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On the Small Sample Distribution of the R/S Statistic*

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Abstract: This paper gives an account of the R/S statistic and its known properties. It assesses the adequacy of the asymptotic distribution of the statistic in the case of samples of small and moderate size, and suggests an improved approximation based on the beta distribution. The results indicate that the proposed beta approximation is superior to the asymptotic distribution for practical purposes. Hence a new table of critical values is presented as an alternative to that of Lo (1991).

I INTRODUCTION

This paper is concerned with the re-scaled adjusted range (R/S) statistic as a means of investigating long-term statistical dependence or persistence in time series data. Introduced by Hurst (1951), R/S analysis was developed by Mandelbrot and Wallis (1969a, 1969b, 1969c) and first used by hydrologists to study persistence in geophysical time series. Although it was also used by Mandelbrot (1971) to examine persistence in asset returns, the R/S statistic has only recently been utilised by economists. However, there are already a few notable applications to financial time series (e.g., Peters (1989, 1991, 1992, 1994); Lo (1991); Moody and Wu (1995)) and at least one to political opinion poll series (Byers *et al.* (1996)).[1]

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1. The financial time series used tend to be very long. For example, although Peters used some moderate sample sizes of the order of 300 or 400, Lo used daily price series with between 1,500 and 6,000 observations, while Moody and Wu used tick-by-tick data and had samples of hundreds of thousands of observations.

The case for persistence to be placed among the practical concerns of economists has been stated strongly by Haubrich and Lo (1989), Haubrich (1990) and Lo (1991). Indeed, as Lo (1991) points out, long-range dependence was an implicit hypothesis in several early theories of trade and business cycles, and it is consistent with the Granger (1966) discovery of the typical spectral shape of an economic variable. Furthermore, it has important implications for several theories in modern financial economics, such as those concerned with optimal consumption and portfolio decisions and the pricing of derivative securities, as well as for standard tests of the capital asset pricing model and the arbitrage pricing theory.

R/S analysis offers a relatively straightforward means of investigating the issue of long-term dependency. It is based on simple graphical plots and Hurst exponents; it is claimed that it can detect long-range dependence in highly non-normal time series with large skewness and kurtosis, and non-periodic cycles; and the R/S statistic it uses is characterised by almost sure convergence for stochastic processes with infinite variance.[2] However, it is known to be adversely affected when there is short-term dependence in the series (see McLeod and Hipel (1978), and Hipel and McLeod (1978)), and it was this deficiency that led Lo (1991), and then Moody and Wu (1995), to propose modifications of the R/S statistic.

Unfortunately, certain shortcomings remain in the calculation and plots of the R/S statistic used in standard R/S analysis. Moreover, although the asymptotic distribution of the statistic is known, there is very little information available on its distribution for small samples and the adequacy of the asymptotic distribution as an approximation in the small sample situation. Therefore we investigate the small sample behaviour of the R/S statistic, having particular regard to the quality of the asymptotic approximation to its finite sample distribution and of an alternative approximation based on a beta distribution. We conclude that our beta procedure gives a better approximation to the finite sample distribution of the R/S statistic than the known asymptotic distribution does. Hence we provide a new table of critical values as an alternative to that given by Lo (1991).

The paper is organised as follows. Section II describes the standard and modified R/S statistics and outlines their main properties. Section III focuses on issues relating to the small sample distribution of the R/S statistic and its approximation by means of a beta distribution. Section IV reports on our experimental work and findings, and Section V presents our table of critical

2. An introduction to long-range dependence is given in the new textbook by Campbell *et al.* (1997, Ch. 2). In particular, their footnote 21, pages 62-63, provides a straightforward account of the behaviour of the R/S statistic and a simple example that shows why it takes on large values in the presence of persistence.

values for the R/S statistic. The paper ends with a short summary and conclusion in Section VI.

II THE R/S STATISTIC

Hurst was a civil engineer interested in reservoir storage and, in particular, the ideal reservoir size. Thus he was concerned with inflows and outflows of water. For economic applications, the idea of flows has immediate relevance to many variables and problems, such as those involving stock returns or flows in to and out of unemployment per period, and it may be useful to bear such examples in mind. However, we proceed along more general lines, denoting a series of one period changes in any variable by $\{x_t\}_{t=1}^{T}$, and its sample mean and standard deviation over the period t=1,...,T by m and s, respectively. The partial or cumulative sums of deviations of the x_t from their mean are defined as

$$p(k) = \sum_{t=1}^{k}(x_t - m), \; k = 1,\ldots,T.$$

Clearly, when $x_k (k > 1)$ exceeds m, the kth partial sum will increase, and when x_k is less than m, the kth partial sum will decrease. The range of the partial sums is

$$r = \max_k[p(k)] - \min_k[p(k)],$$

and the re-scaled adjusted range, i.e. the R/S statistic, for the time period t=1,...,T is just the ratio of r to s. Specifically, the R/S statistic is calculated as

$$R/S = \frac{r}{s} = \frac{\max_k[p(k)] - \min_k[p(k)]}{\sqrt{\frac{1}{T}\sum_{t=1}^{T}(x_t - m)^2}}.$$

The adjustment of the range relates to subtraction of the mean, and the re-scaling to division by the maximum likelihood estimate of the standard deviation.

The R/S statistic is of a rather complex form which makes the analysis of its small sample behaviour difficult, even in the null situation in which the x_t are independently and identically distributed (i.i.d.). Indeed, its sampling distribution for finite samples in this case remains unknown except for the expectation. The asymptotic properties of R/S are easier to handle, though they are by no means uncomplicated.

Using simple experimentation, Hurst (1951) derived an approximation for the expected value of R/S for large samples in the i.i.d. case, namely $E[R/S] \approx \sqrt{\frac{\pi}{2}T}$. This result was central to Hurst's early applications of R/S analysis. Round about the same time, Feller (1951) derived the asymptotic distribution of R/S under the i.i.d. null hypothesis, using a result due to Doob (1949), and thereby confirmed the accuracy of Hurst's approximation. Implicit in the work of Feller is that $(T^{-\frac{1}{2}})R/S \xrightarrow{d} V$, where the distribution function of V, as given explicitly by Lo (1991), is

$$F_v(v) = 1 + 2\sum_{k=1}^{\infty}(1-4k^2v^2)e^{-2(kv)^2}.$$

The probability density, and hence the moments of V, may therefore be derived; and (non-trivial) calculation shows that $E[V] = \sqrt{\frac{\pi}{2}}$ and $E\left[V^2\right] = \frac{\pi^2}{6}$. It follows that for large T,

$$E[R/S] = \sqrt{\frac{\pi}{2}T} \approx 1.2533\sqrt{T} \text{ and}$$

$$V[R/S] = \left(\frac{\pi^2}{6} - \frac{\pi}{2}\right)T \approx 0.0741T.$$

These results are well known. However, to our knowledge, higher moments have not been reported in the literature. Our calculations show that the standard skewness and kurtosis coefficients are, respectively,[3]

$$\sqrt{\beta_1}[R/S] = \frac{1}{\left(\frac{\pi^2}{6} - \frac{\pi}{2}\right)^{3/2}}\left(3\sqrt{\frac{\pi}{8}}\varsigma(3) - \sqrt{\frac{\pi^5}{8}} + \sqrt{\frac{\pi^3}{2}}\right) \approx 0.6132,$$

where $\varsigma(.)$ denotes the Riemann Zeta function, and

$$\beta_2[R/S] = \frac{1}{\left(\frac{\pi^2}{6} - \frac{\pi}{2}\right)^2}\left(\frac{\pi^4}{30} + \frac{\pi^3}{2} - \frac{3\pi^2}{4} - 3\pi\varsigma(3)\right) \approx 3.4178.$$

