The Economic Specification of the Neoclassical Production Function — a case study

MICEAL ROSS

A production function may be defined as the mathematical expression of the technological information which relates the quantities of inputs to quantities of outputs. As such, the concept is perfectly general and a specific function may be given as a single point, a set of points, a single continuous or discontinuous function, or a system of equations. The neoclassical production function is essentially a single continuous function with continuous first and second order partial derivatives or a system of such equations. Addressing itself directly to marginal analysis and the neoclassical function this paper excludes consideration of other functions of major importance in economic analysis. Foremost among these are the linear and point functions which underlie most of the studies involving linear programming and game theory and the fixed proportions functions used in Input-Output analysis.

In neoclassical theory, as formulated by Hicks and others,[1] the firm's decision-makers are assumed to possess two sets of data: first, complete knowledge of the production technology available to them and second, product demand information. The first assumption implies that a given production process has some "true" functional representation, perhaps of a stochastic nature, which involves some definite set of variables either in a single equation or a set of equations and that this underlying relationship is known. Given the assumptions of profit maximisation and rationality the economic implications follow as simple mathematical tautologies for any given price conditions.[2]

However in real world situations this ideal so seldom, if ever, obtains that some economists have come to regard the neoclassical production model as completely non-operational[3] as an analytical tool for studying the behaviour of the complex technology of the modern production process. They point to empirical research

1. An excellent brief statement of the Hicksian production function and its underlying assumptions is presented in T. H. Naylor and J. M. Vernon, *Microeconomics and decision models of the firm*, New York, 1970, pp. 70–73.

2. M. Ross, "Some management aspects of production functions", to be submitted to the *Irish Journal of Agricultural Economics and Rural Sociology*.

3. T. H. Naylor and J. M. Vernon, *op. cit.*, p. 87; also M. Shubik, "A curmudgeon's guide to Microeconomics", *Review of Economic Literature*, Vol. VIII, No. 2, 1970, p. 411.

to justify their pessimism.[4] Here it is usually found that the economic, physical and biological logic underlying the function is largely unknown, as is the logic of entrepreneurial decision-making within a complex business organisation. In such a situation the economist will hypothesise that the function can be approximated by some given algebraic form, with several unknown parameters to be estimated from the available data. The chief difficulty with this procedure is that the economic inferences which are drawn from the estimated parameters often depend critically upon the algebraic form chosen. It frequently happens that alternative forms fit the data equally well but have very different implications for the most profitable level of inputs. Clearly, where the basic logic is only hypothesised there is the further danger that relevant variables and relationships will be omitted through ignorance; and biassed estimates of the structure of the process, and its response to price situations, obtained.

Even if the basic logic were known with certainty, the empirical construction of a model will frequently require a compromise between what is computationally feasible and what is theoretically desirable. For instance, some of the variables, known to be relevant, may be unobservable. Again, data availability and estimation resources may limit the number of separate variables that can be considered. In addition, the functional relationship to be fitted needs to be manageable, both in terms of estimation and of testing. Often a worthwhile compromise may not be possible. Neoclassical theory assumes furthermore that the entrepreneur possesses full knowledge of product demand information as well as advance information on the level of inputs not under his control (e.g. weather, genetic make-up). Empirical applications are typically characterised by risk and uncertainty. Prices have to be forecast, and the levels of uncontrollable inputs predicted. The degree of risk and uncertainty involved will indicate whether a marginal analysis approach is warranted.

As mentioned earlier, where production logic is unknown the form of equation selected automatically imposes certain constraints on the basic relationships and implies the economic optima. If the margins of error in predicting price and output levels are likely to be considerable, it has been argued, the selection of an equation may compound the difficulties. Hildreth[5] suggested that a procedure which obtains maximum likelihood estimates of discrete points on the production function may be preferable under these conditions.

Again, there are limits to the mathematical procedures for optimising continuous functions, assuming that all equations are indeed continuous. If the model embraces complex choice or allocation problems linear programming or related approaches are probably most appropriate in determining optimum levels of inputs and output; and these should be used in preference to continuous functions.[6]

4. The interested reader is referred to T. H. Naylor and J. M. Vernon, *op. cit.*, pp. 132 *et seq.*, for a more complete statement.

