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Multi-fidelity support vector machines classifiers exploiting discretization error estimators for structural reliability

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ABSTRACT: In this paper, we propose two approaches to build support vector machines classifiers for structural reliability. By exploiting a posteriori error estimators, the discretization error is controlled and bounds on the exact probability of failure can be obtained. The two approaches are illustrated on crack propagation two-dimensional mechanical problem.

1. INTRODUCTION

As industrial structures may be subjected to uncertain loads and as their geometric or material properties may be uncertain, a deterministic approach for design is limited and reliability analysis is preferred. During this analysis, a performance function G is associated to a failure scenario. This function is usually defined as the difference between resistance and solicitation. A negative limit state function corresponds to failure and a strictly positive state function corresponds to safety. If the uncertainties are modeled as random variables, it is therefore possible to compute sensitivity factors, probability of failure or reliability indexes. In this work, we focus on the probability of failure.

The Finite Element Method (FEM) is a widely spread numerical method that allows to simulate and predict the mechanical response of a structure. Thus, Monte Carlo estimators (Metropolis and Ulam (1949)) can be easily employed to estimate the probability of failure from calls to the finite element code for different realizations of the random variables. However, the poor convergence rate of Monte Carlo estimators leads to huge computational costs. This is the reason why variance reduction techniques (Rashki et al. (2018); Giles (2008); Au and Beck (2001)) and meta-modeling methods have been proposed. Some methods consists in building a meta-model $\hat{G}$ that would be a satisfying cheap approximation of G through Kriging (Echard et al. (2011)), response surfaces (Schoefs (2008)), neuronal networks (Teixeira et al. (2021),) ... Support vector machines (SVM) have been thoroughly developed in reliability analysis, see Most (2007); Pan and Dias (2017); Song et al. (2013); Bourinet et al. (2011); Basudhar and Missoum (2010). They enable to metamodeling the limit state $G = 0$, which is the only information necessary for the computation of the probability of failure.

Taking into account the discretization error introduced by the FEM (Babuška and Rheinboldt (1978)) is a major concern in reliability analysis (Mell et al. (2020); Ghavidel et al. (2020)). Indeed, since FEM outputs are used to build the meta-model, the discretization error pollutes the estimation of the probability of failure. However, few

works take into account this discretization error (Morse et al. (2019); Alvin (2000); Yi et al. (2020)).

In this paper, we propose to exploit a posteriori error estimators (Babuška and Rheinboldt (1978)) during the construction of an adaptive SVM-based classifier. We give two algorithms that compute the probability of failure. The first one builds two classifiers in parallel and enables to compute upper and lower bound of the probability of failure. The second algorithm update the classifiers with points obtained on different meshes, in order to guarantee the correct classification. It results in a multi-fidelity meta-model because its construction relies on computations done on two different meshes.

Section 2 defines the reliability mechanical problem. Section 3 give the principles of support vector machines classifiers. Section 4 present the two new algorithms and numerical examples are given in Section 5.

2. Definition of the reliability mechanical problem

In this section, we present the mechanical framework. We also give the discretized formulation of the mechanical problem we solve thanks to the FEM. Finally, we derive bounds on the performance function from a posteriori error estimators based on the error in constitutive relation.

2.1. Continuous mechanical problem

The structure occupies the open domain Ω and is subjected to a force $\underline{f}_{vol}$, a pressure $\underline{F}$ on $\partial_F\Omega$. On $\partial_u\Omega$, the displacement $\underline{u}_d$ is imposed. We consider small strains and linear elasticity characterized by the Hooke tensor $\mathbb{H}$. $\underline{\underline{\sigma}}$ is the Cauchy stress tensor, $\underline{u}$ is the displacement field and $\underline{\underline{\varepsilon}}(\underline{u}) = \frac{1}{2}(\underline{\underline{grad}}(\underline{u}) + \underline{\underline{grad}}^T(\underline{u}))$ is the linearized strain tensor.