3. The standard skewness coefficient is the ratio of the third moment about the mean to the cube of the standard deviation; the standard kurtosis coefficient is the ratio of the fourth moment about the mean to the square of the variance. The symbols used here, i.e., $\sqrt{\beta_1}$ and β_2, are the conventional ones for the skewness and kurtosis coefficients, respectively.

These features of slight positive skewness and leptokurtosis are evident in plots of the density function of V; see Figure 1 which also includes a normal distribution (dashed) with the same mean and variance as V (a similar diagram, but also containing the distribution function, is given by Lo (1991, p. 1292)).

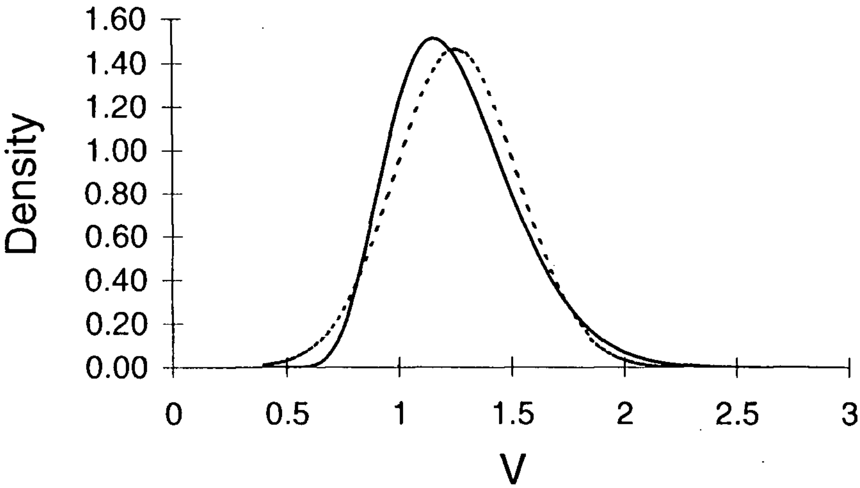

Figure 1: *Density of V and Normal Distribution*

When the x_t are generated by an AR(1) process, Lo (1991) has shown that $(T^{-\frac{1}{2}})R/S \xrightarrow{d} \xi V$, where $\xi = \sqrt{\frac{1+\rho}{1-\rho}}$ and ρ is the first-order autocorrelation coefficient. Thus for AR(1) data the mean of R/S may be seriously biased, depending on the value of ρ, and inferences may be seriously misleading if based on the use of critical values obtained from the standard asymptotic distribution. In an attempt to account for this type of effect in detecting long-range dependence using a statistic whose limiting distribution is invariant to many forms of short-term dependence, Lo (1991) suggested modifying the denominator of the R/S statistic as follows:

$$\overline{R/S} = \frac{r}{\overline{\sigma(q)}} = \frac{\max_k[p(k)] - \min_k[p(k)]}{\overline{\sigma(q)}},$$

where

$$\overline{\sigma(q)^2} = s^2 + 2\sum_{j=1}^{q}\omega_j(q)\overline{\gamma_j} = \frac{1}{T}\sum_{t=1}^{T}(x_t - m)^2 + \frac{2}{T}\sum_{j=1}^{q}\omega_j(q)\left[\sum_{t=j+1}^{T}(x_t - m)(x_{t-j} - m)\right]$$

and

$$\omega_j(q) = 1 - \frac{j}{q+1}, q < T.$$

The intuition underlying this modification is straightforward: if there is short-term dependence in $\{x_t\}_{t=1}^{T}$, then the range r should not simply be normalised by the standard deviation, but the non-zero autocovariances, estimated by the usual sample estimator, should also be taken into account. The weighting function, which is the same as that used by Newey and West (1987), always produces a positive $\overline{\sigma(q)^2}$, but little is known about how best to pick q, though some guidance is provided by the Monte Carlo study of Andrews (1991). Lo (1991, Theorem 3.1) proves that $(T^{-\frac{1}{2}})\,\overline{R/S}$ has the same limiting distribution as $(T^{-\frac{1}{2}})$R/S.

However, Moody and Wu (1995) have recently argued that Lo's re-scaling factor, $\overline{\sigma(q)}$, has a significant downward bias for small T which distorts $\overline{R/S}$. To overcome this problem they propose an unbiased re-scaling factor, $\overline{\overline{\sigma(q)}}$, that corrects for mean biases in r due to short-term dependencies without inducing the distortions in small samples that R/S and $\overline{R/S}$ do. Specifically,

$$\overline{\overline{\sigma(q)^2}} = \left[1 + \frac{2}{T^2}\sum_{j=1}^{q}\omega_j(q)(n-j)\right]\overline{\overline{s^2}} + \frac{2}{T}\sum_{j=1}^{T}\omega_j(q)\left[\sum_{t=j+1}^{T}(x_t - m)(x_{t-j} - m)\right],$$

where

$$\overline{\overline{s^2}} = \frac{1}{T-1}\sum_{t=1}^{T}(x_t - m)^2$$

is the standard unbiased estimator of the variance. Although a formal proof has not been given, one may speculate that, given the consistency of $\overline{\overline{\sigma(q)}}$ under the conditions given by Lo (1991), $T^{-\frac{1}{2}}$ times the Moody and Wu modified R/S statistic

$$\overline{\overline{R/S}} = \frac{r}{\overline{\overline{\sigma(q)}}}$$

also has the same limiting distribution as $T^{-\frac{1}{2}}$ times the previous two variants of the statistic. Moody and Wu point out that Lo's $\overline{R/S}$ gives different results depending on the choice of q, but their own variant of the statistic also depends on q and may well suffer from the same sensitivity to the choice of its value.