5. G. C. Hildreth, "Point estimates of ordinates of concave functions," *Journal of the American Statistical Association*, Vol. 49, 1954.

6. E. O. Heady and J. L. Dillon, *Agricultural Production Functions*, Iowa, 1961, p. 72.

If there is considerable output and price uncertainty these alternative approaches may be not only computationally simpler, but also of sufficient accuracy for decision-making purposes.

This paper takes the view that these objections have considerable validity. Accordingly, the problems of the economic specification are considered in a context where the neoclassical model is most likely to be operational, i.e., the data are derived from scientific experiment, the model is relatively simple, and the time-lags between input and output are sufficiently short to enable reasonable forecasts of prices to be made. Although the focus in this study is on production functions, most of the discussion applies with equal force to other neoclassical models (cf. Prais's discussion of demand relations, or Pratschke's specification of Engel functions).[7] Similarly, the illustrative material's base in agriculture does not mean that the analysis is not equally valid in other industrial contexts. Greater data availability has made agriculture a rich source of examples, as the perusal of any econometric textbook will substantiate.

The Experiment

Except where instigated by an economist very few scientific experiments seek to define continuous production functions—or response curves.[8] The case study presented here was unusual in that the experiment proved amenable to such analysis.[9] This Danish experiment[10] sought to determine the effect of varying levels of protein and energy rations on the performance of fattening pigs from weaners (44 lb.) to market weight (198 lb.).

The protein, derived mainly from skim milk and a protein-rich mixture (two parts soyabean meal, and one part meat and bone meal)[11], was fed as a constant level at one of the three following levels:—

	Low	*Normal*	*High*
Skim milk (lb.)	1·65	3·3	4·63
Protein mix (oz.)	1·94	3·8	5·47

The fixed daily quantity of barley was stepped up after every 11 lb. of the pig's growth. Each increase was designed to cater for the increasing requirements of the

7. S. J. Prais, "Non-linear estimation of the Engel curves", *Review of Economic Studies*, Vol. 20, 1953; also J. L. Pratschke, *Income-Expenditure Relations in Ireland 1965–1966*. Dublin: Economic and Social Research Institute. Paper No. 50, 1969.

8. It may be confusing that what are essentially regression lines to the statistician takes on different names in different applications and in different disciplines e.g. production functions, engel-, cost-, demand-, supply curves to the economist, response curves to the agronomist, reaction equations to the chemist etc.

9. E. Vestergaard Jensen, "Forskellige protein og fodernormer til svin i driftsokonomisk belysning", *Tolvmandsbladet Nr. 1*, Copenhagen, 1958.

10. H. Clausen, "Forsog SV 698 Sj. 11", *Bilag til oversikten, Forsogslaboratoriets efterarsmode*, Copenhagen, 1956.

11. Barley also contains some protein: see footnote 17.

growing animal, and was at a rate calculated to maintain the pig on the required (stepped) feeding plane. The five feeding planes ranged from an average intake of 3·6 fodder units (f.u.) per day, up to 6·0 f.u. The barley was added to the fixed

TABLE 1: *Descriptions of Main Aspects of Treatments*

Treatment Number	*Intensity of Feeding*	*Protein Standard*	*Per pig daily*		*Percentage Protein in the ration at*	
			Total f.u. average	*Protein f.u. fixed*	*44 lbs*	*198 lbs*
1	Weak	low	3·64	·41	11·7	7·7
2	Weak	normal	3·77	·82	17·0	8·8
3	Weak	high	3·72	1·15*	18·5	9·7
4	Moderate	low	4·17	·41	10·5	7·6
5	Moderate	normal	4·21	·82	14·5	8·7
6	Moderate	high	4·14	1·15	17·7	9·5
7	Fairly strong	low	4·76	·41	9·7	7·5
8	Fairly strong	normal	4·78	·82	13·0	8·5
9	Fairly strong	high	4·89	1·15	15·6	9·3
10	Strong	low	5·40	·41	9·3	7·4
11	Strong	normal	5·40	·82	12·1	8·4
12	Strong	high	5·62	1·15	14·5	9·1
13	Very Strong	low	5·51	·41	8·9	7·4
14	Very Strong	normal	5·73	·82	11·3	8·3
15	Very Strong	high	5·97	1·15	13·3	9·0

*This group was fed ·94 f.u. of protein up to 55 lb weight.

protein ration to bring daily intake up to the required plane. The combination of a fixed protein ration with a stepped scale for barley resulted in a decline in the richness of the ration as the pig grew older, as indicated in Table 1. Such an arrangement caters for the well known fall in a pig's protein requirements with increasing liveweight.

