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Non-stationary Response Autocorrelation Function of Nonlinear Oscillators Endowed with Fractional Derivative Elements

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ABSTRACT: An approximate analytical technique is developed for determining both the response evolutionary power spectrum (EPS) and the corresponding non-stationary autocorrelation function of nonlinear oscillators endowed with fractional derivative elements. Specifically, first, a stochastic averaging/linearization treatment is employed for deriving an input-output relationship in the joint time-frequency domain. The derived relationship between the excitation and the response evolutionary power spectra can be construed as an extension of earlier results in the literature to account for fractional derivative elements in the oscillator equation of motion. Further, the response non-stationary autocorrelation function can be also evaluated readily based on Priestley's EPS theory. A bilinear hysteretic oscillator with fractional derivative elements is considered as a numerical example. The reliability of the developed technique is demonstrated by comparisons with pertinent Monte Carlo simulation data.

1. INTRODUCTION

Fractional calculus has been used widely over the last few decades in various fields of science and engineering (e.g., Sabatier et al., 2007; Miller and Ross, 1993). Indicatively, the necessity for more accurate modeling of viscoelastic materials has led to the development of enhanced mechanics theories that exploit the concept of non-integer order (fractional) derivatives (e.g., Makris and Constantinou, 1992; Di Paola and Zingales, 2012).

Further, pertinent solution approaches have been developed over the years for determining the stochastic response and assessing the reliability of linear and nonlinear systems with fractional derivative elements (e.g., Jesus and Tenreiro Machado, 2009; Spanos and Malara, 2014; Pinnola, 2016; Pirrotta et al., 2021; Kougioumtzoglou et al., 2022; Fragkoulis and Kougioumtzoglou, 2023; Zhang et al., 2023).

Nevertheless, developing techniques in the joint time-frequency domain and determining the response evolutionary power spectrum (EPS) of nonlinear oscillators with fractional derivatives is a topic that has received only limited attention in the literature (e.g., Kougioumtzoglou and Spanos, 2016).

In this paper, an approximate analytical technique is developed for determining both the response EPS and the corresponding response non-stationary autocorrelation function of nonlinear/hysteretic oscillators with fractional derivative elements. Specifically, first, a stochastic averaging/linearization treatment is employed for deriving an input-output relationship in the joint time-frequency domain. The derived relationship between the excitation and the response EPS can be construed as an extension of earlier results in the literature (Kougioumtzoglou, 2013) to account for fractional derivative elements in the oscillator equation of motion. Notably, the expression for the nonlinear system response EPS employs the concept of conditional EPS. To elaborate further, the nonlinear system response EPS can be construed as the sum of the response EPS corresponding to equivalent, response amplitude dependent, linear systems, appropriately weighted by the non-stationary response amplitude probability density function (PDF). Furthermore, the response non-stationary autocorrelation function can be also evaluated readily based on Priestley's EPS theory (Priestley, 1988). A nonlinear bilinear hysteretic oscillator with fractional derivative elements is considered as a numerical example. The reliability of the developed technique is demonstrated by comparisons with pertinent Monte Carlo simulation (MCS) data.

2. MATHEMATICAL FORMULATION

2.1. *Stochastic averaging/linearization of nonlinear oscillators with fractional derivative elements*

The equation of motion of a stochastically excited nonlinear oscillator with fractional derivative elements is given by

$$\ddot{x}(t) + \beta \mathscr{D}_C^{\alpha} x(t) + z(t, x, \dot{x}) = w(t), \tag{1}$$

where $x(t)$ denotes the response displacement and a dot over a variable denotes differentiation with respect to time. Further, $z(t, x, \dot{x})$ is an arbitrary nonlinear function that can also account for hysteretic modeling, and $w(t)$ is a Gaussian zero-mean non-stationary stochastic process with a broad-band EPS $S_w(\omega, t)$. Furthermore, β is a constant and $\mathscr{D}_C^{\alpha}(\cdot)$ denotes the Caputo fractional derivative of order α ($0 < \alpha < 1$) defined as (Podlubny, 1998)

$$\mathscr{D}_C^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^{\alpha}} d\tau. \tag{2}$$

Next, assuming relatively light damping, the oscillator follows a pseudo-harmonic response behavior described by (Roberts and Spanos, 1986)

$$x(t) = A(t) \cos(\omega(A)t + \psi(t)) \tag{3}$$

and

$$\dot{x}(t) = -\omega(A)A(t) \sin(\omega(A)t + \psi(t)). \tag{4}$$

In Eqs. (3-4), $A(t)$ and $\psi(t)$ denote the response amplitude and phase, respectively, that vary slowly with respect to time and are considered constant over one cycle of oscillation.

