# Critical Supercurrents and Self-Organization in Quantum Hall Bilayers

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We present a theory of interlayer tunneling in a disordered quantum Hall bilayer at total filling factor one, allowing for the effect of static vortices. In agreement with recent experiments [Phys. Rev. B **80**, 165120 (2009); Phys. Rev. B **78**, 075302 (2008)], we find that the critical current is proportional to the sample area and is comparable in magnitude to observed values. This reflects the formation of a Bean critical state as a result of current injection at the boundary. We predict a crossover to a critical current proportional to the square-root of the area in smaller samples. We also predict a peak in the critical current as the electron density varies at fixed layer separation.

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The quantum Hall bilayer at total Landau level filling $\nu_T = 1$ has been a strong contender [1–3] in the search for Bose-Einstein condensation in excitons because electron-hole binding can be stabilized by the quenching of kinetic energy in a Landau level. Strong interlayer coherence has indeed been observed [4]. A direct demonstration of superfluidity would be a form of the Josephson effect [5]—a finite interlayer current $I$ at negligible interlayer voltage $V$. This was observed recently in four-terminal measurements [6,7]. (In two-terminal geometries [8], this is seen as a zero-bias peak in $dI/dV$.) This low-voltage regime terminates and dissipation increases dramatically above a critical current $I_c$.

However, a key question has emerged over these results as evidence for excitonic superfluidity. Experiments [7,9] show that the critical current $I_c$ is proportional to the sample area. Within the model of a homogeneous excitonic superfluid, this scaling can only be explained if the tunnel splitting is several orders of magnitude smaller than expected [10]. For realistic splittings, tunneling is estimated to occur within a few microns of the contact [Eq. (5)] so that $I_c$ should *not* depend on the sample length in the direction of the current [10–12]. A similar puzzle arises when counterflowing currents are injected into a bilayer short circuited at one edge. These currents are expected to be excitonic supercurrents that decay within a few microns but they traverse the samples in experiments [13,14].

In this Letter, we present a theory which produces a critical current [Eq. (8)] that is proportional to the area of the sample and of the correct order of magnitude, given reasonable estimates for the parameters. A disorder-induced lengthscale, $L_d$, emerges in our theory [Eq. (6)]; this scale has no counterpart in the clean system [10]. Our results are also consistent with the observed dependence of $I_c$ on the magnetic length $l$ and on an in-plane magnetic field. A key test of our theory is the prediction that $I_c$ should scale with the square-root of the sample size for samples smaller than $L_d$.

The essential feature of our work is that we allow for static vortices in the exciton superfluid, which will be nucleated by strong charge disorder [15–18]. Such vortices play a crucial role for the critical current. They pin any injected supercurrents and sustain dissipationless states, in much the same way that disorder pins magnetic flux in superconductors [19,20], or charge in charge-density waves [21]. However, there is a significant difference in the bilayer. The depinning force comes from the injected charge current which cannot penetrate the bulk of the quantum Hall state. So, the depinning force is applied *only* at the sample boundary. Given this geometry, it is a surprising feature of our results that the critical current $I_c$ can scale with the sample area.

We will first present numerical results showing that, in one dimension, currents injected at the boundary decay linearly in space in the disordered state (Fig. 1), analogous to the Bean critical state in a superconductor. We argue that these special critical states are generated by current injection from the boundary, and hence are selected in the

