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Notes and Comments

Notes on Aggregation

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Précis: Brown and Chang(1976) consider questions of output and capital aggregation in the context of a neo-classical general equilibrium model. *Inter alia,* they derive a condition, using Hicks' composite commodity theorem, for capital input aggregation under the assumption of distinct depreciation rates. Operating with this same assumption, we provide conditions for capital output and total output aggregation; our conditions can be interpreted in terms of equal labour shares in net output and of uniform organic composition of capital.

1. Brown and Chang (1976) consider the problems of output and capital aggregation within the context of a neo-classical general equilibrium production model; to be specific, they assume an economy characterised by no joint production, constant returns to scale, one non-produced factor and neo-classical sectoral production functions. Their approach to the aggregation problems is based on Hicks' (1946) Composite Commodity Theorem which states that "if the prices of a group of goods change in the same proportion, that group of goods behaves just as if it were a single commodity" (p. 313). Abstracting initially from depreciation, Brown and Chang prove that "when labour shares in all sectors are equal for all rates of interest . . ., the economy reduces to a single aggregate sector" (p. 1187); for, under the equal shares condition, all (output) prices change in the same proportion so that output aggregation can be performed and, by the absence of depreciation, all rentals change in the same proportion so that capital aggregation can be performed. Once depreciation is introduced, the aggregation problems naturally become more complex, particularly if different

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capital goods are permitted to depreciate at different rates[1]; for, even if (output) prices change in the same proportion (permitting output aggregation), gross rentals will not change in the same proportion which implies that capital input aggregation cannot be performed. On the other hand, conditions can be found for equi-proportional rates of change of gross rentals[2] (thereby permitting capital input aggregation). However, output prices do not change in the same proportion, so that output aggregation does not follow.

As remarked above, Brown and Chang (1976) have provided a condition for capital input aggregation under the assumption of distinct depreciation rates. Operating with this same assumption, we now provide conditions for capital output aggregation and for total output aggregation.

2. We use the same notation and definitions as Brown and Chang. The price equations are

$$p_j = wa_{0j} + \sum_{i=1}^{n} (r + \delta_i)p_i a_{ij} \qquad j = 0, 1, \ldots, n \tag{1}$$

Differentiating (1) logarithmically and assuming cost minimisation, we obtain

$$p_j^* = w^*\theta_{0j} + \sum_{i=1}^{n} (p_i^* + r^* \rho_i)\theta_{ij} \qquad j = 0, 1, \ldots, n \tag{2}$$

or, in matrix notation,

$$p_0^* = w^*\theta_{00} + \phi^{0\prime}(p^* + r^*\rho) \tag{3}$$

$$p^* = w^*\phi_0 + \phi'(p^* + r^*\rho) \tag{4}$$

3. We now derive a result on capital output aggregation. From (4)

$$(I - \phi')p^* = w^*\phi_0 + \phi' r^* \tag{5}$$

Substituting

$$\phi_0 = (I - \phi')u_n \tag{6}$$

in (5), we obtain

$$(I - \phi')\,(p^* - w^*u_n)/r^* = \phi'\rho \tag{7}$$

whence

1. This is not a very satisfactory assumption, as it does not enable us to deal properly with fixed capital.

2. See Brown and Chang (1976), Theorem 8.

$$(I - \phi')\epsilon = \phi' \tag{8}$$

where $\epsilon = (p^* - w^* u_n)/r^*$.

Clearly, if $\epsilon = \alpha u_n$ where α is a scalar, the proportional rates of change of capital output prices will be equal so that capital output aggregation is possible.

Theorem 1: $\epsilon = \alpha u_n$ iff $\phi_0 = \beta\phi'\rho$, where $\beta = \alpha^{-1}$.
Proof: Necessity: From (8)

$$\alpha(I - \phi')u_n = \phi'\rho \qquad \text{or}$$

$$(I - \phi')u_n = \alpha^{-1}\phi'\rho \tag{9}$$

Recalling (6), we conclude that $\phi_0 = \beta\phi'\rho, \beta = \alpha^{-1}$.
Sufficiency: From (8), $(I - \phi')\,\epsilon = \phi'\rho = \alpha\phi_0$. Therefore,

$$\epsilon = \alpha(I - \phi')^{-1}\phi_0 \tag{10}$$

Substituting for ϕ_0 from (6), we obtain $\epsilon = \alpha u_n$.

