Economic and Social Review, Vol. 11, No. 2, January, 1980, pp. 113–125.

Notes on the Behaviour of Prices

J. E. WOODS*
Trinity College, Dublin

Précis: We discuss the behaviour of prices in an n-sector, circulating capital model with no joint production, of the type considered by Sraffa in Part I of his book. Instead of following Sraffa's approach, which uses the notion of proportions of labour to means of production in the various "layers" of means of production, we base our analysis directly on the characteristic roots and vectors of the input-output matrix, A. A straightforward corollary of our main result applies to the case of uniform organic composition of capital, familiar in aggregation theory. We use two mathematical results – the Cauchy determinant and the Perron-Frobenius theorem on semi-positive indecomposable matrices.

1. In these notes, we shall discuss the behaviour of prices in an n-sector, circulating capital model with no joint production of the type considered by Sraffa (1963) in Part I of his book. Sraffa's analysis, it will be recalled, is conducted within the framework of a stationary state – "No changes in output and . . . no changes in the proportions in which different means of production are used by an industry are considered, so that no question arises as to the variation or constancy of returns. The investigation is concerned exclusively with such properties of an economic system as do not depend on changes in the scale of production" (p.v). Sraffa can thus concentrate attention on the relations between prices, wages, profits and the conditions of production, which Garegnani (1970, p. 427) later described as "the proper object of value theory" to the relative exclusion of the "circumstances governing changes in the outputs of commodities" (ibid, p. 428). Sraffa exhibits the dependence of prices and the value of capital on income dis-

*I am grateful to the referee for his very helpful comments on earlier versions on this paper.

tribution – "The reversals in the direction of movement of relative prices, in the face of unchanged methods of production, cannot be reconciled with *any* notion of capital as a measurable quantity independent of distribution and prices" (ibid, p. 38) – and solves Ricardo's (1951, p. 43) problem of an "invariable standard measure of value which should itself be subject to none of the fluctuations to which the other commodities are exposed".

Certain properties of Sraffa's systems are by now well known and are briefly discussed below. However, the main purpose of the paper is to analyse the effect of changes in the rate of profit on relative prices. Such analysis has usually been performed by examining a particular matrix inverse. However, the analysis can be clarified by the simple device of operating in terms of the characteristic roots and vectors of a matrix rather than of an associated matrix inverse. The main result of this paper is presented in Section 2.4.

2.1 We begin with a brief discussion of the quantity equations of the Sraffa system. Assuming that the gross outputs of the various commodities are exchanged at the end of the common production period (say, a year) with a physical surplus of at least one commodity (over replacement needs), we have

$$x_i = \sum_{j=1}^{n} x_{ij} + f_i \qquad i = 1, \ldots, n \tag{1}$$

where x_i denotes the gross output of commodity i obtained at the end of this year, x_{ij} the quantity of commodity i advanced as means of production to industry j for use next year and f_i ($\geqslant 0$) the surplus over inter-industry requirements. Under the stationary state assumption, the gross outputs of commodity i at the end of this year and of next year are identical. Hence, we may define

$$a_{ij} = x_{ij}/x_j \tag{2}$$

without in any way introducing a constant returns to scale assumption[1]. Equation (1) becomes

$$x_i = \sum_{j=1}^{n} a_{ij}x_j + f_i \qquad i = 1, \ldots, n \tag{3}$$

or, in matrix notation,

$$x = Ax + f. \tag{4}$$

Intuitively, a "productive" economy is one which can yield a surplus over

1. The question of returns to scale in Sraffa's analysis has been discussed recently in *The Journal of Economic Literature*; the March 1977 number contains papers by Burmeister, Levine and Eatwell, who provide an extensive bibliography.

and above inter-industry requirements (i.e., each component of f is non-negative, and f itself is non-zero and semi-positive, written $f \geqslant \underline{0}$). We may enquire into the conditions for a productive economy – that is, the conditions under which a semi-positive vector, f, is associated with a semi-positive vector, x. If each commodity is required either directly or indirectly in the production of every commodity, the economy and the matrix A are said to be indecomposable. Clearly, in an indecomposable economy, the gross output of each commodity is positive if there is a surplus (or net output) of at least one commodity; hence if $f \geqslant \underline{0}$, $x > \underline{0}$. There are a number of necessary and sufficient conditions for a productive economy and a productive matrix (reported, for example, in Pasinetti, 1977, or Woods, 1978) of which we shall use the following: the economy and the matrix A are productive if, and only if, the Frobenius characteristic root of A is less than one.