One hundred and twenty weaners purchased in the market place[12] were allocated at random over the fifteen treatments (three Protein multiplied by five feeding plane), with four males and four females per treatment. The published results set out in Table 2 give the average performances of each group. However Professor Hjalmar Clausen of the Danish National Research Station for Animal Husbandry kindly supplied details of the performance of the individual pigs within each group. This, then, is the raw material of the study.

12. This means that previous feeding practices and the genetic qualities of the pigs were uncontrolled inputs in the experiment.

TABLE 2: *Principal Results of the Experiment* (treatment means)

Treatment Number and Code[1]	*Daily lb growth*	*Conversion Rate*	*Days to Bacon*[2]	*Killing out loss*	*Percentage lean in whole side*	*Average Backfat Thickness*
				%		(mm)
1 W—L	1·12	3·23	137	28·4	56·1	33·7
2 W—N	1·24	3·03	124	26·6	58·4	32·4
3 W—H	1·18	3·16	131	26·7	59·8	29·4
4 M—L	1·27	3·30	122	27·0	53·7	35·5
5 M—N	1·43	2·94	108	26·7	57·3	30·5
6 M—H	1·41	2·94	109	27·3	56·9	31·2
7 FS—L	1.43	3.33	108	27·0	50·3	37·6
8 FS—N	1·58	3·02	97	25·3	55·7	33·1
9 FS—H	1·61	3·04	96	25·9	56·2	31·1
10 S—L	1·52	3·56	102	27·4	47·7	38·5
11 S—N	1·72	3·14	90	26·5	54·0	36·2
12 S—H	1·80	3·12	86	26·3	53·7	36·0
13 VS—L	1·55	3·55	99	26·3	48·7	37·51
14 VS—N	1·77	3·23	87	24·5	52·6	37·8
15 VS—H	1·90	3·14	81	25·1	54·2	36·1

[1]The initials of the level of feeding and protein standard as set out in Table I.
[2]From 44 lb. to 198 lb.

Logical and Statistical Functions

If the production logic were fully known the logical function could be fitted to the data and used to predict the entire production surface. Where the underlying logic can only be hypothesised (the normal case), a statistical function will be obtained which approximates the "true" function over the range of observed data. A statistical production function will only prove an error-free guide to decision-making if certain severe conditions are met: (1) all inputs involved in production must be included; (2) observations must cover the relevant range of inputs and outputs, and (3) the parameters must be estimated without error. Such an ideal is unobtainable. The argument of the sections which follow is that, with reasonable precautions in the analysis of the data, coupled with discretion in interpretation, a workable compromise may be achieved.

The first step is to determine the purpose of the study since this will play a role in determining the statistical approach to be adopted. The elucidation of the basic logic is best undertaken using a differential equation approach.[13] For the prediction of economic optima, fitting a polynomial may be preferred. In this latter case the

13. See E. O. Heady and J. L. Dillon, *op. cit.*, p. 204, footnote 7.

fitted parameters may not correspond individually to the true structural representation; but the overall effect may be an accurate reproduction of the true function, at least within the range of observations.

The specification of a statistical function entails three major and interlocking decisions (1) Is a single equation or set of equations more appropriate? (2) What are the relevant variables? (3) What is the most appropriate mathematical form of the equation(s)?

The type of function

(*a*) The cumulative growth rate function.

