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Sums and Products of Indirect Utility Functions

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Abstract: There are relatively few known demand systems that are theoretically satisfactory and practically implementable. This paper considers the possibility of deriving more complex demand systems from simpler known ones by considering sums and products of the component indirect utility functions, an approach that does not seem to have been exploited previously in the literature. While not all sums and products of valid utility functions need yield new valid utility functions, it is possible to usefully extend the range of available utility functions. Some of the demand systems that result are interesting and potentially useful: the simpler (in a parameter parsimony sense) for applied general equilibrium studies and for theoretical explication, while more complex systems have potential for the analysis of real world consumption data.

I INTRODUCTION

An indirect utility function $U(\mathbf{p}, y)$, where $\mathbf{p}$ is a vector of prices and y is income, and the demand equations derived from it through Roy's identity

$$q_i = -\frac{\partial U}{\partial p_i} / \frac{\partial U}{\partial y} \tag{1}$$

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satisfy demand theory, or utility maximisation, provided $U(\mathbf{p}, y)$ meets stringent criteria. These are that U be homogeneous of degree zero in income and prices (**p**), non-decreasing in y, non-increasing in **p**, and convex or quasi-convex in **p.** Then the demand equations satisfy the required constraints of aggregation, homogeneity, Slutsky symmetry and negativity. These criteria for the validity of indirect utility functions are very restrictive on the choice of functional forms, even with constraints placed on the parameters occurring in the forms. There are relatively few known functions U that satisfy validity conditions for all, or even for all plausible, values of prices and income and some of them are very basic. This paper investigates building more complex demand systems from simple known ones by considering sums and products of component utility functions. It is true that not all sums and products of valid utility functions necessarily result in new valid utility functions, but some do.

The basic combination devices, which will be described in Section II, are quite simple, but at least as far as this author knows, they have not been exploited previously in the literature in order to expand the range of valid demand systems. Some of the simpler systems that result and that will be described in Section III, may not be as flexible as might be desired for the analysis of real world survey or time series data on consumer expenditures on commodities. However, they may still be useful for applied general equilibrium studies and for theoretical explication. More complex systems, to be investigated in Section IV, are more flexible, with greater potential for analysis of consumer demand data. Perhaps as might be expected, some systems derived in these sections turn out to be rediscoveries of already known ones instead of being new. However, even the way in which they arise as combinations of simple components is of interest in itself in showing them as sub sets of wider classes.

II DEMAND EQUATIONS FROM SUMS AND PRODUCTS OF UTILITIES

Suppose we have two (indirect) utility functions U_1 and U_2 satisfying all validity criteria and convex, rather than quasi-convex, in prices. Then the criteria obviously apply to $U_1 + U_2$ (the sum of two convex functions is convex, although the sum of two quasi-convex functions is not necessarily quasi-convex) and indeed to $(1 - \lambda)U_1 + \lambda U_2$, where λ is a positive constant, and corresponding demand systems can be derived. Let $w_{1i} = w_{1i}(\mathbf{p}, y)$ and $w_{2i} = w_{2i}(\mathbf{p}, y)$ be the sets of demand equations, in budget share form, resulting from application of Roy's identity to U_1 and U_2 respectively. Then by applying (1) to $(1 - \lambda)U_1 + \lambda U_2$ and simplifying, the demand equations corresponding to this sum of utilities turn out to be

$$w_{si} = w_{1i} \frac{(1-\lambda)\dfrac{\partial U_1}{\partial y}}{(1-\lambda)\dfrac{\partial U_1}{\partial y} + \lambda\dfrac{\partial U_2}{\partial y}} + w_{2i} \frac{\lambda\dfrac{\partial U_2}{\partial y}}{(1-\lambda)\dfrac{\partial U_1}{\partial y} + \lambda\dfrac{\partial U_2}{\partial y}}, \tag{2}$$

or the original individual demand formulae weighted by (apart from constants) the derivatives of utilities with respect to income. The subscript s denotes the utilities were summed.

Turning to products of utility functions, the convexity of the functions need not imply the convexity of the product. However, if the logs of the utility functions were also convex, then the fact that the sum of utilities gives a valid utility would suffice for the product of utilities, because an increasing convex function (the antilog) of a convex function is convex. Some such functions do exist and for these $U_1^{1-\lambda}U_2^{\lambda}$ is a valid utility function. In the next section, one specially simple class of utility functions with this property will be considered. However, although convexity of the log utilities is sufficient for convexity of the product, it is not always necessary and in Section IV the issue of the convexity of more complicated products will be returned to. Application of Roy's identity to the product gives

$$w_{mi} = w_{1i} \frac{(1-\lambda)\dfrac{\partial \log U_1}{\partial \log y}}{(1-\lambda)\dfrac{\partial \log U_1}{\partial \log y} + \lambda\dfrac{\partial \log U_2}{\partial \log y}} + w_{2i} \frac{\lambda\dfrac{\partial \log U_2}{\partial \log y}}{(1-\lambda)\dfrac{\partial \log U_1}{\partial \log y} + \lambda\dfrac{\partial \log U_2}{\partial \log y}}, \tag{3}$$

the individual demand formulae weighted by the elasticities of utilities with respect to income. The subscript m denotes the utilities were multiplied.