We introduce the space of kinematically admissible fields $\mathrm{KA} = \left\{\underline{u} \in \left(\mathrm{H}^1(\Omega)\right)^d,\ \underline{u} = \underline{u}_d \text{ on } \partial_u\Omega\right\}$ and note KA^0 the associated vectorial space. We also defined the space of statically admissible stress fields:

$$\mathrm{SA} = \{\underline{\underline{\tau}} \in \left(\mathrm{L}^2(\Omega)\right)^{d\times d}_{\mathrm{sym}};\forall \underline{v} \in \mathrm{KA}^0,$$
$$\int_\Omega \underline{\underline{\tau}} : \underline{\underline{\varepsilon}}(\underline{v})\, d\Omega = \int_\Omega \underline{f}_{vol}\cdot\underline{v} d\Omega + \int_{\partial_F\Omega} \underline{F}\cdot\underline{v} dS\}. \quad (1)$$

The error in constitutive relation is $e_{CR_\Omega}(\underline{u},\underline{\underline{\sigma}}) = \|\underline{\underline{\sigma}} - \mathbb{H} : \underline{\underline{\varepsilon}}(\underline{u})\|_{\mathbb{H}^{-1},\Omega}$ where $\|\underline{\underline{\kappa}}\|_{\mathbb{H}^{-1},\Omega} = \sqrt{\int_\Omega \left(\underline{\underline{\kappa}} : \mathbb{H}^{-1} : \underline{\underline{\kappa}}\right) d\Omega}$ (see Ladevèze and Leguillon (1983)).

We model the uncertainties by random variables gathered in the vector $X : \zeta \in Z \to X(\zeta) = x \in \mathbb{R}^q$ where Z is the Universe and $x \in \mathbb{R}^q$ is a realization of the random variable. p is the joint distribution of X. Uncertainties may concern the applied loads or displacement ($\underline{f}_{vol}$, $\underline{F}$, $\underline{u}_d$, $\partial_u\Omega$, $\partial_F\Omega$), the material properties ($\mathbb{H}$) or the geometry (Ω). To avoid heavy notations, we do not specifically write the dependence on X (or ζ) in the rest of the paper. The following equalities are defined written almost surely:

$$\begin{cases} \text{Find } \underline{u} \text{ and } \underline{\underline{\sigma}} \text{ such that} \\ \underline{u} = \underline{u}_d \text{ on } \partial\Omega\bigcap\partial_u\Omega \\ \underline{\mathrm{div}}(\underline{\underline{\sigma}}) + \underline{f}_{vol} = \underline{0} \text{ on } \Omega \text{ and } \underline{\underline{\sigma}}\underline{n} = \underline{F} \text{ on } \partial_F\Omega \\ \underline{\underline{\sigma}} = \mathbb{H} : \underline{\underline{\varepsilon}}(\underline{u}) \text{ on } \Omega \end{cases} \quad (2)$$

The exact solution $\left(\underline{u}_{ex}, \underline{\underline{\sigma}}_{ex}\right)$ exists and is unique. In this paper, we consider that the performance function G_{ex} can be written $G_{ex} = R - S(\underline{u}_{ex})$ where R is the resistance (deterministic or random) and S is the solicitation. We assume that S is a linear form of the displacement. If it is not the case (Von Mises equivalent stress for examples), bounds given in (6) are not guaranteed. The probability of failure reads $P_{f,ex} = \int_{G_{ex}(x)\le 0} p(x)dx$. The exact solution is unknown and discretization techniques are used to approximate the solution.

2.2. Discretized problem

The finite element method consists in searching the solution into a subspace of finite dimension $\mathrm{KA}_H = \left\{\underline{u} \in \left(\mathrm{H}^1(\Omega_H)\right)^d,\ \underline{u} = \underline{u}_d \text{ on } \partial_u\Omega_H\right\}$. The discretized problem reads:

$$\text{Find } \underline{u}_H \in \mathrm{KA}_H \text{ such that } \underline{\underline{\sigma}}_H = \mathbb{H} : \underline{\underline{\varepsilon}}(\underline{u}_H) \text{ and}$$
$$\int_{\Omega_H} \underline{\underline{\sigma}}_H : \underline{\underline{\varepsilon}}(\underline{v}_H)\, d\Omega = \int_{\Omega_H} \underline{f}_{vol}\cdot\underline{v}_H d\Omega + \int_{\partial_F\Omega_H} \underline{F}\cdot\underline{v}_H dS \quad (3)$$