Finally, with regard to small sample properties of the R/S statistic, Anis and Lloyd (1976), using a theorem of Spitzer (1956), have derived the expectation

$$E[R/S] = \frac{\Gamma\left(\frac{1}{2}(T-1)\right)}{\sqrt{\pi}\,\Gamma\left(\frac{T}{2}\right)} \sum_{k=1}^{T-1} \sqrt{\frac{T-k}{k}},$$

where $\Gamma(.)$ is the gamma function, for the case of i.i.d. normal variables. Of course, the Feller result on the asymptotic expectation emerges as a special case of this as $T \to \infty$. Anis and Lloyd also show that their result applies to the case of symmetrically correlated normal variables, while Hipel and McLeod (1978), on the basis of Monte Carlo simulations, claim that it remains a good approximation for independently distributed variables of various non-normal forms. This result has been used to supplement the Hurst/Feller variant of E[R/S] for practical work with small samples.

III THE SMALL SAMPLE DISTRIBUTION OF R/S

Unfortunately, apart from the Anis and Lloyd (1976) result on the expectation, very little is known about the small sample properties of R/S. Therefore the main purpose of this paper is to provide some information about them. In particular, one of our concerns is to assess how good an approximation the asymptotic distribution of R/S is in the finite sample situation. Using Monte Carlo simulation methods, we conclude that for the cases examined, the asymptotic approximation is not good, even for samples as large as 500, and therefore that the use of asymptotic critical values, such as those provided by Lo (1991, Table II), may be misleading. Another concern is to explore ways of improving the description of the sampling distribution of R/S for small samples. Highly satisfactory results are obtained for the cases examined using a beta approximation based on the first four moments. We would suggest therefore that for practical purposes, our beta-approximate critical values are preferable to those based on the asymptotic distribution. Full details of the experiments and results are given in Section IV; and details of the construction of the table of critical values are given in Section V. The present section concludes with a brief description of the beta approximation methodology.

Let X be described by a beta distribution of the first kind, i.e. $x \in (0, 1)$ and $x \sim \beta(p, q)$, where $\beta(..)$ denotes the beta density function and p and q are its parameters. The first four moments of X are well known; and the first two of them are given explicitly and used below.[4] The idea in fitting theoretical forms to empirical distributions is to equate certain theoretical moments with their sample counterparts so as to be able to solve for the unknown parameters of the theoretical distribution. Although there are just two unknown beta parameters to be estimated here, the exercise is complicated by the need to re-scale R/S to the range (0, 1).

The minimum value of R/S can be shown to be unity (Mandelbrot, 1972), which means that the minimum of $(T^{-\frac{1}{2}})R/S$ which for convenience we choose to work with, is zero asymptotically, though referring to Figure 1, it can be seen that 0.5 constitutes an effective minimum for the limiting distribution of $(T^{-\frac{1}{2}})R/S$. Similarly, Mandelbrot gives the maximum value of R/S as $\frac{T}{2}$ and hence that of $(T^{-\frac{1}{2}})R/S$ is $\frac{\sqrt{T}}{2}$; but again from Figure 1 an effective maximum for the limiting distribution is, say, 3. Re-scaling of R/S could be done using its theoretical bounds, but use of "effective" or approximate bounds yields superior results. For specified finite sample sizes the "effective" maximum and minimum values for R/S might be determined directly from experimentally generated approximations to the sampling distribution, but we prefer to determine them as part of the beta fitting procedure as follows. It should be noted that although the methodology is explained in terms of R/S, it applies equally to $(T^{-\frac{1}{2}})R/S$. Let

$$x = \frac{R/S - L}{U - L},$$

where U is the approximate maximum of R/S and L is the approximate minimum. Taking expectations of both sides of this equation, treating x as a realisation from a beta distribution, and replacing E[R/S] by its sample estimator m(R/S), we have

$$\frac{p}{p+q} = \frac{m(R/S) - L}{U - L}.$$

4. The third and fourth moments of $\beta(p, q)$ are, respectively, $\sqrt{\beta_1} = \frac{2(q-p)\sqrt{p+1+1}}{(p+q+2)\sqrt{pq}}$ and $\beta_2 = \frac{6[(p+q+1)(q-p)^2 - (p+q+2)pq]}{pq(p+q+2)(p+q+3)} + 3$. For details of a beta fitting procedure using the first four moments similar to the procedure used here, see, for example, Harrison (1972). A referee has suggested an alternative means of solution for p, q, U and L and this will be explored in any future research.

By similar reasoning, we may write

$$\frac{\sqrt{pq}}{(p+q)\sqrt{p+q+1}} = \frac{s(R/S)}{U-L},$$

where s(R/S) denotes the standard deviation of the empirical sampling distribution of R/S for a given sample size. These last two equations can be solved for U and L, given values of p and q. The p and q values may be obtained in the usual manner by equating the theoretical skewness and kurtosis coefficients of the beta distribution to the computed skewness and kurtosis measures for the empirical sampling distribution, which process is independent of scale; see Harrison (1972) for further details. In this way we may fit a beta distribution with the same skewness and kurtosis as the experimentally determined finite sample sampling distribution, and scaled in accordance with the observed mean and standard deviation. Critical values for R/S corresponding to required significance levels may then be obtained from the beta tables using the relation

$$R/S^{\alpha} = L + (U-L)x^{\alpha},$$

where α is the significance level.

IV EXPERIMENTS AND RESULTS

In order to investigate the finite-sample distribution and properties of the R/S statistic under the null hypothesis of independence, the method of Monte Carlo simulation was used. Three forms of probability density function were used to generate sample data. First the standard normal distribution, an obvious choice in view of the fact that the only available exact small sample result relates to this case. Second, the uniform distribution on the interval (0, 1), another symmetric, but bounded distribution by contrast to the normal; and third, the asymmetric log-normal distribution which, although bounded below, can produce extreme positive values. Specifically, the log-normal variate was derived as ten raised to the power of a standard normal variable. In each case, sample sizes of 5 to 100, inclusive, in steps of 5, and 110 to 200, inclusive, in steps of 10, and also 225 to 500, inclusive, in steps of 25, i.e., 42 different sample sizes, were taken. The observations were produced using the random number generator of L'Ecuyer (1988) which combines two different random number sequences with different periods so as to obtain a new sequence whose period is the least common multiple of the two periods. This generator has a period which is approximately 2.3×10^{18}, more than adequate for the simulation work that was carried out. It is strongly recommended by Press *et al.* (1992), who provide full details, and it is not known to have failed any statistical tests of randomness. For each of the 126 combinations of population and sample size, the value of $(T^{-\frac{1}{2}})R/S$ was calculated, and the process replicated 100,000 times to yield an empirical approximation to the sampling distribution. Full details of the computer programs and machine used are described in Treacy (1997).