III SIMPLE HOMOTHETIC COMPONENT UTILITY FUNCTIONS

Consider the simple class of utility functions

$$U(\mathbf{p}, y) = \frac{y}{P(\mathbf{p}, \rho)} \tag{4}$$

where the price index $P(\mathbf{p}, \rho)$ is a weighted mean of order $\rho(\leq 1)$ in the prices, defined as

$$P(\mathbf{p}, \rho) = (\Sigma \phi_j p_j^{\rho})^{1/\rho}, \tag{5}$$

with positive ϕ_j summing to unity.[1] For example, taking ρ equal to 1 gives an arithmetic mean $\overline{p}_a = (\Sigma\phi_j\, p_j)$; taking it equal to –1 gives a harmonic mean $\overline{p}_h = (\Sigma\phi_j / p_j)^{-1}$ and taking it zero gives, via a limiting argument (see, for example, Diewert, 1993), a geometric mean $\overline{p}_g = \Pi\, p_j{}^{\phi_j}$. This class is of course well known (Pollak, 1971) and the corresponding direct utility functions are the simple sums of a power of commodity quantities and are often called the Bergson family. The demand functions following from (4) are

$$w_i = \frac{\phi_i p_i{}^{\rho}}{\Sigma\phi_j\, p_i{}^{\rho}} \tag{6}$$

with income elasticity unity, own price elasticity equal to $-1 + \rho(1 - w_i)$ and cross price elasticity (with respect to commodity k) equal to $-\rho w_k$. Some choices of ρ yield familiar simple systems; the value 0 gives the Bergson, or constant budget share, demands, with own-price elasticity minus one and cross-price elasticity zero. Taking $\rho = 1$ gives Leontief demands in that the ratios of quantities of commodities are always in fixed proportions, irrespective of prices or income.

As regards the validity of $U(\mathbf{p}, y)$ as an indirect utility function, compliance with requirements is obvious except for convexity in prices and this is easily, if tediously, verified by showing that the Hessian matrix of U with respect to prices is nonnegative definite for $\rho \le 1$. Equivalently, $P(\mathbf{p}, \rho)$ is concave in prices with negative semidefinite Hessian. However, unlike utility functions in general, the log of (4) is also convex in prices because

$$\log U(\mathbf{p}, y) = \log y - \log P(\mathbf{p}, \rho)$$

and concavity of P, with the fact that the log function is concave and increasing, implies log P is concave and therefore log U is convex. It follows that the product of any pair of utility functions U_1 and U_2 from the class (4) also satisfies convexity and formula (3) applies.

Indeed for this class (4) there is another way of deriving a new valid utility function from two of its members. If U_1 and U_2 are the component utilities, with corresponding price indices P_1 and P_2 belonging to the class (5), it is evident that the utility function

$$U = \frac{y}{(1 - \lambda)P_1 + \lambda P_2},$$

which could also be written as the weighted sum of the utilities U_1 and U_2,

[1] $P(\mathbf{p}, \rho)$ could also be described as a CES (constant elasticity of substitution) price index with elasticity equal to $1/(1 - \rho)$.

$$U = \frac{(1-\lambda)P_1}{(1-\lambda)P_1 + \lambda P_2} U_1 + \frac{\lambda P_2}{(1-\lambda)P_1 + \lambda P_2} U_2$$

is also valid. Applying Roy's identity to it gives the demand functions

$$w_{wsi} = w_{1i} \frac{(1-\lambda)P_1}{(1-\lambda)P_1 + \lambda P_2} + w_{2i} \frac{\lambda P_2}{(1-\lambda)P_1 + \lambda P_2} \tag{7}$$

which are the individual demand formulae weighted (apart from constants) by the price indices, or the reciprocals of the derivatives of utilities with respect to income.