Because of the discretization error introduced by the FEM, $S(\underline{u}_H) \neq S(\underline{u}_{ex})$ so $G_H = R - S(\underline{u}_H) \neq$

G_{ex}. Using a posterior error estimators based on the constitutive error relation (Ladevèze (2008)) and techniques to build statically admissible stress fields (Parés et al. (2006); Pled et al. (2011); Ladevèze and Leguillon (1983)), it is possible to obtain bounds on the exact value G_{ex}.

Let defined the following space:

$$\widetilde{\mathrm{SA}} = \{\underline{\underline{\tau}} \in \left(\mathrm{L}^2(\Omega)\right)^{d\times d}_{\mathrm{sym}} ; \forall \underline{v} \in \mathrm{KA}^0, \int_\Omega \underline{\underline{\tau}} : \underline{\underline{\varepsilon}}(\underline{v})\, d\Omega = S(\underline{v})\} \tag{4}$$

The adjoint problem is

$$\begin{aligned} &\text{Find } \left(\widetilde{\underline{u}}_{ex}, \widetilde{\underline{\underline{\sigma}}}_{ex}\right) \in \mathrm{KA}^0(\Omega) \times \widetilde{\mathrm{SA}}(\Omega) \text{ such that} \\ &e_{CR_\Omega}(\widetilde{\underline{u}}_{ex}, \widetilde{\underline{\underline{\sigma}}}_{ex}) = 0 \end{aligned} \tag{5}$$

This problem is solved using the FEM on a mesh that can differ from the mesh used to solve the direct problem. The obtained solution is noted $\widetilde{\underline{u}}_{\widetilde{H}}$. We note $\hat{\underline{\underline{\sigma}}}_H$ (respectively $\hat{\widetilde{\underline{\underline{\sigma}}}}_{\widetilde{H}}$) the statically admissible stress field built from $\underline{\underline{\sigma}}_H$ (resp. $\widetilde{\underline{\underline{\sigma}}}_{\widetilde{H}}$). Then, the following upper and lower bounds of G_{ex} can be computed:

$$G^- \leq G_{ex} \leq G^+ \tag{6}$$

with

$$G_- := G_m - \frac{1}{2} e_{CR_\Omega}(\underline{u}_H, \hat{\underline{\underline{\sigma}}}_H) e_{CR_\Omega}(\widetilde{\underline{u}}_{\widetilde{H}}, \hat{\widetilde{\underline{\underline{\sigma}}}}_{\widetilde{H}}) \tag{7}$$

and

$$G_+ := G_m + \frac{1}{2} e_{CR_\Omega}(\underline{u}_H, \hat{\underline{\underline{\sigma}}}_H) e_{CR_\Omega}(\widetilde{\underline{u}}_{\widetilde{H}}, \hat{\widetilde{\underline{\underline{\sigma}}}}_{\widetilde{H}}) \tag{8}$$

and

$$G_m = G_H - \int_\Omega \frac{1}{2} (\hat{\widetilde{\underline{\underline{\sigma}}}}_{\widetilde{H}} + \mathbb{H} : \underline{\underline{\varepsilon}}(\widetilde{\underline{u}}_{\widetilde{H}})) : \mathbb{H}^{-1} : (\hat{\underline{\underline{\sigma}}}_H - \mathbb{H} : \underline{\underline{\varepsilon}}(\underline{u}_H)) d\Omega \tag{9}$$

Note that it is possible that $G_H \notin [G^-; G^+]$ and that G_m might be a better approximation of G_{ex} than G_H.

3. Support Vector Machines (SVM) classification for reliability

In this Section, we briefly explain the construction of SVM classifiers. More details can be found in Vapnik (2013). The objective is to build a classifier $D : \mathbb{R}^q \to \{-1; 1\}$ from n observations $(\mathbf{x}_i, y_i)_{i=1..n}$.