In the following, applying a statistical linearization treatment (Fragkoulis et al., 2019), an equivalent to Eq. (1) linear system is defined as

$$\ddot{x}(t) + (\beta_0 + \beta(A))\dot{x}(t) + \omega^2(A)x(t) = w(t), \tag{5}$$

where $\beta_0 = 2\zeta_0\omega_0$ is a damping coefficient, with ω_0 denoting the natural frequency of the corresponding linear oscillator and ζ_0 represents the damping ratio. As shown in Fragkoulis et al. (2019), the amplitude-dependent equivalent damping and stiffness elements are given by

$$\beta(A) = \frac{S(A)}{A\omega(A)} + \frac{\beta}{\omega^{1-\alpha}(A)} \sin\left(\frac{\alpha\pi}{2}\right) - \beta_0 \tag{6}$$

and

$$\omega^2(A) = \frac{F(A)}{A} + \beta\omega^{\alpha}(A) \cos\left(\frac{\alpha\pi}{2}\right), \tag{7}$$

where

$$S(A) = -\frac{1}{\pi} \int_0^{2\pi} z(A\cos\phi, -A\omega(A)\sin\phi) \times \sin\phi \, d\phi, \tag{8}$$

$$F(A) = \frac{1}{\pi}\int_0^{2\pi} z(A\cos\phi, -A\omega(A)\sin\phi) \times \cos\phi d\phi, \quad (9)$$

and $\phi(t) = \omega(A)t + \psi(t)$. Note that in deriving Eqs. (6) and (7) an error between Eqs. (1) and (5) has been defined and a minimization procedure has been applied in the mean square sense.

Next, noticing that $\beta(A)$ and $\omega(A)$ depend on the non-stationary response amplitude A, and thus they can be construed also as non-stationary stochastic processes, the corresponding time-varying mean values can be evaluated by applying the expectation operator to Eqs. (6-7) (Kougioumtzoglou and Spanos, 2009). This yields

$$\beta_{eq}(t) = -\beta_0 + \int_0^\infty \frac{S(A)}{A\omega(A)} p(A,t)dA + \beta\sin\left(\frac{\alpha\pi}{2}\right)\int_0^\infty \frac{1}{\omega^{1-\alpha}(A)} p(A,t)dA \quad (10)$$

and

$$\omega_{eq}^2(t) = \int_0^\infty \frac{F(A)}{A} p(A,t)dA + \beta\cos\left(\frac{\alpha\pi}{2}\right)\int_0^\infty \omega^\alpha(A) p(A,t)dA. \quad (11)$$

Clearly, the non-stationary response amplitude PDF $p(A,t)$ is required for the evaluation of the time-varying equivalent elements in Eqs. (10) and (11). In this regard, $p(A,t)$ is expressed in the form (Fragkoulis et al., 2019)

$$p(A,t) = \frac{GA}{c(t)}\exp\left(-\frac{GA^2}{2c(t)}\right), \quad (12)$$

where $G = \frac{\sin\left(\frac{\alpha\pi}{2}\right)}{\omega_0^{1-\alpha}}$ and $c(t)$ denotes a time-dependent coefficient to be determined. The rationale behind the choice of the time-dependent Rayleigh PDF of Eq. (12) relates to the fact that the linear oscillator stationary response amplitude PDF is a Rayleigh one (see also Spanos, 1978; Spanos and Lutes, 1980). In Kougioumtzoglou and Spanos (2009) it was further shown that Rayleigh PDF representation with a time-dependent coefficient is suitable for nonlinear oscillators also and under evolutionary stochastic excitations as well. Furthermore, it was demonstrated by Fragkoulis et al. (2019) that the generalization shown in Eq. (12) is valid for nonlinear oscillators endowed with fractional derivative terms. Specifically, employing a stochastic averaging treatment of Eq. (5), substituting Eq. (12) into the associated Fokker-Planck equation