This brief discussion of the quantity equations enables us to introduce the fundamental productivity condition.

2.2 We now turn our attention to the price equations. Sraffa's prices are not scarcity prices, determined by an equilibrium of supply and demand; they do not, in Sraffa's words, "depend as much on the demand side as on the supply side" (ibid, p. 9). Sraffa's analysis "contains no reference to market prices" (ibid, p. 9). Rather, it is concerned with what we might now call a long-run equilibrium, where the prices of commodities imply a uniform rate of profit on the supply prices of capital goods. Sraffa describes "the ratios which satisfy the conditions of production" (ibid, p. 8) as "values" or "prices" for the sake of brevity, but states that "Such classical terms as 'necessary prices', 'natural prices' or 'prices of production' " (ibid, p. 9) would be acceptable. The distinction being drawn by Sraffa between "natural" and "market" prices is one of long standing, having been made by Adam Smith *inter alios.*

The assumption of a uniform rate of profit, which can be justified by an appeal to the working of competitive forces, is not at all an extreme one. Indeed, reasons must be adduced for non-uniformity of the rate of profit in the long run. Of course, a short-run equilibrium, of the type analysed by Debreu (1959), for example, in which (market) prices are determined by an equilibrium of supply and demand, need not imply a uniform rate of profit. However, little justification is provided for the study of such short-run or temporary equilibria, characterised as they must be to some extent by transitory phenomena. Attempts to "generalise" such an analysis to the long run, primarily by the introduction of forward markets, are doomed to failure simply because the rentals (obtained like all other prices from a demand and supply equilibrium) will in general be incompatible with a uniform rate of profit on the supply prices of the capital goods.

Sraffa's prices, satisfying the conditions of production, can be described by the equation

$$p' = w\ell' + (1 + r)p'A, \qquad (5)$$

A being the matrix already encountered in (4), ℓ the vector of direct labour input coefficients, p the price vector, r the rate of profit and w the wage rate.

Comparing a subsistence economy with one which produces a surplus, Sraffa observes that, whereas in the former "all commodities ranked equally, each of them being found both among the products and the means of production . . . and each played a part in the determination of prices", in the latter "there is room for a new class of products which are not used, whether as instruments of production or as articles of subsistence, in the production of others. These products have no part in the determination of the system. Their role is purely passive" (ibid, p. 10). Hence, Sraffa distinguishes between basic and non-basic commodities: "The criterion is whether a commodity enters (no matter whether directly or indirectly) into the production of *all* commodities. Those that do we shall call *basic,* and those that do not, non-basic commodities" (ibid, p. 8). Changes in the conditions of production of basic commodities affect all prices, whereas changes in the conditions of production of non-basic commodities affect only some prices. The (economic) assumption that all commodities are basic implies the (mathematical) condition that A is indecomposable.

Sraffa "follows the usual practice of treating the whole of the wage as variable", recognising that "it involves relegating the necessaries of consumption to the limbo of non-basic commodities" (ibid, p. 10). However, we do not need to follow Sraffa in rejecting the argument that the wage can be divided into two parts – the "subsistence" part, consisting of "the goods necessary for the subsistence of the worker [which] would continue to appear with the fuel, etc., among the means of production" (ibid, pp. 9-10), and the "surplus" part, consisting of a share of the surplus product – with only the surplus part as variable. The method by which the subsistence part of the wage can be incorporated into the technical coefficients is described, for example, in Chapter 7, Section 7.1.2 of Woods (1978) in the context of a discussion of the von Neumann model. Hence in (5), A is interpreted as the matrix of commodity input coefficients inclusive of the subsistence wage (which must be advanced) and w is interpreted as the surplus wage[2].

