

A Note on Reswitching and Capital Reversing*

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Precis: It is well known that, in the context of a stationary economy, reswitching of techniques implies capital reversing and that capital reversing can occur without reswitching. The aim of this paper is to discuss these two conclusions in the context of an economy experiencing steady growth at an arbitrary feasible rate. Some implications for the production function and aggregate growth theory are drawn. The discussion is conducted within the framework of a fixed-coefficients, two-sector model (after Garegnani), the mainly geometrical argument being based on the wage curve and the consumption-growth curve.

I

Introduction

CONSIDER a fixed-coefficients two-sector economy without joint production such as the one studied by Garegnani (1970, pp. 407–436). A system or technique of production is characterised by a method of production for each commodity. If we assume that there is only one technique and that the methods of production are employed at such intensities as to form an “integrated consumption-good industry”, we can derive further information from the wage curve. This follows from the simple realisation that the net output of the economy is a quantity of the consumption good. A wage curve is illustrated in Figure 1 (in the Garegnani model, the wage curve can take one, and only one, of the following forms: it can be bowed outwards, as illustrated, or bowed inwards or a straight line). The intercept OW measures not only the wage rate when the rate of profits is zero, but also

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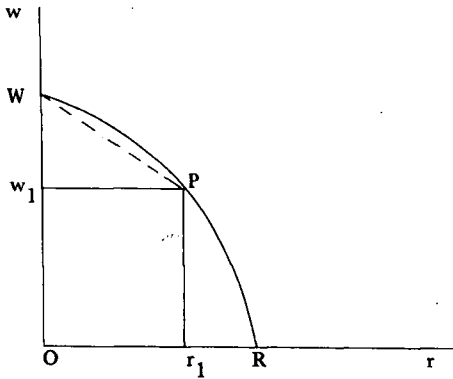


Figure 1. A wage curve.

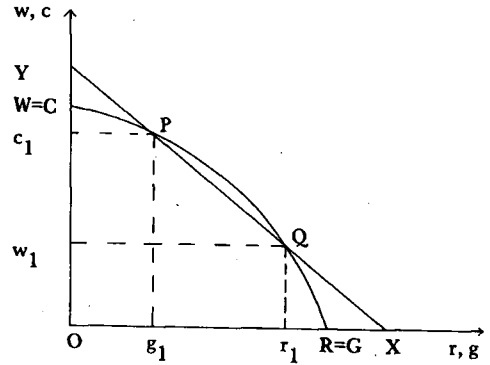


Figure 2: A wage curve and consumption-growth curve.

the net output of the technique (which is just a quantity of the consumption good) at any feasible value of r . Also, as indicated by Garegnani, $\tan w_1 PW$ measures the value of capital per head when $r = r_1$ and $w = w_1$.

Now, if we maintain the assumptions above, but permit alternative techniques (each with a different capital good), we introduce the possibility that the reswitching and capital reversing phenomena will occur (Harcourt, 1972). By the assumption of "integrated consumption-good industries" (or stationary states), we can compare the net outputs of different techniques simply by reading the vertical intercepts of the respective wage curves. Also, the values of capital per man at switch points can be compared by evaluating the tangents of appropriate angles.

The implications for capital theory of the reswitching and capital reversing phenomena have been discussed by, among others, Pasinetti (1966 and 1969) and Garegnani (1970), with Harcourt (1972) providing a summary and references.

If we confine our attention to stationary states (or "integrated consumption-good industries"), as most contributors to the literature seem to have done, we can conclude *inter alia* that:

- (i) reswitching implies capital reversing, and
- (ii) capital reversing can occur without reswitching.

It is a straightforward matter to illustrate these conclusions pictorially.

In the remainder of this paper, we wish to discuss these two conclusions in the context of a growing economy. An immediate implication of the assumption of a growing economy is that the net output of the economy does not consist only of a quantity of the consumption good. Hence, the net output of a growing economy cannot be read from the vertical intercepts of the wage curves. Further analytical tools are required and it is for this reason that we introduce the consumption-growth curve. Just as the wage curve $w = f(r)$ is derived from the price equations,

$$\begin{aligned} 1 &= a_{31}w + a_{21}(r+d)p \text{ and} \\ p &= a_{32}w + a_{22}(r+d)p, \end{aligned} \quad (1)$$

where the consumption good, which is the numeraire, is indexed by 1, the capital good by 2 and labour by 3,

w = the wage rate,

r = the rate of profits,

p = the price of the capital good,

d = the common rate of depreciation,

c = the output of the consumption good per man,

m = the number of machines per man, and

g = the rate of growth,

so the consumption-growth curve, $c = f(g)$, is derived from the quantity equations,

$$\begin{aligned} 1 &= a_{31}c + a_{32}(g+d)m \text{ and} \\ m &= a_{21}c + a_{22}(g+d)m. \end{aligned} \quad (2)$$

