Distribution*	*Mean*	*Standard*
q(N/m)	*Normal*	μ_1	*1400*
l(m)	*Normal*	μ_2	*0.12*
A_S(m^2)	*Normal*	9.82×10^{-4}	5.82×10^{-5}
A_C(m^2)	*Normal*	*0.04*	*0.0048*

Table 5. *Distribution of the prior distribution parameters.*

Variables	*Distribution*	*Lower*	*Upper*
μ_1	*Uniform*	1.8×10^4	2.2×10^4
μ_2	*Uniform*	*11*	*13*

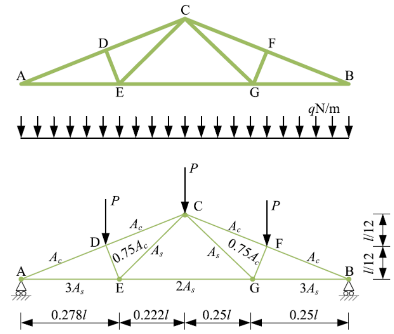

Fig. 2 *Schematic diagram of roof truss model.*

Table 6. *The results of reliability updating of the roof truss model*

	MC-NRA-RU	*MC-ARA-RU*	*SS-NRA-RU*	*SS-ARA-RU*
$\Pr(Z)$	*0.0212*	*0.0212*	*0.0212*	*0.0209*
$\Pr(F\cap Z$	6.974×10^{-5}	6.760×10^{-5}	6.360×10^{-5}	7.194×10^{-5}
$\Pr(F\|Z)$	3.270×10^{-3}	3.254×10^{-3}	3.005×10^{-3}	3.423×10^{-3}
N	1×10^8	5×10^6	6×10^7	6×10^4

This example studies the reliability updating problem of a roof truss model. Since there are two likelihood functions in total, by introducing two auxiliary variables, two inequality limit state functions can be obtained in total. It can be seen from Table 6 that the results of these methods are still consistent. Among these methods, SS-ARA-RU still requires the least performance function calls, where 1×10^4 samples per subset are set in SS and a total of six subsets are used. This example verifies again that the proposed two kinds of reliability updating methods in the presence of DPU are correct, and SS-ARA-RU is more recommended employed in engineering practice to save the computational cost.

4. CONCLUSIONS

Considering the data scarcity in engineering problems, this paper studies the reliability updating problem in the presence of DPU. Two reliability updating models in the presence of DPU are firstly constructed. Then, based on the constructed models, two SS-based reliability updating algorithms are the proposed, namely, SS-NRA-RU and SS-ARA-RU. It is proved that NRA-based methods have nearly the same solutions with ARA-based methods, while ARA-based methods have higher efficiency, especially for SS-ARA-RU method. Therefore, this paper can provide an efficient solution to reliability updating problems in the presence of DPU.

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