

Generalised Means of Simple Utility Functions with Risk Aversion

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Abstract: The paper examines the properties of a generalised mean of simple utilities each displaying risk aversion, that is, with first derivative positive and second derivative negative. It shows the mean is itself a valid utility function and argues that simple component utilities, each of which may have quite restricted risk aversion properties, can be parsimoniously combined through the generalised mean formula to give a much more versatile utility function.

I INTRODUCTION

Several areas of economics and finance employ a utility function $u(x)$, where x is income or wealth, to formulate approaches to decision making under uncertainty. Utility is assumed to increase with wealth so that $u'(x)$ is > 0 and the decision maker is presumed to be risk averse so that the function is concave, that is $u''(x) \leq 0$. The Arrow-Pratt coefficients

$$R_A(x) = -\frac{u''(x)}{u'(x)} \quad \text{and} \quad R_R(x) = -\frac{xu''(x)}{u'(x)}$$

serve as measures of absolute and relative risk aversion respectively. Types of behaviour under risk can be characterised by how these coefficients change with increasing x . For example, absolute risk aversion might decrease and

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relative risk aversion increase with rising wealth, as Arrow (1971) thought would often be plausible. Without prior information on the matter, it seems desirable that a utility function be flexible enough to represent different types of behaviour depending on the values assigned to the parameters of the function. However, reasonable parsimony of the number of parameters requiring estimation is usually also desirable and this can conflict with flexibility.

The most frequently employed utility functions, while quite parsimonious, are not particularly flexible. For example, the frequently employed negative exponential utility $u(x) = 1 - \exp(-\gamma x)$, with γ positive, has constant absolute risk aversion (CARA) equal to γ and increasing relative risk aversion (IRRA) of $x\gamma$. The popular power function utility $u(x) = x^{1-\alpha}$, with $0 \leq \alpha \leq 1$,¹ has decreasing absolute risk aversion (DARA) equal to α/x and constant relative risk aversion (CRRA) of α . Another familiar utility function, the quadratic $u(x) = x - bx^2$, with b positive and $x < 1/2b$ to ensure the derivative positive, has increasing absolute risk aversion (IARA) equal to $2b/(1 - 2bx)$ and therefore IRRA of $2bx/(1 - 2bx)$. So while all three involve just one parameter, their risk aversion measures evolve quite differently with x , so that in situations where there is little prior knowledge, a more versatile utility function is desirable.

Towards the end of his fundamental paper defining absolute and relative risk aversion, Pratt (1964) observed that the sum of concave utility functions is itself concave and a valid utility function. He went on to derive the absolute and relative risk aversions of this utility in terms of the Arrow-Pratt coefficients of the components. A weighted mean of utilities would also be a valid utility and although Pratt did not remark on it, it is clear the risk aversion properties of

$$wu_1(x) + (1 - w)u_2(x) \tag{1}$$

will be close to those of $u_1(x)$ if w is near zero, to those of $u_2(x)$ if w is near unity and a mixture if w is intermediate. So by treating w as an unknown parameter to be estimated, two component utilities, each of which may have quite restricted risk aversion properties, can be combined to give a more versatile utility function. Economists have not followed up Pratt's observation, but in the management science field Bell (1988, 1995) proposed the LINEX function,

$$U(x) = bx - ce^{-\gamma x},$$

¹ Or $u(x) = 1 - x^{1-\alpha}$ if $\alpha > 1$.

which could be seen as the sum of a linear² and an exponential. It is rather extreme in that it is clear that for large x it effectively implies risk neutrality with the Arrow-Pratt coefficients zero. Nakamura (1996) introduced the SUMEX utility function, the sum of two exponentials, although not entirely in a risk aversion context, and indeed that combination would be rather ineffective in increasing flexibility in that regard. Other papers relating to either LINEX or SUMEX include Farquhar and Nakamura (1988), Gelles and Mitchell (1999) and Bell and Fishbourne (2001).

