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Physically Driven Full Probabilistic Uncertainty Propagation in Complex Nonlinear Structures

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ABSTRACT: The safety and reliability evaluation of complex engineering structures under dynamic loading conditions has long been a challenging problem. The coupling of nonlinearity and randomness in high-dimensional or large-degree-of-freedom stochastic systems is the essential difficulty to be circumvented. To this end, the refined analysis techniques to capture structural behaviors and uncertainty quantification and propagation, seemly irrelevant, are essentially the two sides of a coin. This perspective leads to the thought of physically driven full probabilistic uncertainty propagation in complex nonlinear systems. In this paper, some advances in this framework, in particular the probability density evolution method and its extensions and applications, will be outlined. Problems to be further studied are also discussed.

1. INTRODUCTION

Ensuring the safety and reliability of complex engineering structures under various loading conditions, such as seismic events, has long been a major challenge (Lutes & Sarkani 2004). Actually, the behavior of structures subjected to various dynamic actions exhibits strong nonlinearity (Belytschko et al. 2014, Li et al. 2014). Meanwhile, both structural parameters and external excitations involve unignorable randomness (Housner 1947, Li & Chen 2009). Therefore, the refined nonlinear analysis of complex structures and uncertainty quantification and propagation are essentially the two sides of a coin (Li 2021), where tackling the coupling of nonlinearity and randomness in complex high-dimensional nonlinear systems plays a central role in circumventing the above challenge.

Great efforts have been devoted to the two aspects over the past 70 years. In structural analysis, the past 20 years have witnessed the transition of paradigm from the component-structure two-level framework, including the numerical simulation of behaviors of components (e.g., beams, columns, and shear walls) (Takeda et al. 1970) and assembling of components to a structure, to the continuum-mechanics based framework, where the constitutive law (Belytschko et al. 2014, Li et al. 2014), spatial discretization (e.g., element techniques) and time marching algorithm are the three pillars. In the aspect of uncertainty quantification and propagation, both stochastic modeling of structural parameters or parameter fields (Ang & Tang 2006) and stochastic excitations (Housner 1947, Shields & Deodatis 2013), and stochastic mechanics/dynamics, including, e.g., the sampling approaches (Au & Beck 2001), the moment-level approaches (Soize & Ghanem 2004, Zhao & Lu 2021), and the approaches at the level of probability density functions (Zhu 2006, Kougioumtzoglou & Spanos 2012), together with the surrogate models (Sudret 2008) and the emerging machine learning techniques (Karniadakis et al. 2021), were extensively studied.

Unfortunately, the above two aspects were developed somewhat independently without connections, and the techniques for uncertainty propagation considerably lag behind the refined structural behavior analysis. Actually, Freudenthal (1947) stressed that "the discrepancy between the highly refined procedure of modern design and the rather arbitrary manner of choosing the safety factor is seriously hampering the development of more effective design methods based upon a perfect balance of safety and economy". Seventy years later, Li (2017) came up with the thought of the third-generation structural design theory, revealing that the structural behavior analysis and the uncertainty quantification and propagation are the two indispensable wheels driving the updating of structural design theory. Along this line, a new unified framework of physically driven uncertainty propagation in complex nonlinear systems was proposed and developed (Li & Chen 2009, Li 2021). The present paper will review and revisit some advances and problems to be studied in the future.

2. PROBLEM FORMULATION

For a generic solid mechanics problem, including an engineering structure under dynamic actions, the equation of motion on a body $\mathcal{B}$ reads

$$\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{x},t) + \boldsymbol{F}(\boldsymbol{x},t;\xi) = \rho(\xi)\ddot{\boldsymbol{U}}(\boldsymbol{x},t) + c(\xi)\dot{\boldsymbol{U}}(\boldsymbol{x},t), \tag{1}$$

where $\boldsymbol{U}(\boldsymbol{x},t)$ is the displacement at $\boldsymbol{x}$ and time t, the over dot denotes the time derivative, $\boldsymbol{\sigma}$ is the stress tensor, $\nabla\cdot$ denotes the divergence operator, $\boldsymbol{F}(\boldsymbol{x},t;\xi)$ is the body force, $\rho(\xi)$ and $c(\xi)$ are the mass density and viscous damping coefficient, respectively, and the symbol ξ denotes a point in the sample space, characterizing the source of randomness involved in the system. If there is only one single point in the sample space, then it is a deterministic problem, and ξ can be omitted.