Equation (5) consists of n equations with (n+2) unknowns, the n prices, w and r. With a commodity chosen as numeraire, the number of unknowns is reduced to (n+1). Hence, there is one degree of freedom – the price system remains open with respect to income distribution. Given one of the distri-

2. This interpretation is also discussed in Chapter 5 of Pasinetti (1977).

butive parameters – the surplus wage or the rate of profit – the price system and the other distributive parameter can be determined.

On putting the surplus wage $w = 0$ in (5), we obtain

$$p' = (1 + r)p'A. \tag{6}$$

By the Perron-Frobenius theorem on semi-positive indecomposable matrices[3], (6) has a solution $p^* > \underline{0}$ and $R_1 > 0$. From (6), the positive Frobenius root $\lambda^*(A) = 1/(1+R_1)$, whence $R_1 = (1/\lambda^*(A)) - 1$. As established in Section 2.1 above, $\lambda^*(A) < 1$ for a productive system, so that $R_1 > 0$. R_1 can be interpreted as the maximum rate of profit that can be supported by the technology.

On putting $r = 0$ in (5), we obtain

$$p'(I - A) = wl' \tag{7}$$

For a semi-positive indecomposable matrix, A, $(\alpha I - A)^{-1} > [0]$ if and only if $\alpha > \lambda^*(A)$[4]. As $1 > \lambda^*(A)$ for a productive system, it follows that $(I-A)^{-1} > [0]$ and that (7) has a positive solution, $p^{**} = w^{**}\ell'(I-A)^{-1}$.

Consider now an intermediate case where both r and w are non-zero. Then (5) becomes

$$p'[I - (1 + r)A] = wl' \tag{8}$$

By the theorem just quoted, $[I - (1 + r)A]^{-1} > [0]$ if and only if $1/(1+r) > \lambda^*(A) = 1/(1+R_1)$ – that is, if and only if $R_1 > r$. Hence, as long as r lies between 0 and R_1, (8) has a positive solution,

$$p' = wl'[I - (1 + r)A]^{-1} \quad \text{or}$$
$$\tilde{p}' = l'[I - (1 + r)A]^{-1} \tag{9}$$

where $\tilde{p} = p/w$. From (9), we can see that for $R_1 > r \geqslant 0$, $d\tilde{p}_i/dr > 0$.

This more or less summarises the price theory for a circulating capital model with only one available technique, described by the matrix A and the vector ℓ (apart, that is, from the case where ℓ is a characteristic vector of A, which we shall discuss below).

Sraffa devotes Chapter III of his book to the analysis of the effect of changes in the wage on the rate of profit and the prices of individual commodities. "The key to the movement of relative prices consequent upon a change in the wage rate lies in the inequality of the proportions in which labour and means of production are employed in the various industries" (ibid, p. 12). This applies not only to an industry which produces a commodity that will be used directly as a means of production, but also to an

3. See Woods (1978), Chapter 2, Section 2.3.2, or Pasinetti (1977), Mathematical Appendix, or Debreu and Herstein (1953).

4. See Woods (1978), Chapter 2, Sections 2.3.1 and 2.3.2, or Pasinetti (1977), Mathematical Appendix, or Debreu and Herstein (1953).

industry which produces a commodity that is used only to produce another commodity that will be used directly as a means of production. So, it is necessary to analyse the proportions of labour to means of production in the successive "layers" of means of production. By this approach, Sraffa is able to construct the invariable standard of value, the Standard Commodity, which exhibits the property that "the 'balancing' proportion recurs in all successive layers of the industry's aggregate means of production without limit" (ibid, p. 16). Our purpose is to examine the effect of changes in the rate of profit on the prices of production, not to analyse the Standard Commodity. Concepts relevant to the latter problem may not be so relevant to the former. Accordingly, we put to one side Sraffa's concept of the proportion of labour to means of production in the various layers of means of production (a concept embodied in the matrix series expansion on the right-hand side of (9)) in favour of an analysis directly based on the characteristic roots and vectors of the matrix A.