However, concave utilities can be combined into a new concave utility in a much more general way than (1). If component utilities $u_j(x)$ are positive³ and increasing and concave in x , then the generalised mean of utility functions

$$U(x) = \left[\sum_{j=1}^n w_j u_j^{-\lambda}(x) \right]^{-\frac{1}{\lambda}} \quad (2)$$

can be shown to be increasing and concave in x provided $\lambda \geq -1$. The case $\lambda = -1$ gives the weighted arithmetic mean, while $\lambda = 0$ and $\lambda = -1$ give weighted geometric and harmonic means respectively. The idea of this paper is that simple component utilities $u_j(x)$ can be combined through the generalised mean formula (2) to give a much more flexible utility function. For example, the quadratic utility $u_1(x) = x - bx^2$, which displays IARA could be combined through formula (2) with the simple utility $u_2(x) = \log x$, which displays DARA, which could be expected to give a utility capable of displaying either property. Taking w as an unknown parameter and $\lambda = -1$ gives

$$U(x) = w(x - bx^2) + (1 - w) \log x,$$

which is a case of (1), while taking $\lambda = 0$ gives

$$U(x) = (x - bx^2)^w (\log x)^{1-w}.$$

and there are an infinity of other possibilities through choice of λ . Indeed, λ could be treated as an unknown parameter to give a three parameter utility function

$$U(x) = \left[w(x - bx^2)^{-\lambda} + (1 - w)(\log x)^{-\lambda} \right]^{-\frac{1}{\lambda}}.$$

² The linear, risk neutral, function $u(x) = bx$ is concave and can be employed in (1).

³ Utilities are sometimes written in negative form in economic literature since Arrow-Pratt measures are unaffected. For example, the exponential utility may appear as $-\exp(-\gamma x)$. But for non-integer values of λ or $1/\lambda$, (2) could take imaginary values if a $u_j(x)$ is negative, so the positive form $1 - \exp(-\gamma x)$ is required. Again, $u(x) = \log x$ will assume $x > 1$, which is hardly a difficulty, although if it was, $\log(1 + x)$ could be substituted.

Since, in all these cases, $U(x)$ will resemble $u_1(x)$ for w near zero and $u_2(x)$ for w close to unity, it can display the risk aversion properties of either component as well as properties that neither possess. This theme will be discussed further in Section III.

Combination of simple utility functions is not the only possible approach to obtaining a versatile utility function and a few multi-parameter functions have been proposed in the literature as capable of representing a wide range of risk aversion properties. These include the power risk aversion form of Xie (2000), the explicit marginal utility form of Meyer and Meyer (2005), the generalisation of Xie's form by Conniffe (2007a) and perhaps the HARA (hyperbolic absolute risk aversion)⁴ form. However, the combination device can add many extra candidates to this rather small set and, as will be discussed, these may have special advantages in some circumstances.

II CONCAVITY AND RISK AVERSION OF $U(x)$

That $U(x)$, as given by (2), is a valid utility function for $\lambda \geq -1$ provided all the $u_j(x)$ are, can be deduced from the proof (Conniffe, 2007b) of the global regularity of an indirect consumption utility of the same form with x replaced by a vector of commodity prices. However, it is easily shown directly by differentiation and the derivatives are required anyway to obtain expressions for risk aversion. Differentiating (2)

$$U'(x) = \left[\sum w_j u_j^{-\lambda} \right]^{-\frac{1}{\lambda}-1} \left[\sum w_j u_j^{-\lambda-1} u'_j \right] \quad (3)$$

which is positive, given each first derivative $u'_j(x)$ is positive. Differentiating again

$$U''(x) = \left[\sum w_j u_j^{-\lambda} \right]^{-\frac{1}{\lambda}-1} \left[\sum w_j u_j^{-\lambda-1} u''_j \right] - (\lambda + 1) \left[\sum w_j u_j^{-\lambda} \right]^{-\frac{1}{\lambda}-2} \left[\sum w_j u_j^{-\lambda} \sum w_j u_j^{-\lambda-2} u_j'^2 - \left(\sum w_j u_j^{-\lambda-1} u'_j \right)^2 \right] \quad (4)$$

and the top term on the right hand side of equation (4) is clearly negative, provided each second derivative $u''_j(x) \leq 0$. The sign of the bottom term is less obvious. Let

$$y_j^2 = w_j u_j^{-\lambda} \quad \text{and} \quad z_j = u'_j / u_j.$$

⁴ The HARA utility lacks full capacity to represent DRRA.

