

Binomial Option Pricing and the Conditions for Early Exercise: An Example using Foreign Exchange Options

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Abstract: In this paper are derived simple and general conditions under which the value of an American option will exceed that of its European counterpart. These conditions are developed using the binomial option pricing framework. The derivation is motivated and illustrated by the example of foreign exchange options.

I INTRODUCTION

Foreign exchange (FX) or currency options were first traded in Philadelphia in 1982 and since then have become common on all the world's major options exchanges. In the details of their operation, FX options are similar to all other options, with calls giving the right to buy the currency at the agreed price, while puts confer the right to sell.

FX options have previously been discussed in the Irish context in a paper by Bradley and Walsh (1986). They illustrate the way in which FX options can be used as hedging instruments. However, as they note, an analytic solution to the question of the value of such options (by Garman and Kohlhagen, 1983) relates to a European option. Such an option may only be exercised on its maturity date. While there is a very large over the counter market in European FX options, most FX options traded on the world's options exchanges are of the American variety, which permit exercise of the option at any time between the writing of the option and the maturity date. As a result the

Garman-Kohlhagen pricing formula for a European FX option provides a lower bound on the possible values of an American FX option.

In other words the possibility of "early exercise" may make the American FX option more valuable than the European. In this paper we address the question: "under what circumstances will an American FX option be more valuable than its European counterpart?" This is equivalent to asking under what circumstances early exercise of the option will be worthwhile. More generally we provide a method by which this question can be answered in respect of American options written on any asset. The original contribution which this paper makes lies in deriving these conditions using the framework of the binomial option pricing method. These conditions are, in consequence, stated in terms of the parameters of this method. They thus have the merit of being both simple and widely applicable.

II EARLY EXERCISE

It is a straightforward matter to prove that, if an American option is written on a stock which pays no dividends, then the holder of an American call option would never wish to exercise it before its maturity date: hence, in this case the Black-Scholes (1973) formula provides the correct valuation (the proof was originally given by Merton, 1973). However, this is not true of American put options on non-dividend paying stock. Here early exercise may be profitable, and thus the Black-Scholes formula will understate the true value of the put.

Always, of course, the forward value of a non-dividend paying stock will be at a premium to its spot value.¹ However, it is possible to show that if the forward value of such a stock were at a discount to its present value, it would not pay to exercise a put early but it would pay to exercise a call early. In the case of foreign currencies, however, the forward exchange rate is given by the interest rate parity theorem, and can therefore be at a premium or a discount to the spot rate, depending whether the domestic or foreign risk free interest rate is higher. Thus, for foreign exchange options, some calls (where the forward rate is at a discount) and some puts (where the forward rate is at a premium) may be exercised early. However, there are circumstances in which a call on a currency which has a forward rate at a premium may also be exercised early. To clarify the exact circumstances under which FX calls and puts will be exercised early, we turn first to the binomial option pricing model.

1. This is not true of stocks which pay dividends, and here both calls and puts may be exercised early.

III THE BINOMIAL OPTION PRICING MODEL

In the original Black-Scholes (1973) option pricing model, the stock price was assumed to follow a lognormal diffusion process in continuous time. In the binomial option pricing model, introduced by Cox, Ross and Rubinstein (1979) and Rendleman and Bartter (1979) the stock price is modelled as following a multiplicative binomial process in discrete time. Here the time between the writing of the option and its maturity date is divided into N discrete periods, during each of which the price of the underlying asset is assumed to make a single move, either up or down. The magnitude of these movements is given by the multiplicative parameters u and d . Hence at the end of the first period the share price will have moved from S to either uS or dS . The present value of the share price at the end of the first period can be obtained by discounting either of these values by the one period risk free rate, which we denote by q . u , d and q are obtained from the parameters of the continuous time model as

$$\begin{aligned} u &= \exp \{ \sigma(T/N)^{1/2} \} \\ d &= 1/u \\ q &= r^{T/N} \end{aligned}$$

where σ is the annualised standard deviation of the share (or other asset) price, T is the time to maturity of the option, and r is the annualised riskless rate of interest. The derivations of u , d , and q are given by Cox, Ross and Rubinstein (1979, pp. 247-249). N must be chosen to ensure that $d < q < u$.