Selected summary results from this experiment, namely, the mean, standard deviation, skewness coefficient, kurtosis coefficient, minimum and maximum of the empirical sampling distributions for twelve selected sample sizes, are given in Tables 1, 2 and 3. In order to aid analysis of these

Table 1: *Estimated Moments of Sampling Distributions: Normal Population*

T	*Mean*	*Std. Dev.*	*Skewness*	*Kurtosis*	*Min.*	*Max.*
25	1.051	0.234	0.423	2.921	0.426	2.063
50	1.105	0.252	0.510	3.107	0.434	2.379
75	1.128	0.258	0.559	3.248	0.434	2.427
100	1.146	0.262	0.565	3.275	0.450	2.558
150	1.164	0.264	0.572	3.311	0.487	2.671
200	1.176	0.266	0.585	3.349	0.446	2.618
225	1.180	0.268	0.599	3.395	0.486	2.879
250	1.183	0.267	0.598	3.395	0.463	2.605
275	1.187	0.268	0.595	3.363	0.484	2.655
300	1.189	0.268	0.597	3.359	0.506	2.655
400	1.197	0.269	0.595	3.376	0.486	2.664
500	1.202	0.270	0.598	3.363	0.493	2.759

Table 2: *Estimated Moments of Sampling Distributions: Uniform Population*

T	*Mean*	*Std. Dev.*	*Skewness*	*Kurtosis*	*Min.*	*Max.*
25	1.065	0.245	0.463	3.004	0.354	2.186
50	1.115	0.258	0.548	3.215	0.412	2.674
75	1.140	0.262	0.579	3.321	0.420	2.789
100	1.157	0.265	0.591	3.348	0.443	2.637
150	1.173	0.268	0.605	3.343	0.474	2.621
200	1.183	0.268	0.598	3.384	0.477	2.697
225	1.188	0.269	0.591	3.366	0.474	2.699
250	1.189	0.268	0.601	3.393	0.486	2.916
275	1.192	0.270	0.611	3.413	0.447	2.746
300	1.194	0.270	0.598	3.368	0.481	2.727
400	1.204	0.270	0.593	3.372	0.507	2.658
500	1.208	0.271	0.618	3.406	0.491	2.748

Table 3: *Estimated Moments of Sampling Distributions: Log-normal Population*

T	*Mean*	*Std. Dev.*	*Skewness*	*Kurtosis*	*Min.*	*Max.*
25	1.016	0.157	0.836	4.239	0.507	2.044
50	1.044	0.172	0.808	4.009	0.511	2.173
75	1.057	0.180	0.806	3.960	0.533	2.089
100	1.066	0.184	0.792	3.910	0.526	2.236
150	1.076	0.191	0.793	3.870	0.555	2.231
200	1.082	0.195	0.799	3.867	0.525	2.324
225	1.084	0.196	0.787	3.842	0.501	2.364
250	1.087	0.198	0.774	3.808	0.535	2.319
275	1.089	0.200	0.781	3.818	0.522	2.288
300	1.092	0.201	0.779	3.792	0.495	2.254
400	1.096	0.203	0.752	3.725	0.534	2.373
500	1.100	0.207	0.772	3.774	0.554	2.498

tabulated results, Figures 2, 3, 4 and 5 are provided to show graphically how the estimated first four moments of the sampling distributions of $(T^{-\frac{1}{2}})R/S$ vary with sample size *T*. Figures 6, 7 and 8 show the actual form of the approximate sampling distributions for a sample size of 200 taken from the normal, uniform and log-normal distributions, respectively. The graphs

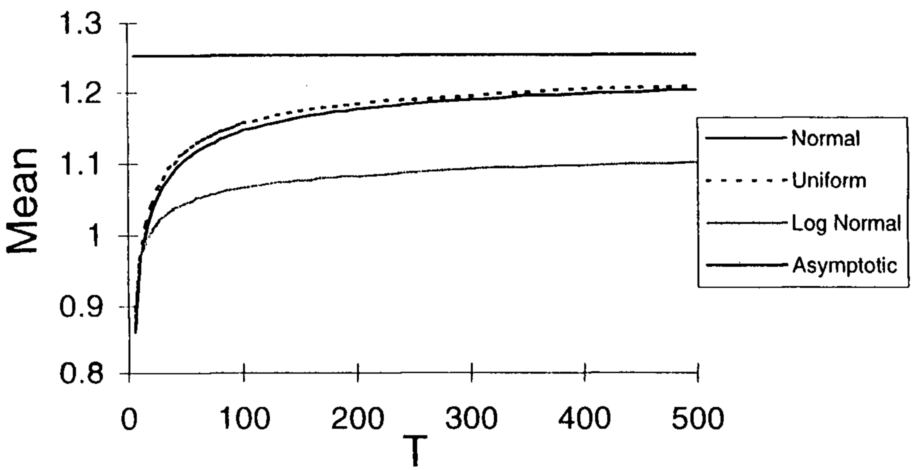

Figure 2: *Means of Empirical Sampling Distributions*

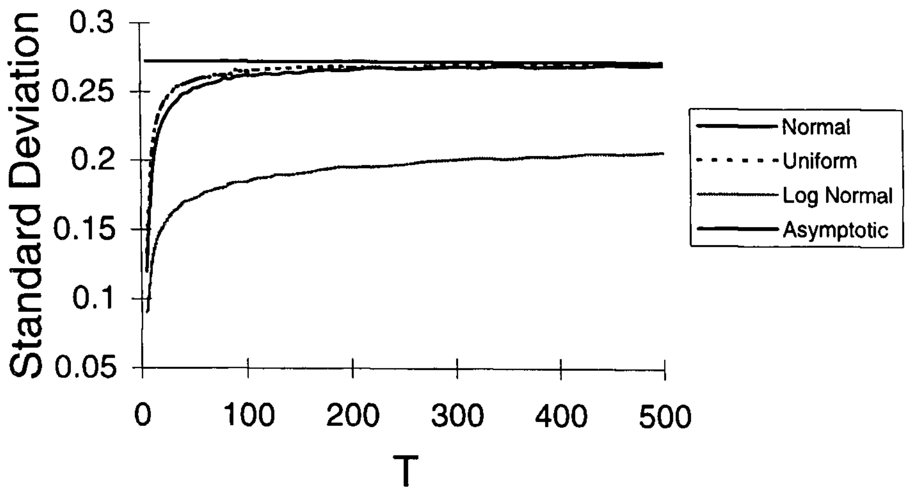

Figure 3: *Standard Deviations of Empirical Sampling Distributions*

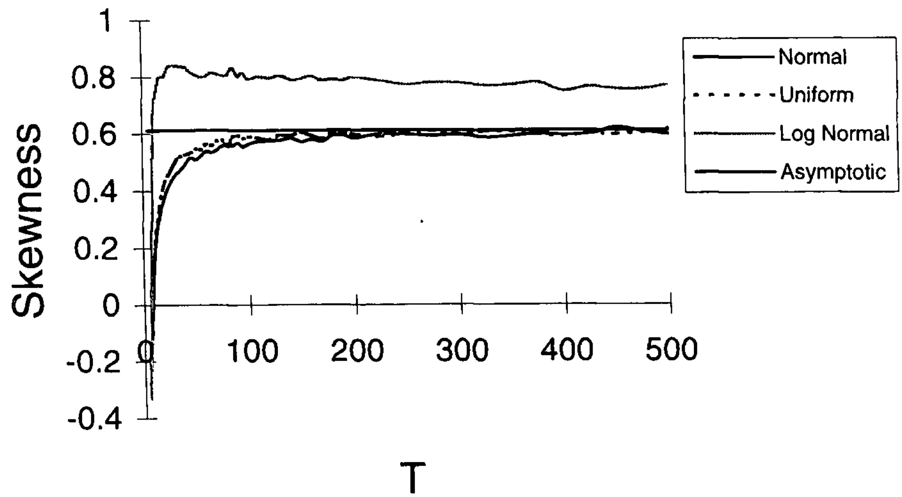

Figure 4: *Skewness Coefficients of Empirical Sampling Distributions*

plotted in all figures are based on the data for all 42 of the sample sizes investigated; and in Figures 2 to 5, inclusive, the plots indicate the corresponding asymptotic value of the moment by a horizontal solid line.