The wage curve and the consumption-growth curve are identical. Eliminate p from (1) to obtain

$$w = [1 - a_{22}(r+d)] / [a_{31} + (a_{21}a_{32} - a_{22}a_{31})(r+d)] \quad (3)$$

and eliminate m from (2) to obtain

$$c = [1 - a_{22}(g+d)] / [a_{31} + (a_{21}a_{32} - a_{22}a_{31})(g+d)]. \quad (4)$$

The consumption-growth curve exists if there is only one consumption good (or, what amounts to the same thing, a fixed basket of consumption goods). If both goods in the Garegnani model are consumed, tedious calculations show that the consumption-growth curve exists only in the golden rule case.

The usefulness of this duality between the wage and consumption-growth curves can be easily demonstrated. Consider an economy with just one technique available, with a wage curve and a consumption-growth curve as illustrated in Figure 2.

Suppose that the economy experiences steady growth at a rate $g_1 \leq G$; considering the figure as a consumption-growth curve, the corresponding consumption per head is c_1 . From the identities,

$$y = w + rk = c + gk, \quad (5)$$

we may derive

$$k = (c - w) / (r - g) \text{ if } r \neq g. \quad (6)$$

Given $g = g_1$ and $c = c_1$, we can derive k if we know r . Suppose $r = r_1$ so that $w = w_1$, using Figure 2 as a wage curve. Then the slope of PQ is equal to k , the value of capital per man. Using either of the identities, we see that net output per man is given by OY and the output-capital ratio is given by OX . With g fixed at g_1 (and c at c_1), we may let r vary over the interval $[0, R]$ and plot, corresponding to each value of r , the value of y and k^1 . So, from the information in the wage curve (and the consumption-growth curve) and given the value of g , we can construct a "production function". This is illustrated in Figure 3.

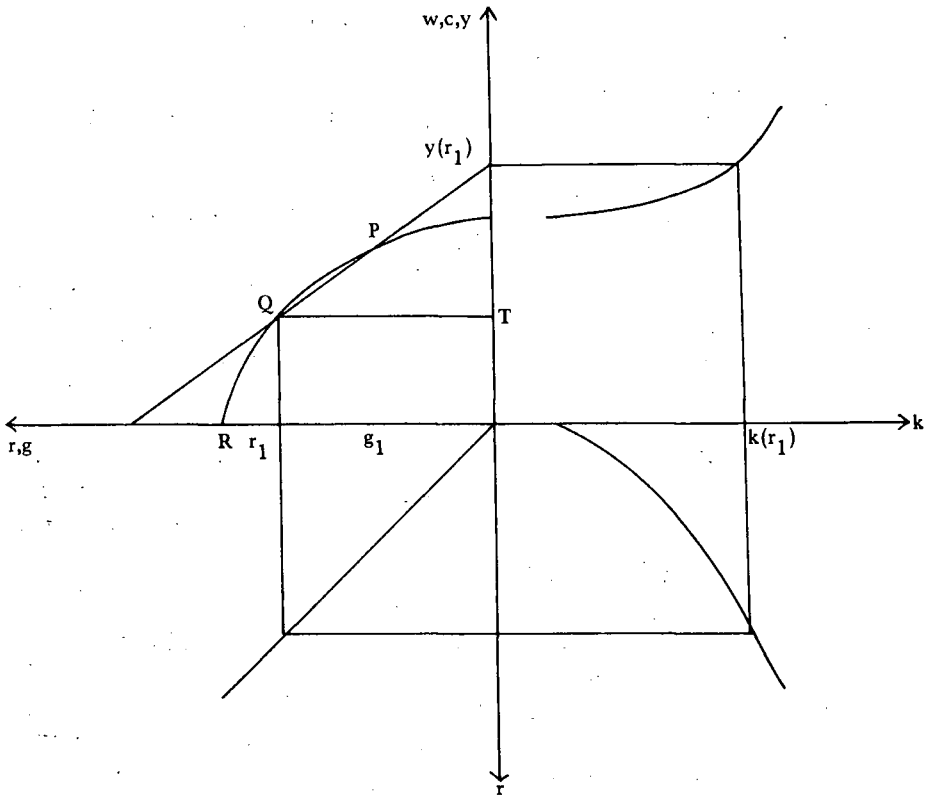


Figure 3: Construction of a "production function" corresponding to a given technique.