The only remaining parameter we require is the probability of an upward or downward movement (p and $1-p$ respectively).² In the case of a share, the Black-Scholes model sets the drift of the underlying Wiener process equal to the risk free rate, r . Accordingly, the value of p must be such that, at the end of one period

$$p \cdot uS + (1-p) \cdot dS = qS \quad (1)$$

implying that

$$p = (q-d)/(u-d). \quad (2)$$

In the case of a foreign exchange option, however, the drift of the process and the discount rate are not identical. For a foreign exchange option the one period discount rate is again

$$q = r^{T/N}$$

2. By which we mean, of course, a probability within the framework of the binomial model. This is only a real world probability if investors are risk neutral.

whereas

$$p = (i - d)/(u - d) \quad (3a)$$

where

$$i = (r/r^*)^{T/N} \quad (3b)$$

where r^* is the foreign riskless rate.

Continuing with the simpler example of an equity option we see that, for a one period binomial model the value of a European call option will be

$$\frac{p}{q} (\max [0, uS - X]) + \frac{1-p}{q} (\max [0, dS - X])$$

where X is the exercise price of the option.

For a two period model the value is

$$\begin{aligned} & (p/q)^2 (\max [0, u^2 S - X]) + 2(1-p)(p/q^2) (\max [0, S - X]) \\ & + ((1-p)/q)^2 (\max [0, d^2 S - X]) \end{aligned}$$

This process generalises to the following equations for the value of a European call and put respectively:

$$c = \sum_{j=a}^N \binom{N}{j} p^j (1-p)^{N-j} [u^j d^{N-j} S - X] e^{-rT} \quad (4a)$$

$$p = \sum_{j=0}^a \binom{N}{j} p^j (1-p)^{N-j} [X - u^j d^{N-j} S] e^{-rT} \quad (4b)$$

where a is the minimum (maximum) number of upward stock movements such that the call (put) finishes in the money. These equations give surprisingly accurate results (when compared with the Black-Scholes formula) for even small values of N , and converge to the Black-Scholes value for $N > 100$.

In the case of an American option early exercise will be a possibility. Accordingly, at any point in time, t , the option's value will be the larger of (a) its immediate exercise value ($S(t) - X$ for a call); and (b) its value if held beyond t , (i.e., not exercised at t). In the binomial model the exercise value at the j th node in the binomial "tree" at the end of the n th period (we label this B_{jn}) is given by (in the case of a call)

$$B_{jn} = u^j d^{n-j} S - X \quad (5a)$$

while the holding value (A_{jn}) is equal to the weighted sum of the discounted

values of the option at the two points or nodes to which it can move in the following period:

$$A_{jn} = \frac{p}{q} (\max [0, A_{j+1, n+1}, B_{j+1, n+1}]) + \frac{(1-p)}{q} (\max [0, A_{j, n+1}, B_{j, n+1}]) \quad (5b)$$

Thus an option will only be exercised early if

$$V_{jn} = B_{jn} > A_{jn} \quad (6)$$

where V_{jn} is the value of the option at the j th node of the process, given by

$$V_{jn} = \max [0, A_{jn}, B_{jn}] \quad (7)$$

We can write the necessary and sufficient condition for early exercise as

$$B_{jn} > (p/q) (\max [0, B_{j+1, n+1}]) + ((1-p)/q) (\max [0, B_{j, n+1}]) \quad (8)$$

for some j th node in the process. That this is a necessary condition is clear from (7), in so far as the value of the option at the j th node will always be at least as great as the RHS of (8). That it is a sufficient condition follows from the fact that $A_{jN} = 0$ for $j = 0 \dots N$ (that is, an option has zero holding value at the end of the N th period) and thus

$$V_{jN} = \max (0, B_{jN}) \text{ for } j = 0 \dots N.$$

In fact, this condition can be simplified. For (8) to hold, B_{jn} must be greater than zero which implies that, in the case of a call option, $B_{j+1, n+1}$ is also greater than zero. We can therefore rewrite (8) as

$$B_{jn} > (p/q) (B_{j+1, n+1}) + ((1-p)/q) (\max [0, B_{j, n+1}]) \quad (9a)$$

In the case of a put (and redefining B_{jn} accordingly), B_{jn} greater than zero implies that $B_{j, n+1}$ is also greater than zero, allowing us to rewrite (8) as

$$B_{jn} > (p/q) (\max [0, B_{j+1, n+1}]) + ((1-p)/q) (B_{j, n+1}). \quad (9b)$$

Further, providing that, in the case of a call, there exists within the binomial process $B_{j, n+1}$ greater than zero (and we can always ensure this by setting

N sufficiently large³), we can write the necessary and sufficient condition for early exercise of an American call as

$$S^* - X > (p/q)(uS^* - X) + ((1-p)/q)(dS^* - X) \quad (10a)$$

where S^* is any value of S such that $dS^* - X > 0$.

