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The effect of pion mass on Skyrme configurations

Conor Houghton and Shane Magee,
School of Mathematics, Trinity College Dublin, Dublin 2, Ireland
Email: houghton@maths.tcd.ie, magees@maths.tcd.ie

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Abstract

In the Skyrme model, atomic nuclei are identified with solitonic configurations. If the pion mass is set to zero, these configurations are spherical shells of energy with a fullerene-like appearance and are well approximated by a simple rational map ansatz. Using simulated annealing, we have calculated minimum energy configurations for non-zero pion mass and have found that they are less round and are less well approximated by the rational map ansatz.

The Skyrme model [1] is a mathematically elegant effective theory which gives an approximate description of nucleons interacting at low energies. It is a pion field theory in which baryons appear as soliton configurations: the baryon number B is given by the topological winding number of the soliton. The model has had some success in describing aspects of nuclear phenomenology such as the nucleon-nucleon potential [2] and the stability of ^{4}He and ^{7}Li [3]. The model is non-renormalizable and, therefore, the approach usually taken is to treat it as a quantum mechanical model of nuclei and to quantize on a finite-dimensional space of classical configurations. For example, in [4], the rotational space of minimum energy $B = 1$ Skyrme configurations is quantized to give a reasonable description of the nucleon and the delta resonance.

The Skyrme field is an SU(2) field $U(\mathbf{x})$ which is identified with the normal pion fields $\boldsymbol{\pi}$ by

$$U = \exp i\boldsymbol{\pi} \cdot \boldsymbol{\sigma}, \tag{1}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the usual Pauli matrices. The energy functional is most conveniently written in terms of the skew-Hermitian current $R_i = \partial_i U U^\dagger$ as

$$E = \int \mathrm{d}^3\mathbf{x} \, \left[-\frac{1}{2} \, \mathrm{Tr} \, (R_i R_i) - \frac{1}{16} \, \mathrm{Tr} \, ([R_i, R_j][R_i, R_j]) - m^2 \, \mathrm{Tr} \, (U - 1) \right]. \tag{2}$$

Here, the coupling constants appearing in front of the first two terms have been scaled to one by a choice of the length and energy scales. The Skyrme model is a modified sigma model and the first term in the energy is the usual sigma-model energy. The second term was added by Skyrme [1] and is needed to prevent Derrick scaling [5] of the minimum energy solutions. The chiral symmetry-breaking third term was introduced in [6]; writing the Lagrangian in terms of the pion fields, $\boldsymbol{\pi}$, shows that m corresponds to the pion mass.

Most classical investigations of the Skyrme model have ignored the effect of the pion mass. However, in the model the classical counterpart of nucleons are rotating configurations and stable rotating Skyrme configuration must have a non-zero pion mass [7]. In fact, the faster the rotation, the larger the pion mass required. In order to produce stable Skyrme configurations that model the nucleon and delta resonance, the pion mass must be set to over twice its experimental value [8].

Skyrme configurations with zero pion mass have a remarkable fullerene-like appearance: their energy density is concentrated along the edges and vertices of a polyhedral skeleton. Furthermore, they are well approximated by the rational map ansatz [9] which restricts Skyrme fields to the form:

$$U = \exp(-if(r)\mathbf{n}(\theta,\phi)\cdot\boldsymbol{\sigma}), \tag{3}$$

where $f(r)$ is a radial profile function and $\mathbf{n}(\theta,\phi)$ is a unit vector with a specific holomorphic form, which is described in [9]. The ansatz fields have a very distinctive structure; the iso-surfaces of trace U, for example, are spheres around the origin.

If this field ansatz is substituted into the Skyrme energy functional, the minimization problem splits into two parts: one for the angular behavior, allowing the angular map $\mathbf{n}(\theta,\phi)$ to be calculated, and a second for the radial behavior, allowing $f(r)$ to be calculated. Only the second of these is affected by the pion mass. If the ansatz is used to approximate minimum energy Skyrme configurations for non-zero pion mass the angular behavior is identical to the angular behavior for zero pion mass. The only change is in the profile function: for non-zero pion mass it has a more rapid fall-off resulting in a smaller, tighter, configuration. However, in [10] it is argued that the rational map ansatz cannot produce good approximations to the minimum energy solutions for large pion mass and large baryon numbers. In fact, it is shown here that the rational map ansatz gives a less accurate approximation for non-zero pion mass and the iso-surfaces of trace U are not spherical.