Data availability rules out defining the function of the individual pig's response to the ration fed:[14]

Cumulative liveweight gain $= f$ (cumulative inputs of protein and energy). Since the daily ration of barley is increased after each 11 lb. of live weight gain, total consumption of the pigs on any treatment level will depend on their speed of growth over these intervals, which in turn will depend on their genetic capacity. The objective of functions of this type is usually to determine optimum marketing weight and/or to specify optimum ration mixes, either over the entire feeding, or over segments of it. This function will be discussed again later.[15]

(*b*) *The multi-equation model*

The various measures of performance can be expressed as a function of the inputs

Days to Bacon	$= f^2$ (inputs of protein and barley)	(2)
Backfat	$= f^3$ (inputs of protein and barley)	(3)
% lean meat	$= f^4$ (inputs of protein and barley)	(4)
Killing out %	$= f^5$ (inputs of protein and barley)	(5)
Conversion Rate	$= f^6$ (inputs of protein and barley)	(6)

From other sources the relationship between backfat[16] and price can be established:

Unit price of pigmeat (P_y) $= f^1$ (Backfat) or f^7 (inputs of protein and barley) where f^1 is substituted in f^3 (7)

From these equations we can define

$$\text{Annual Output of pigmeat } (Y) = \frac{154 f^2 . f^5}{365}$$

$$\text{Annual consumption of Barley } (B) = \frac{154 f^2 . f^6}{365} - S$$ (S is fixed annual input of protein)

14. This is a form particularly favoured by Heady and his association in their many studies of agricultural production functions (E. O. Heady and J. L. Dillon, *op. cit.*, p. 266 *et seq.*).

15. See below, p. 250

16. The percentage lean meat is more important, but cannot be measured directly by the bacon curer at the time of purchase of the fat pig. Equation f^4 may therefore be ignored.

$$\text{Profit} = \text{Revenue} - \text{Cost or } P_y Y - P_b B - P_s S$$

$$= \frac{154}{365} f^2 \cdot f^5 \cdot f^7 - P_b \left(\frac{154}{365} f^2 f^6 - S \right) - P_s S \qquad (8)$$

Ignoring the problem of error terms, and assuming all functions are linear, equation 8 can be differentiated in respect of B and S, and an iterative process employed to determine optimum input levels. If some equations are not linear the computational procedures become extremely difficult. For this reason an approach requiring one, or at most two, equations is preferable. This would require the amalgamation of the various output charcteristics into a composite variable.

(c) *The single equation approach*

(i) The dependent variable

"Days to bacon" defines the duration of the fattening period over which each pig gains 154 lb. This can be standardised between treatments, e.g., the number of fattening periods in a year multiplied by 154 gives total liveweight gain per annum. Applying the appropriate "killing out percentage" (the ratio between deadweight and liveweight at slaughter) we obtained the annual output of pigmeat.

The Independent Variables Relating to Feed Inputs

The adjustment of output requires a similar adjustment of input. Multiplying the annual liveweight gain by the fodder units per lb. of liveweight gain, gave annual consumption in fodder units. Protein, fed at a fixed daily level, is easily converted to an annual basis. Barley made up the difference between total annual consumption and protein. These inputs could be either left in terms of barley and protein mix, or converted into their chemical ingredients.[17] The latter approach had some appeal, since barley contains some protein, and skim milk, etc., some energy value. However, other elements, such as lysine, would have to be considered and also the substitutability of vegetable protein for animal protein, etc. Interpretation of the results would be much simpler if the ingredients were left as they stood, as in Table 3.

17. The value of these feeds are given as

	Barley	*Skim Milk*	*Protein Mixture*
Fodder units per lb	1·0252	0·1707	1·0625
Percentage total digestible protein	6·12	3·33	40·37

TABLE 3: *Annual Output of Pigmeat and Associated Levels of Protein and Barley for Various Levels of Daily intake of Both Inputs**

Pigmeat (Y)	Total Fodder Units	Protein f.u. (S)	Barley f.u. (B)	Backfat thickness mm. (F)
293	1,329	150	1,179	33·7
333	1,376	300	1,076	32·4
315	1,358	420	938	29·4
337	1,522	150	1,372	35·5
382	1,537	300	1,237	30·5
376	1,511	420	1,091	31·2
381	1,737	150	1,587	37.6
434	1,745	300	1,445	33·1
435	1,785	420	1,365	31·1
401	1,971	150	1,821	38·5
460	1,971	300	1,671	36·2
483	2,051	420	1,631	36·0
419	2,011	150	1,861	37·51
489	2,091	300	1,791	37·8
521	2,179	420	1,759	36·1

*following the same order as in previous tables.