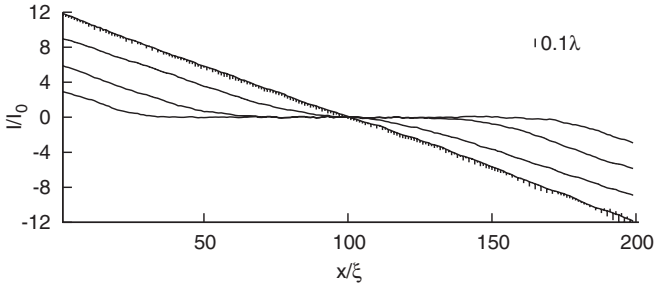


FIG. 1. Profile of counterflow current $I$ (lines) and interlayer voltage ($V \propto \dot{\phi}$, vertical bars, nonzero in top curve only) from (4) on a one-dimensional lattice with current injection at both ends, at a time $\approx 10^4/\lambda$ after the current is switched on. The injected counterflow currents at each boundary are 3, 6, 9, $12I_0$ for the four curves ($I_0 = e\rho_s/\hbar\xi$). The interlayer voltages vanish below a critical current $I_c$. Here, $9 < I_c/I_0 < 12$. $t\xi^2/\rho_s = 0.4$, results averaged over 50 realizations.

bilayer. Based on our numerics, we then present a theory for the critical current, which we generalize to two dimensions and compare with experiments.

Our starting point is the energy functional

$$H_{\rm eff} = \int\left[\frac{\rho_s}{2}(\nabla\phi)^2 - t\cos(\phi+\theta^0)\right]d^D\mathbf{r}, \qquad (1)$$

for the low-energy modes of a bilayer containing pinned vortices. This form follows from the clean model [2] when the phase field of the pinned vortices $\theta^0$ is subtracted out of the superfluid phase $\theta$: $\phi = \theta - \theta^0$. The first term in Eq. (1) is the superfluid stiffness while the second describes the interlayer tunneling. The counterflow supercurrent density above the ground state, $j_{\rm CF}$, and the interlayer tunneling current density, $j_t$, are related to the phase by:

$$j_{\rm CF} = \frac{e\rho_s}{\hbar}\nabla\phi, \qquad j_t = \frac{et}{\hbar}\sin(\phi+\theta^0). \qquad (2)$$

Owing to the strong charge disorder, we expect the incompressible quantum Hall phase to occupy a small fraction of the sample, with the remainder occupied by puddles of compressible electron liquid [15,16]. We suppose that the incompressible phase forms a percolating network of channels of size $l$ separating puddles of size $d_d$, the distance to the dopants. Equation (1) is the effective theory for the channels, whose parameters differ from the bulk values by the area fraction of superfluid

$$t \sim (l/d_d)t^0, \qquad \rho_s \sim (l/d_d)\rho_s^0. \qquad (3)$$

We assume that the vortex field penetrating the channels $\theta_0$ is disordered, with a correlation length of the order the puddle size, $\xi \approx d_d \approx 200$ nm [15].

A time-varying superfluid phase $\phi(t)$ gives rise to an interlayer voltage difference $V$ *via* the Josephson relation $V = \hbar\dot{\phi}/e$. Therefore, a state with a finite interlayer current at zero interlayer voltage is time independent, corresponding to a local minimum of the energy (1). To investigate this possibility, we consider the dissipative model

$$-\lambda\dot{\phi} = \frac{\delta H_{\rm eff}}{\delta\phi} = -\rho_s\nabla^2\phi + t\sin(\phi+\theta^0), \qquad (4)$$

whose stationary solutions $\dot{\phi} = V = 0$ are the local minima of Eq. (1). The stationary equation is the continuity equation stating that the loss of counterflow current (first term) is accounted for by interlayer tunneling (second term). The dissipative dynamics of Eq. (4) is physically a resistive shunt due to interlayer quasiparticle tunneling. Additional dynamical terms are necessary in the finite voltage regime [11,17,18,22]. However, it is the dissipation that determines the long-time limit at low voltages, so that here we may use Eq. (4).

The boundary conditions for Eq. (1) come from the current flows through the sample [10]. For definiteness, we consider a tunneling geometry in which a current $I_t$ is injected into the top layer at one corner and removed from the bottom layer at the opposite corner. These current flows may be written as superpositions of layer-symmetric and layer-antisymmetric currents,

$$I_{\rm in(out)} = \frac{I_t}{2}[(1,1) \pm (1,-1)]$$

where the components refer to the two layers. Thus, the tunneling experiment corresponds to a flow of layer-symmetric current, with equal counterflow currents $I = I_t/2$ injected by both the electron source and drain. In the low-voltage regime, these counterflow currents will be the supercurrents $j_{\rm CF}$ in Eq. (2). Since the symmetric component cannot penetrate deep into the bulk, we can focus on the injection of supercurrents. As we shall see, the injection profile along the boundary is unimportant.