The elastoplastic damage mechanics based constitutive law of concrete reads (Li et al. 2014)

$$\boldsymbol{\sigma}(\boldsymbol{x},t) = [\mathbb{I} - \mathbb{D}(\boldsymbol{x},t;\xi)] : \mathbb{E}_0(\xi) : [\boldsymbol{\varepsilon}(\boldsymbol{x},t) - \boldsymbol{\varepsilon}_{\mathrm{p}}(\boldsymbol{x},t)], \tag{2}$$

where $\mathbb{E}_0(\xi)$ is the fourth-order initial elastic modulus tensor; $\mathbb{I}$ is the fourth-order unit tensor; $\mathbb{D}(\boldsymbol{x},t;\xi)$ and $\boldsymbol{\varepsilon}_{\mathrm{p}}(\boldsymbol{x},t)$ are, respectively, the fourth-order damage and the second-order plastic strain tensors.

In addition, the initial and boundary conditions should be prescribed. For instance, the boundary conditions usually include two parts

$$\left.\begin{aligned} \boldsymbol{\sigma}\cdot\boldsymbol{n} &= \boldsymbol{T}(\boldsymbol{x},\xi,t), \quad \text{for } \boldsymbol{x}\in\partial\mathcal{B}_{\boldsymbol{T}}, \\ \boldsymbol{u} &= \bar{\boldsymbol{u}}(\boldsymbol{x},\xi,t), \quad \text{for } \boldsymbol{x}\in\partial\mathcal{B}_{\boldsymbol{u}}, \end{aligned}\right\} \tag{3}$$

i.e., the force and displacement boundary conditions, $\partial\mathcal{B}_{\boldsymbol{T}} \cap \partial\mathcal{B}_{\boldsymbol{u}} = \varnothing$, $\partial\mathcal{B}_{\boldsymbol{T}} \cup \partial\mathcal{B}_{\boldsymbol{u}} = \partial\mathcal{B}$, and $\partial\mathcal{B}$ is the boundary of $\mathcal{B}$, $\boldsymbol{T}$ is the traction on the boundary and $\bar{\boldsymbol{u}}$ is the prescribed displacement on the boundary.

Eqs. (1) and (2), together with the geometric equations, constitute the governing physical equations for generic solid mechanics problems. If all the quantities are deterministic, i.e., there is only one single point ξ in the sample space, this set of equations under the prescribed initial and boundary conditions can be solved by the numerical methods, e.g., the finite element method (Belytschko et al. 2014) or various mesh free methods. This also provides the physical-mechanical basis for uncertainty propagation and reliability evaluation. Actually, if there is randomness involved either in the external forces and boundary conditions, in the structural parameters in the constitutive law, or both, which is usually the case in engineering practice, then the response quantities are also random in nature. In such cases, the following challenging tasks have to be tackled:

(1) Uncertainty quantification of material properties and excitations;
(2) Uncertainty propagation and stochastic response analysis of nonlinear structures;
(3) Reliability evaluation of structures;
(4) Reliability-based control and optimization of complex structures.

Some advances in these aspects will be outlined and reviewed in the following sections.

3. UNCERTAINTY QUANTIFICATION AND REPRESENTATION

To characterize the sources of uncertainty in complex engineering systems is the probabilistic information basis of uncertainty propagation and reliability evaluation of structures. This encompasses two key aspects: system parameters and external excitations.

3.1. Uncertainties in parameters of concrete

Concrete behavior is featured by strong nonlinearity and unignorable randomness (Li et al. 2014). The uncertainty in the constitutive law of concrete involves the randomness in the behavior of concrete at any given position and the spatial variability, which usually, in turn, is exhibited as the randomness of mechanical parameters and their spatial variability.

