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A Modern Bayesian Approach to Model Updating of Bridges Considering Measurement Uncertainty

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ABSTRACT: Existing bridges were designed using contemporaneous standards, but increased loading due to increased freight demand must be considered in their ongoing safety management. Finite-Element (FE) modelling provides the structural responses under known loads. However, such models typically include many assumptions about structural behaviour. On the other hand, structural health monitoring (SHM) provides valuable data about the actual structural behaviour and potentially the structural condition. Typically, there are significant differences between the FE model prediction and the SHM measurements. Due to the significant prior engineering knowledge about structural behaviour and performance, an effective way to combine new observations with existing models is using Bayesian updating strategies. This paper reviews the current state of the art of Bayesian updating, which has undergone tremendous developments in the last decade or so. We discuss how engineers can benefit from these developments, specifically for updating structural models based on data through the fusion of prior engineering information and structural health monitoring data. The modern Bayesian workflow is applied to a simple case study of a moving load on a beam. The work illustrates the potential benefits of the approach for updating the performance prediction of structures based on data.

1. Introduction

1.1. Overview

Bridges are vital components of transport infrastructure. Due to their essential role, they require careful ongoing management to provide a reliable service. Existing bridges were designed to the codes of practice at the time of their construction but changing traffic loading and new requirements should be considered.

Finite element models (FEM) can be used to determine the internal forces, displacements, mode shapes, frequencies, and other dynamic parameters. There are different sources of uncertainties and simplifications in the modelling and analysis of structures. Uncertainties including model parameters such as geometric and material properties, model assumptions, simplifications of boundary conditions and loads, and structural behaviour affect the structural responses (Sun et al., 2020).

Using measured responses of the structural behaviour from SHM, the structural model can be updated to improve FEM-based performance prediction (Friswell and Mottershead, 1995).

1.2. Finite Element Model Updating

To calibrate the numerical model to the actual behaviour of the structure, Finite Element Model Updating (FEMU) techniques have been developed. In comparison to the actual structural data, the improved FE model can generate results which are closer to the actual situation of the structure. Updated models have been used in different fields of structural engineering like risk assessment, SHM, and control of structures (Hemez and Doebling, 2001; Mustafa et al., 2015).

Probabilistic approaches consider a statistical problem focusing on the model and data measurement uncertainties during the updating process (Bouzas et al., 2022). This approach incorporates modelling assumptions and measured structural behaviour to produce a probability distribution for parameters. Due to the difficulties and uncertainties involved in determining the actual parameters (initial assumptions and measurement), probabilistic methods are often the best choice in model updating (Baisthakur and Chakraborty, 2020).

1.3. Bayesian Model Updating

Bayesian inference uses random variables and probability density functions (PDFs) to define uncertain parameters and Bayes' theorem as a framework for solving the model updating problems (Asadollahi et al., 2018). This method allows combining the empirical evidence and field measurements with prior engineering information into the probabilistic distributions of critical mechanical parameters of bridges (Asadollahi et al., 2018).

Since analytical methods of inferring a multivariate probability density function (PDF) is generally prohibitive, sampling processes are required for Bayesian model updating. Markov Chain Monte Carlo (MCMC) based simulation techniques create a sequence of samples whose distributions converge to the desired distribution using a sequence of random variables (Neal, 1993). Evolutionary MCMC and Delayed Rejection Adaptive Metropolis (Haario et al., 2006)), Transitional MCMC (Ching and Chen, 2007)), Metropolis-Hasting (MH) (Chib and Greenberg, 1995)), and Gibbs sampling (Cheung and Bansal, 2017)) are example developments in MCMC techniques. However, generating the samples using a random walk in which choosing the next step only depends on the current one are inefficient. Simple methods like MH and Gibbs sampling need, a long time to converge to the desired distribution in large scale models with different parameters (Neal, 1993).

Hamiltonian Monte Carlo (HMC) sampling was introduced by Duane et al. (1987). For simulating the samples that follow a target distribution, it employs Hamiltonian dynamics to determine the next state in the Markov Chain. As a result, it consistently explores the probability space, and larger steps are used for moving across the sample space. In comparison to traditional random walk-based MCMC algorithms, HMC often delivers faster convergence with less correlation between samples, and it provides important diagnostics on the model specification (Baisthakur and Chakraborty, 2020). However, the effectiveness of HMC is significantly reduced by the poor selection of two hyperparameters during sampling. Consequently, the No-U-Turn sampler (NUTS) was developed by Hoffman and Gelman to solve the difficulties in adjusting these hyperparameters (Hoffman et al., 2014).