The mechanisms leading to an extensive critical current for the bilayer can be seen most clearly from a one-dimensional model. In Fig. 1, we show current and voltage profiles obtained for the one-dimensional version of Eq. (4) on a lattice of $L = 200$ sites. These results are averaged over realizations of the disorder $\theta_i^0$, which is taken independently on different lattice sites. Thus we take the correlation length $\xi$ as the lattice spacing. The natural unit of current is then $I_0 = e\rho_s/\hbar\xi$. For this illustration, we take the tunneling strength $t\xi^2/\rho_s = 0.4$. In each realization, we start from an initial state in which the $\phi_i$ are random and independent, and equilibrate by integrating forwards in time with the boundary conditions $\partial_x\phi|_1 = \partial_x\phi|_L = 0$. To model the current injection in a tunneling experiment, we then slowly increase the boundary conditions to the final values $\xi\partial_x\phi|_1 = -\xi\partial_x\phi|_L = I/I_0$. For the lowest three values of $I$ used, the dynamics reach a time-independent solution, corresponding to the Josephson regime with vanishing interlayer voltages. For too large $I$, these time-independent solutions break down and the phase winds continuously at late times. This corresponds to the breakdown of the dc Josephson regime and the appearance of a state with finite interlayer voltages. For the counterflow geometry [13,14], this picture suggests that a counterflow current can traverse the sample only for large currents.

The static states in Fig. 1 differ qualitatively from those of the clean model [11,23], $\theta^0 = 0$. In that case, there is a penetration depth for the injected current of

$$\lambda_J \sim \sqrt{\rho_s/t}. \qquad (5)$$

Since the phase angle is periodic this implies a maximum injected current density of $\partial_x\phi \sim \pi/\lambda_J$. In the disordered case shown, however, the injected current decays linearly close to the contacts, with a slope which is independent of the injected current. Thus, an increase in the injected current is accommodated by an increased current penetration into the sample. As can be seen in Fig. 1, this process continues until the currents fill the entire sample. Beyond this point, further increases in current cannot be accommodated by coherent tunneling and an interlayer voltage develops. Since the current decays linearly with a constant slope, the resulting $I_c$ in the one-dimensional model scales with the sample length.

In the clean model, the breakdown of the stationary solutions can be understood in terms of the injection of

phase solitons at the boundary [11,24], which propagate through the sample. Thus, the phase at any point varies in time, and the system develops an interlayer voltage by the a.c. Josephson effect. However, the solitons may be pinned in a disordered system. We now develop a theory of such pinning, which agrees with our numerical work.

We begin by recalling [21,25] the form of the ground states of the random field $XY$ model, Eq. (1), in the regime $\xi \ll \lambda_J$ relevant for the bilayer. In this regime, the ground state consists of weakly pinned ferromagnetic domains with polarized phase. The key idea is that it is energetically costly to have phase twists at scales shorter than the size, $L_d$, of these domains. The energy cost for a phase twist that varies over the scale $L_d$ is $\rho_s L_d^{D-2}$ in $D$ dimensions. The typical tunneling energy of a polarized domain is obtained by summing random energies in the range $\pm t\xi^D$ for its $(L_d/\xi)^D$ correlation areas, giving $t\xi^D(L_d/\xi)^{D/2}$. Thus, for $d < 4$, the phase stiffness wins at short scales, and the domains reach a finite size where the two energies balance. This gives the Imry-Ma scale:

$$L_d \sim \left(\frac{\rho_s}{t\xi^{D/2}}\right)^{(2/(4-D))} = \left(\frac{\lambda_J^2}{\xi^{D/2}}\right)^{(2/(4-D))}. \qquad (6)$$

In this ground state of polarized domains, the average coarse-grained phase over a domain is chosen such that the tunneling energy $H_t$ of each domain is minimized. Since $\delta H_t/\delta\phi(\mathbf{r})$ is the tunneling current at position $\mathbf{r}$, the total tunneling current over the domain vanishes.