The randomness of mechanical parameters of concrete, e.g., the strength, elastic modulus, and ultimate strain, etc., was recognized long time ago (Ang & Tang 2006, Vanmarcke 2010). For instance, the coefficient of variation of the strength of concrete was provided in most design codes. However, to capture the constitutive law of concrete, several parameters are needed, e.g., 4 parameters are needed in the damage model for the uniaxial compressive curves (Li et al. 2014). The empirical relations between different parameters were extensively investigated, say, based on data fitting. However, two issues need to be addressed: (1) Generally, the data of different parameters are not from the same specimen, e.g., the elastic modulus data are usually from prism specimens, but the strength data are usually from cubic or cylindric specimens; and (2) Full probabilistic dependence between the parameters still lacks. To this end, the mesoscale random fracture-based damage mechanics (MRFD) constitutive model of concrete provides a physical basis for the uncertainty quantification of concrete behavior, and the complete stress v.s. strain curves provide data basis (Li et al. 2014). Based on the MRFD model, the parameters of the mesoscale random field can be identified (Li et al. 2021). In addition, it was found that the macroscopic parameters, including the strength, the elastic modulus, the strain at peak stress, and the descent phase parameter, are nonlinearly correlated and could be captured by the vine-copula functions in the full probabilistic sense (Tao et al. 2020).

The spatial variability of concrete constitutive law will greatly affect the response of concrete structures and thus shall be quantified. A representative method is the hierarchical model recommended by the JCSS (Engen et al. 2018), which has been recently extended to involve the random field of concrete in components, where the multi-parameter random field of concrete is captured by an approach by synthesizing the spatial correlation function and the multi-variate copula function (Tao & Chen 2023). A more physically based approach is to model the random field of concrete constitutive law by the two-scale random field of fracture strain of mesoscale spring (Li et al. 2021). Remarkably, it was discovered that the randomness of the constitutive law of concrete would lead to the transition of failure modes, and thus ignoring the randomness may yield misleading results (He et al. 2021, Tao & Chen 2023).

3.2. External excitations

3.2.1. Statistical models

One direct approach to modeling stochastic processes is to capture the power spectral density (PSD). The widely employed models include, e.g., the Kanai-Tajimi model for ground motion (Tajimi 1960) and the Kaimal model for fluctuating wind (Kaimal et al. 1972), etc. Though the PSD models can be extended to non-stationary processes and random fields (Vanmarcke 2010), it should be stressed that only the second-order statistics rather than the full probabilistic information was characterized (Li & Chen 2009).

Generally, to represent the stochastic process or fields in the time domain based on the above models, which is necessary for the nonlinear analysis of structures, the spectral representative method, the Karhunen-Loève expansion, and the proper orthogonal decomposition are widely employed (Li & Chen 2009). To improve the accuracy with a reduced number of basic random variables, the stochastic harmonic function method, and the orthogonal function model were developed (Chen et al. 2013). For the random fields, the frequency-wavenumber joint spectral representation method was newly proposed, e.g., for fluctuating wind speed (Shields & Deodatis 2013, Song et al. 2018). By doing so, the randomness involved in a stochastic process or random field can be characterized by a set of basic random variables. An alternative approach is to construct a linear filter system whose output PSD is consistent with or approximate to the target PSD (Spanos & Zeldin 1996, Luo et al. 2022).

3.2.2. Physical models

The above models could not capture the full probabilistic information, and generally, huge amounts of data are needed. Physically based models can remedy such problems (Li & Chen 2009). To this end, the physical model of stochastic ground motion that reflects the source-to-site propagation mechanism (Wang & Li 2011), the wind field physical model involving the turbulent mechanism and the phase evolution time of gust wind, and the physically based typhoon risk analysis (Hong et al. 2019) were developed. Generally, in such models, the randomness is also finally characterized by some basic random variables.

4. UNCERTAINTY PROPAGATION

4.1. Principle of preservation of probability

For a deterministic continuum system, the laws of mass, moment, and energy conservation are of fundamental significance. When it comes to systems involving uncertainties, besides the above conservation laws, an additional law, i.e., the principle of preservation of probability, holds (Li & Chen 2008). Actually, this principle can be stated as follows: In a preserved stochastic system, the probability is conservative. It can be interpreted in two different ways: the state space description and the random event description (Li & Chen 2008). By advocating the state space description, the classical Liouville equation, Fokker-Planck-Kolmogorov (FPK) equation, and the Dostupov-Pugachev equation were re-derived in a unified manner (Li & Chen 2009), whereas when the random event description is adopted, the new generalized density evolution equation (GDEE) and dimension-reduced probability density evolution equation (DR-PDEE) can be established.