2.3 There are a number of conceptual and notational preliminaries to the statement and proof of the main result in this paper. Our first assumption is that the matrix A has n distinct characteristic roots. This assumption is made primarily for mathematical convenience, the analysis of an n x n matrix with n distinct roots being simpler than that of an n x n matrix with repeated roots. However, we shall suggest later a way of dealing with the case of repeated roots. This assumption, that the matrix A has n distinct roots, is not extreme. From the mathematical viewpoint, productive matrices of order n with repeated roots are exceptional among the set of all productive matrices of order n — this can be seen intuitively by considering polynomials of order n, among which those with repeated roots form a very small proportion. Likewise, productive matrices of order n with repeated roots form a very small proportion of the totality of productive matrices of order n.

We denote the (non-zero) roots of A by $\lambda_1, \lambda_2, \ldots, \lambda_n$ where $\lambda_1 = \lambda^*(A)$ is the Frobenius root of A[5]. Corresponding to each λ_i is an R_i such that $\lambda_i = 1/(1+R_i)$; λ_i and hence R_i can be negative or complex. We denote the "right" characteristic vectors by $x^1, x^2, \ldots, x^n$ and the "left" characteristic vectors by $z^1, z^2, \ldots, z^n$ — that is,

$$Ax^i = \lambda_i x^i; \quad z^{j\prime}A = \lambda_j z^{j\prime} \qquad i, j = 1, \ldots, n \tag{10}$$

We write $\tilde{p}$ in (9) as $\tilde{p}(r)$ to denote dependence of $\tilde{p}$ on r.

2.4 In this sub-section, we shall present the main result of this paper. Let r take n distinct values, $r_1, r_2, \ldots, r_n$, where $r_i \neq R_j$ for all $i, j = 1, \ldots, n$. We have then the following:

5. The case where A has a single zero root is dealt with in the Appendix, Section 3.

Theorem: $\{\tilde{p}(r_1), \tilde{p}(r_2), \ldots, \tilde{p}(r_n)\}$ is a linearly independent set if and only if $1'x^i \neq 0$ for all $i = 1, \ldots, n$.

Proof: Consider

$$\tilde{p}(r)'x^i = 1'[I-(1+r)A]^{-1}x^i \tag{11}$$

By elementary manipulations on the first equation in (1), we obtain

$$[I-(1+r)A]^{-1}x^i = [(1+R_i)/(R_i-r)]\,x^i \tag{12}$$

Substituting (12) into (11), we obtain

$$\tilde{p}(r)'x^i = 1'x^i(1+R_i)/(R_i-r) \tag{13}$$

Necessity: In this half of the proof, we assume that the set $\{\tilde{p}(r_1), \tilde{p}(r_2), \ldots, \tilde{p}(r_n)\}$ is a linearly independent set – that is,

the matrix $P = \begin{bmatrix} \tilde{p}(r_1)' \\ \vdots \\ \tilde{p}(r_n)' \end{bmatrix}$ is non-singular.

Suppose that $1'x^j = 0$ for some j. Clearly $\tilde{p}(r_h)'x^j = 0$ for all $h = 1, \ldots, n$ or $Px^j = \underline{0}$ for $x^j \neq \underline{0}$. The matrix P is thus singular. This is a contradiction. Hence, $1'x^j \neq 0$ for all $j = 1, \ldots, n$.