Suppose $r = r_1$. The corresponding output per man is $y(r_1)$ on the upper vertical axis and the corresponding value of capital per man, $k(r_1)$, is $\tan PQT$. We use the south-west quadrant to project values of r from one axis to another. Extracting $k(r_1)$ from the north-west quadrant, we can construct the relation between k and r shown in the south-east quadrant. All that remains is to relate $y(r_1)$ and $k(r_1)$ which is done in the north-east quadrant.

1. When $r = g$, $k = -dc/dg$ (i.e., the slope of the tangent to the consumption-growth curve at P).

It can be seen from Figure 3 that lower values of capital per man and output per man are associated with lower values of r . The wage curve in Figure 3 represents a technique in which $a_{22}/a_{32} > a_{21}/a_{31}$. Consider another economy with only one available technique in which the corresponding inequality is reversed. Then the wage curve (and the consumption-growth curve) will be bowed inwards, from which it is easy to see that higher values of capital per man and output per man will be associated with lower values of r .

It should be emphasised that Figure 3 has been constructed on the assumption that $g = g_1$. Indeed we should write $y(r_1, g_1)$ and $k(r_1, g_1)$ instead of $y(r_1)$ and $k(r_1)$, respectively. It follows that if $g_1 \neq g_2$, then $y(r_1, g_1) \neq y(r_1, g_2)$ and $k(r_1, g_1) \neq k(r_1, g_2)$ in general. So, for a different growth rate, a different production function must be constructed, but can we arrive at any definite conclusions about the changes in the production function consequent on a change in the rate of growth?

Let us consider the wage curve in either of Figures 2 or 3. It is evident that if $g_2 > g_1$, then $k(r_1, g_2) > k(r_1, g_1)$ and $y(r_1, g_2) > y(r_1, g_1)$ as long as g_2 is feasible. Now these inequalities will be true for any feasible r_1 . So we have $k(r, g_2) > k(r, g_1)$ and $y(r, g_2) > y(r, g_1)$ for all feasible r . Figure 3, it will be recalled, was constructed on the assumption that $g = g_1$. Nevertheless, we could use the diagram to construct another production function corresponding to $g = g_2 > g_1$ (using the same technique).

It might be thought that knowledge of the two inequalities — $k(r, g_2) > k(r, g_1)$ and $y(r, g_2) > y(r, g_1)$ for all feasible values of r — would enable us to arrive at a definite conclusion about the “movement” of the production function consequent on a change in the rate of growth. However, this is not so as can be demonstrated pictorially. It is possible that the production function corresponding to $g = g_2$ may lie either above or below the production function corresponding to $g = g_1$ and still be consistent with the two inequalities. Indeed, there seems to be no *a priori* reason why the two production functions should not intersect (any number of times). Nor should we exclude the (singular) case where the two production functions have the same graph, apart from lower and upper intervals of k . Thus, it seems that the only definite inference we can make is that, in general, there will be upward and/or downward “movement” of the production function consequent on a change in the rate of growth.

Let us now consider the alternative forms of the wage curve in the Garegnani model. In Figure 4, we have the wage curve of a technique in which $a_{22}/a_{32} < a_{21}/a_{31}$. It is clear that if $g_2 > g_1$, $k(r_1, g_2) < k(r_1, g_1)$ and $y(r_1, g_2) < y(r_1, g_1)$ as long as g_2 is feasible. Hence, we have $k(r, g_2) < k(r, g_1)$ and $y(r, g_2) < y(r, g_1)$ for all feasible values of r . Now we could construct production functions in this case corresponding to $g = g_1$ and

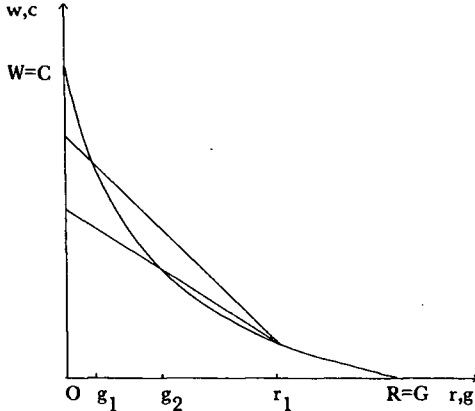


Figure 4: A wage curve (and consumption-growth curve) of a technique for which the physical-capital/labour ratio is less in the capital good industry than in the consumption good industry.