4.2. Generalized density evolution equation

In the stochastic dynamical system governed by Eq. (1), if all the randomness characterized by ξ can be captured finally by the random vector $\boldsymbol{\Theta}(\xi)$, say by the modeling and representation methods in Section 3. If the quantities of interest are $\boldsymbol{Z}=(Z_1,Z_2,\cdots,Z_m)^{\mathrm{T}}$ (e.g., the displacement or stress, strain at any position), then the augmented system $(\boldsymbol{Z},\boldsymbol{\Theta})$ is probability-preserved. Therefore, according to the random event description of the principle of preservation of the probability, for any random event $\{(\boldsymbol{Z},\boldsymbol{\Theta})\in\Omega_{\boldsymbol{Z}_t}\times\Omega_\theta\}$, where $\Omega_{\boldsymbol{Z}_t}$ is the subdomain corresponding to $\Omega_{\boldsymbol{Z}_0}$, an arbitrary subdomain in the state space at the initial time, and Ω_θ is any arbitrary subdomain in the sample space, there is

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}\{(\boldsymbol{Z},\boldsymbol{\Theta})\in\Omega_{\boldsymbol{Z}_t}\times\Omega_\theta\}\\&=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\boldsymbol{Z}_t}\times\Omega_\theta}p_{\boldsymbol{Z}\boldsymbol{\Theta}}(\boldsymbol{z},\boldsymbol{\theta},t)\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{\theta}=0,\end{aligned}\tag{4}$$

which leads to the GDEE (Li & Chen 2009):

$$\frac{\partial p_{\boldsymbol{Z}\boldsymbol{\Theta}}(\boldsymbol{z},\boldsymbol{\theta},t)}{\partial t}+\sum_{i=1}^{m}\dot{Z}_i(\boldsymbol{\theta},t)\frac{\partial p_{\boldsymbol{Z}\boldsymbol{\Theta}}(\boldsymbol{z},\boldsymbol{\theta},t)}{\partial z_i}=0.\tag{5}$$

The initial condition is

$$p_{\boldsymbol{Z}\boldsymbol{\Theta}}(\boldsymbol{z},\boldsymbol{\theta},t)|_{t=0}=\delta(\boldsymbol{z}-\boldsymbol{z}_{\mathrm{ini},0})p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}),\tag{6}$$

where $\boldsymbol{z}_{\mathrm{ini},0}$ is the initial value of $\boldsymbol{Z}$; $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ is the joint PDF of $\boldsymbol{\Theta}$; $\delta(\cdot)$ is Dirac's delta function. After solving Eq. (5) under the condition (6), finally, we have

$$p_{\boldsymbol{Z}}(\boldsymbol{z},t)=\int_{\Omega}p_{\boldsymbol{Z}\boldsymbol{\Theta}}(\boldsymbol{z},\boldsymbol{\theta},t)\mathrm{d}\boldsymbol{\theta}.\tag{7}$$

It should be stressed that the information of $\dot{Z}_i(\boldsymbol{\theta},t)$ that drives the evolution of PDFs as revealed in Eq. (5), comes from the solution of physical equations [Eqs. (1) through (3)], while the source of the probabilistic information is inserted in Eq. (6). Therefore, Eqs. (1) through (7) are together the governing equations for the physically driven uncertainty propagation.

The most remarkable property of Eqs. (5) through (7) is that the coupling of nonlinearity and randomness is, in a way, broken, and thus the dimension of the GDEE fully depends on what quantities are of interest instead of the dimension of the original stochastic system. In particular, in most applications, $m=1$ is adequate. This thoroughly circumvents the difficulty due to high dimensions of the classical equations, e.g., the Liouville and the FPK equation, and provides great flexibility for practical applications to large complex structures.

4.3. *Dimension-reduced probability density evolution equation*

In some cases, it is hard to represent the stochastic processes or random fields by random functions with finite basic random variables, or the number of random variables employed in such an approach is very large. An alternative method is the DR-PDEE (Lyu & Chen 2022).