1.4. Bayesian Updating for Structures

Stochastic updating methods which are used in civil engineering to update models with measurements from SHM are relatively computationally complex and expensive. Multi-dimensional problems which use MCMC methods require shorter steps, so that the highest probability, the region will be kept. However, this procedure causes highly autocorrelated samples, and chain convergence only occurs after many steps. Furthermore, traditional MCMC methods do not provide diagnostics on their performance, unlike HMC.

1.5. Contribution

Bayesian model updating of large-scale structures is rarely done or they use problematic older sampling methods, as noted previously. New developments in Bayesian updating in sampling using HMC, NUTS and probabilistic programming languages can avoid many of these problems. It has larger steps and less correlated results, speeds up convergence to the target distribution, and so makes Bayesian updating more practical for large-scale models. This paper discusses the current state of the art of Bayesian updating focusing on HMC and NUTS, and how engineers can benefit from developments in Bayesian modelling using a large amount of data from SHM. As a demonstration, a simple case study of a point moving load on a beam will be updated using modern Bayesian workflow to show the efficiency of using new developments of Bayesian sampling in practice.

2. Sampling for Bayesian Inference

2.1. Bayesian Inference

Considering the iterative stochastic method, FEMU can be considered a statistical problem. The prior belief of parameters that consider uncertainty using the probability density function (PDF) is updated by measured data (D) using Bayes' theorem:

$$P(\theta \mid D) = \frac{P(D \mid \theta)P(\theta)}{P(D)} \tag{1}$$

where, θ and $P(\theta)$ represent uncertain parameters, and the prior PDF for θ, respectively. D is the measurement or data, the term $P(D \mid \theta)$ is the likelihood function which is the probability of observing D for the specified model parameters (θ). Finally, $P(\theta \mid D)$ is the updated (or posterior) distribution of the model parameters, while $P(D) = \int P(D \mid \theta)P(\theta)\,\mathrm{d}\theta$ is the normalization factor. And it is this normalization factor that is so difficult to integrate, that requires the use of sampling methods.

Priors can be obtained from many different forms, including professional judgment, testing of lab-scale experiments, and prior uncertainty quantification of the relevant parameter(s). A very common approach is to use a conjugate prior distribution (Kim and Song, 2021; Bernardo and Smith, 2009; Lindley, 2005), in which the integration can be done in closed form.

The likelihood function can be achieved by considering the equation for the probabilistic model measurements and model output error. The PDF of the posterior can be calculated using sampling methods (Ereiz et al., 2022).

Having determined the priors and likelihood, the calculation of the posterior (multidimensional) PDF can be very challenging since it involves a large number of parameters. The high-dimensional integral in the denominator must be found by approximating methods or sampling. Sampling methods such as Markov chain Monte Carlo is the most popular solution for this problem (Ereiz et al., 2022).

2.2. Sampling Methodologies

Stochastic methods can generate long sequences of samples for probability distributions in a way that the empirical average converges to the corresponding value (Betancourt, 2019). Creating these samples is very challenging. Using a discrete-time Markov-chain, samples can be produced from successive states. The random walk Metropolis-Hastings (MH) is one of the first widely used MCMC algorithms that was improved and created other methods by using first or higher order gradient. This method moves to a new state near the current sample randomly. Based on the density distribution, if a new point is more probable than the current point movement will be accepted, and if a less probable point obtains movement sometimes will be rejected. MH produces samples with high correlation and low acceptance rates based on random walks. This means that all of the samples that were taken are not effective and a large number of iterations are required, causing computational complexity in large-scale models (Roberts et al., 1997). In complicated problems with the high dimensionality of the target distribution, simple proposals like random walks will have a high rate of rejection and thus require small steps. As a result, the movement through the parameter space will be slow, and exploring the target distribution adequately often fails (Betancourt, 2019).