$$\frac{\partial p(A,t)}{\partial t} = -\frac{\partial}{\partial A}\left\{\left(-\frac{1}{2}(\beta_0 + \beta_{eq}(t))A + \frac{\pi S(\omega_{eq}(t),t)}{2\omega_{eq}^2(t)A}\right)p(A,t)\right\} + \frac{1}{4}\frac{\partial}{\partial A}\left\{\frac{\pi S(\omega_{eq}(t),t)}{\omega_{eq}^2(t)}\frac{\partial p(A,t)}{\partial A} + \frac{\partial}{\partial A}\left(\frac{\pi S(\omega_{eq}(t),t)}{\omega_{eq}^2(t)}p(A,t)\right)\right\}, \quad (13)$$

and manipulating yields the deterministic first-order nonlinear differential equation (Fragkoulis et al. 2019)

$$\dot{c}(t) = -\left(\beta_0 + \beta_{eq}(t)\right)c(t) + \pi G\frac{S(\omega_{eq}(t),t)}{\omega_{eq}^2(t)}. \quad (14)$$

Eq. (14) can be readily solved by a Runge-Kutta numerical integration scheme to compute the time-dependent coefficient $c(t)$, which is substituted into Eq. (12) to determine the non-stationary response amplitude PDF $p(A,t)$.

2.2. *Response evolutionary power spectrum and non-stationary autocorrelation function of nonlinear oscillators with fractional derivative elements*

In this section, a novel closed-form expression is derived for determining the response EPS of nonlinear oscillators with fractional derivative elements, which can be construed as a generalization of earlier results in the literature (Kougioumtzoglou, 2013) to account for fractional derivative terms in the oscillator equation of motion. Next, the corresponding non-stationary response autocorrelation function can be determined readily by resorting to the Priestley's theory of EPS (Priestley, 1988).

Specifically, taking into account the slowly-varying in time nature of the equivalent elements $\omega_{eq}(A)$ and $\beta_{eq}(A)$, considered to be approximately constant over one cycle of oscillation, employing a GHW-based representation of the excitation and response processes and relying on the GHW orthogonality properties, Kougioumtzoglou (2013) derived the relationship

$$S_x(\omega,t) = \int_0^{\infty} S_x(\omega,t|A)p(A,t)\mathrm{d}A. \tag{15}$$

In Eq. (15) $S_x(\omega,t|A)$ denotes the conditional EPS, originally proposed as a concept by Miles (1989; 1993), which can be viewed approximately as the response EPS of a linear oscillator possessing natural frequency equal to $\omega(A)$ and damping element equal to $\beta(A)$. It is given by

$$S_x(\omega,t|A) = \frac{S_w(\omega,t)}{\left(\omega^2(A)-\omega^2\right)^2 + \left(\beta(A)\omega\right)^2}. \tag{16}$$

Thus, the nonlinear response EPS can be construed approximately as the sum of the "linear" response EPS appropriately weighted by the value of the corresponding response amplitude PDF $p(A,t)$. Clearly, Eq. (15) represents a generalization of the celebrated input-output spectral relationship of linear random vibration theory to treat nonlinear systems subjected to non-stationary stochastic processes. Remarkably, considering section 2.1, it is readily seen that Eq. (15) can be used in a rather straightforward manner to treat a broader class of systems; that is, oscillators endowed with fractional derivative elements.

Further, based on Priestley's theory of EPS (Priestley, 1988), the non-stationary response autocorrelation function can be evaluated as

$$R_x(t,\tau) = \int_{-\infty}^{\infty} S_x(\omega,t)\exp(i\omega\tau)\mathrm{d}\omega, \tag{17}$$

where τ denotes the time lag between two time instants.

3. NUMERICAL EXAMPLE

A bilinear hysteretic oscillator with fractional derivative elements is considered for assessing the reliability of the developed semi-analytical technique. In this regard, $z(t,x,\dot{x})$ in Eq. (1) takes the form

$$z(t,x,\dot{x}) = \gamma\omega_0^2 + (1-\gamma)\omega_0^2 x_y z_0, \tag{18}$$

where γ denotes the post- to pre-yield stiffness ratio, x_y is the critical value of the displacement at which yielding occurs, and z_0 represents the hysteretic component corresponding to the elastoplastic characteristic. The latter is described by (Roberts and Spanos, 2003)

$$x_y\dot{z}_0 = \dot{x}\left(1 - H(\dot{x})H(z_0-1) - H(-\dot{x})H(-z_0-1)\right), \tag{19}$$

where $H(\cdot)$ denotes the Heaviside step function.