Then the bottom term of (4) becomes

$$-(\lambda + 1) \left[\sum y_j^2 \right]^{-\frac{1}{\lambda}-2} \left\{ \sum y_j^2 \sum (y_j z_j)^2 - \left[\sum y_j (y_j z_j) \right]^2 \right\}$$

and by the Cauchy-Schwarz inequality the term in chain brackets is positive. So provided $\lambda \geq -1$, $U(x)$ is a valid utility function.

From (3) and (4) the Arrow-Pratt coefficient of absolute risk aversion, $R_A(x)$, is

$$\begin{aligned} &= \frac{\sum w_j u_j^{-\lambda} R_{Aj} \frac{u'_j}{u_j}}{\sum w_j u_j^{-\lambda} \frac{u'_j}{u_j}} + (\lambda + 1) \left[\frac{\sum w_j u_j^{-\lambda} \left(\frac{u'_j}{u_j} \right)^2}{\sum w_j u_j^{-\lambda} \frac{u'_j}{u_j}} - \frac{\sum w_j u_j^{-\lambda} \frac{u'_j}{u_j}}{\sum w_j u_j^{-\lambda}} \right] \\ &= C(x) + (\lambda + 1)D(x) \end{aligned} \tag{5}$$

where R_{Aj} is the coefficient of absolute risk aversion for the j th utility function. Since the coefficient of relative risk aversion R_R is just defined as xR_A , it is

$$R_R(x) = xC(x) + (\lambda + 1)xD(x) \tag{6}$$

For $\lambda = -1$ the formulae simplify to

$$\frac{\sum w_j R_{Aj} u'_j}{\sum w_j u'_j} \quad \text{and} \quad \frac{\sum w_j R_{Rj} u'_j}{\sum w_j u'_j}$$

respectively. If the weights are all equal, this is Pratt's case of a sum of utilities and he examined the sign of the derivatives for sums of the same functional forms of utilities, showing that if each component utility displays DARA, then $U(x)$ displays DARA. More surprisingly, he showed that if each component has exponential form with CARA equal to γ_j , $U(x)$ still displays DARA unless the γ_j are all equal. Again, if each component has the power form displaying CRRA equal to α_j , $U(x)$ displays DRRA unless the α_j are all equal. The Appendix extends these results to unequal weights and briefly discusses matters for values of λ other than $\lambda = -1$. Generally the situation is complicated, although there are some simple results. For example, with λ zero CRRA holds for $U(x)$ unlike the situation for $\lambda = -1$, when DRRA held.

As was said in the Introduction, this paper takes the objective of combination as obtaining a more versatile utility function than the components, implying combination of utilities with different risk aversion properties, rather than with similar ones. So further examination of combinations of the same

functional forms is probably not worth while here. But the finding that the properties can differ depending on the value of λ is of note and carries over to combination of dissimilar components.

III COMBINING COMPONENTS WITH DISSIMILAR RISK PROPERTIES

We will usually want reasonable parsimony of parameters, so probably only two, or at most three, utility functions would be combined. So we commence with a very parsimonious example, but one combining a component displaying IARA with one displaying DARA. The quadratic utility $u(x) = x - bx^2$, was mentioned in the Introduction as an IARA type, but it does have one parameter b and the rather awkward limitation that $x < 1/2b$. Imposing this upper bound on combinations with functions that have no upper bound seems unduly restrictive. But the utility function $u_1(x) = \Phi(x)$, where $\Phi(x)$ denotes the (cumulative) distribution of the standard normal, has similar risk aversion properties to a quadratic, but has no unknown parameters⁵ and imposes no upper bound on x . Since $\Phi'(x) = \phi(x)$, where $\phi(x)$ is the standard normal density and $\Phi''(x) = -x\phi(x)$, $R_{A1} = x$, implying IARA. For $u_2(x) = \log x$, $R_{A2} = 1/x$, implying DARA. Choosing $\lambda = -1$ gives