For a put, given a $B_{j+1, n+1}$ greater than zero we can write the condition as

$$X - S^* > (p/q)(X - uS^*) + ((1-p)/q)(X - dS^*) \quad (10b)$$

where S^* is some value of S such that $X - uS^* > 0$.

These conditions then reduce to, respectively,

$$X < X/q \quad (11a)$$

and

$$X > X/q \quad (11b)$$

In passing we note that (11a) demonstrates that a call on a non-dividend paying stock will never be exercised early given positive interest rates.

Applying these arguments to American FX options we find the following conditions under which early exercise will have some value.

1. *Calls*: (10a) yields the condition (via 3): a call option on a foreign currency will have early exercise value iff

$$\exists S^* \equiv u^j d^{n-j} S(0) \text{ for some pair } \langle j, n \rangle \text{ where } 0 \leq j \leq n, 0 \leq n \leq N$$

such that

$$S^* - X > (iS^* - X)/q \quad (12)$$

where S is the exchange rate.

This condition will obviously hold if the forward rate is at a discount since then $i < 1 < q$. But early exercise will also be feasible if the forward rate is at a premium but i is sufficiently small and q sufficiently large – in other words, if the forward premium is small and the domestic (and thus the foreign) risk free rates are relatively high.

3. In the case of call this means choosing N such that $X < u^{N-1} d^1 S$.

2. *Puts*: (10b) yields the condition: a put option on a foreign currency will have early exercise value iff

$$\exists S^* \equiv u^j d^{n-j} S(0) \text{ for some pair } \langle j, n \rangle \text{ where } 0 \leq j \leq n, 0 \leq n \leq N$$

such that

$$X - S^* > (X - iS^*)/q. \tag{13}$$

If $i < 1 < q$ (i.e., a forward discount) the option will never be exercised early unless both i and q are large. This will occur when the risk free rates are high but differ little in value. If there is a forward premium then the option will always have early exercise value.

Conversely, American calls will have no price premium over European values in cases where the forward rate is at a large premium relative to the domestic risk free rate. Likewise, the American FX put should not be priced above the European put in cases where, for example, the forward rate is at a deep discount relative to the domestic risk free rate.

For certain combinations of parameters, S^* as defined in conditions (12) and (13) will not exist (because only a negative S^* could ensure the inequality). On the other hand, any positive value of S could be generated within the binomial providing that the number of periods, N , in the binomial model were set large enough. Clearly, however, if condition (12) or (13) can only be met by setting N very large then the early exercise value will be negligible. This introduces an important distinction between, on the one hand, options on non-dividend paying stock and, on the other, commodity options broadly defined to include foreign exchange options. Whereas in the former we can say with certainty that all calls will have zero early exercise value, in the latter there are cases in which we can only say that the early exercise value of a particular option is so small as to be worthless. As we might expect, the continuous time approach to option pricing leads to exactly the same result. Let T^- be the instant before the option's maturity date ($T^- = T - dt$): then, defining i' to be the ratio of the instantaneous domestic and foreign rates and q' to be the instantaneous domestic risk free rate, a put will have early exercise value iff

$$\text{prob} \left\{ (S(T^-)) < X \frac{(1-1/q')}{(1-i'/q')} \right\} > 0.$$

Again, all FX put options will meet this condition provided that it does not imply a negative value for $S(T^-)$ but for many of them the probability will be too small to be worth considering.

Armed with these results we can now see that a "blanket rule" such as that

proposed by Bradley and Walsh (1986) for converting European into American FX option values, will be misleading. Bradley and Walsh (1986) attempt to arrive at an approximate value for an American FX call option in terms of a premium over the European valuation.

As a starting point, however, it is suggested that a premium of 15-20% should be added to the Garman and Kohlhagen valuation (Bradley and Walsh, 1986, p. 96).

While this premium level may have been reasonably accurate for the particular sample data on which Bradley and Walsh performed their analysis, as a general procedure for valuing American FX call options, this will give inaccurate results. In particular, in the situations in which we have identified the American option as carrying no value for early exercise, their suggestion will severely over-price the option.