Sufficiency: In this half of the proof, we assume that $1'x^i \neq 0$ for all $i = 1, \ldots, n$. Suppose that the set $\{\tilde{p}(r_1), \tilde{p}(r_2), \ldots, \tilde{p}(r_n)\}$ is linearly dependent. Then there exists a vector $y \neq \underline{0}$ such that

$$Py = \underline{0} \tag{14}$$

where P is the same matrix as in the first half of the proof. Given that the roots of A are distinct, it follows that the corresponding characteristic vectors form a linearly independent set and, therefore, form a basis. The vector y in (14) can then be written as a linear combination of the characteristic vectors of A:

$$y = \sum_{i=1}^{n} \alpha_i x^i \tag{15}$$

Substituting (15) into (14) and using (13), we obtain

$$\begin{aligned} P\Big(\sum_{i=1}^{n} \alpha_i x^i\Big) &= \sum_{i=1}^{n} \alpha_i P x^i \\ &= \sum_{i=1}^{n} \alpha_i (1+R_i) 1'x^i \begin{bmatrix} 1/(R_i-r_1) \\ 1/(R_i-r_2) \\ \vdots \\ 1/(R_i-r_n) \end{bmatrix} = \underline{0} \end{aligned} \tag{16}$$

Let

$$\left.\begin{aligned} \beta_i &= \alpha_i(1+R_i)l'x^i \qquad i=1,\ldots,n \\ b' &= (\beta_1,\beta_2,\ldots\ldots,\beta_n) \\ Q &= [q_{ij}]\,,\, q_{ij} = 1/(R_i-r_j),\, i,j=1,\ldots,n \end{aligned}\right\} \tag{17}$$

Then (16) becomes

$$b'Q = \underline{0}' \tag{18}$$

Note that $y \neq \underline{0}$ implies that $\alpha_i \neq 0$ for at least one i; then the hypothesis that $l'x^i \neq 0$ for all i enables us to infer that β_i is non-zero when α_i is non-zero. Hence, if $y \neq \underline{0}$ in (14), $b \neq \underline{0}$ in (18).

Now, as can be seen from Appendix I, Q is a non-singular matrix (detQ is the Cauchy determinant). Hence, $b = \underline{0}$ in (18); then $y = \underline{0}$ in (14). We conclude that the set $\{\tilde{p}(r_1), \ldots, \tilde{p}(r_n)\}$ is linearly independent. This completes the proof.

2.5 We may call $\{\tilde{p}(r_1), \tilde{p}(r_2), \ldots\ldots, \tilde{p}(r_n)\}$ the price set and that subspace of R^n spanned by the vectors in this set the price space. The theorem states that the price space has dimension n if, and only if, 1 is not orthogonal to each "right" characteristic vector.

As a corollary to the theorem, we can prove that the price space has dimension (n–g) if, and only if, 1 is orthogonal to g of the "right" characteristic vectors.

If $l'x^i = 0$ for $i = n-g+1, \ldots, n$, say, then from (13) we see that $Px^i = \underline{0}$ for $i = n-g+1, \ldots, n$. So the column nullity of P is equal to g. The column rank is equal to n minus the column nullity; hence, the price space has dimension (n–g). Conversely, if the price space has dimension (n–g), the column nullity of P is equal to g. Then $l'x^i = 0$ for $i = n-g+1, \ldots, n$, say; otherwise the price space would not have dimension (n–g)[6].

2.6 The results in Sections 2.4 and 2.5 provide us with the basis for a general analysis of the behaviour of prices. At one extreme, when the price space has dimension 1, we can say that prices vary in a regular or systematic way; changes in r do not result in any changes in the vector of relative prices. At the other extreme, when the price space has dimension n, we can say that prices vary in an unsystematic way; n distinct values of r will yield a linearly independent price set. As this last statement is true for any n distinct values of r, we are justified in stating that the variation in prices is unsystematic.