$$U(x) = w\Phi(x) + (1-w)\log x \quad (7)$$

a utility function with one parameter w . From formula (5)

$$R_A = \frac{wx^3\phi(x) + 1 - w}{wx^2\phi(x) + (1-w)x}$$

For fixed x

$$\frac{\partial R_A}{\partial w} = \frac{(x^2 - 1)\phi(x)}{(wx\phi(x) + (1-w))^2}$$

which is positive for $x > 1$, so that relative risk aversion increases with w as would be expected. For fixed w and putting $\theta = w/(1-w)$

$$R'_A = \frac{\partial R_A}{\partial x} = \frac{\theta^2 x^4 \phi^2(x) - \theta x \phi(x)(x^2 - 1)(x^2 - 2) - 1}{x^2 (\theta x \phi(x) + 1)^2} \quad (8)$$

Since the third term in the numerator of (8) is negative, it is obvious that

⁵ $\Phi(x)$ is concave only for $x > 0$, but that is no difficulty when x is income or wealth.

DARA holds if w , and therefore θ , is small enough. If θ is large (w near 1) the derivative is clearly positive, at least for low x , and IARA holds with $R'_A \rightarrow 1$ as $\theta \rightarrow \infty$. However, unless θ is huge R'_A will be negative for large x because the second term is negative if $x > \sqrt{2}$ and the positive first term is small if $\phi^2(x)$ is. So this combination of u_1 and u_2 can display either IARA or DARA and transitions from one state to another can occur with increasing wealth x . The point of transition could be obtained by setting the numerator of (8) to zero, but because of the $\phi(x)$ terms the equation is complicated. But it is clear that for modest values of θ the transition from IARA to DARA will occur at quite low x since a standard normal variate has a probability of only .0001 of exceeding 3.72 and $\phi(3.72)$ is correspondingly small. This could be remedied, if desired, by introducing a standard deviation parameter σ to the cumulative normal distribution. Then $u_1 = \Phi(x/\sigma)$ and data sets implying a higher transition point can be adequately fitted, without forcing w towards unity, by estimating a second parameter σ .

Combination of the same utilities with $\lambda = 0$ gives

$$U(x) = \Phi^w(x)(\log x)^{1-w} \tag{9}$$

and rather tedious differentiation shows some similar risk aversion properties to those just outlined for (7). For example, if w is not large DARA will hold in the higher range of x . But all properties are not the same. Provided w is not almost unity, it is clear that as x becomes very large $\Phi(x) \rightarrow 1$ and (9) is effectively

$$U(x) = (\log x)^{1-w},$$

implying

$$R_R(x) = \frac{w}{\log x} + 1,$$

and DRRA. But in these circumstances (7) is effectively $w + (1-w) \log x$, giving CRRA. Taking $\lambda = 1$ would give the harmonic mean utility

$$U(y) = \frac{1}{\frac{w}{\Phi(y)} + \frac{1-w}{\log y}},$$

which also gives DRRA for large x and $w < 1$. There are many other possibilities. Keeping λ as a parameter to be estimated would give a two-parameter utility

$$U(y) = \left[w\Phi(y)^{-\lambda} + (1-w)(\log y)^{-\lambda} \right]^{-\frac{1}{\lambda}}$$

with the obvious possible extension to a three-parameter utility by allowing a σ parameter in Φ . Again, w could be set to a predetermined value, retaining a two parameter function in λ and σ .