The conditions for early exercise derived above can be extended to any American option. The binomial model can be used to value options written on a variety of financial instruments. This can be accomplished by setting

$$p = (i - d)/(u - d)$$

where i is now defined to be the one period "cost of carry" of the underlying asset. Clearly the one period cost of carry of a non-dividend paying stock is q : for a foreign currency option it is as given in (3b). For an option on a futures contract, $i=1$ and S , in this case, is the futures price. For a commodity, i will be the one period physical cost of carry associated with holding a long spot position (see Barone-Adesi and Whaley (1987) for a discussion of the cost of carry issue). Consequently, Equations (12) and (13) provide general conditions for early exercise of American options within the binomial option pricing model.

IV PRICING AMERICAN FX OPTIONS USING THE BINOMIAL METHOD

As an alternative to *ad hoc* adjustments to the Garman Kohlhagen model, accurate pricing of American FX options can be carried out using the binomial model.⁴ Cox and Rubinstein (1985) provide a clear exposition of the use of the binomial to price share options. To apply the method to FX options it is

4. The binomial method is widely used to value options on FX, stocks, futures and so on, but it is by no means the only method used (see, for example, Barone-Adesi and Whaley, 1987; Brennan and Schwartz, 1977; Geske and Johnson, 1984; Parkinson, 1977; Shastri and Tandon, 1987).

only necessary to replace (2) and (3a) as discussed earlier. In Table 1 we present the values of a sample of nine month American FX put options (estimated using the above method and $N=50$), together with their European values (using both the Garman and Kohlhagen formula and the binomial with $N=50$). We also show the implicit forward rate and the domestic risk free rate and the percentage difference between the American and European values. The exchange rate (and also the option's exercise price) is set at 1.2 domestic units per foreign unit: thus we can think of this as being a likely scenario for Irish pound/pound sterling options, with the option price quoted in Irish pence per pound sterling (although it should be noted that this example uses a very high hypothetical exchange rate volatility).

Table 1: *Option Values for a Sample of Foreign Exchange Put Options*

<i>Forward Rate</i>	<i>Domestic Risk Free Rate</i>	<i>Option Value</i> (pence per £ sterling)			<i>Premium for American over European option</i> (<i>b-a</i> as % of <i>a</i>)
		<i>Garman Kohlhagen</i>	<i>European Binomial</i> (<i>a</i>)	<i>American Binomial</i> (<i>b</i>)	
<i>(a) Forward discount with low risk free rates</i>					
1.170	.05	20.8	20.7	20.7	0.0
1.176	.05	20.6	20.5	20.5	0.0
1.182	.05	20.4	20.3	20.3	0.0
1.188	.05	20.2	20.1	20.2	0.0
<i>(b) Forward premium</i>					
1.224	.05	19.1	19.0	19.3	1.6
1.218	.05	19.3	19.2	19.5	1.6
1.212	.05	19.5	19.4	19.6	1.0
1.206	.05	19.7	19.6	19.8	1.0
<i>(c) Forward discount with high risk free rates</i>					
1.165	.15	19.5	19.3	19.7	2.1
1.170	.15	19.2	19.1	19.5	2.1
1.176	.15	19.0	18.9	19.4	2.6
1.182	.15	18.9	18.8	19.3	2.7

Assumed annual standard deviation of exchange rate = .5.

A comparison between the Garman and Kohlhagen values and the binomial European put values shows that, even with only 50 periods, the binomial model is accurate to within one-tenth of a penny.

Note that the American option is always more valuable when the forward rate is at a premium (greater than 1.2), has no premium over the European value if the risk free rates are small and the forward discount is large, but regains a premium in cases where the forward discount occurs together with high risk free rates (as in the last few examples in the table).

The variability of the American premium over the European is clear from the final column of the table, where the percentage difference ranges from zero to just over 2 per cent in these particular examples.

V CONCLUSION

We have demonstrated the circumstances under which the price of American FX puts and calls will exceed that for their European equivalents. Because it will not always be profitable to exercise an American FX option early, such options will not always be worth more than their European counterparts. As a result, any rule based on increasing the European price by a certain percentage in order to approximate the American option price will certainly give misleading results in many situations.

In fact, there is no need for the use of any such *ad hoc* methods, in so far as it is a straightforward matter to value accurately all American FX options by an adaptation of the binomial method used for equity options. In this paper we have demonstrated how the same binomial framework can be used to determine the conditions under which an American option written on any asset will carry a premium for its early exercise.

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