Other Independent Variables

Part of the specification problem is the selection of variables. Some of these, such as labour, management, housing, environment and breed were controlled by the experiment. Experimental design attempted to ensure that uncontrollable variables, such as disease, previous feeding experience,[18] and genotype of the pig, were as far as possible randomly distributed. Their influence on production is recognised by adding u (a normally distributed error term with an expected value of zero) to the generalised formula. Sex was the only other relevant variable explicitly taken into account in the results. An analysis of variance on these results indicated that sex had a highly significant influence on all measures of performance. It was not included as an explicit variable in specifying the function; instead separate functions were estimated for each sex. The exclusion of the controlled variables means that the results, strictly speaking, only apply to the statistical population to which they refer. It is a matter for discretion to decide how much they might apply under other conditions.

The Form of the Function

In the generalised statement $Y = f(B, S) + u$, f can take on any one of an infinite number of possible forms. Knowledge of the production process is often

18. Animals fed at a low plane initially often exhibit an accelerated growth rate later compared to those fed normally when both are on the same plane.

sufficient to limit the search to a small subset of equation forms which have logical implications compatible with reasonable *a priori* restrictions. Even here some potential candidates can be eliminated, because of the statistical difficulties involved in deriving their parameters. Some others have terms which cannot be transformed readily into linear regression equations and hence, can only be estimated by laborious iterative processes.[19]

In mathematics, when the form of a function is unknown it can be approximated over the range of observations by a Taylor series expansion. In general, the more terms evaluated the better the estimates, though with stochastic data a higher degree of accuracy may be misleading. Any expansion can be reduced to polynomial form. Expanding the simple first degree series to one and two terms gives, respectively, the linear and quadratic cross product equation below, while the Cobb-Douglas is the first term expansion using a log transformation. Similarly the square root function is the first two terms of the square root transformation. These forms[20] are most commonly used, i.e.:

1. Linear: $Y = a + b_1S + b_2B$
2. Cobb-Douglas: $Y = aS^{b1}B^{b2}$
3. Quadratic Cross Product: $Y = a + b_1S + b_2B - b_3S^2 - b_4B^2 + b^5SB$
4. Square Root Cross Product: $Y = a - b_1S - b_2B + b_3\sqrt{S} + b_4\sqrt{B} + b^5\sqrt{SB}$

Table 4 gives the results of fitting them to the three sets of data.

The linear form, although not a neoclassical function, is included to show the strength of the claim by linear programmers, etc., that the assumption of linearity is often very plausible.

The Selection of the Equation Form

(*a*) Statistical criteria

Where reliable information is available on the basic production logic, it supersedes the statistical criteria which otherwise would be decisive. Taking the latter first, the statistical measures of goodness of fit used in this study are various formulations of R^2, the correlation coefficient, viz. R^2, $\bar{R}^2$, $(R^2)'$ and $(\bar{R}^2)'$. $\bar{R}^2$ is R^2

19. One such equation, the Spillman-Mitscherlich, takes the form $Y = M(1 - R_s^s)(1 - R_b^b)$ where R is ratio (less than unity) by which the marginal product of *S* and *B* decline with increasing inputs of *S* and *B* and *M* is the maximum response possible from increasing both factors. At high input levels R^s and R^b become insignificant so that *Y* approximates closely to *M*. Fitting the form involves an iterative process which postulates a value for *M* before calcaulting *Rs* and *Rb* and revising the *M* value until no further changes improve the results. It is also of interest as being the earliest form of production function and was suggested by Von Thunen.

20. Evaluating $f(x)$ at $x = a$ to one term gives $y = f(a) + f^1(a)\frac{x - a}{1}$ which is reduced to $[f(a) + f^1(a)(-a)] + f^1(a)(x)$ or $y = k + mx$ (the linear form), see E. O. Heady and J. L. Dillon, *op. cit.*, p. 204 *et seq.*

TABLE 4: *Regression Coefficients and related statistics for selected forms of production functions based on unweighted physical data*