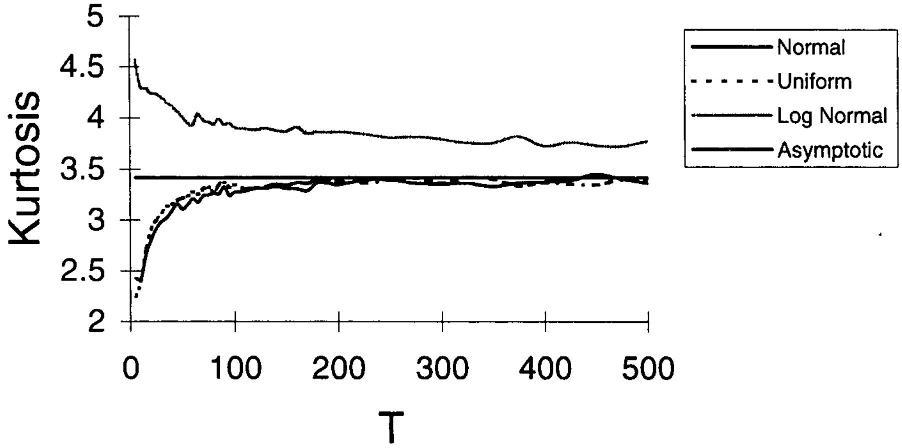

Figure 5: *Kurtosis Coefficients of Empirical Sampling Distributions*

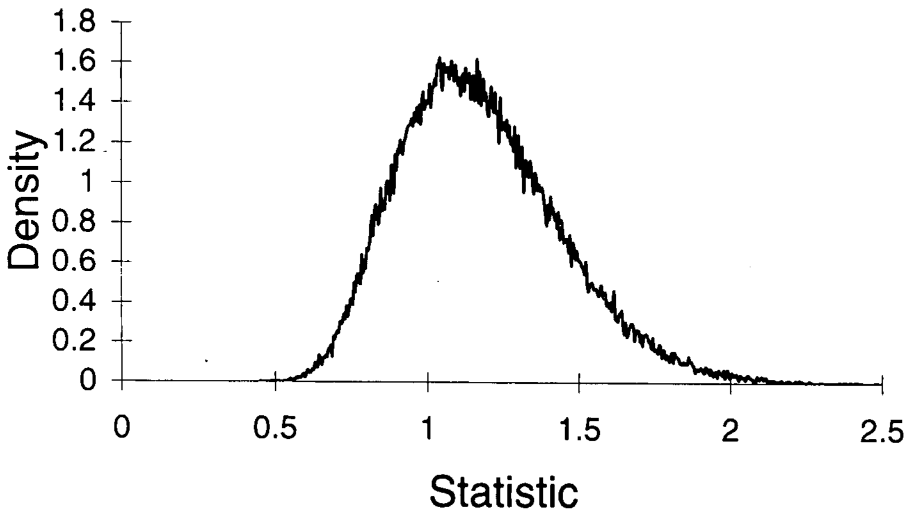

Figure 6: *Empirical Sampling Distribution: Normal Population, T=200*

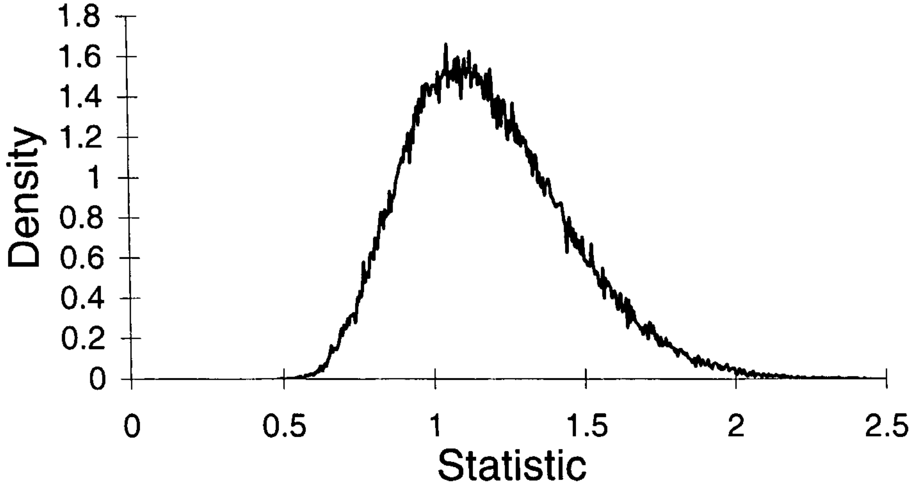

Figure 7: *Empirical Sampling Distribution: Uniform Population, T=200*

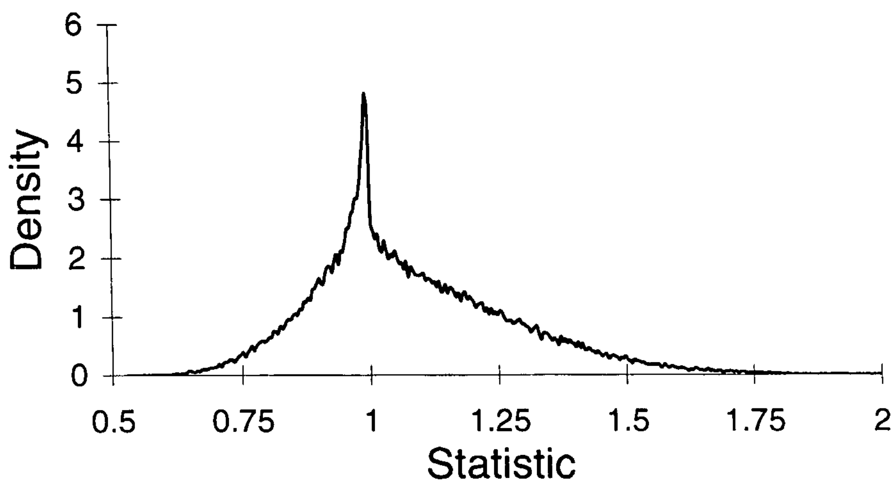

Figure 8: *Empirical Sampling Distribution: Log-normal Population, T=200*

Tables 1 to 3, and Figures 2 to 5 especially, reveal a very close agreement between the behaviour of the two sets of descriptive statistics relating to the samples drawn from the normal and uniform populations. The similarity of the means for these two populations accords with the findings of Hipel and McLeod (1978). The results also show that for the symmetric populations the rate of convergence to the asymptotic value is noticeably slower in the case of the mean than it is for the other three moments, whose asymptotic convergence rates are similar. For example, the second, third and fourth moments of the sampling distribution are certainly close to their asymptotic values for a sample of size 200, and yet the mean is still not close to its asymptote for a sample as big as 500.