4. We now derive a result for the aggregation of all outputs (i.e., capital goods and the consumption good). Define

$$\epsilon_0 = (p_0{}^* - w^*)/r^* \tag{11}$$

Then (3) can be rewritten as

$$p_0{}^* - w^* = -(1 - \phi_{00})w^* + \phi^{0\prime}(p^* + r^*\rho) \tag{12}$$

Substituting

$$\phi^{0\prime}u_n = 1 - \theta_{00} \tag{13}$$

into (12) and using the definitions of ϵ and ϵ_0, we obtain

$$\epsilon_0 = \phi^{0\prime}(\epsilon + \rho) \tag{14}$$

Clearly, if $\epsilon = \alpha u_n$ and $\epsilon_0 = \alpha$, the proportional rates of change of both capital goods prices and the price of the consumption good will be equal; hence, total output aggregation will be possible.

Theorem 2: $\begin{bmatrix} \epsilon_0 \\ \epsilon \end{bmatrix} = \alpha u_{n+1}$ iff $\begin{bmatrix} \theta_{00} \\ \phi_0 \end{bmatrix} = \beta \begin{bmatrix} \phi^{0\prime}\rho \\ \phi'\rho \end{bmatrix}, \beta = \alpha^{-1}$.

Proof: Necessity: If $\epsilon = \alpha u_n$, $\phi_0 = \beta\phi'\rho$ from Theorem 1. If $\epsilon_0 = \alpha$, we have from (14) $\alpha = \phi^{0\prime}(\alpha u_n + \rho)$ which on using (13) becomes $\alpha = \alpha(1 - \theta_{00}) + \phi^{0\prime}\rho$ (i.e., $\theta_{00} = \beta\phi^{0\prime}\rho$).

Sufficiency: If $\phi_0 = \beta\phi'\rho$, $\epsilon = \alpha u_n$ from Theorem 1. If $\theta_{00} = \beta\phi^{0\prime}\rho$, we have from (14) $\epsilon_0 = \phi^{0\prime}(\epsilon + \rho) = \phi^{0\prime}\alpha u_n + \alpha\theta_{00}$ which on substituting for $\phi^{0\prime}u_n$ becomes $\epsilon_0 = \alpha(1 - \theta_{00}) + \alpha\theta_{00} = \alpha$.

5. In the special case where depreciation rates are equal, it follows from Theorem 1 that all prices will change in the same proportion if and only if labour's share in the value of gross output is the same in all industries. In the context of a discrete technology, Woods (1979) has shown that this equal labour shares condition is equivalent to the condition that the price vector is proportional to the vector of direct labour inputs. This latter condition holds if and only if the vector of direct labour inputs is a characteristic vector of the matrix of input-output coefficients corresponding to the Frobenius root; in other words, there is uniform organic composition of capital.

Theorems 1 and 2 are more general as they take account of different rates of depreciation. To provide some economic interpretation of our results above, let us first of all define

$$f_j = (p_j - \sum_{i=1}^{n} \delta_i p_i a_{ij})/p_j = (wa_{0j} + \sum_{i=1}^{n} rp_i a_{ij})/p_j \qquad j = 0, 1, \ldots, n \tag{15}$$

to be the ratio of the value of net output to the value of gross output. (15) can be written in matrix notion as

$$\begin{bmatrix} f_0 \\ f \end{bmatrix} = \begin{bmatrix} \theta_{00} \\ \phi_0 \end{bmatrix} + \begin{bmatrix} \phi^{0\prime}\rho \\ \phi'\rho \end{bmatrix} \tag{16}$$

Then from Theorem 2 and (16), we have

$$\begin{bmatrix} \epsilon_0 \\ \epsilon \end{bmatrix} = \alpha u_{n+1} \text{ iff } \begin{bmatrix} \theta_{00} \\ \phi_0 \end{bmatrix} = \beta \begin{bmatrix} \phi^{0\prime}\rho \\ \phi'\rho \end{bmatrix} \text{ iff } \begin{bmatrix} f_0 \\ f \end{bmatrix} = \frac{(1+\beta)}{\beta} \begin{bmatrix} \theta_{00} \\ \phi_0 \end{bmatrix}.$$

This means that all output prices change in the same proportion if and only if labour's share in the value of net output is the same for all industries. This provides us with an easily applicable test for the use of Hicks' Composite Commodity Theorem in aggregation. Furthermore, when this condition is satisfied, $(\sum_{i=1}^{n} rp_i a_{ij})/wa_{0j} = \beta^{-1}$ for all j (i.e., there is uniform organic composition of capital).

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