3.1. Linear classifier

If data is linearly separable, the classifier D can be built from function $f(x) = \mathbf{v}^T\mathbf{x} + a$ with $a \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^q$ where $\mathbf{v}^T\mathbf{x}$ is the scalar product between $\mathbf{v}$ and $\mathbf{x}$. Thus $D(\mathbf{x}) = \mathrm{sign}(f(\mathbf{x}))$. The margin m between observations and the hyperplane $\Delta = \{\mathbf{x} \in \mathbb{R}^q \text{ such that } f(\mathbf{x}) = 0\}$ is $m = \min_{i=1..n}\left(\frac{|\mathbf{v}^T\mathbf{x}_i + a|}{||\mathbf{v}||}\right)$. Parameters $\mathbf{v}$ are a sought to maximise the margin m. In order to obtain a unique solution, the optimization problem is written with the variables $\mathbf{w} = \frac{\mathbf{v}}{m||\mathbf{v}||}$ and $b = \frac{a}{m||\mathbf{v}||}$ and the primal formulation reads :

$$\begin{aligned} &\text{Find } \mathbf{w} \text{ and } b \text{ such that } \frac{1}{2}||\mathbf{w}||^2 \text{ is minimum} \\ &\text{and } y_i(\mathbf{w}^T\mathbf{x}_i + b) \geq 1 \ \forall i = 1..n \end{aligned} \tag{10}$$

The associated Lagrangien is $\mathscr{L}(w, b, \alpha) = \frac{1}{2}||w||^2 - \sum_{i=1}^{n} \alpha_i \left(y_i(\mathbf{w}^T\mathbf{x}_i + b) - 1\right)$ where α_i are Lagrange multipliers. It enables to define the following dual formulation:

$$\begin{aligned} &\text{Find } \alpha_i \text{ with } i \in [1; n] \text{ such that} \\ &\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j y_i y_j \mathbf{x}_i^T\mathbf{x}_j - \sum_{i=1}^{n}\alpha_i \text{ is minimum and} \\ &\sum_{i=1}^{n} \alpha_i y_i = 0 \text{ and } \alpha_i \geq 0 \ \forall i = 1..n \end{aligned} \tag{11}$$

Both primal and dual formulations can be solved with standard quadratic programming solvers.

3.2. Non-linear classifier

In case the data is not linearly separable, the scalar product $\mathbf{x}_i^T\mathbf{x}_j$ is replaced by $\kappa(\mathbf{x}_i, \mathbf{x}_j)$ representing a measure of the influence of $\mathbf{x}_i$ on $\mathbf{x}_j$. In this paper, we use a Gaussian kernel $\kappa(\mathbf{x}_i, \mathbf{x}_j) = \exp\left(\frac{||\mathbf{x}_i - \mathbf{x}_j||}{2\sigma^2}\right)$ where σ is an hyperparameter usually obtained by cross-validation. The dual formulation of the optimization problem reads:

$$\begin{aligned} &\text{Find } \alpha_i \text{ pour } i \in [1; n] \text{ such that} \\ &\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j y_i y_j \kappa(\mathbf{x}_i^T\mathbf{x}_j) - \sum_{i=1}^{n}\alpha_i \text{ is minimum} \\ &\sum_{i=1}^{n} \alpha_i y_i = 0 \text{ and } 0 \leq \alpha_i \leq C \ \forall i = 1..n \end{aligned} \tag{12}$$

where C is the penalty. The smaller C, the more misclassification is allowed. In this paper, we do not authorize misclassification and choose a very large value for C such that α_i are not bounded.

4. MULTI-FIDELITY SUPPORT VECTOR MACHINES CLASSIFIERS EXPLOITING DISCRETIZATION ERROR ESTIMATORS

In this section, we propose to improve the estimation of the probability of failure by exploiting a posteriori error estimators during the construction of the classifier. We give two algorithms based on Pan and Dias (2017). The first one consists in building a multifidelity classifier from calls to the solver on two different meshes. The second one aims at building two classifiers bounding the exact unknown limit state.