Gibbs sampling converts sampling multi-dimension problems to choosing a new sample conditionally for each dimension (Lee, 2022).

To examine the effective number of samples, the Gelman-Rubin statistic (denoted $\hat{R}$) was introduced (Gelman and Rubin, 1992). This diagnostic analyzes the difference between multiple chains to evaluate MCMC convergence. The variances between-chains and within-chains will be estimated for each parameter and if large differences exist, it shows non-convergence (Betancourt, 2019).

HMC leads the Markov chain to the higher posterior density regions using the gradient of the log-likelihood. It needs transitions that follow the high probability mass, and automatically generate the desired exploration direction for the target distribution using a differential equation. The computation may be more complicated but the acceptance rate is higher even with a large step size. Trying to find the next movement to a new point using a numerical integrator (typically leapfrog integrator) while staying in the regions with the most contribution to the integral, step sizes can be increased in sample space so the correlation will be reduced. Consequently, the number of samples that are used to approximate the integral decreases, and convergence to the target occurs more rapidly (Betancourt, 2019).

Choosing an inappropriate leapfrog algorithm's step size and number of steps can decrease the HMC's efficiency. A very useful algorithm called No U-Turn Sampling (NUTS) automatically tunes HMC using a recursive approach without the need for human input or expensive tuning cycles (Hoffman et al., 2014). As a result, HMC-NUTS is a highly efficient sampler for exploring the multidimensional log-likelihood surface (Nishio and Arakawa, 2019; Gelman et al., 2020).

2.3. Probabilistic Programming Languages

Probabilistic programming is an approach where probabilistic models are defined and estimated automatically. There are a range of Probabilistic Programming Languages (PPLs) such as *BUGS*, *WinBUGS*, *JAGS* (Just Another Gibbs Sampler), *Stan*, and *PyMC*. Thus, in contrast to much of the literature in structural engineering, there is no longer a need for bespoke programming of samplers or log-likelihood functions.

PyMC is a PPL written in python providing users with access to a modern framework of Bayesian inference algorithms (e.g. NUTS and ADVI). Models are created in user-friendly syntax, and visualization is done using the related package *ArviZ* for the post-processing of Bayesian models.

2.4. Principled Bayesian Workflow

Bayesian analysis needs a generative model for parameters and data which can define a joint probability distribution between them. After choosing priors and likelihood, it is very beneficial to do the predictive simulation and check the priors. Prior predictive checking yields the model performance, given the prior distributions. It simulates using the model instead of measured data. The model outputs can be examined to observe the relationship between the outcome and prior distributions selected so that the outcomes stay within a feasible region (Martin et al., 2022).

The current state-of-the-art in fitting models in Bayesian inference uses HMC-NUTS sampling. Indeed, NUTS facilities the solution of complex models without requiring specialist knowledge of the sampler settings, opening up advanced Bayesian modelling to a wider audience than previously possible (Gelman et al., 2020).

After running samples, chains should be evaluated. Divergence during sampling indicates a poorly specified model that would benefit from re-parametrization (Martin et al., 2022). There shouldn't be a substantial difference between different chains run, and all chains should look like a "furry caterpillar“. Chains meeting this requirement indicate that the choice of model and priors are appropriate. Furthermore, the $\hat{R}$ statistic will indicate the effectiveness of the sampling for each model parameter.

After checking posterior samples and checking inference errors, posterior predictive checks should be performed. Firstly, plots can be checked to identify if the inferred posterior distribution of parameters is of the reasonable form (e.g. uni-modal) (Martin et al., 2022). Secondly, the actual data can be overlaid with the posterior model predictions to confirm the reasonableness of the estimated model, using *ArviZ* or other visualization programs.

3. APPLICATION

The application of the current state of the art in Bayesian Updating and using the above-mentioned workflow in structural engineering is shown using a simple beam example.