Actually, without loss of generality, Eq. (1) can be expressed in the form of the state equation

$$\dot{\boldsymbol{Y}}(t)=\boldsymbol{A}(\boldsymbol{Y},t)+\boldsymbol{B}(\boldsymbol{Y},t)\boldsymbol{\eta}(\xi,t), \tag{8}$$

where $\boldsymbol{Y}(t)=\left(\boldsymbol{U}^{\mathrm{T}}(\boldsymbol{x},t),\dot{\boldsymbol{U}}^{\mathrm{T}}(\boldsymbol{x},t)\right)^{\mathrm{T}}$ is the state variable vector, and $\boldsymbol{\eta}(\xi,t)$ is the random excitation vector. Then for any, say the ℓ-th, component of interest, $Y_\ell(\xi,t)$, by advocating the random event description of the principle of preservation of probability, there is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}\{Y_\ell(\xi,t)\in\Omega_{Y,t}\}=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{Y,t}}p_{Y_\ell}(y,t)\,\mathrm{d}y=0. \tag{9}$$

By the procedure similar to the Kramers-Moyal expansion, we are led to

$$\frac{\partial p_{Y_\ell}(y_\ell,t)}{\partial t}=\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\frac{\partial^m[\alpha_m^{(\mathrm{int})}(y_\ell,t)\,p_{Y_\ell}(y_\ell,t)]}{\partial y_\ell^m}, \tag{10}$$

where $\alpha_m^{(\mathrm{int})}(y_\ell,t)$ are the first, second, third, … conditional derivate moments with respect to $Y_\ell(t)$, respectively. Further, Lyu and Chen (2022) rigorously proved that if the process $Y_\ell(\xi,t)$ is path-continuous, then the third and higher orders of conditional derivate moments turn out to be zeros, that is,

$$\frac{\partial p_{Y_\ell}(y_\ell,t)}{\partial t}=-\frac{\partial\alpha_{Y_\ell}^{\mathrm{int}}(y_\ell,t)\,p_{Y_\ell}(y_\ell,t)}{\partial y_\ell}+\frac{1}{2}\frac{\partial^2\beta_{Y_\ell}^{\mathrm{int}}(y_\ell,t)\,p_{Y_\ell}(y_\ell,t)}{\partial y_\ell^2}. \tag{11}$$

Eq. (11) is the so-called DR-PDEE, and the first and the second conditional derivate moments are the intrinsic drift and diffusion coefficients, respectively,

$$\alpha_{Y_\ell}^{\mathrm{int}}(y_\ell,t)=\lim_{\Delta t\to 0}\frac{1}{\Delta t}E[\Delta Y_\ell\,|\,Y_\ell=y_\ell], \tag{12}$$

and

$$\beta_{Y_\ell}^{\mathrm{int}}(y_\ell,t)=\lim_{\Delta t\to 0}\frac{1}{\Delta t}E[(\Delta Y_\ell)^2\,|\,Y_\ell=y_\ell]. \tag{13}$$

Alternative derivation methods of the DR-PDEE can be found in Chen and Lyu (2022) and Lyu and Chen (2022). (It is called GE-GDEE in these papers.) The DR-PDEE asserts that provided a stochastic process is path-continuous, the evolution of its PDF only contains the drift and diffusion terms, no matter whether the process and the original system represented by state variable are Markovian or non-Markovian (Lyu & Chen 2022, Luo et al. 2023), the sources of randomness are from the system or excitations (Chen & Lyu 2022).

Again, it is worth stressing that the dimension of Eq. (11) depends only on what quantities are of interest. The crucial crux is that the intrinsic drift and diffusion functions in Eqs. (12) and (13) are determined in turn by solving Eq. (8), which is the physical governing equation. Consequently, DR-PDEE is an alternative formulation for the physically driven uncertainty propagation.