By contrast, in the case of samples drawn from the highly skewed log-normal distribution, all four moments of the sampling distribution are different from both their asymptotic values and the corresponding values of the moments in the normal and uniform cases. The mean converges to its asymptotic value at an even slower rate for the log-normal population than it does for the symmetric populations; and unlike in the latter cases, the behaviour of the mean does not conform well to the exact expectation provided by the Anis and Lloyd (1976) result given above. We would suggest, therefore, that the claim that the mean of the R/S statistic is robust, as put forward by Mandelbrot and Wallis (1969c) and Hipel and McLeod (1978), for example, may be overstated. The standard deviation also converges to its asymptotic value at a markedly slower rate than it does in the cases of the symmetric populations. The skewness and kurtosis coefficients behave similarly which indicates that the sampling distribution remains more positively skewed and leptokurtic than it does in the symmetric population cases, and than what it is asymptotically, even for the largest of our sample sizes. Moreover, the decreasing mode of convergence of the third and fourth moments is in stark contrast to the tendency for the other moments to increase towards their asymptotic value as sample size increases.

The general features of slight positive skewness and leptokurtosis of the approximate sampling distributions associated with the normal and uniform populations, for nearly all of the sample sizes considered, echo the properties of the asymptotic distribution discussed in Section II. However, this similarity is not so apparent in the case of the empirical sampling distributions associated with the log-normal population. This may be seen by comparing Figures 6 to 8 with Figure 1. Although the results presented in Figures 6 to 8 are in each case representative of those for most other sample sizes, for small samples in which $T < 15$, the empirical distributions are quite irregular. The most striking feature of the distributions arising from the log-normal population compared with those from the symmetric populations, as illustrated in

Figure 8, is the heavy concentration of probability mass — i.e., the spike — at about the value unity. This phenomenon is the subject of on-going research, but an explanation is suggested by Treacy (1997).

V APPROXIMATIONS AND TABLES

Critical values for the R/S statistic are available to practitioners in the table provided by Lo (1991, p. 1288). These critical values were derived by Lo from the asymptotic distribution. The first task in this section is to use our experimentally derived finite-sample results to assess the quality of the asymptotic distribution as a small sample approximation. Thus, using the eleven asymptotic critical values from Lo's table reproduced in Table 4 to define twelve intervals, the asymptotic probability associated with each interval was compared to the relative frequency of occurrence of $(T^{-\frac{1}{2}})R/S$ values in the corresponding interval on the empirical sampling distribution.

Table 4: *Lo's Critical Values for V*

Pr(V<v)	*0.05*	*0.1*	*0.2*	*0.3*	*0.4*	*0.5*	*0.6*	*0.7*	*0.8*	*0.9*	*0.95*
v	0.861	0.927	1.018	1.090	1.157	1.223	1.294	1.374	1.473	1.620	1.747

This was done for the distributions arising from all three populations and all sample sizes. From these data, standard chi-squared goodness-of-fit statistics were computed, and these are given in Table 5 for selected sample sizes between 25 and 500. As can be seen from this table, the χ^2 values are

Table 5: χ^2 *Values of Goodness of Fit of V to the Empirical Sampling Distributions*

T	*Normal*	*Uniform*	*Log-normal*
25	8.2×10^5	7.3×10^5	1.3×10^5
50	4.0×10^5	3.6×10^5	8.7×10^4
75	2.8×10^5	2.3×10^5	7.2×10^4
100	2.0×10^5	1.6×10^5	6.3×10^4
150	1.3×10^5	1.1×10^5	5.3×10^4
200	9.8×10^4	8.2×10^4	4.9×10^4
225	9.0×10^4	6.8×10^4	4.7×10^4
250	7.7×10^4	6.5×10^4	4.4×10^4
275	7.0×10^4	6.1×10^4	4.3×10^4
300	6.7×10^4	5.7×10^4	4.1×10^4
400	4.8×10^4	3.9×10^4	3.8×10^4
500	4.2×10^4	3.3×10^4	3.5×10^4

extremely large and all are highly significant statistically. This may not be surprising given, especially, the disparity between the asymptotic mean and the behaviour of the mean of the empirical sampling distributions. This poor fit of the asymptotic distribution is illustrated in Figures 9 and 10.

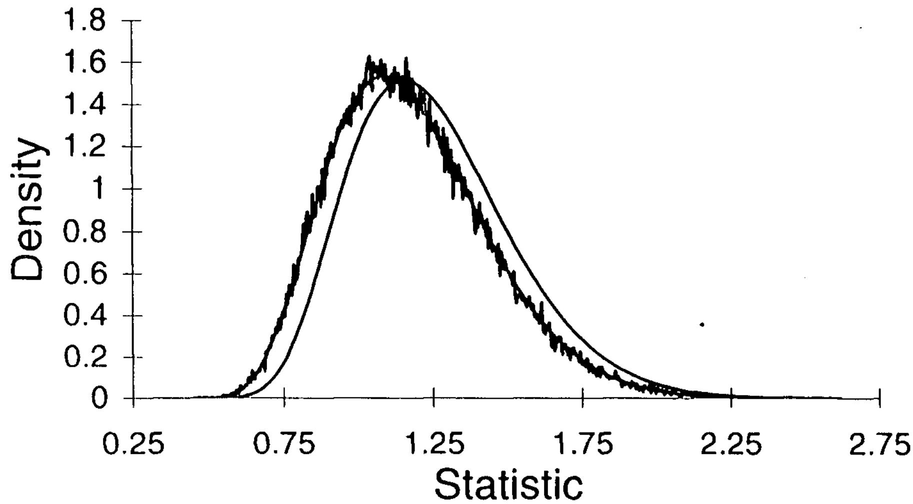

Figure 9: *Empirical Sampling Distribution, Asymptotic and Beta Approximations: Normal Population, T=200*

With a view to improving the approximation to the small-sample distributions, a beta distribution, which encompasses a wide variety of forms, was investigated as an alternative using the fitting methodology based on the first four moments outlined in Section III. However, as the unusual empirical distribution for the log-normal case, illustrated in Figure 8, can not be adequately captured by a beta distribution, the investigation focused solely on the symmetric distributions. From this point on, then, the paper confines itself to the case of R/S statistics calculated from observations drawn from symmetric distributions.