4.1. First approach: guaranteed state classifier

In this approach, two mesh sizes h_{max} and h_{min} are defined a priori. The design of experiment *DOE* and the Monte Carlo population U are generated during the initialization. For each point x_i of the DOE we compute $G_+(x_i)$ and $G_-(x_i)$ for $h = h_{max}$. If $G_+G_- > 0$, $\text{sign}(G_H(x_i)) = \text{sign}(G_{ex}(x_i))$ and this observation can be used for the construction of the classifier. On the contrary, if $G_+G_- < 0$, the discretization error pollutes the observations. Therefore the FE simulation is done again on the finer mesh $h = h_{min}$ and we use the result obtained on the fine mesh for the construction of the classifier.

Once the classifier is built, the Monte Carlo population is separated into two sub-populations and the probability of failure is estimated as illustrated in algorithm 1.

To control the meta-modelling error, new observations can be added to improve the classifier. This is done as suggested in Pan and Dias (2017) with the learning function ξ being the ratio between the number of points of U located inside the margin of the classifier over the size of U. If $\xi > \eta_1$, the learning criterion is not satisfied. Therefore, the enrichment of the meta-model is done by adding observation at a new point $\mathbf{x_{new}}$ defined by:

$$\mathbf{x_{new}} = \underset{\mathbf{x}\in U}{\text{argmin}} \frac{s(\mathbf{x})\max(d)}{d(\mathbf{x})\max(s)} \tag{13}$$

where $s(\mathbf{x})$is the distance between $\mathbf{x}$ and the separator. $d(\mathbf{x})$ is the distance between $\mathbf{x}$ and the closest point in the DOE.

The Monte Carlo population is enlarged if the coefficient of variation COV is too large, which enables to control the sampling error.

The first approach is described in algorithm 2.

Algorithm 1: Evaluation SVM

Separate U into two sub-populations
$U_f = \{x_i \in U | D(x_i) = -1\}$ and
$U_s = \{x_i \in U | D(x_i) = +1\}$;
Compute the probability of failure $P = \frac{\text{card}(U_f)}{n_{MC}}$;
Compute COV $= \sqrt{\frac{1-P}{P \times n_{MC}}}$;
Compute the learning function ξ;

Algorithm 2: Guaranteed state approach

Generate the Monte Carlo population U (size n_{MC}) ;
Generate the design of experiment *DOE* (size n_{DOE});
for $i = 1..n_{DOE}$ **do**
  Compute $G_+(x_i)$ and $G_-(x_i)$ for $h = h_{max}$;
  if $G_+(x_i)G_-(x_i) < 0$ **then**
    Compute $G_+(x_i)$ and $G_-(x_i)$ for $h = h_{min}$;
  end
  Append observations with $y_i = \text{sign}(G_m(x_i))$;
end
Build the classifier D from observations ;
Run the algorithm **Evaluation SVM**;
while $\xi > \eta_1$ *or* COV $> \eta_2$ **do**
  if $\xi > \eta_1$ **then**
    Select the next learning point x_{new} ;
    Compute $G_-(x_{new})$ and $G_+(x_{new})$ for $h = h_{max}$;
    if $G_+(x_{new})G_-(x_{new}) < 0$ **then**
      Compute $G_+(x_{new})$ and $G_-(x_{new})$ for $h = h_{min}$;
    end
    Append observations with $\text{sign}(G_m(x_{new}))$;
    Build the classifier D from observations ;
  else
    Enlarge Monte Carlo population ;
  end
  Run the algorithm **Evaluation SVM**;
end

The proposed approach can be done with more than two mesh sizes (the user may define a family of nested meshes for instance). It is possible that

even for the finest mesh, $G^+G^- < 0$. Note that minimal finest mesh has to be defined in order to avoid infinite loop in the case $G_{ex}(x_i)$ (the point is located exactly on the limit state).