These utilities $u_1(x) = \Phi(x)$ and $u_2(x) = \log x$ are not of particular practical significance and were just chosen to show that combination of two simple and parametrically parsimonious functions could display quite a variety of risk aversion behaviour. There are many other candidate components and the examination in this paper is far from comprehensive. In reality though, the commonest utilities appearing in the economic or finance literature are the CARA exponential form and the CRRA power form, both simple single parameter functions. But even if circumstances are such that a strong assumption such as CARA or CRRA seems plausible, it must often, if not always, be desirable to test the assumption. Combining the utility which embodies the assumption, $u_1(x)$, with another, $u_2(x)$, which does not and estimating

$$U(x) = \left[wu_1(x)^{-\lambda} + (1-w)u_2(x)^{-\lambda} \right]^{-\frac{1}{\lambda}},$$

with any value of λ , would permit testing the null hypothesis of $w = 1$. The power of such a test would depend on what the true utility is if it is not $u_1(x)$ and how well chosen $u_2(x)$ is. In testing CARA, $u_1(x)$ is the exponential utility and if it was believed that the alternative model displays IARA, then $u_2(x)$ a quadratic or $\Phi(x)$ would be appropriate. But if, as is more likely, the alternative to the CARA utility is a DARA utility, these would be bad choices as they cannot display DARA. However, taking $u_2(x)$ as a power or log form would give a test with discriminatory power. Again, if a CRRA utility is hypothesised, but an IRRA situation is feared, a combination of a power function and an exponential utility (which displays IRRA) would be one appropriate combination, while if DRRR rather than IRRA is feared, the exponential could be replaced a utility displaying DRRR such as $\log \log x$ or $\log x$ to a power. Clearly much depends on the amount assumed that is known about the possible patterns of risk aversion. If little is known a very flexible utility function is desirable and, as mentioned in the Introduction, some do exist in the literature. Other flexible functions could be constructed by this paper's approach by combining across components displaying the full range of risk behaviour. On the other hand, if there is enough prior knowledge to narrow the range considerably and hypothesise a 'true' utility, appropriate combination could select a 'precision' test utility.

IV DISCUSSION

The previous section has argued that the combination device has considerable potential to create useful utility functions and so deserves further investigation with applications to practical problems. But this raises the question of why the device has not already been investigated and employed, at least in the $\lambda = -1$ case. That a sum of concave utilities is concave is obvious enough and was noted by Pratt (1964) and in management science Bell's (1988) LINEX function could be seen as the sum of utilities with dissimilar risk aversion properties, a linear and an exponential function. The explanation is not that flexible multi-parameter forms already in the economic literature are perceived as greatly superior. These are relatively recent and have not yet been frequently employed.

There has to be a perceived problem to motivate a search for a solution and the vast majority of economic and finance literature authors seem perfectly happy with simple single parameter utilities, usually either the exponential CARA form or the power CRRA form. For example, in investment portfolio analysis the exponential form is very prominent, probably because, assuming a normal distribution for return, expected utility maximisation is then equivalent to familiar mean-variance analysis with a simple investor indifference curve. Again, in macroeconomics a huge volume of research on the consumption function has almost always featured the power utility. It is certainly convenient for deriving results and, on occasions, it has been claimed to have empirical support. Also, various authors have sought to impose extra constraints on utility functions besides monotonicity and concavity on the grounds of increasing their behavioural 'plausibility'. Thus Pratt and Zeckhauser (1987) defined 'proper risk aversion' utility functions, Kimball (1993) 'standard risk aversion' utilities and Caballe and Pomanski (1996) 'mixed risk aversion' functions. But the power function is a valid member of all classes. For example, 'mixed risk aversion' requires the derivatives of the utility to alternate in sign, which is true for the power function.

On the other hand, Xie (2000) has discussed the dangers implicit in assuming this CRRA utility and has argued for a more flexible form. Also, studies on the 'equity premium puzzle' have found it difficult, if not impossible, to reconcile observed data with a power utility function. Meyer and Meyer (2005) have argued that replacing the power utility by one permitting DRRA can provide one avenue towards a resolution of the puzzle. The paper by Roche (2006) using Xie's function reaches a similar conclusion. Future research in various fields may replace simple utility functions like the exponential or power by more versatile forms and then the combination device described in this paper may have a useful role to play.