The initially at rest oscillator is subjected to non-stationary stochastic excitation described by the non-separable EPS of the form (Spanos and Solomos, 1983; Fragkoulis et al., 2019)

$$S_w(\omega,t) = S_0\left(\frac{\omega}{5\pi}\right)^2 \exp(c_0 t)t^2 \exp\left(-\left(\frac{\omega}{5\pi}\right)^2 t\right). \tag{20}$$

The parameter values used for the non-separable excitation EPS are $S_0 = 1$ and $c_0 = 0.15$, while the natural frequency and the damping coefficient of the oscillator are $\omega_0 = 12.57$ and $\beta_0 = 0.05$, respectively. Further, the post- to pre-yield stiffness ratio is $\gamma = 0.2$, whereas $x_y = 0.016$. Lastly, the fractional order of the oscillator is $\alpha = 0.5$. For comparison of the analytical results with MCS data, realizations compatible with the excitation EPS of Eq. (20) are produced by utilizing the spectral representation methodology ($10,000$ samples); see, e.g., Liang et al. (2007). The L1-algorithm is used for the numerical integration of Eq. (1) and for determining response realizations (e.g. Koh and Kelly, 1990). Further, EPS estimates based on the ensemble of the response realizations are obtained by employing a formula derived in Spanos et al. (2005) that relies on the theory of generalized harmonic wavelets.

Next, considering Eq. (18), Eq. (8) becomes

$$S(A) = \begin{cases} \frac{4x_y}{\pi}\left(1-\frac{x_y}{A}\right), & A > x_y \\ 0, & A \leq x_y \end{cases} \tag{21}$$

whereas Eq. (9) takes the form

$$F(A)=\begin{cases} \frac{A}{\pi}\left(\Lambda-\frac{1}{2}\sin(2\Lambda)\right), & A>x_y \\ A, & A\leq x_y \end{cases} \tag{22}$$

with $\Lambda=\arccos\left(1-\frac{2x_y}{A}\right)$. Also Eqs. (10) and (11) become

$$\begin{aligned}\beta_{eq}(t)=&-\beta_0+\frac{\beta G\sin\left(\frac{\alpha\pi}{2}\right)}{c(t)}\\ &\times\int_0^\infty \frac{A}{\omega^{1-\alpha}(A)}\exp\left(-\frac{GA^2}{2c(t)}\right)\mathrm{d}A\\ &+\frac{4x_y\omega_0^2(1-\gamma)G}{\pi c(t)}\\ &\times\int_{x_y}^\infty \frac{1-\frac{x_y}{A}}{\omega(A)}\exp\left(-\frac{GA^2}{2c(t)}\right)\mathrm{d}A\end{aligned} \tag{23}$$

and

$$\begin{aligned}\omega_{eq}^2(t)=&\omega_0^2-(1-\gamma)\omega_0^2\left\{\exp\left(-\frac{Gx_y^2}{2c(t)}\right)\right.\\ &-\frac{\pi}{Gc(t)}\int_{x_y}^\infty\left(\Lambda-\frac{1}{2}\sin(2\Lambda)\right)A\\ &\left.\times\exp\left(-\frac{GA^2}{2c(t)}\right)\mathrm{d}A\right\}\\ &+\frac{\beta G\cos\left(\frac{\alpha\pi}{2}\right)}{c(t)}\\ &\times\int_0^\infty \omega^\alpha(A)A\exp\left(-\frac{GA^2}{2c(t)}\right)\mathrm{d}A,\end{aligned} \tag{24}$$

respectively.

Next, applying the herein proposed technique, the response EPS is determined and is plotted in Figure 1. This is also compared and found in satisfactory agreement with the corresponding MCS-based estimate plotted in Figure 2. It is seen that the technique is capable of capturing the salient aspects of the nonlinear system response EPS. In fact, it succeeds in predicting, approximately, the evolution in time of the effective natural frequency of the bilinear hysteretic oscillator, starting from the linear regime at early time instants and reaching nonlinear response behavior as the intensity of the excitation increases with time.

Lastly, applying Eq. (17) yields the non-stationary response autocorrelation function that is plotted in Figure 3.