We now consider the presence of an injected current. The injected counterflow will cause the phase to twist away from its equilibrium value, leading to finite tunneling currents. We assume that the configuration remains smooth on the scale $L_d$, and hence average Eq. (4) over each domain. The tunneling term becomes $(t\xi^D/L_d^D)(L_d/\xi)^{D/2} f(\bar{\phi})$, where $\bar{\phi}$ is the deviation of the coarse-grained phase from its equilibrium value, and the range of $f(\bar{\phi})$ is typically $[-1, 1]$. For a dissipationless state ($\dot{\phi} = 0$), the coarse-grained $\bar{\phi}$ should obey:

$$-L_d^2\nabla^2\bar{\phi} + f(\bar{\phi}) = 0. \qquad (7)$$

The source term in Eq. (7) describes the loss of injected current due to tunneling in a domain. As discussed above, current injection induces counterflow currents and hence phase twists. Since it is energetically costly to introduce phase twists in a domain, the domain at the boundary will respond by rotating uniformly, increasing its tunneling current, thereby reducing the counterflow current. This process continues until the tunneling in the domain saturates, so that $|f| \sim 1$. The residual counterflow currents will be transmitted further into the sample, causing the domains there to rotate in a similar way. Thus, we argue that forcing at a boundary leads to a self-organized critical state, in which the driven part of the system sits at the threshold $|f| \sim 1$.

In one dimension, this argument means that Eq. (7) would give an average counterflow current $\rho_s\nabla\bar{\phi}$ that decreases linearly from the boundary. It predicts a linear $I/I_0$ in the saturated regions, with a slope $-\xi^{-1}(t\xi^2/\rho_s)^{4/3}$. This is qualitatively consistent with the numerical results (Fig. 1). Note that, in one dimension, Eq. (7) describes a harmonic chain with random static friction [26]. The process above is simply the transmission of forces when the chain is pushed at its ends.

As we now describe, the generalization of this argument to two dimensions will account for the critical current of the bilayer, as measured by Tiemann *et al.* [7]. (We see similar behavior in 2D simulations, albeit with large disorder fluctuations.) In our scenario of saturated domains, the static state only breaks down when the final domain in the sample exceeds threshold. Therefore, we may determine the critical current by setting $f = 1$ everywhere in Eq. (7). Integrating over space, we see that the critical current, defined as the total injected current at threshold, effectively counts the number of domains in the sample, and is independent of the precise geometry. We find, for its order of magnitude,

$$I_c \sim \frac{e\rho_s}{\hbar}\frac{S}{L_d^2}, \qquad (8)$$

where $S$ is the sample area in two dimensions and length in one dimension. Note that this is a natural form for an extensive critical current, composed of the system size, the characteristic length $L_d$, and the microscopic current scale $e\rho_s/\hbar$. The same form can be seen in Eq. (13) of Ref. [18]. However, it is not clear whether the result there applies to a bilayer driven at its boundary.

For Eq. (8) to apply, the sample should comprise many domains. If a dimension $L_x$ is smaller than the domain size $L_d$ then the second term in Eq. (7) should be multiplied by $\sqrt{L_d/L_x}$, because the total tunneling current of the domain is cut off at the sample width $L_x$. This gives

$$I_c \sim \frac{e\rho_s}{\hbar}\sqrt{\frac{L_x}{L_d}}\frac{L_y}{L_d} \quad (\text{quasi-1D } L_x \ll L_d \ll L_y). \qquad (9)$$