3.1. Beam Model

Figure 1 illustrates a simple beam with a length of 20 m, flexural rigidity ($EI = 200$ MNm^2) and a moving point load ($P = 100$ kN). When the moving point load crosses the beam, the deflection that occurs along the beam can be calculated. Based on Macaulay's Method (Caprani (2009)) a closed-form expression of the deflection (δ) of the beam is found:

$$\delta = \frac{PL^3}{6EI}\left[a^3(1-x) - \langle a-x\rangle^3 + a(1-x)^3 - a(1-x)\right]. \tag{2}$$

The notation $\langle a-x\rangle$ represents a Macaulay bracket, which is equal to zero when the term inside it is negative and takes its value when the term inside it is positive (Caprani, 2009). P is the point load, a represents the location where the deflection is sought, and x is the location of the moving point load on the beam. In this application, it is assumed that the beam length L and flexural rigidity EI are uncertain.

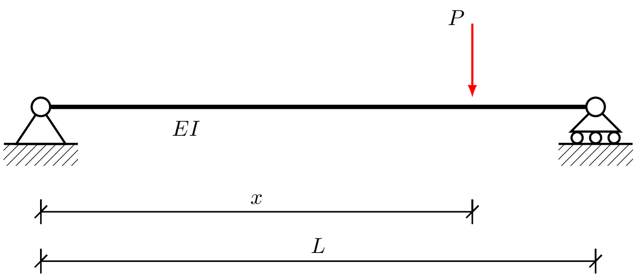

Figure 1: Beam and moving point load.

3.2. *Generating Data*

In this application, it is assumed that five sensors are located along the beam, at $0.25L$, $0.4L$, $0.5L$, $0.6L$, and $0.75L$. These sensors 'record' the deflection while the moving point load crosses the beam (δ_m,), using Equation (2). Artificial Gaussian noise was added to the deflections to reflect realistic measurement conditions:

$$\delta_m = \delta + \varepsilon \tag{3}$$

where $\varepsilon \sim \mathcal{N}(0.0, 0.005)$, that is an unbiased additive noise with 5 mm standard deviation. Readings were taken with a sample rate such that 200 readings are made for each sensor during the movement. One traverse of the point load was used to update the uncertain parameters of the beam using *PyMC* and HMC-NUTS sampling. An example of the theoretical and simulated measurements is shown in Figure 2.

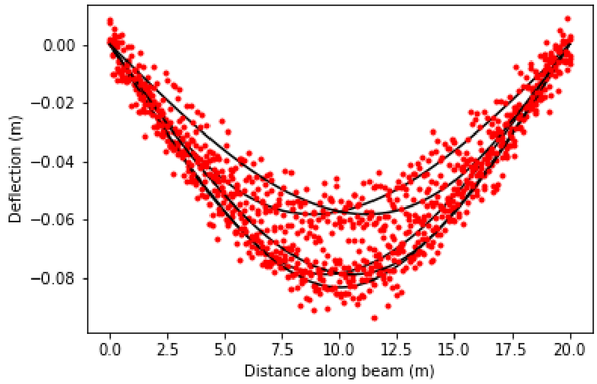

Figure 2: Function and data results plot.

3.3. *Beam Model Updating*

Typical of engineering problems, there is much available prior knowledge that can assist in the definition of plausible prior distributions for uncertain parameters. Even better, using the advanced sampling techniques described earlier, it is not necessary to select conjugate priors, and the most appropriate physical prior distributions can be selected. In this example, EI and L are considered uncertain parameters with Beta and Normal distribution, respectively. Although the normal distribution has physically unreasonable negative support, it is a maximum entropy distribution with a small variance around the nominal value of length, L, and so remains plausible. In contrast, we demonstrate the use of a non-conjugate prior for EI that respects engineering bounds. Proposing the engineering bounds (based on reasonable limits for the elastic modulus of concrete) $EI_{\min} = 150$ MNm^2 and $EI_{\max} = 250$ MNm^2, EI can be parametrized using beta-distributed η on $(0,1)$, and then $EI = EI_{\min} + \eta(EI_{\max} - EI_{\min})$. The distribution parameters (mean, μ and standard deviation, σ) are presented in Table 1. Then, EI can be considered a Deterministic in *PyMC*.

The likelihood function is described by a Normal distribution with zero mean and a variance for the error between model output and generated data. A HalfNormal distribution with a scale parameter of 0.01 was used for the standard deviation, as this has non-negative support.

The model structure is shown in Figure 3. A prior

Table 1: The prior distributions.

