A Note on Discount Rates and Present Values

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It is a long-established proposition that, under profit maximising conditions, the interest elasticity of investment will vary positively with the length of life of projects. Thus, Shackle (1, Chap. XI) proves that the present value of an income stream is more sensitive to changes in the rate of discount the further into the future that stream extends. But Shackle's proof is in terms of continuous time, with present value expressed as an integral of a time function. Since, however, the literature on the application of discounted-cash-flow techniques to investment decisions (and its public sector analogue, cost-benefit analysis) treats time as a discrete variable, it is perhaps worthwhile proving the theorem in terms more in accord with this literature. We shall do this for an income stream, though of course the proof is equally valid for a cost stream, and shall take two cases, one where the income flow is constant from period to period, and the other where it varies.

I

Let A be a constant annuity which accrues discretely each period from period one to the end of the asset's life. The discount rate per period is r (always positive).

The present value of this stream if the asset life equals n periods is then

$$\sum_{i=1}^{n} \frac{A}{(1+r)^i}$$

A change in the discount rate to $(r+dr)$ will change the present value to

$$\sum_{i=1}^{n} \frac{A}{(1+r+dr)^i}$$

To measure the effect of a change in the discount rate take the ratio of the new and old present values, as follows:

$$\frac{\sum_{i=1}^{n} \dfrac{A}{(1+r+dr)^i}}{\sum_{i=1}^{n} \dfrac{A}{(1+r)^i}}$$

$$= \frac{\sum\limits_{i=1}^{n} \dfrac{1}{(1+r+dr)^i}}{\sum\limits_{i=1}^{n} \dfrac{1}{(1+r)^i}}$$

For positive dr, this ratio is less than unity: for negative dr, it is greater than unity. We shall proceed on the basis of a positive dr, and shall demonstrate that the ratio is smaller (the proportional change larger) the longer the life of the asset.

Take another asset with a life of $(n+1)$ periods. The equivalent ratio for this is

$$\frac{\sum\limits_{i=1}^{n+1} \dfrac{1}{(1+r+dr)^i}}{\sum\limits_{i=1}^{n+1} \dfrac{1}{(1+r)^i}}$$

and the theorem can be presented, since we are assuming positive dr, as

$$\frac{\sum\limits_{i=1}^{n} \dfrac{1}{(1+r+dr)^i}}{\sum\limits_{i=1}^{n} \dfrac{1}{(1+r)^i}} > \frac{\sum\limits_{i=1}^{n+1} \dfrac{1}{(1+r+dr)^i}}{\sum\limits_{i=1}^{n+1} \dfrac{1}{(1+r)^i}}$$

Re-arrangement yields

$$\frac{\sum\limits_{i=1}^{n+1} \dfrac{1}{(1+r)^i}}{\sum\limits_{i=1}^{n} \dfrac{1}{(1+r)^i}} > \frac{\sum\limits_{i=1}^{n+1} \dfrac{1}{(1+r+dr)^i}}{\sum\limits_{i=1}^{n} \dfrac{1}{(1+r+dr)^i}}$$

or

$$1 + \frac{\dfrac{1}{(1+r)^{n+1}}}{\sum\limits_{i=1}^{n} \dfrac{1}{(1+r)^i}} > 1 + \frac{\dfrac{1}{(1+r+dr)^{n+1}}}{\sum\limits_{i=1}^{n} \dfrac{1}{(1+r+dr)^i}}$$

Inversion and cancelling yields

$$\sum\limits_{i=1}^{n} (1+r)^i < \sum\limits_{i=1}^{n} (1+r+dr)^i$$

i.e. $(1+r) < (1+r+dr)$, which is known to be true.

II

Now for the case where the income stream is variable. Let Ai be a vector of receipts (all positive) which again accrue discretely each period to the end of the asset's life. The other notation is the same as for the previous example. The theorem to be proved is then

$$\frac{\sum\limits_{i=1}^{n}\dfrac{A_i}{(1+r+dr)^i}}{\sum\limits_{i=1}^{n}\dfrac{A_i}{(1+r)^i}} \quad > \quad \frac{\sum\limits_{i=1}^{n+1}\dfrac{A_i}{(1+r+dr)^i}}{\sum\limits_{i=1}^{n+1}\dfrac{A_i}{(1+r)^i}}$$

Re-arrangement yields

$$\frac{\sum\limits_{i=1}^{n+1}\dfrac{A_i}{(1+r)^i}}{\sum\limits_{i=1}^{n}\dfrac{A_i}{(1+r)^i}} \quad > \quad \frac{\sum\limits_{i=1}^{n+1}\dfrac{A_i}{(1+r+dr)^i}}{\sum\limits_{i=1}^{n}\dfrac{A_i}{(1+r+dr)^i}}$$

or

$$1+\frac{\dfrac{A_{n+1}}{(1+r)^{n+1}}}{\sum\limits_{i=1}^{n}\dfrac{A_i}{(1+r)^i}} \quad > \quad 1+\frac{\dfrac{A_{n+1}}{(1+r+dr)^{n+1}}}{\sum\limits_{i=1}^{n}\dfrac{A_i}{(1+r+dr)^i}}$$

Inversion and cancelling yields

$$\sum_{i=1}^{n} A_i(1+r)^{n+1-i} \quad < \quad \sum_{i=1}^{n} A_i(1+r+dr)^{n+1-i}$$

Again, this is clearly true for positive dr.

Therefore, by induction we can say that, the longer the asset life, the greater the relative effect on present value of any change in the discount rate.

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Reference:
1. G. L. S. Shackle, *Uncertainty in Economics* (Cambridge U.P., 1955).