Here, the minimum energy Skyrme configurations are calculated numerically for a range of values of the pion mass m. This is performed using simulated annealing: an established method of generating minimum energy Skyrme configurations [11, 12]. However, simulated annealing can be computationally expensive and this limits the possible baryon numbers that can be considered. High pion mass is an advantage here because Skyrme configurations with high pion mass are much smaller than their counterparts with zero pion mass. For example, using a value of $m = 3.8$, configurations up to $B = 9$ can be annealed on a 100^3 lattice with lattice spacing of 0.06 Skyrme units without significant edge effects. The pion mass value of $m = 3.8$ is approximately eight times the experimental value and three times that used in [8] to ensure stable rotating solutions. It is high enough to give a clear qualitative demonstration of pion mass effects; the best value of the pion mass to use in a putative Skyrme model of nuclear physics is an open question.

One disadvantage of using a higher pion mass is that tighter Skyrme configurations have very high field derivatives. These are less well approximated by the lattice Taylor expansion used in the lattice simulation. For example, a good test of numerical accuracy is to calculate the baryon number which should be an integer. If we adopted the 0.12 lattice spacing used in [11] and [12], the error in baryon number is approximately 8% for the $m = 3.8$ solution compared with 2% for the $m = 0$ case. A reduction in lattice spacing to 0.06 reduces the error to less than 2%, and this is the lattice spacing we have used to produce the results presented here.

Using simulated annealing, we have calculated minimum energy configurations for the Skyrme model with pion mass $m = 3.8$. Baryon density iso-surfaces are shown in Fig. 1 for $B = 7$ and 9, and Fig. 2 for $B = 8$. By repeating the minimization for a range of pion masses, we have verified that the effect seen here varies in quantity, but not in its general form, as the pion mass changes.

The $B = 7$ configuration has the same shape for zero and non-zero pion mass. With this exception, the distinguishing feature of the solutions found so far is that the configuration is flatter. This can be understood physically: the region inside the fullerene-like shell has a Skyrme field close to the anti-vacuum value, $U \approx -\mathbf{1}$. Such field values have a high pion-mass density and so the interior of the shell contributes to the energy if the pion mass is non-zero. A flatter configuration has a small interior volume for a particular surface area. Figure 3 shows contours of $\mathrm{trace}\, U$ for planar cross-sections through the $B = 9$ configuration. Unlike the zero-pion mass configuration, the contours are not circular. The rational map ansatz relies on a spherical polar decomposition of $\mathbf{R}^3$. It is possible that another decomposition of space is appropriate for Skyrmions with non-zero pion mass. For example, it is already known that Skyrmions with non-zero pion mass are related to Skyrmions with zero-pion mass in hyperbolic space [13].

For baryon numbers seven and nine, the point symmetry group of the minimum energy solution remains unchanged; the situation is less clear-cut for $B = 8$. We originally obtained a minimum energy result for $B = 8$ that had the same point group as its $m = 0$ counterpart. This, however, seems to belong to a family of minima with similar energy. In [14], using a different numerical scheme, a different minimum energy solution was obtained. Furthermore, the addition of the pion mass has the effect of considerably raising the energy barrier between minima and reducing the portion of Skyrme configuration space that can be searched by the algorithm in a reasonable time.

This is a common hurdle in simulated annealing applications and is generally dealt with using what is called the multi-start strategy, performing many runs, each using a different initial conditions. Although, in principle, simulated annealing will find a global minimum with a sufficiently slow cooling schedule, in practice, even a very slow schedule may not resolve local minima with proximate energy values. In our implementation, the initial condition is important when minimizing $B = 8$ configurations. We have therefore tried initializing the algorithm with many different starting configurations. Three minima have been found which our scheme cannot resolve within numerical accuracy: these are shown in Fig. 2. In addition to a solution with the same point group as the $m = 0$ equivalent, there are also other solutions which resemble the alpha-clusters discussed in [15].

One outstanding problem with attempts to model nuclei as Skyrmions is that for many values of B the semi-classical quantization of the classical minimum does not give the correct quantum spin and iso-spin numbers [16, 17]. This is a result of the large amount of symmetry possessed by the classical minimum. It might have been hoped that the minimum energy solutions with non-zero pion mass would be less symmetric. This does, indeed, seem to be the case; however, for $B = 7$, where the discrepancy between computed and experimental quantum numbers is most glaring, there is no difference in shape between the zero and non-zero pion mass solutions. It has been suggested that a quasi-classical quantization which also takes greater account of the effect of rotation might yield the correct quantum numbers. A prescription for such a quantization is given in [18] and numerical work is continuing in this direction.