4.4. *Numerical implementation*

For some simple systems, the analytical solution of GDEE exists (Li & Wang 2023), which provides important benchmarks verifying the GDEE. For generic systems, the GDEE needs to be solved numerically. To this end, the partition of the probability space and the selection of representative point are crucial to the accuracy (Li & Chen 2009). The discrepancy is the rational indicator to evaluate the efficacy of the point set, for the Koksma-Hlawka inequality governs the worst error of numerical integrals with the discrepancy. Chen and Zhang (2013) proposed the extended F- (EF-) discrepancy to take account for the non-uniformly distributed cases and the assigned probability of each representative point. However, calculating the EF-discrepancy is an NP-hard problem. To circumvent this problem, the generalized F- (GF-) discrepancy that considers the marginal distribution of each dimension involving the assigned probabilities is proposed (Chen & Zhang 2013), and the extended Koksma-Hlawka inequality in terms of the GF-discrepancy was established (Chen & Chan 2019). In addition to independent random variables, the iterative screening-rearrangement method has been developed to select representative points for dependent random variables (Yang et al. 2020).

To numerically solve the GDEE, refined numerical procedures based on the finite difference method (Li & Chen 2009), the finite element method, the mesh-free method, and the deep learning have been developed so far (Papadopoulos & Kalogeris 2016). Further, to improve the accuracy, the ensemble evolution numerical method of solving GDEE was developed recently (Tao & Li 2017).

As for the DR-PDEE, for linear systems and some nonlinear systems, the closed-form or semi-analytical expression of the intrinsic drift and diffusion functions are available (Sun & Chen 2022, Luo et al. 2023). In generic cases, numerical methods are needed. The intrinsic drift functions can be constructed by physically informed data-driven approaches, i.e., the locally weighted smooth scatters and copula algorithm (Chen & Lyu 2022). The DR-PDEE, as an FPK-like partial differential equation, can be solved via the path integral method (Lyu & Chen 2022).

It is noted that both GDEE and DR-PDEE are rooted in the essence that the evolution of probability density is physically driven, and thus both belong to the physically driven probability density evolution method (PDEM).

5. APPLICATIONS AND EXTENSIONS

5.1. Probabilistic response

The physically driven PDEM, including both GDEE and DR-PDEE, provides powerful tools for uncertainty propagation analysis of complex systems. Actually, the GDEE has been applied in various types of engineering structures, including super high-rise concrete structures, large span structures, and even aero plane structures, etc. (Li et al. 2018, Saraygord & Pourtakdoust 2018).

Alternatively, the DR-PDEE was also applied in huge nonlinear structures involving both randomness in structural parameters and stochastic excitations (Chen & Lyu 2022), and offshore wind turbine systems (Luo et al. 2022), etc. Numerical results show that the DR-PDEE could yield results in the order of $10^{-6}\sim10^{-5}$ level on the tails of PDF.

5.2. Dynamic reliability

Reliability plays a pivotal role in engineering design, risk assessment, and decision-making. Though the first-passage reliability problem has been extensively studied, it remains a great challenge for high-dimensional dynamical systems. Actually, the theoretically seemly sound diffusion process method encounters the so-called curse of dimensionality and the limitation due to Gaussian white noise excitations (Lutes & Sarkani 2004). The alternative widely used method based on level-crossing processes might be of unguaranteed accuracy due to the Poisson or Markov assumption (Vanmarcke 2010).

By advocating the random event description of the principle of preservation of probability, it is straightforward to evaluate the first-passage reliability of a structure by imposing an absorbing boundary condition corresponding to the failure criterion on the GDEE (Li & Chen 2009). This method has now been widely applied. In particular, the physical synthesis method for evaluating the global reliability of structures based on GDEE accommodating multi-scale failure criteria was developed (Li 2021).

However, for the DR-PDEE, it was proved that directly imposing an absorbing boundary condition on the DR-PDEE will not yield the correct first-passage reliability. Alternatively, introducing an absorbing boundary process (ABP) and establishing the DR-PDEE of the ABP will yield the first-passage reliability (Lyu & Chen 2022). In other words, this implies the non-exchangeability of imposing absorbing boundaries and dimension reduction. It was remarkable that solving the DR-PDEE of ABPs with the computational efforts in the order of magnitude of several hundreds of deterministic analyses could capture small failure probabilities in the order of $10^{-5} \sim 10^{-6}$. By this method, the first-passage reliability evaluation of huge complex concrete structures with the degrees of freedom being 277,404 became reality (Lyu et al. 2023). In addition, it is worth pointing out that due to the above non-exchangeability, it should be cautious in adopting the "pure" data-driven approaches for the first-passage evaluation because these approaches are usually equivalent to the DR-PDEE imposed by the absorbing boundary condition rather than the DR-PDEE of ABPs.