To illustrate the use of the combination formulae, we take as components the demand functions resulting from the earlier mentioned examples of $\rho = 1$, 0 and –1. From (6) these are

$$w_{ai} = \frac{\gamma_i p_i}{\Sigma \gamma_j p_j}, \quad w_{gi} = \alpha_i \quad \text{and} \quad w_{hi} = \frac{\delta_i / p_i}{\Sigma \delta_j / p_j},$$

where the parameters ϕ_j have been replaced by γ_j, α_j and δ_j respectively. The subscripts a, g, and h (from the arithmetic, geometric and harmonic mean natures of the price indices in the parent utilities) identify the components.

Even with these three, there are quite a few potential demand systems. Taking the utility functions two at a time, there are three possibilities and the three combination methods via (2), (3) and (7) make nine demand systems. But how much more flexibility do they give? Since the component systems have unitary income elasticities, the combination systems will too because the weights in the combination formulae are functions of prices and not income. So we are only considering greater flexibility in response to price changes. Taking the combinations of w_{ai} and w_{gi} by formulae (2), (3) and (7) gives the demand systems

$$w_{si} = \frac{(1-\lambda)\alpha_i \bar{p}_a + \lambda \gamma_i p_i \dfrac{\bar{p}_g}{\bar{p}_a}}{\lambda \bar{p}_g + (1-\lambda)\bar{p}_a} \tag{8}$$

$$w_{mi} = (1-\lambda)\,\alpha_i + \lambda \frac{\gamma_i p_i}{\bar{p}_a} \tag{9}$$

and

$$w_{wsi} = \frac{(1-\lambda)\alpha_i \bar{p}_g + \lambda \gamma_i p_i}{(1-\lambda)\bar{p}_g + \lambda \bar{p}_a} \tag{10}$$

respectively, where $\bar{p}_a$ and $\bar{p}_g$ and $\bar{p}_h$ are functions of γ_j and α_j, respectively. These are systems with 2n – 1 parameters and they do permit a greater range of economic behaviour in response to price than the parent systems. For the w_{ai} system all goods had to be price inelastic and cross-price elasticities of all goods relative to a particular price had to be equal. For the w_{gi} system own-price elasticity had to be –1 and cross-price elasticities zero. But for (10), for example, the own-price elasticity is

$$- w_{wsi} - \alpha_i(1 - \alpha_i)\, \frac{(1 - \lambda)\bar{p}_g}{w_{wsi}[(1 - \lambda)\bar{p}_g + \lambda\, \bar{p}_a]}\,,$$

so that price elastic goods are possible and the cross-price elasticity with respect to price k is

$$- w_{wsk} - \alpha_i \alpha_k\, \frac{(1 - \lambda)\bar{p}_g}{w_{wsi}[(1 - \lambda)\bar{p}_g + \lambda\bar{p}_a]}\,,$$

which need not be constant over commodities. The three systems (8), (9) and (10) are distinct, because the requirement to become the same is $\bar{p}_g = \bar{p}_a$ and, as is well known, a geometric mean is always less than an arithmetic mean unless all commodities have the same price. But the systems have an evident similarity – own price appears explicitly and linearly in all, while the other prices (and own price) occur implicitly through the price indices. The feature of only own price appearing explicitly and others implicitly is, however, shared with many other demand systems.

Corresponding results follow from combinations of w_{gi} and w_{hi} and are

$$w_{si} = \frac{(1 - \lambda)\alpha_i\bar{p}_h + \lambda \dfrac{\delta_i}{p_i}\,\bar{p}_h\bar{p}_g}{\lambda\bar{p}_g + (1 - \lambda)\,\bar{p}_h},$$

$$w_{mi} = (1 - \lambda)\alpha_i + \lambda\,\frac{\delta_i\bar{p}_h}{p_i}$$

and

$$w_{wsi} = \frac{(1 - \lambda)\alpha_i\bar{p}_g + \lambda \dfrac{\delta_i}{p_i}\,\bar{p}_h{}^2}{(1 - \lambda)\bar{p}_g + \lambda\bar{p}_h},$$

respectively. Again, systems equality would require $\bar{p}_g = \bar{p}_h$, but a harmonic mean is always less than a geometric mean unless prices are equal. The reciprocal of own price appears explicitly and linearly in all three, while the other prices feature only through the price indices. Again, the demand systems for combinations of w_{ai} and w_{hi} are easily obtainable and are found to explicitly feature both own price and its reciprocal, while other prices again feature only through price indices.