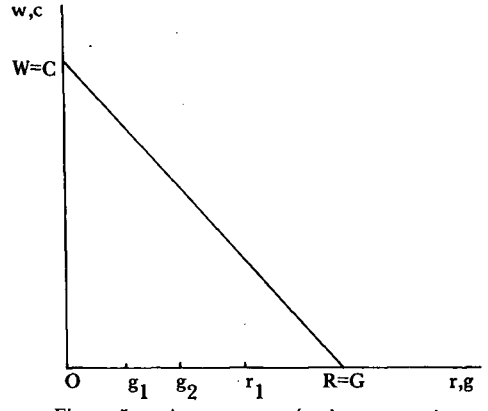


Figure 5: A wage curve (and consumption-growth curve) of a technique with equal physical-capital/labour ratios.

$g = g_2$ (using the method outlined above and illustrated in Figure 3), comparing them in order to arrive at conclusions about the “movement” of the production function consequent on a change in the growth rate. However, our arguments have already been well rehearsed above when we examined the case of a technique in which $a_{22}/a_{32} > a_{21}/a_{31}$. An upward or downward “movement” of the production function is consistent with the two inequalities — $k(r, g_2) < k(r, g_1)$ and $y(r, g_2) < y(r, g_1)$ for all feasible values of r . Again, our only inference is that the direction of “movement” of the production function is ambiguous.

The third possible form of the wage curve is illustrated in Figure 5. This occurs when $a_{22}/a_{32} = a_{21}/a_{31}$. Evidently the change in the rate of growth does not alter either k or y . That is, $k(r_1, g_2) = k(r_3, g_4)$ where r_1 and r_3 are feasible values of r and g_2 and g_4 are feasible values of g .

We have established our terminology and notation by reference to the one-technique case. We now turn our attention to the implications of the existence of more than one technique.

II

In this section, we examine the validity of conclusion (i) in the context of a growing economy. Our argument is illustrated by Figure 6 in which there are two techniques, denoted by (α) and (β) , which have switch-points (\bar{r}, \bar{w}) and (\bar{r}, \bar{w}) .

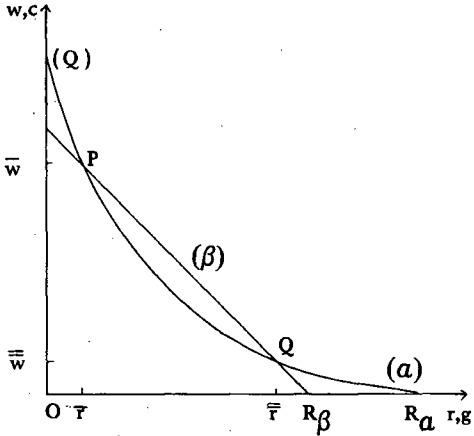


Figure 6: An example of reswitching

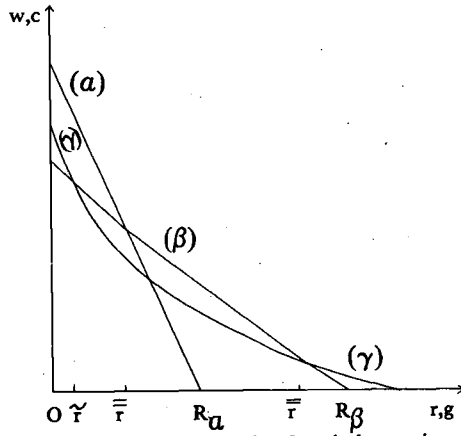


Figure 7: An example of capital reversing

Using (5) and (6) above, we have

$$k_a(\bar{r}, g) = (c_a - \bar{w}) / (\bar{r} - g) ; k_\beta(\bar{r}, g) = (c_\beta - \bar{w}) / (\bar{r} - g) \quad (g \neq \bar{r}) \quad (7)$$