The fit of a beta distribution to the empirical distributions arising from the normal and uniform populations was assessed in the same manner as was the asymptotic distribution. The resulting χ^2 values that were obtained are presented in Tables 6 and 7. An impressive improvement in the quality of the fit is indicated. The extent of this is emphasised graphically in Figures 9 and 10, in each of which the beta distribution fits the sample data so well that it

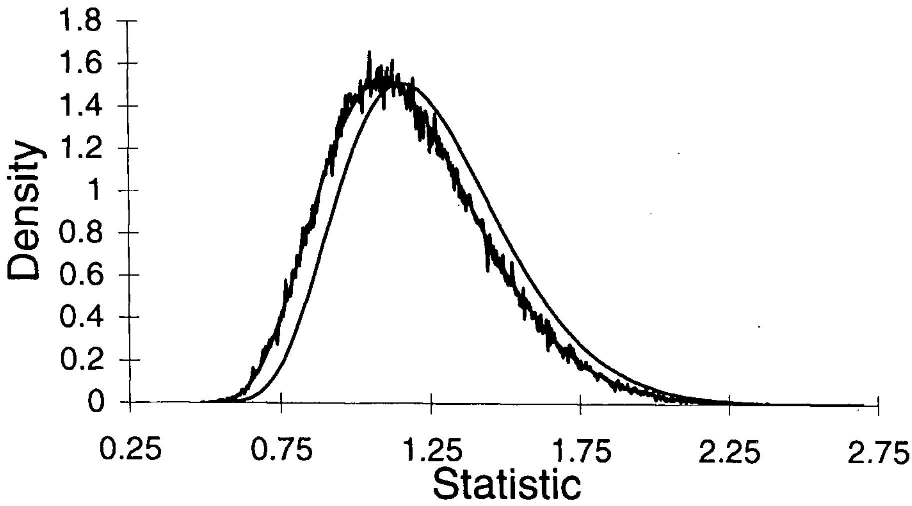

Figure 10: *Empirical Sampling Distribution, Asymptotic and Beta Approximations: Uniform Population, T=200*

Table 6: χ^2 *Goodness of Fit of Beta Distributions: Normal Population*

T	χ^2	p	q	L	U
25	15.7	4.43	10.60	0.46	2.50
50	35.1	4.72	14.88	0.46	3.14
75	97.6	5.24	21.23	0.46	3.85
100	14.3	5.42	23.33	0.46	4.11
150	19.6	5.78	27.46	0.46	4.53
200	43.4	5.89	30.54	0.46	4.88
225	47.2	6.15	36.44	0.45	5.49
250	65.8	6.20	36.95	0.46	5.52
275	5.0	5.74	30.14	0.48	4.91
300	44.5	5.57	28.43	0.49	4.78
400	43.1	5.96	32.98	0.48	5.20
500	22.7	5.60	28.97	0.49	4.87

is obscured by the plot of the data, while the asymptotic distribution lies well to the right. The implication of this, of course, is that the actual size of a one-tailed (right-sided) test, which is the natural procedure to employ to detect positive persistence, would be smaller, the probability of error type two larger and the power smaller, if the asymptotic distribution is used for inference.

Table 7: *Goodness of Fit of Beta Distributions: Uniform Population*

T	χ^2	p	q	L	U
25	8.8	4.56	12.32	0.43	2.77
50	31.9	5.10	19.36	0.45	3.66
75	61.7	5.67	27.18	0.44	4.48
100	98.5	5.65	28.50	0.46	4.69
150	39.7	5.06	23.76	0.50	4.35
200	22.5	5.97	33.74	0.46	5.25
225	21.8	5.96	32.40	0.46	5.11
250	19.8	6.04	35.05	0.47	5.38
275	15.3	5.94	35.60	0.47	5.50
300	28.9	5.71	30.17	0.48	4.96
400	84.2	5.99	32.98	0.48	5.21
500	12.4	5.53	30.93	0.51	5.13

Given the superiority of the beta approximation, it would seem worthwhile to derive an associated table of critical values for use in practical applications.

The first four moments of the statistic are very similar in terms of their behaviour over different sample sizes in the case of the normal and uniform populations (see Figures 2 to 5). Therefore the normal case was chosen as representative, and a mathematical function was used to describe the observed behaviour as a first step in the construction of a table of critical values. Of course, for the mean, the exact result of Anis and Lloyd (1976) is available which, as mentioned above, applies to the normal population, but which also conforms well to our empirical results for the uniform population. The function used for the second and higher moments may be referred to as the estimated moment function, and this was chosen to be an inverse tangent function of the form:

$$y_i = A_i \arctan(f_i T),\ i = 2,3,4,$$

where y_i is the ith estimated moment of $(T^{-\frac{1}{2}})R/S$, and A_i and f_i are parameters to be determined. Since the inverse tangent has an asymptotic value of $\frac{\pi}{2}$, the A_i parameters were determined by dividing the relevant asymptotic moment of the statistic by $\frac{\pi}{2}$. Thus, for example, $A_3 = 0.6132(\frac{\pi}{2})^{-1}$. The f_i values were determined by ordinary least squares regression of $\tan(\frac{y_i}{A_i})$ on T. Summary results from these regressions are given in Table 8. It can be seen from the R^2 and t statistics that excellent fits and highly significant results are obtained for the standard deviation and skewness coefficient. In the case of the kurtosis coefficient the fit is somewhat lower but the significance of the estimate of f_i remains very high.

Table 8: *Estimated Parameters of Moment Functions and Regression Statistics*

Moment	A_i	f_i	R^2	t
Std. Dev.	0.173	0.1680	0.98	86.3
Skewness	0.390	0.0838	0.92	34.9
Kurtosis	2.176	0.1700	0.74	17.4

Using the exact mean and the estimated moments corresponding to the sample sizes required, values of p, q, U and L were obtained as described earlier, and critical values from the associated beta approximate sampling distributions derived. The values used for the moments are given for a few selected sample sizes in Table 9, along with the parameters used in the final beta fitting. The beta approximate critical values for the statistic are presented in Table 10, together with the corresponding asymptotic values given in the last row for purposes of comparison. The first column of Table 10 refers to sample size, the first row, excluding the first entry, refers to the significance level, and the numbers in the body of the table are the critical values of the statistic for the associated sample size and probability level. Again, only selected sample sizes, and only standard probability levels, are included in Table 10; fuller information is available from the authors.