The unitary income elasticities common to all these systems is obviously a serious inflexibility in some contexts. However, it is sometimes considered a desirable property in applied general equilibrium studies. Datta and Dixon (2000) have proposed a demand system with commodity quantities given by equations with own price appearing explicitly and linearly in all, while the other prices (and own price) occur implicitly through two price indices, $\bar{p}_a$ and that obtainable by taking $\rho = 2$ in (5).[2] (They choose to minimise on unknown parameters by taking $\gamma_i = \phi_{ii} = 1/n$.) So (8), (9) or (10) may have applicability in applied general equilibrium studies, perhaps with imposition of a similar parsimony in parameters if desired, although presumably, there will be occasions when more than a single parameter is desired. The other combination families would be appropriate if there are situations where "linearity" in the reciprocal of own price may be desirable instead of, or as well as, "linearity" in own price.

IV MORE COMPLEX COMPONENTS

In seeking to relax the limitation to unitary income elasticities associated with the demand systems of the previous section, it is useful to commence with what might seem an almost trivial variation on a combination already considered. The demand systems w_{gi} and w_{ai} followed from the utility functions

$$U_g = \frac{y}{\Pi p_i^{\alpha_i}} \quad \text{and} \quad U_a = \frac{y}{\Sigma \gamma_j p_j},$$

with sum and product both valid utility functions. Consider replacing U_a by

$$U_a^* = 1 - \frac{\Sigma \gamma_j p_j}{y}.$$

Obviously, applying Roy's identity to U_a^* must give the same result as applying it to U_a, that is, the demand system w_{ai}. However, as will be seen, the new demand systems obtained by employing combination formulae (2) and (3) are importantly different from (8) and (9).

[2] Actually the Datta and Dixon demand equations follow from application of combination formula (7) to the systems w_{ai} and the "system" resulting from taking $\rho = 2$ in (6) and corresponding equations would result from application of (2) or (3). However, with this value of ρ the convexity of the utility (4) fails and the validity of the combinations require other justification.

U_a^* is convex in prices, so the sum of U_g and U_a^* is convex and (2) gives

$$w_{si} = \frac{(1-\lambda)\alpha_i y^2 + \lambda\gamma_i p_i \bar{p}_g}{(1-\lambda) y^2 + \lambda\Sigma\gamma_j p_j \bar{p}_g},$$

where $\bar{p}_g$ is again the geometric mean $\Pi p_j^{\alpha_j}$. Defining $\gamma_i\lambda/(1-\lambda)$ as v_i makes this

$$w_{si} = \frac{\alpha_i y^2 + v_i p_i \bar{p}_g}{y^2 + \Sigma v_j p_j \bar{p}_g}, \tag{11}$$

where the v_j, unlike the γ_j, are not constrained to sum to unity. Equivalently, the parameters in U_a^* could have been originally defined as v_j, where all were independent and Roy's identity could have been applied to the simple sum (or mean) of utilities.

Turning to combination based on the product of U_g and U_a^* a complexity needs discussion Although U_a^* is convex in prices its log is not, so the type of proof of convexity of the product employed in the previous section does not apply. However, convexity might still hold over at least some ranges of variable values. Applying (3), with the reparametrisation already mentioned, and noting the elasticity of U_g with respect to y is unity and that of U_a^* is $\Sigma v_i p_i/(y - \Sigma v_j p_j)$ gives

$$w_{mi} = \frac{\alpha_i(y - \Sigma v_j p_j) + v_i p_i}{y}, \tag{12}$$

which is the famous linear expenditure system (LES). The LES satisfies validity for positive values of parameters and y > $\Sigma\gamma_j p_j$. So two simple homothetic demand systems, one of which does not have its log utility convex, have been combined to give a system permitting non-unitary income elasticities, which is valid for all prices and incomes of practical interest.[3]

Actually for this LES case, and indeed for a larger class containing it, this derivation from a product of utilities is identical to the method of "translation" (Gorman, 1975), whereby income y in a utility function is replaced by $y - \Sigma v_j p_j$, that is income minus a fixed cost, altering demand equations from $q_i = q_i(\mathbf{p}, y)$ to $q_i = v_i + q_i(\mathbf{p}, y - \Sigma v_j p_j)$. Now any utility of the form (4), if multiplied by U_a^*, gives $(y - \Sigma v_j p_j)/P(\mathbf{p})$, which is a translation of the utility function $y/P(\mathbf{p})$ and leads to the corresponding translation of the homothetic demand system. The suggestion from the derivation here, however, is that there may be other

[3] The possibility that the LES may not apply at low incomes is not considered a serious difficulty in the literature.

interesting and widely valid demand systems obtainable from products of utilities besides when the component utilities are such that the product comprises a translation. We will consider a relevant case shortly.