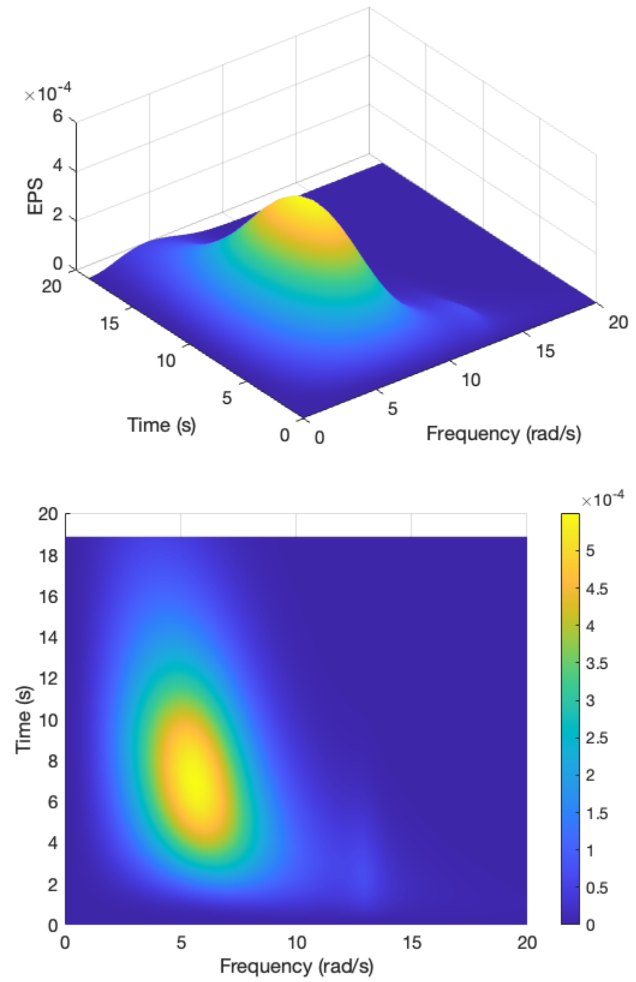

Figure 1: Nonlinear response EPS of a bilinear hysteretic oscillator with fractional derivative elements ($\omega_0=12.57$, $\beta_0=0.05$, $\alpha=0.5$, $\gamma=0.2$, $x_y=0.016$, $S_0=1$); analytical solution.

4. CONCLUDING REMARKS

An approximate analytical technique has been developed for determining both the response EPS and the response non-stationary autocorrelation function of nonlinear/hysteretic oscillators endowed with fractional derivative elements. This has been done in two steps. First, a recently developed stochastic averaging/linearization treatment (Fragkoulis et al., 2019) has been employed for deriving an approximate expression for the nonlinear oscillator non-stationary response amplitude PDF. Second, the non-stationary response amplitude PDF has been used for extending an input-output EPS relationship in the