Intermediate between these two extremes are the cases where the price

6. The result on rank and nullity is derived in Birkhoff and MacLane (1965), Chapter VIII, Section 7.

space has dimension k, $n > k > 1$. Consider the orthogonal complement of the price space[7] – that is, the set of vectors w such that

$$\tilde{p}(r)'w = 0 \text{ for all } \tilde{p}(r) \tag{19}$$

in the price set. This orthogonal complement has dimension (n–k). Equation (19) seems to suggest some degree of dependence among prices; for, by the assumption that the price space has dimension k, we can find (n–k) linearly independent w vectors such that

$$\tilde{p}(r)'[w^1, w^2, \ldots\ldots, w^{n-k}] = \underline{0}'_{n-k}$$

$$\text{or} \quad \tilde{p}(r)'W = \underline{0}'_{n-k} \tag{20}$$

Partition W into W_1, a non-singular square matrix of order (n–k), and W_2, an (n–k) x k matrix; partition $\tilde{p}(r)$ correspondingly into $\tilde{p}^1(r)$ with (n–k) components and $\tilde{p}^2(r)$ with k components. Equation (20) can then be written as

$$[\tilde{p}^1(r)' \quad \tilde{p}^2(r)'] \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \underline{0}'_{n-k}$$

whence

$$\tilde{p}^1(r)' = -\tilde{p}^2(r)'W_2W_1^{-1} \tag{21}$$

As relative prices are determined by the conditions of production and the rate of profit, the dependence of a subset of relative prices on the complementary subset, as exhibited in (21), would seem to suggest a corresponding dependence in the conditions of production. However, we initially assumed that all commodities are basic, that each commodity is used directly or indirectly in the production of every commodity. It may be that the assumption of basic commodities is not the most fundamental assumption that can be made in production theory.

2.7 In this sub-section, we concentrate on the case where the price space has dimension 1. This occurs if and only if $\dot{1}'x^i = 0$ for all $i = 2, \ldots, n$. Hence $\dot{\tilde{p}}(r)'x^i = 0$ for all $i = 2, \ldots, n$. By elementary manipulations on (10) we see that $z^{j\prime}x^i = 0$ for all $i \neq j$. Then $\tilde{p}(r)'x^i = 0$ for all $i = 2, \ldots, n$ implies that $\tilde{p}(r)$ is proportional to z^1 for all feasible r where

$$\lambda_1 z^{1\prime} = z^{1\prime}A \quad \text{or} \quad z^{1\prime} = (1+R_1)z^{1\prime}A, \tag{22}$$

$z^1 > \underline{0}$ by the Perron-Frobenius theorem. Then from (5), we see that 1 is

7. See Birkhoff and MacLane (1965), Chapter VII, Sections 8-11.

also proportional to z^1. Thus l is a "left" characteristic vector of A corresponding to the Frobenius root $\lambda_1 = 1/(1+R_1)$, as is $\tilde{p}(r)$ for $R_1 > r \geq 0$[8].

The vector of (Marxian) values, v, is obtained by putting r = 0 and w = 1 (as the normalisation) in (5). Then we obtain

$$\begin{aligned} v' &= l' + v'A \\ \text{or } v' &= l'(I - A)^{-1} \end{aligned} \tag{23}$$

Clearly, from (23), if l is a "left" characteristic vector of A corresponding to λ_1, then so is v. Elementary manipulations yield

$$v' = l'(1+R_1)/R_1 \tag{24}$$

So, in the case where $l'x^i = 0$ for all $i = 2, \ldots, n$, the price space has dimension 1. Then the vector of relative prices is constant for all feasible r (i.e., $R_1 \geq r \geq 0$). Further, the vector of relative prices is a "left" characteristic vector of A corresponding to λ_1, as are the vectors of direct labour inputs and (Marxian) values.

From (23), we interpret v as the vector of quantities of embodied labour required to produce one unit of each commodity. The Frobenius root of A has the largest absolute value. As $1 > \lambda_1$, we can express $(I - A)^{-1}$ as an infinite matrix series. So (23) becomes

$$\begin{aligned} v' &= l'[I + A + A^2 + \ldots\ldots] \\ \text{or } v' &= l' + l'[A + A^2 + \ldots\ldots] \end{aligned} \tag{25}$$

l' is the vector of direct labour inputs (or, in Marxian terminology, the vector of "variable" capital) and $l'[A + A^2 + \ldots]$ is the vector of indirect labour inputs (or "constant" capital). As v and l are proportional, from (24), we see that the ratio of "constant" to "variable" capital is the same for each commodity. In other words, we have uniform organic composition of capital.