$$y_a(\bar{r}, g) = \bar{w} + \bar{r}k_a(\bar{r}, g) ; y_\beta(\bar{r}, g) = \bar{w} + \bar{r}k_\beta(\bar{r}, g) \quad (8)$$

$$k_a(\bar{r}, g) = (c_a - \bar{w}) / (\bar{r} - g) ; k_\beta(\bar{r}, g) = (c_\beta - \bar{w}) / (\bar{r} - g) \quad (g \neq \bar{r}) \quad (9)$$

$$y_a(\bar{r}, g) = \bar{w} + \bar{r}k_a(\bar{r}, g) ; y_\beta(\bar{r}, g) = \bar{w} + \bar{r}k_\beta(\bar{r}, g) \quad (10)$$

where $k_a(\bar{r}, g)$ = the value of capital per man in technique (a) given \bar{r} and g
 $y_a(\bar{r}, g)$ = the value of output per man from technique (a), given \bar{r} and g , etc.

We shall consider three cases, corresponding to g 's lying in the intervals $[0, \bar{r})$, (\bar{r}, \bar{r}) and $(\bar{r}, R]$.

CASE 1 Let $g = g_1 < \bar{r}$. From (7) – (10) above, we conclude that

$$\begin{aligned} c_a(\bar{r}, g_1) &> c_\beta(\bar{r}, g_1) \\ k_a(\bar{r}, g_1) &> k_\beta(\bar{r}, g_1) \\ y_a(\bar{r}, g_1) &> y_\beta(\bar{r}, g_1) \end{aligned} \quad (11)$$

and

$$\begin{aligned} c_a(\bar{r}, g_1) &> c_\beta(\bar{r}, g_1) \\ k_a(\bar{r}, g_1) &> k_\beta(\bar{r}, g_1) \\ y_a(\bar{r}, g_1) &> y_\beta(\bar{r}, g_1) \end{aligned} \quad (12)$$

Capital reversing occurs at the switch-point (\bar{r}, \bar{w}) in this case, just as it would in the stationary state case. We could construct a production function corresponding to $g = g_1$; it would certainly exhibit the same sort of “perverse” behaviour as the production function in the familiar stationary state case. With many techniques, this involves a slight modification of the

method outlined in the previous section and illustrated in Figure 3. First of all, construct the wage curve of each technique. The outer envelope of these wage curves is the wage frontier. For each feasible value of r , find the corresponding technique on the frontier and calculate the value of net output per man and the value of capital per man of the technique at that value of r and the given value of g . At a switch-point between two techniques, any convex combination of the two techniques can be used. When this procedure has been followed for each feasible value of r , the production function can be constructed. If reswitching occurs, a technique will contribute two or more segments to the production function, these segments being separated by segments contributed by other techniques. The reswitching phenomenon creates well-known problems for some economists; from the point of view of *production*, two identical plants do *not* represent two identical plants (Solow (1955), p. 101).

CASE 2 Let $g = g_2$ and $\bar{r} < g_2 < \bar{\bar{r}}$. Again using (7) – (10), we have

$$\begin{aligned} c_a(\bar{r}, g_2) &< c_\beta(\bar{r}, g_2) \\ k_a(\bar{r}, g_2) &> k_\beta(\bar{r}, g_2) \\ y_a(\bar{r}, g_2) &> y_\beta(\bar{r}, g_2) \end{aligned} \quad (13)$$

and

$$\begin{aligned} c_a(\bar{\bar{r}}, g_2) &< c_\beta(\bar{\bar{r}}, g_2) \\ k_a(\bar{\bar{r}}, g_2) &< k_\beta(\bar{\bar{r}}, g_2) \\ y_a(\bar{\bar{r}}, g_2) &< y_\beta(\bar{\bar{r}}, g_2) \end{aligned} \quad (14)$$

It is clear that, in this case, capital reversing does not occur at either (\bar{r}, \bar{w}) or $(\bar{\bar{r}}, \bar{\bar{w}})$. The production function corresponding to $g = g_2$ appears “well-behaved”.