4.2. Second approach: double classifier

In this second approach, we consider a unique mesh size and we propose to build 2 classifiers: the first one separates points certainly safe $U_{cs} = \{x_i \in U | G_-(x_i) > 0\}$ from the rest; the second one separates the points leading certainly to failure $U_{cf} = \{x_i \in U | G_+(x_i) < 0\}$ from the rest. The subroutine 3 indicates how the classifiers are exploited to obtain two bounds on the probability of failure P_+ and P_-.

Algorithm 3: SVM classifiers

Define the populations $U_{cs} = \{x_i \in U | D_-(x_i) = +1\}$ and $U_{cf} = \{x_i \in U | D_+(x_i) = -1\}$;
Compute the failure probabilities $P_+ = \frac{\text{card}(U_{cf})}{n_{MC}}$; $P_- = 1 - \frac{\text{card}(U_{cs})}{n_{MC}}$ and $P = \frac{P_+ + P_-}{2}$;
Compute the coefficient of variation $\text{COV} = \sqrt{\frac{1-P}{P n_{MC}}}$;
Compute the learning functions ξ_+ and ξ_-;

As in the previous approach, the learning criterion enables to control the meta-modeling error and the coefficient of variation is used to control the sampling error. Since upper and lower bounds $G_-(x_{new})$ and $G_+(x_{new})$ are always computed together, both classifiers are updated from this observations even if only one classifier triggered the learning criterion. The final procedure is given in Algorithm 4.

This second approach enables to steer the estimation of the probability of failure by an objective of precision. If the ratio $\frac{2(P_+ - P_-)}{P_+ + P_-}$ is larger than the desired precision, the procedure could be done again on a new mesh size $h_{next} = \frac{h}{2}$. Only the uncertain population $U_{uc} = U - U_{cs} \bigcup U_{cf}$ will have to be classified.

5. NUMERICAL ILLUSTRATIONS

In this section, we apply the two proposed approaches on the two-dimensional mechanical problem of a cracked plate. Only two random variables are considered to ease the illustration of the limit states.

Algorithm 4: Double classifier

Generate the Monte Carlo population U (size n_{MC}) ;
Generate the design of experiment DOE (size n_{DOE});
for $i = 1..n_{DOE}$ **do**
  Compute $G_+(x_i)$ and $G_-(x_i)$;
  Append the observations with $y_+ = \text{sign}(G_+)$ and $y_- = \text{sign}(G_-)$;
end
Build the classifiers D_+ and D_- from y_+ and y_- ;
Run the algorithm **SVM classifiers**;
while $\xi_+ > \eta_1$ *or* $\xi_- > \eta_1$ *or* $\text{COV} > \eta_2$ **do**
  if $\xi_+ > \eta_1$ *or* $\xi_- > \eta_1$ **then**
    if $\xi_+ > \eta_1$ **then**
      Select the next learning point x_{new} ;
      Compute $G_-(x_{new})$ and $G_+(x_{new})$;
      Append the observations with $y_+ = \text{sign}(G_+(x_{new}))$ and $y_- = \text{sign}(G_-(x_{new}))$;
      Build the classifiers D_+ and D_- ;
    end
    if $\xi_- > \eta_1$ **then**
      Select the next learning point x_{new} ;
      Compute $G_-(x_{new})$ and $G_+(x_{new})$;
      Append the observations with $y_+ = \text{sign}(G_+(x_{new}))$ and $y_- = \text{sign}(G_-(x_{new}))$;
      Build the classifiers D_+ and D_- ;
    end
  else
    Enlarge the Monte Carlo population;
  end
  Run the algorithm **SVM classifiers**;
end

5.1. Cracked plate

Let consider a rectangular plate ($w = 7mm$, $L = 16mm$) with a horizontal crack of length $a \in [2;5]$ following a Beta-distribution $B(2,2)$. We consider isotropic elasticity with Poisson coefficient $\nu = 0.3$ and Young modulus $E = 210$ GPa. This plate is subjected to traction. The failure scenario is the crack opening according to Griffith criterion: $G = K_{lim} - K_I$ with the deterministic resistance $K_{lim} = 22MPa\sqrt{mm}$. the stress intensity factor K_I is a linear functional of the displacement computed thanks to the integral on a crown of inner radius $R_i = 1mm$ and outer radius $R_e = 1.5mm$ (Stern et al. (1976)). The direction of the force is given by $\theta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$ following a Beta-distribution $B(3,2)$. This mechanical problem is illustrated in Figure 1 and the joint

distribution of the two independent random variables is given in Figure 2.