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APPENDIX

For the component utility functions $u_j(x) = 1 - \exp(-\gamma_j x)$, which have $R_{A_j} = \gamma_j$, and with $\lambda = -1$, (5) gives

$$R_A = \frac{\sum w_j \gamma_j^2 e^{-\gamma_j x}}{\sum w_j \gamma_j e^{-\gamma_j x}}.$$

Differentiating this gives

$$\begin{aligned} R'_A &= -\frac{\sum w_j \gamma_j^3 e^{-\gamma_j x}}{\sum w_j \gamma_j e^{-\gamma_j x}} + \frac{\left(\sum w_j \gamma_j^2 e^{-\gamma_j x}\right)^2}{\left(\sum w_j \gamma_j e^{-\gamma_j x}\right)^2} \\ &= -\frac{1}{\left(\sum w_j \gamma_j e^{-\gamma_j x}\right)^2} \left[\sum w_j \gamma_j^3 e^{-\gamma_j x} \sum w_j \gamma_j e^{-\gamma_j x} - \left(\sum w_j \gamma_j^2 e^{-\gamma_j x}\right)^2 \right] \end{aligned}$$

and by writing

$$y_j = w_j^{1/2} \gamma_j^{3/2} e^{-\frac{1}{2}\gamma_j x} \quad \text{and} \quad z_j = w_j^{1/2} \gamma_j^{1/2} e^{-\frac{1}{2}\gamma_j x}$$

It is clear that the term in square brackets is $\sum y_j^2 \sum z_j^2 - (\sum y_j z_j)^2$ and therefore positive by Cauchy-Schwarz. So R'_A is negative and the combination displays DARA unless all γ_j are equal when R'_A is zero and CARA holds.

Again for $u_j(x) = x^{1-\alpha_j}$, $0 \leq \alpha_j \leq 1$, $R_{R_j} = \alpha_j$ and with $\lambda = -1$ (6) gives

$$R_R = \frac{\sum w_j \alpha_j (1-\alpha_j) x^{-\alpha_j}}{\sum w_j \alpha_j x^{-\alpha_j}}$$

Then

$$\begin{aligned} R'_R &= -\frac{\sum w_j \alpha_j^2 (1-\alpha_j) x^{-\alpha_j-1}}{\sum w_j (1-\alpha_j) x^{-\alpha_j}} + \frac{\left(\sum w_j \alpha_j (1-\alpha_j) x^{-\alpha_j}\right) \left(\sum w_j \alpha_j (1-\alpha_j) x^{-\alpha_j-1}\right)}{\left(\sum w_j (1-\alpha_j) x^{-\alpha_j}\right)^2} \\ &= -\frac{1}{x \left(\sum w_j (1-\alpha_j) x^{-\alpha_j}\right)^2} \left[\sum w_j \alpha_j^2 (1-\alpha_j) x^{-\alpha_j} \sum w_j (1-\alpha_j) x^{-\alpha_j} \right. \\ &\quad \left. - \left(\sum w_j \alpha_j (1-\alpha_j) x^{-\alpha_j}\right)^2 \right] \end{aligned}$$

and by writing

$$y_j = w_j^{1/2} \alpha_j (1 - \alpha_j)^{1/2} x^{-\frac{1}{2}\alpha_j} \quad \text{and} \quad z_j = w_j^{1/2} (1 - \alpha_j)^{1/2} x^{-\frac{1}{2}\alpha_j}$$

the Cauchy-Schwarz inequality shows DRRA holds unless all the α_j are equal.

But these results depend on $\lambda = -1$. For other values of λ matters can be complicated because the second term of (6) does not vanish. However, there are some simple results, although different from those for $\lambda = -1$. For example, with λ zero, (6) becomes

$$R_R = \frac{\sum w_j R_{Rj} \frac{u'_j}{u_j}}{\sum w_j \frac{u'_j}{u_j}} + \left[\frac{\sum w_j \left(\frac{u'_j}{u_j} \right)^2}{\sum w_j \frac{u'_j}{u_j}} - \sum w_j \frac{u'_j}{u_j} \right]$$

When each utility is of the power form, CRRA holds for each with $R_{Rj} = \alpha_j$ and also each u'_j/u_j is proportional to $1/x$. So x cancels out of the formula for R_R , R'_R is zero and CRRA holds for $U(x)$ also, unlike the situation for $\lambda = -1$.