Parameter	Distribution	Parameters	Mode	Support
EI (MNm^2)	Beta	$\alpha = 2, \beta = 3$	183.3	$(150, 250)$
L (m)	Normal	$\mu = 20, \sigma = 0.01$	20	$(-\infty, +\infty)$

predictive check shows that the model priors cover the range of the readings well, e.g.. Figure 4.

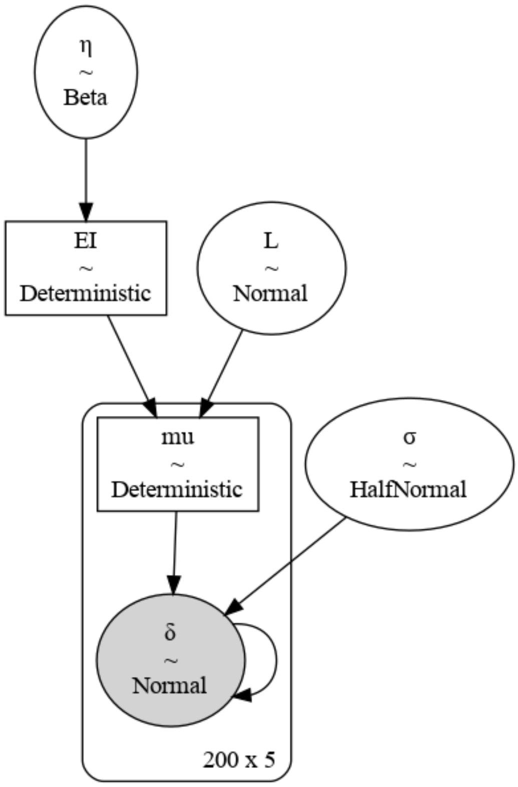

Figure 3: Probabilistic model structure via GraphViz.

HMC-NUTS inference is run on the model using *PyMC* and the resulting posterior density and trace plots for EI and L are shown in Figure 5. Trace plots demonstrate good results between and within the chains. The $\hat{R}$ statistic is close to 1.0 for all parameters indicating good convergence of the chains. Kruschke plots of the posterior parameter densities are shown in Figure 6.

3.4. Posterior Predictive Checks

Posterior predictive checks (PPC) are useful ways to understand the model. PPC or sampling from the posterior, using traces to draw different samples of parameters and generate data. Figure 7

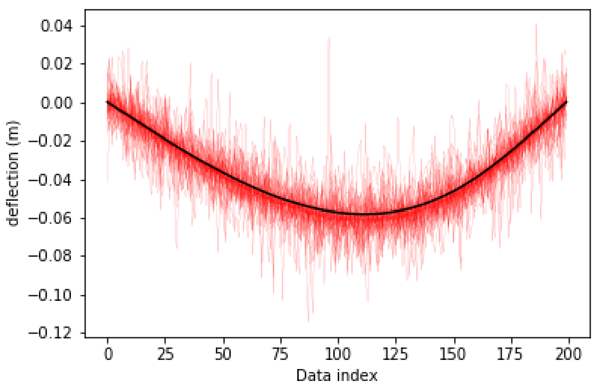

Figure 4: Prior predictive check (50 samples) for the sensor at $0.75L$.

shows the distributions of deflections: observed, posterior mean, and samples from the posterior.

4. CONCLUSIONS

Recent developments in Bayesian methods can help structural engineers improve model updating. Using HMC-NUTS sampling, engineers are free to use non-conjugate distributions, and can have important diagnostics to improve model building. In most structural engineers' problems, there is a well prescribed model, but with uncertain parameters. Priors can use engineering knowledge or experiments.

The deflection of a simple beam with a moving point load was used as a demonstration. The prior and posterior predictive checks show the model is acceptable, and the parameter estimates care accurate. The updating procedure was done without any user input in choosing the best algorithm and their settings for best convergence and model-fitting.

Overall then, following a modern Bayesian workflow can help structural engineers to update priors with SHM data, and the updated model with posterior parameter distributions can better predict the uncertain real structure behaviour.