CASE 3 Let $g = g_3$ and $\bar{\bar{r}} < g_3 \leq R_\beta$. Using (7) – (10), we have

$$\begin{aligned} c_a(\bar{r}, g_3) &> c_\beta(\bar{r}, g_3) \\ k_a(\bar{r}, g_3) &< k_\beta(\bar{r}, g_3) \\ y_a(\bar{r}, g_3) &< y_\beta(\bar{r}, g_3) \end{aligned} \quad (15)$$

and

$$\begin{aligned} c_a(\bar{\bar{r}}, g_3) &> c_\beta(\bar{\bar{r}}, g_3) \\ k_a(\bar{\bar{r}}, g_3) &< k_\beta(\bar{\bar{r}}, g_3) \\ y_a(\bar{\bar{r}}, g_3) &< y_\beta(\bar{\bar{r}}, g_3) \end{aligned} \quad (16)$$

Capital reversing occurs in this case, but note at (\bar{r}, \bar{w}) and not at $(\bar{\bar{r}}, \bar{\bar{w}})$ as in the first case. Similar comments regarding the production function apply

in this case as in the first case; the production function will exhibit "perverse" behaviour, but in this case at (\bar{r}, \bar{w}) .

It is clear from the three cases considered above that quite drastic changes in the production function can result from very small changes in g . For example, a change in g from $\bar{r}-\delta$ to $\bar{r}+\delta$, where δ is arbitrarily small and positive, will eliminate capital reversing at (\bar{r}, \bar{w}) . Likewise, a change in g from $\bar{r}-\delta$ to $\bar{r}+\delta$ will introduce capital reversing at (\bar{r}, \bar{w}) .

III

In this section, we examine conclusion (ii) in the context of a growing economy. Our argument is illustrated by Figure 7. In this example, we have three techniques, (α) , (β) and (γ) , two switch-points, (\bar{r}, \bar{w}) and (\bar{r}, \bar{w}) , and no reswitching. With regard to choice of technique, the switch-point (\bar{r}, \bar{w}) between (β) and (γ) can be ignored as both techniques are dominated by (α) over the interval $[0, \bar{r})$. However, for other reasons, this switch-point should not be ignored, as will become clear.

CASE 4 Let $g = g_4 < \bar{r}$. From (7) – (10), we have

$$\begin{aligned} c_\alpha(\bar{r}, g_4) &> c_\beta(\bar{r}, g_4) \\ k_\alpha(\bar{r}, g_4) &> k_\beta(\bar{r}, g_4) \\ y_\alpha(\bar{r}, g_4) &> y_\beta(\bar{r}, g_4) \end{aligned} \quad (17)$$

and

$$\begin{aligned} c_\beta(\bar{r}, g_4) &< c_\gamma(\bar{r}, g_4) \\ k_\beta(\bar{r}, g_4) &< k_\gamma(\bar{r}, g_4) \\ y_\beta(\bar{r}, g_4) &< y_\gamma(\bar{r}, g_4). \end{aligned} \quad (18)$$

So we have capital reversing at (\bar{r}, \bar{w}) .

CASE 5 Let $g = g_5$ and $\bar{r} < g_5 < \bar{r}$. From (7) – (10), we have

$$\begin{aligned} c_\alpha(\bar{r}, g_5) &> c_\beta(\bar{r}, g_5) \\ k_\alpha(\bar{r}, g_5) &> k_\beta(\bar{r}, g_5) \\ y_\alpha(\bar{r}, g_5) &> y_\beta(\bar{r}, g_5) \end{aligned} \quad (19)$$

and

$$\begin{aligned} c_\beta(\bar{r}, g_5) &> c_\gamma(\bar{r}, g_5) \\ k_\beta(\bar{r}, g_5) &> k_\gamma(\bar{r}, g_5) \\ y_\beta(\bar{r}, g_5) &> y_\gamma(\bar{r}, g_5). \end{aligned} \quad (20)$$

It is clear that there is no capital reversing in this case. As there is no re-switching, by construction, it follows that the production function in this case will be more well-behaved than the production function of Case 4.

CASE 6 Let $g = g_6$ and $\bar{r} < g_6 \leq R_a$. Using (7) – (10), we have

$$\begin{aligned} c_a(\bar{r}, g_6) &< c_\beta(\bar{r}, g_6) \\ k_a(\bar{r}, g_6) &> k_\beta(\bar{r}, g_6) \\ y_a(\bar{r}, g_6) &> y_\beta(\bar{r}, g_6) \end{aligned} \quad (21)$$

and

$$\begin{aligned} c_\beta(\bar{r}, g_6) &> c_\gamma(\bar{r}, g_6) \\ k_\beta(\bar{r}, g_6) &> k_\gamma(\bar{r}, g_6) \\ y_\beta(\bar{r}, g_6) &> y_\gamma(\bar{r}, g_6) \end{aligned} \quad (22)$$

So, there is no capital reversing in this case, as in Case 5.