Similarly, for a sample containing only a single domain,

$$I_c \sim \frac{e\rho_s}{\hbar}\sqrt{\frac{L_x L_y}{L_d^2}} \quad (\text{for } L_x, L_y \ll L_d). \qquad (10)$$

To compare Eq. (8) with the experiments, we start from the microscopic theory [27] for the zero-temperature values of the stiffness, $\rho_s^0$, and order parameter, $m_x = \langle\cos\phi\rangle$, for a homogeneous bilayer. The latter renormalizes the tunneling strength $t^0$ so that $t^0 = \Delta_0 m_x/2\pi l^2$ where $\Delta_0$ is the single-particle tunnel splitting. This theory of Gaussian fluctuations around a Hartree-Fock state should be reasonable for a large range of layer separation $d$ except close to the critical layer separation for the loss of interlayer coherence. Using Eq. (3) we estimate $\rho_s \approx 20$ mK at layer separation $d \approx l$, and the microscopic current scale $e\rho_s/\hbar \approx 0.5$ nA.

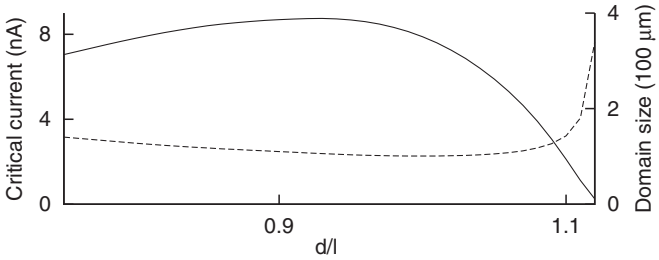


FIG. 2. Predicted critical current (solid, left axis) and domain size (dotted, right axis), for area $S = 0.1$ mm$^2$, $d = 28$ nm, $\xi(\approx d_d) = 200$ nm, and bare tunneling $\Delta_0 = 150$ $\mu$K. The stiffness and tunneling renormalization are taken from theory [27], and scaled by the network geometrical factor (see text).

Figure 2 shows our estimates for the domain size $L_d$ and the critical current $I_c$, as functions of the ratio of interlayer separation to magnetic length, $d/l$. We see that the domains ($L_d^2 \lesssim 0.01$ mm$^2$) are indeed not larger than the samples' areas of (0.01–1) mm$^2$ so that the results are consistent with area scaling for $I_c$. Moreover, $I_c$ has the correct order of magnitude compared to the observed values [7] of 0.1–10 nA.

Interestingly, $I_c$ has a peak as a function of $d/l$ which is also suggested in the experimental data [7]. This feature appears robust: it arises from the increase in $L_d$ as $d/l$ is reduced, caused by the increase in $\rho_s$. However, the peak position depends on the precise variation of model parameters with $d/l$. For example, the variation in $\rho_s$ may cause some variation in vortex density [15] and hence $\xi$, pushing the peak to smaller values of $d/l$.

Finally, we consider the effect of an in-plane magnetic field $B_\parallel$. This introduces a length scale $l_\parallel = hc/eB_\parallel d$, the length of a loop enclosing a flux quantum in the cross-section of a bilayer. Within our theory, the system should be insensitive to $B_\parallel$ unless $l_\parallel < \xi$. At such fields, circulating tunneling currents are set up between the two layers, reducing the *net* tunneling current in a domain. Therefore, coherent tunneling should be suppressed for $B_\parallel > hc/ed\xi \sim 0.7$ T, consistent with experiments [8] where enhanced tunneling decreases above 0.5 T.

In conclusion, we have presented a theory of the critical interlayer currents in a disordered quantum Hall bilayer with static pinned vortices. We find that, because the current is injected at the boundary, coherent tunneling saturates in the current-carrying region, leading to a Bean critical state. This results in an extensive critical current for sufficiently large samples (in contrast to the clean limit [10] where area scaling holds for small samples). The magnitude of the critical current is consistent with experiments. We predict that area scaling does not hold when the samples become smaller than the phase-pinned domains, and also that the critical current peaks in the interlayer coherent phase.

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