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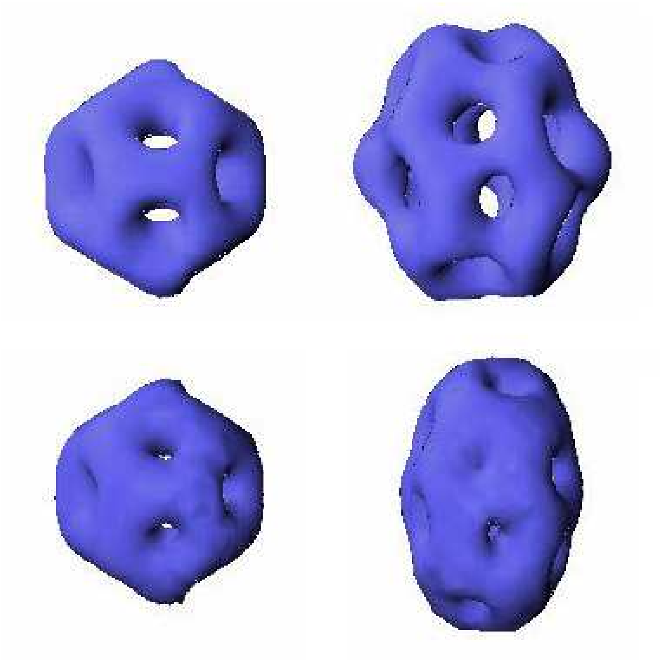

Figure 1: Iso-surface plots at baryon density $\mathcal{B}$ = .017 and pion mass m=3.8 for baryon numbers seven and nine. The top row are the annealed solutions obtained from rational map initial conditions while the bottom row are their corresponding rational map ansatz approximations.

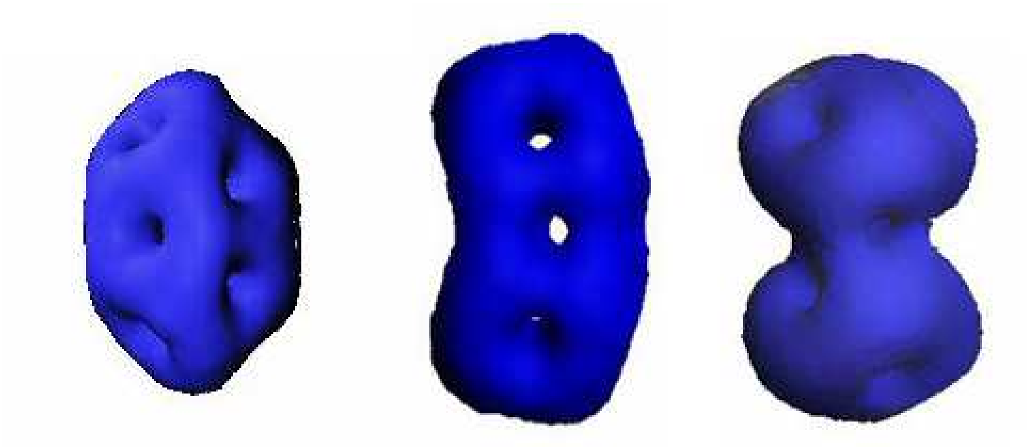

Figure 2: Three local $B = 8$ minima for pion mass $m = 3.8$, obtained from three different initial configurations, from left, a $B = 8$ rational map, a $B = 4$ rational map wound twice around the target sphere, and eight separated $B = 1$ Skyrmions. These configurations have energies, from left, of 1.9, 1.91 and 1.917 in $12\pi^2 B$ Skyrme units

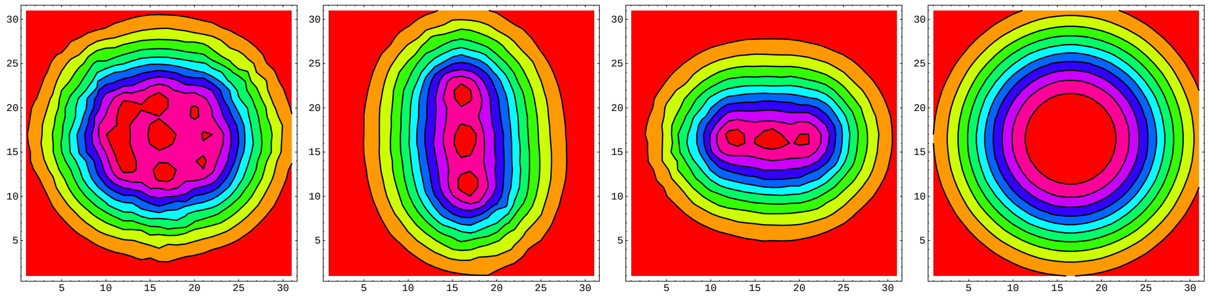

Figure 3: Contour plot of trace U of an annealed $B = 9$ configuration with $m = 3.8$ and a lattice spacing of .06 Skyrme units. The first three plots are cross-sections through the origin in the x, y and z directions, while the plot on the very right is the spherical cross-section for the rational map configuration.

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