5.3. Extreme value distribution

To capture the extreme value distribution (EVD) is an alternative approach to evaluate the first-passage reliability. In particular, it is superior to the above absorbing boundary method when the boundary is random in nature (Li & Chen 2009). Further, when the failure criterion is some kind of compound criterion, i.e., the logic combination of more than one criterion, the principle of equivalent extreme value event can be constructed (Li et al. 2007). The EVD, including the EVD of the equivalent event, can be captured by the GDEE via constructing a virtual stochastic process (Li & Chen 2009).

Time-variant EVDs provides another option for the time-variant dynamic reliability evaluation

$$\begin{aligned} R(t) &= \Pr\{U_{\text{int}}(\tau) \leqslant b,\ \text{for}\ 0 \leqslant \tau \leqslant t\} \\ &= \Pr\{Z_{\text{ext}}(t) \leqslant b\} = F_{Z_{\text{ext}}}(b), \end{aligned} \tag{14}$$

where b is a certain boundary; $Z_{\text{ext}}(t) = \max_{0 \leqslant \tau \leqslant t}\{U_{\text{int}}(\tau)\}$ is the extreme value of the quantity of interest $U_{\text{int}}(t)$, and $F_{Z_{\text{ext}}}(\cdot)$ is its cumulative distribution function (CDF). However, obtaining its CDF is challenging either analytically or numerically. Recently an augmented Markov vector approach was developed, offering precise numerical solutions for time-variant EVDs (Chen & Lyu 2020). Numerical examples demonstrate its effectiveness, even for systems with special distributions like jumps or concentrated probabilities, e.g., Poisson-excited systems. Furthermore, a class of Volterra integral equations has been developed to analytically or numerically determine the time-variant EVDs of path-continuous Markov processes (Lyu et al. 2021). However, the extension to high-dimensional problems is still ongoing.

5.4. Optimal control

In order to mitigate the vibration and increase the resistance of structures, structural control is an effective pathway. Note that dynamic excitations, such as earthquakes and winds, are stochastic and are of complex properties in terms of amplitude and frequency. The deterministic and even the classic stochastic optimal control theory may underperform in such scenarios. However, the generality of the PDEM provides a new possibility for optimal control (Peng & Li 2019).

Unlike classic stochastic control policies that aim to improve structural performance in the sense of the second-order moment, the PDEM-based optimal control allows the mitigation of vibration in the sense of probability density. Furthermore, the generalized optimal control law has been devised based on the PDEM and the probabilistic controllability indexes, which can serve as a unified formula of optimal control law in the passive, active, semi-active, and hybrid control (Peng & Li 2019). Determining the locations and parameters of control devices is another crucial topic in vibration control of structural under stochastic actions. In this regard, the reliability-based optimization of the parameters of control devices has also been studied based on the PDEM. For example, Peng et al. (2021) conducted reliability-based optimization of parameters of sliding implant-magnetic bearings to reduce the seismic response of structures. Numerical examples indicated that the reliability-based optimization of control device parameters can help to improve the seismic reliability of the structure.

5.5. Reliability-based design optimization

In the current design methodology, the reliability of the structure is implicit, which means that the reliability can only be determined by a reliability analysis after the design phase. In this context, Reliability-based design optimization (RBDO) provides feasibility to explicitly consider the reliability in structural design (Schuëller & Jensen 2008, Li 2021).

The RBDO problem can usually be formulated as a nonlinear programming problem, where the dynamic reliability is involved in the objective or constraint functions. A general formulation of the RBDO problem is given by

$$\begin{aligned} &\min \ f[\boldsymbol{\kappa}, P_{\mathrm{F}}(\boldsymbol{\kappa})] \\ &\text{s.t. } g_i[\boldsymbol{\kappa}, P_{\mathrm{F}}(\boldsymbol{\kappa})] \le 0, \text{ for } i = 1, \ldots, n_{\mathrm{c}}, \end{aligned} \tag{15}$$

where $P_{\mathrm{F}}(\boldsymbol{\kappa}) = 1 - R(T; \boldsymbol{\kappa})$ is the probability of failure, i.e., first-passage probability, in which T is the end point of the time interval of interest; $f[\boldsymbol{\kappa}, P_{\mathrm{F}}(\boldsymbol{\kappa})]$ is the objective function; $g_i[\boldsymbol{\kappa}, P_{\mathrm{F}}(\boldsymbol{\kappa})]$ denotes the i-th constraint function; n_{c} is the number of constraint functions; $\boldsymbol{\kappa} = (\kappa_1, \cdots, \kappa_{n_{\mathrm{d}}})^{\mathrm{T}}$ is the n_{d}-dimensional design vector. Evidently, the probability of failure is a function of the design variables.