The case where the price space has dimension 1 is undoubtedly very special. It is special in the sense of being unlikely or exceptional. It is special in the sense that readily interpretable results can be derived – if the price space has dimension 2, for example, the condition of uniform organic composition of capital is lost. The condition that each technique should exhibit uniform organic composition of capital is fundamental to the construction of Samuelson's Surrogate Production Function (see Samuelson, 1962, and Garegnani, 1970). This condition is also essential to Brown and

8. When $r = R_1$, $w = 0$ so that we obtain

$$p(R_1)' = (1 + R_1)p(R_1)'A.$$

Hence, $p(R_1)$ is also a "left" characteristic vector of A corresponding to the Frobenius root.

Chang's (1976) work on aggregation. They argue that capital aggregation can be performed (using Hicks' Composite Commodity theorem) when labour shares are equal for all values of r. Woods (1979) has shown that this equal labour shares condition is equivalent to the uniform organic composition of capital condition. We are justified in devoting attention to this special case, for much of modern aggregation theory seems to depend on it.

2.8 In conclusion, we discuss the importance of the assumption that A has n distinct roots. It might be thought that the essential property captured by A in this case is that of having n linearly independent characteristic vectors. It is known that an n x n matrix with repeated roots can have n linearly independent characteristic vectors. It might then be thought that the theorem in Section 2.4 is valid for a matrix with repeated roots and n linearly independent characteristic vectors. To examine this, let us suppose that A has a once-repeated root, λ_j, with associated R_j, x^j and x^{j+1} being the associated characteristic vectors. We may rewrite (11)–(13), substituting the vectors x^j and x^{j+1}, to obtain

$$\tilde{p}(r)'x^j = l'x^j(1+R_j)/(R_j-r) \tag{26}$$

$$\tilde{p}(r)'x^{j+1} = l'x^{j+1}(1+R_j)/(R_j-r) \tag{27}$$

Clearly, if $x = x^j - (l'x^j/l'x^{j+1})x^{j+1}$, then from (26) and (27)

$$\tilde{p}(r)'x = 0 \text{ for all } r \tag{28}$$

— that is, the matrix P is singular and the proof is no longer valid. It seems that the essential property captured by A under the assumption of n distinct roots is that of having only one characteristic vector associated with each root.

APPENDIX I

1. In the context of the Cauchy determinant (see Bellman, 1970, p. 193), consider the matrix

$$B = [b_{ij}],\ b_{ij} = 1/(\mu_i + \upsilon_j), \quad i, j = 1, \ldots, n \tag{29}$$

$$\det B = \left[\prod_{1\leqslant i<j\leqslant n} (\mu_i - \mu_j)(\upsilon_i - \upsilon_j)\right] \Big/ \left[\prod_{1\leqslant i,j\leqslant n} (\mu_i + \upsilon_j)\right] \tag{30}$$

Comparing the matrices B in (29) and Q in (17), we see from (30) that $\det Q \neq 0$ so that Q is non-singular.

2. A result on the matrix T (to be defined below) will be used to analyse the case where the matrix A has a zero root. Let

$$\begin{aligned} T = [t_{ij}]\,;\ t_{ij} &= 1/(\theta_i + \phi_j) \quad i = 1, \ldots, n-1, \\ t_{nj} &= 1 \qquad\qquad\quad j = 1, \ldots, n. \end{aligned} \tag{31}$$

Following the same procedure as in the Cauchy determinant, we obtain

$$\det T = \left[\prod_{1 \leqslant r < s \leqslant n-1} (\theta_r - \theta_s) \prod_{1 \leqslant h < k \leqslant n} (\theta_h - \theta_k) \right] \Big/ \left[\prod_{\substack{1 \leqslant i \leqslant n-1 \\ 1 \leqslant j \leqslant n}} (\theta_i + \phi_j) \right] \tag{32}$$

