Using Absolute Deviations to Compute Lines of Best Fit

J. P. HOUCK R. D. HUNT

There is some renewed interest among agricultural economists and others in the old technique of using minimized *absolute* deviations in computing lines of best fit. H. B. Jones and J. C. Thompson, in a recent article in *Agricultural Economics Research,* discussed philosophical and practical questions raised when fitted lines using squared and unsquared deviations are compared[5]. However, economic literature (including the Jones-Thompson paper) as well as elementary statistics and econometric literature seems to contain no concise description of the straight-forward and simple procedure for calculating straight lines of best fit which minimize the sum of absolute deviations from the empirical observations to the fitted line. In some work at the University of Minnesota, we find that lines fitted by minimizing the sum of absolute deviations can be useful in certain circumstances.

Our purpose is to present, without proving, a simple step-by-step method for calculating such lines of best fit for the two-variable case. An illustrative example is appended. Those wishing to investigate the historical development and mathematical basis for these methods are directed to the References section, especially [2] [3] [4] [7] [8]. The article by Kaarst is particularly helpful.

It is our view that the minimized sum of absoluted deviations (MSAD) can be quite useful for linear trend fitting. Computing a linear trend generally does not involve assumptions about the error term's statistical distribution. Therefore, unless some quadratic loss function is postulated, minimizing absolute deviations has as much theoretical appeal as minimizing squared deviations and rests on a simpler conceptual base. Selection of the MSAD procedure over least squares might hinge on the existence of one or more extreme or "unusual" data points which strongly influence the position of a least squares line. The influence of these extreme points is reduced with the MSAD approach. Moreover, an MSAD line is easy to compute with a desk calculator, perhaps easier than simple least squares.

Consider a set of n observations on Y and X to which a linear function is to be fitted. The fitted function is to have the form

$$Y = a + bX$$

where a and b are parameter estimates.

Suppose that, instead of an ordinary least squares line of best fit, a MSAD line is desired.

Two versions of this line can be computed [6]. The first is the MSAD line constrained to pass through a pre-selected point such as the means, the medians, or the origin. The second is the unconstrained MSAD line. Unlike the ordinary least squares line, the unconstrained MSAD straight line does not, in general, pass through the point of means.

The Constrained MSAD Line

1. Select the reference point (Y_0, X_0) through which the MSAD line is to pass. For each of the n observations, compute

$$y_i = (Y_i - Y_0) \qquad i = 1,2,3, \ldots n$$
$$x_i = (X_i - X_0)$$

2. Calculate the n ratios, y_i/x_i. Then rank these ratios in ascending algebraic order, beginning with largest negative or the smallest positive number.

3. Sum the absolute values of x_i; $\Sigma_n \mid x_i \mid$

4. To the value—$\Sigma_n \mid x_i \mid$ add the absolute value $2 \mid x_i \mid$ from the ratio having the lowest rank. Then add the absolute value $2 \mid x_i \mid$ from the ratio with the next lowest rank. Continue until the algebraic sum changes sign from negative to positive.

5. Note the actual value of the ratio (y_k/x_k) at which the sign change occurs. This ratio is the slope, b, of the constrained MSAD line. This line will pass through both the pre-selected point and the point at which the ratio (x_k/y_k) is computed. Having calculated b then

$$a = Y_0 - bX_0.$$

As Kaarst proves, the absolute sum of the error terms (or deviations) in this constrained case is

$$S = \Sigma_n \mid y_i - bx_i \mid$$

When S is plotted on the vertical axis and b is plotted on the horizontal axis the result is an open polygon, convex downward [6]. The minimum point on this polygon is located directly above the point $b = x_k/y_k$, where the slope of S switches from negative to positive.[1]

The Unconstrained MSAD Line

Finding the unconstrained MSAD straight line of best fit involves a simple iterative procedure based on the method described above for the constrained line. The procedure is:

1. Compute a constrained MSAD line, but select one of the data points as the initial reference point (fewer iterations are needed if the selected point is on or near a freehand line of best fit.)

2. This constrained line will pass through at least one other data point in the sample. Select this point as the next reference point, and compute another MSAD line.

3. This second line will pass through still another data point. Using this point as the reference, re-compute.

4. Continue until the fitted line reflects back through a data point already used. This line is then the unconstrained MSAD line of best fit. By judicious selection of the initial reference point on or near a freehand trend line, only one or two iterations usually will be needed.

When the constraint that the MSAD straight line must pass through a pre-selected point is removed, then two parameters, a and b, must be estimated directly. Then, in this case

$$S = \Sigma_n \mid Y_i - bX_i - a \mid$$

This is an open polyhedron in three dimensions [6]. The iterative procedure described above is then a systematic search for the minimum of S using various traces of S, each associated with a given value[2] of a.

Goodness of Fit

Measuring the goodness of fit of an MSAD line is not as straight-forward as with the least squares technique. This is because the total sum of absolute deviations of Y_i about a point, say the mean or median, cannot be partitioned unambiguously into that portion accounted for by the fitted line and that portion

1. If the polygon S has a flat horizontal base rather than a single minimum point, then b is indeterminant between the values at the ends of the base.

2. Similar indeterminant results for a and b also can arise in the unconstrained case.

not accounted for by the fitted line, as can be done with squared deviations about the sample mean in least squares. However, we suggest the following coefficient as a measure of how well an MSAD line fits the sample observations on Y_i.

$$f = 1 - \Sigma_n \mid Y_i - \hat{Y}_i \mid / \Sigma_n \mid Y_i - \overset{\circ}{Y} \mid$$

Where $\hat{Y}_i$ are values of Y along the fitted MSAD line associated with the sample X_i and $\overset{\circ}{Y}$ is the median value of Y.

The median is selected as the reference point since, for any given set of numbers, the sum of the absolute deviations is smaller when measured from the median than from any other number [9]. The coefficient, f, can range between zero and $+ 1 \cdot 0$. If there is no systematic association between X and Y in the data, then the minimizing of absolute deviations will yield $\hat{Y}_i = \overset{\circ}{Y}$ and f will be zero. On the other hand, if the fit is perfect and $\hat{Y}_i = Y_i$, then f will be $+ 1 \cdot 0$. Intermediate values of f will then indicate how well the MSAD line fits the data relative to a scale of zero to $+ 1 \cdot 0$. As a criterion for judging goodness of fit, the coefficient f probably should not be compared with the least squares r^2 for the same set of data. Its use should be confined to comparisons of MSAD lines of best fit. For example, the coefficient f can be used to compare constrained *vs* unconstrained MSAD lines or constrained MSAD lines fitted through various pre-selected points.

An example

Consider the following set of five observations on X and Y respectively = (1,1), (2,4), (3,5), (4,6) and (5,7). If a constrained MSAD straight line is fitted through the point (1,1), its slope, b, is 1·67 and its Y-axis intercept, a, is $- 0 \cdot 67$. The f coefficient is ·67. However, when the unconstrained procedure is used, the slope is $+ 1 \cdot 00$ and the Y-axis intercept is $+ 2 \cdot 00$. The f coefficient increases to ·75. The reader may wish to verify the results of this illustrative example using the procedures outlined previously.

In conclusion

Extending these methods beyond two variables is not easy though it has been done mathematically [7] [8]. The multiple variable problem may be reduced to a linear programming exercise [4]. This use is demonstrated by Arrow and Hoffenberg in an interindustry demand study [1], but beyond this the use of this technique in economic research does not appear to be widespread.

References

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