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NOTES AND COMMENTS

Change in a Product Between Two States as the Symmetrical Sum of Changes in Each of its Factors

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This paper is concerned with the algebraic problem of expressing the change in a product purely in terms of changes in its factors; denoting two states by 1 and 2 (indicated by superscripts) and defining variables x_i ($i = 1,2, \ldots ,k$), the aim is to find coefficients f_i such that the following identity obtains:

$$\prod_{i=1}^{k} x_i^2 - \prod_{i=1}^{k} x_i^1 \equiv \sum_{i=1}^{k} f_i (x_i^2 - x_i^1). \qquad (1)$$

It is evident that f_i must be a homogeneous sum product of all the variables x_j^2 and x_j^1 ($j \neq i$) of dimension ($k-1$); that the sum of the coefficients in each f_i must be 1; and that the solution is not unique without the imposition of conditions. In order to define suitable conditions, consider first the case of two variables:

$$x_1^2 x_2^2 - x_1^1 x_2^1 = f_1(x_1^2 - x_1^1) + f_2(x_2^2 - x_2^1). \qquad (2)$$

One solution to Equation (2) is

$$f_1 = (x_2^1 + x_2^2)/2; \; f_2 = (x_1^1 + x_1^2)/2; \qquad (3)$$

i.e., when these values are substituted into (2), all terms on the right hand side which are mixed in states 1 and 2 (namely, $x_1^2 x_2^1$ and $x_1^1 x_2^2$), cancel out. But this property also obtains in, for example,

*As this article went to Press we learned to our sorrow of the death of Dr. Geary – (*Editors*)

$$f_1 = x_2^2;\ f_2 = x_1^1, \tag{4}$$

which shows that the solution is not unique. However, (3) is the more attractive form, being characterised by symmetry both within f_1 and f_2 and between f_1 and f_2. As will be seen, the imposition of such symmetries in the general case of Equation (1) leads to a unique solution.

In the two-variable case, the symmetric solution given by Equation (3) is also the most useful for an application of the result to index numbers. For example, defining x_1 as price, x_2 as quantity (and $x_1 x_2$, therefore, as value) for a given commodity, and letting the two states differ in time or place, then by introducing a summation on each side of Equation (2), we can divide the sum product of a set of commodities symmetrically and consistently into the sum of contributions due to price and quantity. This resembles I. Fisher's Ideal index number treatment in logarithmic form, but the expression here is far simpler and is accordingly recommended for trial; the unitary changes in price or quantity between the two states are the summations of the expressions on the right of (2) divided by the summed value at state 1.

Although economic interpretation is less evident for k greater than two, the general expression is of interest and will now be discussed. In addition to the three properties of f_i already mentioned, it is clear that the coefficients of the products in f_i which are in (k-1) variables of the same state must all have the value 1/k (e.g., with k=4, the coefficients of $x_2^2 x_3^2 x_4^2$ and $x_2^1 x_3^1 x_4^1$ in f_1 must both be 1/4). Other products, which are mixed in states (e.g., $x_2^2 x_3^1 x_4^2$), must cancel out. In each f_i there will be 2^{k-1} product terms which can be divided binomially into k sets as follows:

$$2^{k-1} = \sum_{n=0}^{k-1} \frac{(k-1)!}{n!\,(k-n-1)!}\ , \tag{5}$$

where n may be regarded as the number of variables in state 1 and (k-n-1) the corresponding number in state 2, of which there will indeed be, for any given n,

$$\frac{(k-1)!}{n!\,(k-n-1)!} \tag{6}$$

in every f_i. From symmetry, each product in a given set must have the same coefficient, c_n, and, also from symmetry, the set of coefficients must be the same in all f_i.

As noted, the sum of coefficients in all f_i is unity and the first and last of these has the value 1/k; in fact, the general solution is given by also equating the sum of coefficients of each n-set to 1/k, i.e.,

$$c_n \frac{(k-1)!}{n!(k-n-1)!} = \frac{1}{k} \quad , \; n = 0,1, \ldots ,(k-1) \tag{7}$$

or

$$c_n = \frac{n!(k-n-1)!}{k!} \quad , \; n = 0,1, \ldots ,(k-1) \tag{8}$$

This can easily be proved as follows. On the right hand side of (1) there will be an equal number of + and − terms, with every term in all the variables

$$x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k} \tag{9}$$

(the superscripts a_j being 1 or 2, permuted in all possible ways). When the a_j are all 1 or all 2, the associated coefficients in every f_i are c_0 and c_{k-1}, which, as seen above, are equal to $1/k$. On the right of (1) the positive coefficient of any term (9) with n variables in state 1 is $(k-n)c_n$ and the negative coefficient of the same term is $n\, c_{n-1}$. Hence we must show that

$$(k-n)c_n - nc_{n-1} = 0 \tag{10}$$

except for $n = 0$ or $n = k$. From (8)

$$\frac{c_n}{c_{n-1}} = \frac{n!(k-n-1)!}{k!} \frac{k!}{(n-1)!(k-n)!} = \frac{n}{(k-n)} \tag{11}$$

so that (10) is true.

The following are the values of c_n for $k = 2, \ldots ,6$ and, in parentheses, the number of terms which symmetrically have the same coefficient:

k \ n	0	1	2	3	4	5
2	1/2(1)	1/2(1)				
3	1/3(1)	1/6(2)	1/3(1)			
4	1/4(1)	1/12(3)	1/12(3)	1/4(1)		
5	1/5(1)	1/20(4)	1/30(6)	1/20(4)	1/5(1)	
6	1/6(1)	1/30(5)	1/60(10)	1/60(10)	1/30(5)	1/6(1)

The binomial character of the solution is evident, as is the fact that, being symmetrical, it is unique. As an illustration of the solution in the four-variable case, the expression for f_1 is as follows, with analogous expressions for the other coefficients:

$$f_1 = \frac{1}{4}(x_2^2 x_3^2 x_4^2) + \frac{1}{12}(x_2^1 x_3^2 x_4^2 + x_2^2 x_3^1 x_4^2 + x_2^2 x_3^2 x_4^1) + \frac{1}{12}(x_2^2 x_3^1 x_4^1 + x_2^1 x_3^2 x_4^1 + x_2^1 x_3^1 x_4^2) + \frac{1}{4}(x_2^1 x_3^1 x_4^1). \tag{12}$$