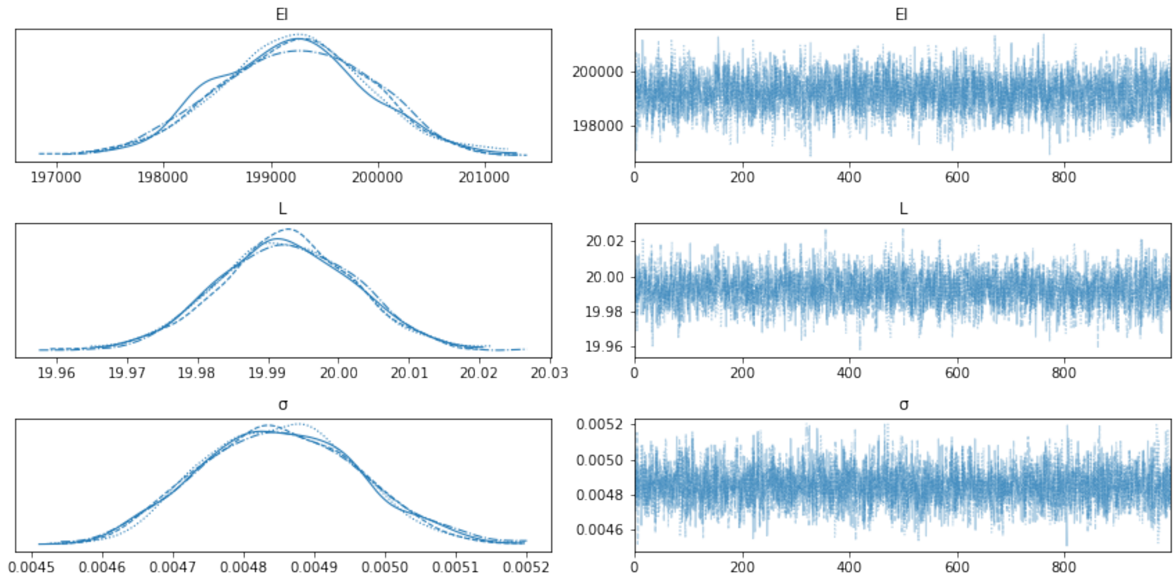

Figure 5: Posterior density and trace plots of each chain.

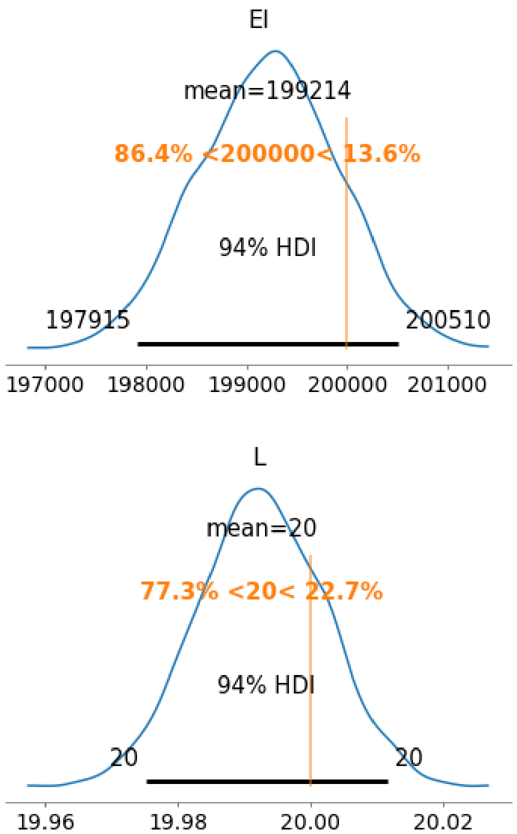

Figure 6: Posterior plots

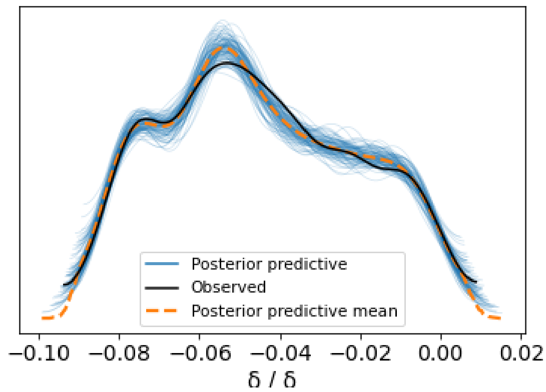

Figure 7: The observed data overlaid on posterior predictive.

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