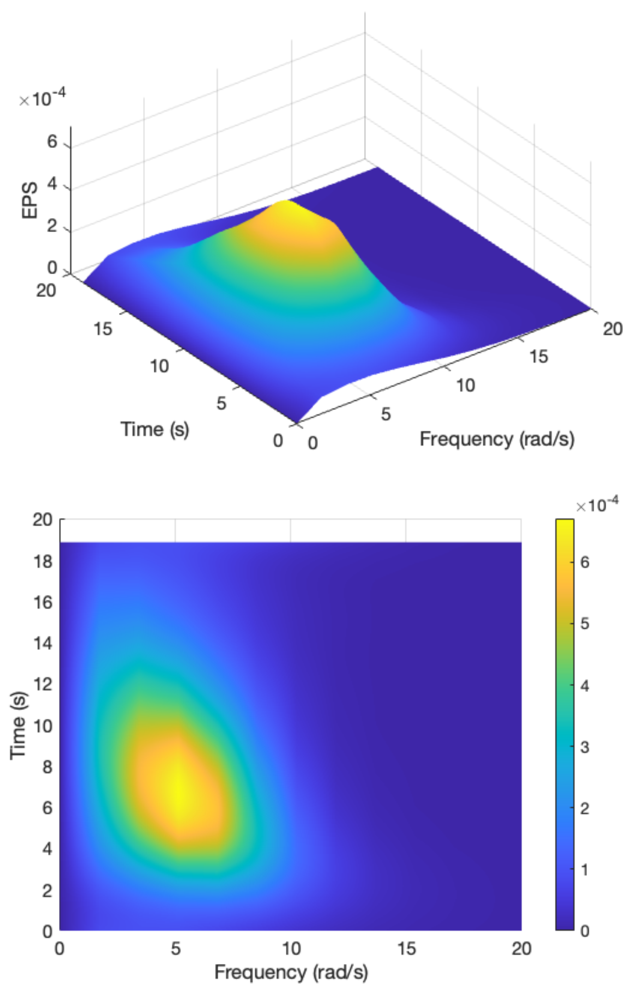

Figure 2: Nonlinear response EPS of a bilinear hysteretic oscillator with fractional derivative elements ($\omega_0 = 12.57$, $\beta_0 = 0.05$, $\alpha = 0.5$, $\gamma = 0.2$, $x_y = 0.016$, $S_0 = 1$)*; MCS-based estimate* ($10,000$ *realizations*).

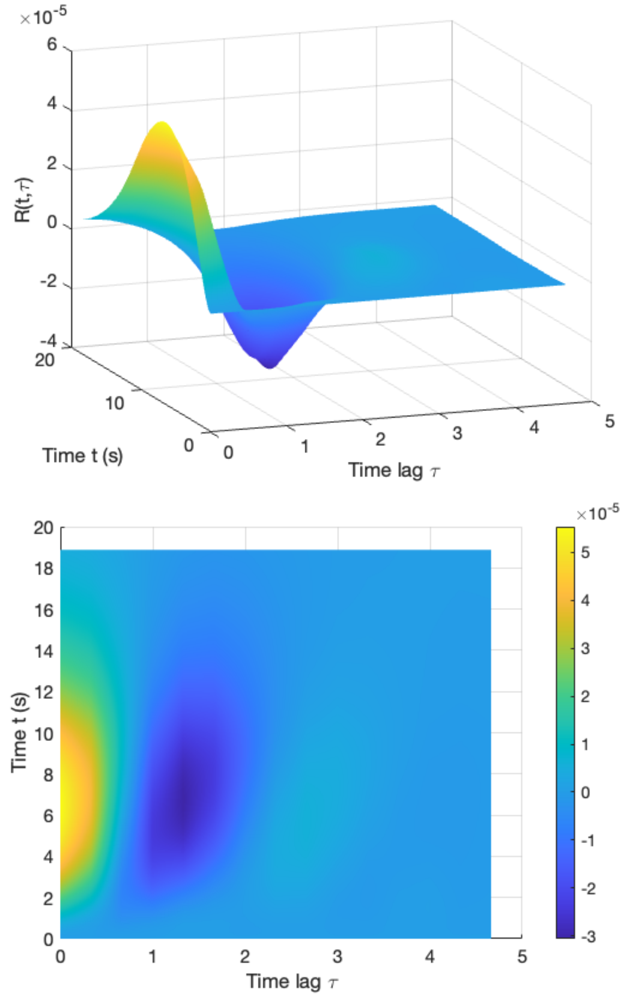

Figure 3: Non-stationary response autocorrelation function of a bilinear hysteretic oscillator with fractional derivative elements ($\omega_0 = 12.57$, $\beta_0 = 0.05$, $\alpha = 0.5$, $\gamma = 0.2$, $x_y = 0.016$, $S_0 = 1$).

joint time-frequency domain (Kougioumtzoglou, 2013) to account for fractional derivative elements in the oscillator equation of motion. Lastly, the non-stationary response autocorrelation function can be readily evaluated by resorting to Priestley's EPS theory. A nonlinear bilinear hysteretic oscillator with fractional derivative elements has been considered as an indicative numerical example. The reliability of the obtained results has been demonstrated by comparisons with pertinent MCS data.

5. ACKNOWLEDGMENT

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6. REFERENCES

Di Paola, M. and Zingales, M. (2012). "Exact mechanical models of fractional hereditary materials." *Journal of Rheology*, 56(5), 983–1004.

Fragkoulis, V. C. and Kougioumtzoglou, I. A. (2023). "Survival probability determination of nonlinear oscillators with fractional derivative elements under evolutionary stochastic excitation." *Probabilistic Engineering Mechanics*, 103411.

Fragkoulis, V. C., Kougioumtzoglou, I. A., Pantelous, A. A., and Beer, M. (2019). "Non-stationary response statistics of nonlinear oscillators with fractional derivative elements under evolutionary stochastic excitation." *Nonlinear Dynamics*, 97(4), 2291–2303.