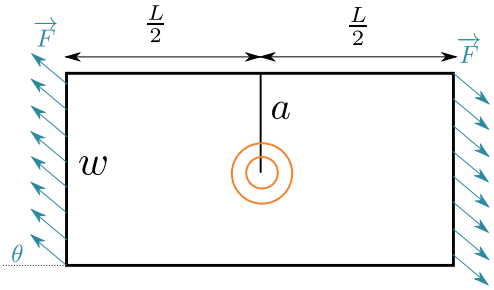

Figure 1: cracked plate

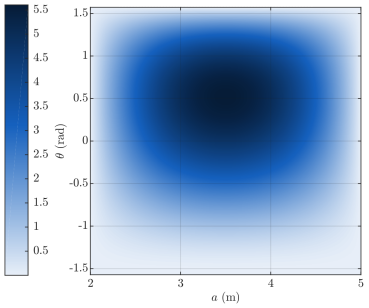

Figure 2: joint distribution of a and θ

We set $\eta_1 = 10^{-4}$ and $\eta_2 = 0.02$. We consider a factorial design of experiment with $n_{DOE} = 12$ as all random variables are bounded. 5 Monte Carlo populations are considered.

5.2. *Standard mono-fidelity approach*

The standard monofidelity approach from Pan and Dias (2017) was done on an overkill mesh of size $h_{overkill} = 0.02$ to obtain a reference probability of failure P_{ref}=5.89 10^{-3} with the first Monte Carlo population. This simulation required 69 calls to the finite element solver that last 16 184 seconds. The classifier built during the computation is referred as overkill limit state and is plotted in blue in the figures in the next subsections.

5.3. *Multi-fidelity classifier*

We apply Algorithm 2 to construct the multi-fidelity classifier with $h_{max} = 0.5$ and $h_{min} = 0.1$. The limit state obtained for the first Monte Carlo population is plotted in Figure 3. We observe that the fine mesh is used only close to the limit state, which illustrates the multi-fidelity strategy. The limit state obtained with the multifidelity approach is close to the reference (blue).

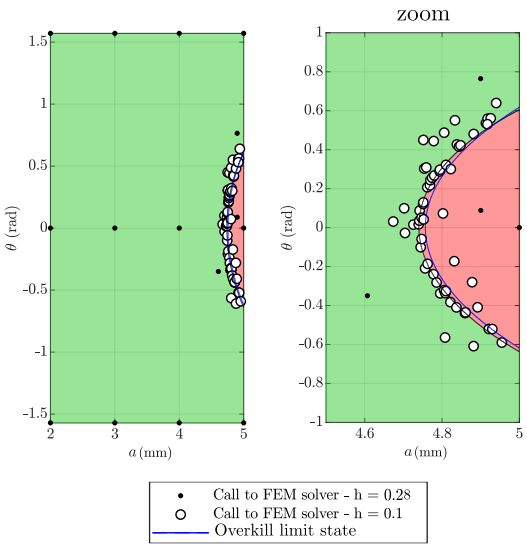

Figure 3: Left: limit state obtained for the first approach; Right: zoom

In Table 1, we give the probabilities of failure, number of call to the FE code and computational times for the five populations. The time necessary for the error estimation t_{err} is very large compared to the time to build and solve the FE problem t_{EF}. However, since $G_m(x_i)$ is a better approximation of $G_{ex}(x_i)$ than $G_H(x_i)$, the estimation of the probability of failure is better than with the standard monofidelity approach for an equivalent total computational time.