Table 9: *Exact Mean, Estimated Moments and Final Beta Parameters*

T	*Mean*	*Std Dev*	*Skewness*	*Kurtosis*	*p*	*q*	*L*	*U*
25	1.052	0.232	0.439	2.916	4.07	9.76	0.475	2.434
50	1.105	0.252	0.522	3.163	5.15	18.02	0.443	3.420
75	1.130	0.259	0.552	3.248	5.47	22.49	0.444	3.952
100	1.145	0.262	0.567	3.290	5.61	25.21	0.449	4.278
150	1.164	0.265	0.582	3.333	5.73	28.31	0.457	4.655
200	1.175	0.267	0.590	3.354	5.79	30.03	0.464	4.866
225	1.180	0.268	0.592	3.361	5.81	30.63	0.466	4.941
250	1.183	0.268	0.595	3.367	5.82	31.12	0.469	5.002
275	1.186	0.269	0.596	3.371	5.83	31.53	0.471	5.052
300	1.189	0.269	0.598	3.375	5.84	31.87	0.473	5.096
400	1.197	0.270	0.602	3.386	5.87	32.84	0.479	5.218
500	1.203	0.270	0.604	3.392	5.88	33.44	0.483	5.294

As a final check, the quality of the beta distributions underlying the table of critical values as approximations to the simulated sampling distributions was examined, by means of goodness-of-fit statistics and plots. The numerical results are given in Table 11, and a representative plot based on the uniform population is given in Figure 11. The superiority of the final beta approximation compared to the asymptotic approximation is clear.

Table 10: *Beta Approximate Critical Values of* $T^{-\frac{1}{2}}R\,/\,S$

T	*0.01*	*0.025*	*0.05*	*0.1*	*0.9*	*0.95*	*0.975*	*0.99*
20	0.617	0.657	0.698	0.754	1.334	1.427	1.505	1.592
25	0.619	0.662	0.706	0.764	1.366	1.465	1.551	1.648
30	0.622	0.668	0.713	0.773	1.390	1.493	1.584	1.688
35	0.626	0.673	0.720	0.781	1.408	1.515	1.609	1.718
40	0.630	0.678	0.725	0.787	1.422	1.532	1.628	1.741
45	0.634	0.682	0.730	0.793	1.434	1.546	1.644	1.760
50	0.638	0.687	0.735	0.798	1.444	1.557	1.658	1.776
60	0.644	0.694	0.743	0.806	1.461	1.576	1.679	1.800
70	0.650	0.700	0.749	0.813	1.473	1.590	1.694	1.818
80	0.655	0.705	0.755	0.819	1.483	1.601	1.707	1.833
90	0.659	0.710	0.759	0.824	1.491	1.610	1.717	1.845
100	0.663	0.714	0.764	0.828	1.498	1.618	1.726	1.854
120	0.669	0.720	0.770	0.835	1.509	1.630	1.739	1.870
140	0.674	0.726	0.776	0.841	1.517	1.640	1.749	1.881
160	0.679	0.730	0.781	0.846	1.524	1.647	1.758	1.890
180	0.682	0.734	0.784	0.850	1.530	1.653	1.764	1.898
200	0.686	0.737	0.788	0.853	1.534	1.658	1.770	1.904
225	0.689	0.741	0.791	0.857	1.539	1.664	1.776	1.910
250	0.692	0.744	0.794	0.860	1.544	1.668	1.780	1.915
275	0.695	0.747	0.797	0.863	1.547	1.672	1.785	1.920
300	0.697	0.749	0.800	0.865	1.550	1.675	1.788	1.924
350	0.701	0.753	0.804	0.869	1.555	1.681	1.794	1.930
400	0.704	0.756	0.807	0.873	1.560	1.685	1.799	1.935
450	0.707	0.759	0.810	0.876	1.563	1.689	1.803	1.940
500	0.709	0.761	0.812	0.878	1.566	1.692	1.806	1.943
∞	0.755	0.809	0.861	0.927	1.620	1.747	1.862	2.001

Table 11: *Goodness of Fit Statistics for Final Beta Distributions: Symmetric Populations*

T	*Normal*	*Uniform*
25	28.8	213.6
50	36.4	97.8
75	92.9	109.0
100	10.6	160.7
150	24.6	121.3
200	48.4	82.1
225	43.3	137.2
250	70.7	76.7
275	10.6	64.2
300	46.2	85.1
400	44.1	142.1
500	23.9	47.3

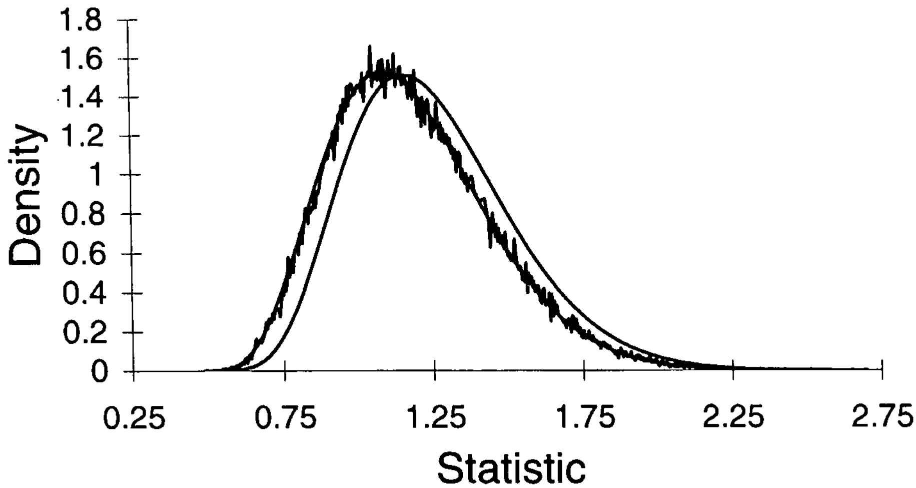

Figure 11: *Empirical Sampling Distribution, Asymptotic and Final Beta Approximations: Uniform Population, T=200*

VI SUMMARY AND CONCLUSION

This paper has discussed the Hurst R/S statistic and its properties, and has indicated some of its uses in economics. The adequacy of the asymptotic distribution as an approximation to the sampling distribution of the statistic in small samples has been examined and an alternative beta approximation suggested and assessed. A table of beta approximate critical values has also been provided as an alternative to those of Lo (1991). The main conclusions are as follows:

First, the asymptotic distribution is not a good approximation in the small sample case, and even for samples as large as 500, whether observations are taken from symmetric or asymmetric populations. In particular, the asymptotic mean appears to differ markedly from the actual mean of the small sample distribution, even in the case of samples from symmetric populations where the other moments converge to their asymptotic values reasonably quickly. The implication of this general finding is serious, namely, that if asymptotic critical values are used for practical testing purposes, there is a sizeable probability that inferences will be misleading.

Second, our results cast doubt on the widely held view that the R/S statistic is robust to the form of population from which observations are taken. In particular, heavily skewed populations, such as the log-normal, appear to give rise to very different behaviour of the moments of the sampling distribution as sample size varies, as well as to the functional form of the sampling distribution, which would not be as well approximated by a beta distribution. This matter is the subject of continuing research.

Third, our suggested four-moment beta approximation is significantly better than the asymptotic distribution as an approximation to the finite-sample sampling distribution of the R/S statistic, at least for situations in which the parent population is symmetric. This being the case, we would suggest that our new table of critical values offers a rather more reliable basis for R/S inference in the case of symmetric populations than the table of asymptotic critical values currently available in the literature.

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