Both (11) and (12) are weighted sums of w_{ai} and w_{gi}, which textbooks (Deaton and Muellbauer, 1980, pp.144-145, for example) often interpret as a "poor" person's and a "rich" person's budget shares respectively. The weights are functions of income and so the demand systems (11) and (12) permit non-unitary income elasticities. However, the weights are somewhat different in nature in the two cases. In (12) they are simply the ratio of fixed cost to income, and one minus this ratio, and this is a property the LES shares with all systems derivable from the Gorman polar form (Gorman, 1961) of cost function, $y = A(\mathbf{p}) + P(\mathbf{p})U$, where $A(\mathbf{p})$, a possibly more general fixed cost[4] than $\Sigma\gamma_j p_j$, is linearly homogenous in prices. But as (11) may be written

$$w_{si} = \alpha_i \frac{y^2}{y^2 + \Sigma\nu_j p_j \overline{p}_g} + \frac{\nu_i p_i}{\Sigma\nu_j p_j} \frac{\Sigma\nu_j p_j \overline{p}_g}{y^2 + \Sigma\nu_j p_j \overline{p}_g},$$

it is clear the weights are more complicated functions of income and prices. So (11) does not follow from the Gorman polar form family,[5] which is unsurprising since, unlike (12), it is not a translation of w_{gi}

Now consider the combination of U_g and the utility function

$$1 - \sum \gamma_j \left(\frac{p_j}{y}\right)^{\beta j},$$

which would reduce to U_a^* if all $\beta_i = 1$. Again this satisfies convexity, but its log does not. Applying Roy's identity to it gives Houthakker's (1960) indirect addilog system (IAD)

$$w_i = \frac{\gamma_i\, \beta_i \left(\frac{p_i}{y}\right)^{\beta i}}{\sum \gamma_j\, \beta_j \left(\frac{p_j}{y}\right)^{\beta j}}.$$

There are 2n – 1 independent parameters, since numerator and denominator can be divided by any constant, but we can reparametrise as before by $\nu_i = \gamma_i\lambda/(1-\lambda)$, getting rid of λ and treating the γ_i as n independent parameters.

[4] Evidently choice of $A(\mathbf{p}) = \Sigma\nu_j p_j$, instead of $A(\mathbf{p}) = 0$, in the Gorman polar form is equivalent to translation of the utility function (4). Some, though not all, demand systems in this paper could have been derived by manipulation of cost rather than (indirect) utility functions.

[5] A referee has suggested the potential value of considering further combinations of utilities within members of the Gorman family, but developing on this is beyond the scope of the present paper.

Based on the sum of utilities we get

$$w_{si} = \frac{\alpha_i + \dfrac{\nu_i \beta_i}{p_i}\left(\dfrac{p_i}{y}\right)^{\beta_i+1}\bar{p}_g}{1 + \sum \dfrac{\nu_j \beta_j}{p_j}\left(\dfrac{p_j}{y}\right)^{\beta_j+1}\bar{p}_g} \tag{13}$$

and the product gives

$$w_{mi} = \frac{\alpha_i\left\{1 - \sum \nu_j \left(\dfrac{p_j}{y}\right)^{\beta_j}\right\} + \nu_i \beta_i \left(\dfrac{p_i}{y}\right)^{\beta_i}}{1 - \sum \nu_j (1-\beta_j)\left(\dfrac{p_j}{y}\right)^{\beta_j}}. \tag{14}$$

Note that neither of these systems are translations or would follow from a Gorman polar form of cost function. Although there is no doubt about the validity of (13), that of (14) requires demonstration. By the theoretically elementary, although algebraically rather lengthly, approach of obtaining the Hessian and examining when it is positive semi-definite, the convexity can be demonstrated (Conniffe, 2002) for the case of positive parameters[6] and $y > \Sigma\gamma_j p_j^{\beta_j} y^{1-\beta_j}$. The latter condition, as with the LES, amounts to y not too small and indeed (14) could be considered a generalisation of the LES to which it reduces if all $\beta_i = 1$, just as (13) reduces to (11) in the same situation. Both (13) and (14) are of course expressible as weighted sums of the constant budget share and IAD demand systems. As these systems possess 3n – 1 parameters, they are more flexible in various respects than the LES as is further discussed in relation to (14) in Conniffe (2002).

It is obviously possible to obtain many more candidate demand systems by these approaches. For example, we do not have to automatically choose U_g as one of the component utilities, as we have done throughout this section. But perhaps enough has already been presented to suggest the usefulness of using sums and products of indirect utility functions as a mechanism to at least indicate promising candidates for new demand systems. As was said in the Introduction, currently there are relatively few known demand systems that are both theoretically satisfactory and practically applicable.

[6] Some relaxation of the requirement that be positive is possible, but only with further restrictions on ranges of prices and income.

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