P_f ($\times 10^{-3}$)	Nb calls		$t_{EF}(s)$	$t_{err}(ks)$
	h_{max}	h_{min}		
6.7	73	58	326	16.2
6.8	71	56	389	20.4
7.0	66	53	326	15.1
6.6	90	74	414	17.1
6.9	93	73	362	14.9

Table 1: Multi-fidelity classifier: results for 5 Monte Carlo populations

5.4. *Bounding classifiers*

We apply Algorithm 4 to construct the two classifiers in parallel with $h_1 = 0.28$ and also with

$h_2 = 0.1$. The limit states obtained for the first Monte Carlo population are plotted in Figure 4. We observe that the reference limit state is always between the two classifiers. We also observe that applying the algorithm with $h_2 = 0.1$ enables to reduce the size of the uncertain population.

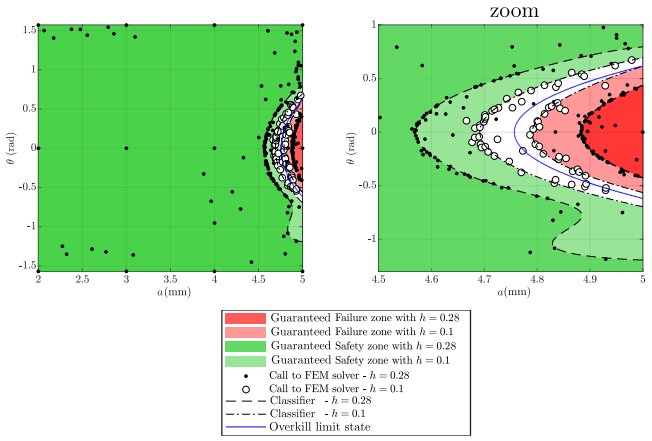

Figure 4: Limit states obtained for the second approach

We also give the bounds on the probability of failure and number of calls to the FE code in Table 2. The use of the finer mesh h_2 improves the bounding of the probability of failure.

h_1			h_2		
Nb calls	P_-	P_+	Nb calls	P_-	P_+
194	$9.3\ 10^{-3}$	$2.3\ 10^{-2}$	65	$4.0\ 10^{-3}$	$1.0\ 10^{-2}$
171	$9.2\ 10^{-3}$	$2.2\ 10^{-2}$	52	$4.2\ 10^{-3}$	$1.0\ 10^{-2}$
110	$9.4\ 10^{-3}$	$2.2\ 10^{-2}$	52	$3.8\ 10^{-3}$	$1.0\ 10^{-2}$
164	$9.6\ 10^{-3}$	$2.2\ 10^{-2}$	61	$4.0\ 10^{-3}$	$1.0\ 10^{-2}$
156	$9.2\ 10^{-3}$	$2.2\ 10^{-2}$	69	$3.9\ 10^{-3}$	$1.1\ 10^{-2}$

Table 2: Bounding classifiers: bounds of the probability of failure

In Table 3, we give the computational time for this approach. Once again, the error estimation procedure is expensive but it enables to provide bounds P^+ and P^- on P_{ex} instead of a unique approximation P.

6. Conclusions and prospects

In this paper, we present two strategies to estimate the probability of failure using SVM classifiers and controlling the discretization error by exploiting a posteriori error estimators. The first strategy consists in exploiting the bounding $G^- \leq G_{ex} \leq G^+$ to built a multifidelity classifier: the computational effort is made close to the limit state by using the fine mesh only when required and by authorizing the use of results on coarse mesh if the discretization error does not pollute the sign of G. The second strategy consists in building two classifiers from the bounds G^+ and G^- to exhibit the uncertain population composed of points for which the discretization error prevents the classification into safe or failure domain. Future work will consist in articulating the two criteria (on the learning process and on the size of the Monte Carlo population) to adapt the precision and balance the three different errors: discretization error, metamodeling error and sampling error.

h_1		h_2	
$t_{EF}(s)$	$t_{err}(ks)$	$t_{EF}(s)$	$t_{err}(ks)$
223	5.293	202	11.069
244	6.499	188	11.580
151	3.938	206	13.017
289	7.553	277	19.177
303	7.878	519	27.717

Table 3: Bounding classifiers: Numerical cost for 5 Monte Carlo populations

7. References

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