Technique (a) cannot support a growth rate greater than R_a . Hence, if g is greater than R_a , we can ignore technique (a) which leaves us with the example analysed in Section II. So Cases 7 (where $g = g_7$ and $R_a < g_7 < \bar{r}$) and 8 (where $g = g_8$ and $\bar{r} < g_8 \leq R_\beta$) have already been discussed.

The analysis of this section confirms the conclusion of the previous section regarding the effect of changes in g on the production function. Furthermore, consideration of Cases 4 and 5 leads us to conclude that what happens off the wage frontier (or the consumption-growth frontier) may well be important. Techniques (β) and (γ) have a switch-point (r, w) which is irrelevant when examining choice of technique because the switch-point occurs in an interval in which both techniques are dominated by technique (a). However, knowledge of the wage curves of (β) and (γ) over the interval $(0, \bar{r})$ is crucial in deriving expressions (18) and (20) which establish relations between values of net output per man and capital per man of (β) and (γ) at switch-points on the wage frontier.

IV

Conclusions

We may use the term aggregate production function to describe the relation between the value of net output per man and the value of capital per man. It is evident from the above that the production function is dependent on the rate of growth. We can conclude that if there is a change in the rate of growth, the production function will change. This will occur if each technique can sustain the lower and higher rates of growth; it will also occur if a particular technique cannot sustain the higher growth rate, for then the technique will contribute to the production function at the lower growth rate, but not at the higher growth rate. For example, in Figure 5, if initially

$g = g_6$ where $\bar{r} < g_6 \leq R_a$, then (a) will contribute to the production function, whereas if $g = g_7$ where $R_a < g_7 \leq R_\beta$, (a) will not contribute to the production function. More specifically, we can see from an examination of Cases 1 – 3 in Section II and Cases 4 and 5 in Section III that the *shape* of the production function may well change if there is a change in the rate of growth; the production function corresponding to $g = g_1$ (in Case 1) will generally have a different shape from that corresponding to $g = g_2$ (in Case 2). Similar inferences may be drawn by comparison of other Cases (2 and 3, 1 and 3, and 4 and 5).

A familiar piece of modern neo-classical comparative macro-dynamic analysis runs as follows: in a one-sector model, an increase in the (natural) rate of growth will result in a decrease in output per man, capital per man, the output-capital ratio, the wage rate, the wage-rental ratio, and an increase in the rental rate (Wan, 1971)². These conclusions may be derived by examining a diagram such as the one used by Wan (1971, p. 45).³ The equilibrium position is obtained by locating the intersection of the curve kn/s and the production function $f(k)$. It is assumed that when the (natural) rate of growth changes, the production function is not affected. However, we have shown that this assumption is not generally valid. Indeed, from our analysis in Section I, we know that in general the “movement” of the production function consequent on a change in the growth rate will not be unambiguous. Suppose that the economy has a number of techniques available, some with wage curves characterised by Figure 2, others with wage curves characterised by Figure 4, and finally those with wage curves characterised by Figure 5. Suppose that the economy has a rate of growth $g = g_1$. We can construct the corresponding production function (call it f_1) using the method outlined on page 92. Now suppose that there is another economy with the same techniques available, but the rate of growth $g = g_2$ (that is, if we do not want to say, suppose that the economy achieves steady growth at a rate $g = g_2$ with the same techniques). We can construct the corresponding production function (call it f_2). What will be the relation between the two production functions? From our analysis in Section I, we know that over some intervals f_2 will lie above f_1 and over other intervals f_2 will lie below f_1 . This is what was meant when we stated above that the “movement” of the production function would not be unambiguous.

If we are willing to stay within the confines of a true one-commodity model (for example, a corn-input, corn-output model referred to by Pasinetti (1966)), then the comparative macro-dynamic results stated above are valid. Such a model is obviously of limited applicability and it is legitimate to enquire how far the results for the one-commodity (or leets/butter/jelly)

2. These results are taken from Wan (1971), p. 46.

3. See Figure 26 on p. 45 of Wan (1971).

model carry over to models with distinct capital and consumption goods. Our purpose here is not to re-examine the Cambridge controversies; rather, it is to point out that the one-sector neo-classical model (as commonly employed) is defective in its own terms.

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