In principle, Eq. (15) can be solved by any numerical optimization algorithm. However, the first-order algorithms, i.e., the gradient-based algorithms, are preferred due to the fact that they require fewer function calls. As a result, the sensitivity or gradient of the first-passage probability is a requisite, and the sensitivity analysis is transformed into repeated dynamic reliability analyses. For design variables which are distribution parameters of the random variables, the sensitivity can be evaluated by the PDEM with the change of probability measure (COM) synthesized method without introducing any extra structural analysis (Chen et al. 2020), i.e., the sensitivity of the probability of failure is a byproduct of the reliability analysis. For design variables which are deterministic but controllable variables, the COM cannot be used. To this end, methods based on surrogate models in augmented space and important representative points have been developed (Yang et al. 2022). The efficiency and efficacy of these methods have been verified by the dynamic reliability-based design optimization of linear and nonlinear frame structures under seismic excitations.

The above discussions are focused on general dynamic-RBDO (DRBDO) problems. Further, dynamic-reliability-based topology optimization (DRBTO) can also be conducted within the framework of the PDEM. In this case, the adjoint variable method for transient sensitivity analysis can be employed together with the PDEM, which can make the computational costs insensitive to the number of design variables.

6. FURTHER RESEARCH EFFORTS

Currently, the physically driven PDEM, i.e., GDEE and DR-PDEE, have shown promising applications and advancements in the field of full probabilistic uncertainty propagation. However, there are still some aspects to be studied.

6.1. Physics-informed uncertainty quantification of processes and fields

Full probabilistic quantification of uncertainty in structural parameters shall be further improved, particularly the physically based identification together with big data techniques. In addition, refined physically based models still need to be developed for temporal-spatial random fields for earthquakes, fluctuating wind and ocean waves, and concurrent multi-processes. Third, the simultaneous quantification and propagation of aleatory and epistemic uncertainties from random variables (Chen & Wan 2019) to temporal-spatial random fields are also of paramount importance.

6.2. Multiscale stochastic physics and mechanics

As stressed, the refined structural behavior analysis provides the physical basis for the uncertainty propagation and reliability evaluation of real-world structures. The phase field models (Wu 2017), peridynamic models (Silling & Lehoucq 2010), and the recently proposed nonlocal macro-meso-scale damage model (Lu & Chen 2020) threw new light. However, refined approaches capturing the embedded stochastic scale physics with high accuracy and efficiency are still in urgent need.

6.3. Global-reliability-based design optimization

As discussed above, RBDO provides a rational framework to explicitly involve uncertainty in the design phase. However, the research on RBDO and especially DRBDO is mainly focused on structural components and specific failure modes. It is necessary to extend the current RBDO incompatible with the physical synthesis method for global reliability assessment, in particular embracing the topology optimization.

7. CONCLUDING REMARKS

This paper elucidates the basic thought of physically driven uncertainty propagation in complex systems, and reviews and revisits the advances. It is stressed that the refined structural behavior analysis and the uncertainty quantification and propagation are the indispensable two sides of a coin, which are the two wheels of structural design theory. In this framework, the advances in the full probabilistic quantification of structural parameters and dynamic actions, the governing equations for uncertainty propagation, i.e., the generalized density evolution equation and the dimension-reduced probability density evolution equation, the reliability evaluation, and the reliability-based structural optimization are reviewed. The studies laid a solid foundation for the third-generation structural design theory, featured by fusing the solid-mechanics-based refined structural behavior analysis, the full probabilistic uncertainty quantification and propagation, and global-reliability-based optimization design. The problems along this line to be further studied are also discussed.

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