Jesus, I. S. and Tenreiro Machado, J. (2009). "Development of fractional order capacitors based on electrolyte processes." *Nonlinear Dynamics*, 56(1), 45–55.

Koh, C. G. and Kelly, J. M. (1990). "Application of fractional derivatives to seismic analysis of base-isolated models." *Earthquake engineering & structural dynamics*, 19(2), 229–241.

Kougioumtzoglou, I. A. (2013). "Stochastic joint time–frequency response analysis of nonlinear structural systems." *Journal of Sound and Vibration*, 332(26), 7153–7173.

Kougioumtzoglou, I. A., Ni, P., Mitseas, I. P., Fragkoulis, V. C., and Beer, M. (2022). "An approximate stochastic dynamics approach for design spectrum based response analysis of nonlinear structural systems with fractional derivative elements." *International Journal of Non-Linear Mechanics*, 146, 104178.

Kougioumtzoglou, I. A. and Spanos, P. D. (2009). "An approximate approach for nonlinear system response determination under evolutionary stochastic excitation." *Current science*, 1203–1211.

Kougioumtzoglou, I. A. and Spanos, P. D. (2016). "Harmonic wavelets based response evolutionary power spectrum determination of linear and non-linear oscillators with fractional derivative elements." *International Journal of Non-Linear Mechanics*, 80, 66–75.

Liang, J., Chaudhuri, S. R., and Shinozuka, M. (2007). "Simulation of nonstationary stochastic processes by spectral representation." *Journal of Engineering Mechanics*, 133(6), 616–627.

Makris, N. and Constantinou, M. (1992). "Spring-viscous damper systems for combined seismic and vibration isolation." *Earthquake engineering & structural dynamics*, 21(8), 649–664.

Miles, R. (1989). "An approximate solution for the spectral response of duffing's oscillator with random input." *Journal of Sound and Vibration*, 132(1), 43–49.

Miles, R. (1993). "Spectral response of a bilinear oscillator." *Journal of sound and vibration*, 163(2), 319–326.

Miller, K. S. and Ross, B. (1993). *An introduction to the fractional calculus and fractional differential equations*. Wiley.

Pinnola, F. P. (2016). "Statistical correlation of fractional oscillator response by complex spectral moments and state variable expansion." *Communications in Nonlinear Science and Numerical Simulation*, 39, 343–359.

Pirrotta, A., Kougioumtzoglou, I., Di Matteo, A., Fragkoulis, V., Pantelous, A., and Adam, C. (2021). "Deterministic and random vibration of linear systems with singular parameter matrices and fractional derivative terms." *Journal of engineering mechanics*, 147(6), 04021031.

Podlubny, I. (1998). *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Vol. 198. Academic press.

Priestley, M. B. (1988). "Non-linear and non-stationary time series analysis." *London: Academic Press*.

Roberts, J. and Spanos, P. (1986). "Stochastic averaging: an approximate method of solving random vibration problems." *International Journal of Non-Linear Mechanics*, 21(2), 111–134.

Roberts, J. B. and Spanos, P. D. (2003). *Random vibration and statistical linearization*. Courier Corporation.

Sabatier, J., Agrawal, O. P., and Machado, J. T. (2007). *Advances in fractional calculus*, Vol. 4. Springer.

Spanos, P. D. (1978). "Non-stationary random vibration of a linear structure." *International Journal of Solids and Structures*, 14(10), 861–867.

Spanos, P. D. and Malara, G. (2014). "Nonlinear random vibrations of beams with fractional derivative elements." *Journal of Engineering Mechanics*, 140(9), 04014069.

Spanos, P. D., Tezcan, J., and Tratskas, P. (2005). "Stochastic processes evolutionary spectrum estimation via harmonic wavelets." *Computer Methods in Applied Mechanics and Engineering*, 194(12-16), 1367–1383.

Spanos, P.-T. and Solomos, G. P. (1983). "Markov approximation to transient vibration." *Journal of Engineering Mechanics*, 109(4), 1134–1150.

Spanos, P.-T. D. and Lutes, L. D. (1980). "Probability of response to evolutionary process." *Journal of the Engineering Mechanics Division*, 106(2), 213–224.

Zhang, Y., Kougioumtzoglou, I. A., and Kong, F. (2023). "A wiener path integral technique for determining the stochastic response of nonlinear oscillators with fractional derivative elements: A constrained variational formulation with free boundaries." *Probabilistic Engineering Mechanics*, 103410.