3. Let us reconsider the theorem under the assumption that A has one zero root, say, $\lambda_n = 0$ (renumbering if necessary). Then

$$Ax^n = \lambda_n x^n = \underline{0} \tag{33}$$

and

$$[I - (1+r)A]\, x^n = x^n \tag{34}$$

If $r \neq R_j$, $j = 1, \ldots, n-1$, the matrix $[I - (1+r)A]$ is non-singular. Hence, from (11) and (34), we have

$$\tilde{p}(r)'x^n = 1'[I - (1+r)A]^{-1}x^n = 1'x^n \tag{35}$$

The "Necessity" proof remains valid in the presence of a zero root. However, the proof of "Sufficiency" requires modification. In particular, (16) becomes

$$\sum_{i=1}^{n-1} \alpha_i (1+R_i) 1'x^i \begin{bmatrix} 1/(R_i - r_1) \\ 1/(R_i - r_2) \\ \vdots \\ 1/(R_i - r_n) \end{bmatrix} + \alpha_n 1'x^n \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \underline{0} \tag{36}$$

Let

$$\left.\begin{aligned} &\gamma_i = \alpha_i(1+R_i)1'x^i, \quad i = 1, \ldots, n-1; \quad \gamma_n = \alpha_n 1'x^n \\ &c' = (\gamma_1, \ldots, \gamma_n) \\ &M = [m_{ij}], \quad m_{ij} = 1/(R_i - r_j) \qquad i = 1, \ldots, n-1 \\ &\qquad\qquad\quad\ m_{nj} = 1 \qquad\qquad\qquad j = 1, \ldots, n. \end{aligned}\right\} \tag{37}$$

Then (36) becomes

$$c'M = \underline{0}' \tag{38}$$

Comparing the matrices M in (37) and T in (31), we see from (32) that detM $\neq 0$. Hence, M is non-singular, so that $c = \underline{0}$ in (38) and $y = \underline{0}$ in (14).

REFERENCES

BELLMAN, R. A., 1970. *Introduction to Matrix Analysis* (second edition), London: McGraw-Hill.

BIRKHOFF, G. and S. MacLANE, 1965. *A Survey of Modern Algebra* (third edition), New York: Macmillan.

BROWN, M. and W. W. CHANG, 1976. "Capital Aggregation in a General Equilibrium Model of Production", *Econometrica*, Vol. 41, pp. 1179-1200.

DEBREU, G., 1959. *Theory of Value*, New Haven: Yale University Press.

DEBREU, G. and I. N. HERSTEIN, 1953. "Non-negative Square Matrices", *Econometrica*, Vol. 21, pp. 597-607.

EATWELL, J., 1977. "The Irrelevance of Returns to Scale in Sraffa's Analysis", *Journal of Economic Literature*, Vol. 15, pp. 61-68.

GAREGNANI, P., 1970. "Heterogeneous Capital, the Production Function and the Theory of Distribution", *Review of Economic Studies*, Vol. 38, pp. 407-436.

PASINETTI, L. L., 1977. *Lectures in the Theory of Production*, London: Macmillan.

RICARDO, D. (Sraffa edition), 1951. *On the Principles of Political Economy and Taxation*, London: Cambridge University Press.

SAMUELSON, P. A., 1962. "Parable and Realism in Capital Theory: The Surrogate Production Function", *Review of Economic Studies*, Vol. 29, pp. 193-206.

SRAFFA, P., 1963. *Production of Commodities by Means of Commodities*, London: Cambridge University Press.

WOODS, J. E., 1978. *Mathematical Economics*, London: Longman Group.

WOODS, J. E., 1979. "On a Recurring Condition in Capital Theory", forthcoming in *Zeitschrift für Nationalökonomie*.