Equation Values			Independent Variable			
Goodness of fit (1)	*Constant Term*	*Name of variable (2)*	*Value of Coefficient*	*S.E. of Coefficient*	*'t' value*	*Level of (3) Significance*
			LINEAR			
			both sexes			
·8821	−15·6204	*S*	·38475	·02104	18·29	****
(·8801)		*B*	·21222	·00775	27·38	****
			LINEAR			
			male			
·9211	−26·6605	*S*	·34568	·02581	13·40	****
(·9184)		*B*	·22944	·00920	24·94	****
			female			
·8663	·9870	*S*	·42282	·03038	13·92	****
(·8616)		*B*	·19113	·01160	16·47	****
			COBB-DOUGLAS			
			Both Sexes			
·9047	·4051	*S*	·25372	·012156	30·87	****
(·9031)		*B*	·75401	·02445	30·83	****
			male			
·9421	·2974	*S*	·22148	·014212	15·58	****
(·9401)		*B*	·82225	·027850	29·52	****
			female			
·9020	·6023	*S*	·28508	·016727	17·04	****
(·8986)		*B*	·67428	·034621	19·48	****
			QUADRATIC CROSS PRODUCT			
			Both Sexes			
·9102	−79·485	*S*	·750016	·188968	3·97	****
(·9062)		*B*	·254743	·095907	2·66	****
		S^2	−·001040	·000238	4·37	****
		B^2	·000030	·000028	1·07	
		SB	·000151	·000074	2·05	***

TABLE 4—*continued*

Equation Values		*Independent Variable*				
Goodness of fit (1)	*Constant Term*	*Name of variable (2)*	*Value of Coefficient*	*S.E. of Coefficient*	*'t' value*	*Level of (3) Significance*
			males			
·9403	−190·672	*S*	·545522	·231225	2·36	****
(·9348)		B^2	·435941	·124361	3·51	****
		S^2	−·000542	·000295	1·84	*
		B	−·000078	·000036	2·16	***
		SB	·000073	·000092	0·79	
			females			
·9204	−38·125	*S*	·932952	·248792	3·75	****
(·9130)		*B*	·180618	·120602	1·50	****
		S^2	−·001475	·000310	4·76	****
		B^2	−·000019	·000036	0·54	
		SB	·000220	·000097	2·28	***
			SQUARE ROOT			
			males			
·9414	−770·441	*S*	−·252273	·328456	0·77	
(·9360)		*B*	−·254205	·202413	1·26	
		$\sqrt{S}$	13·9566	14·6531	0·95	
		$\sqrt{B}$	34·1481	17·9386	1·90	****
		$\sqrt{SB}$	·147763	·227839	0·65	****
			females			
·9232	−238·713	*S*	−1·22606	·346125	3·54	****
(·9161)		*B*	·062711	·203388	0·31	
		$\sqrt{S}$	32·9105	15·8321	2·08	*
		$\sqrt{B}$	·308931	17·6532	0·20	
		$\sqrt{SB}$	·545814	·242295	2·25	****

1. First value relates to R^2, second value i.e. within brackets related to adjusted R^2.
2. *S* means Soya bean meal and other protein rich feed
   B means barley; both measured in fodder units.
3. **** significant at 0·1%; *** at 1%; ** at 5%; * at 10%.

adjusted for the number of degrees of freedom. Since the Cobb-Douglas is normally estimated in logarithmic form

$$\log Y = a + b_1 \log S + b_2 \log B$$

the resultant R^2 relates to actual and calculated values of log *Y*. To achieve comparability with other R^2 based on *Y* a new R^2 is calculated,[21] based on actual

21. See J. L. Pratschke, "Adjusted and unadjusted R^2—Further Evidence from Irish data" *ESRI Memorandum Series*, No. 67.

and calculated values of Y itself, i.e., $(\bar{R}^2)'$. Adjusted in the normal way for degrees of freedom, this becomes $(R^2)'$. The results, which do not appear in Table 4, can be compared with the $\bar{R}^2$ for the quadratic and square root polynomials:—

TABLE 5: *Results from various formulations of the correlation coefficient applied to the Cobb Douglas production function compared with those for other forms of equations*

Equation for	R^2	$\bar{R}^2$	$\bar{R}^{2\prime}$	$\bar{R}^{2\prime}$	$\bar{R}^2$	$\bar{R}^2$
		Cobb-Douglas			*Quadratic*	*Sq. Root*
both sexes	·9047	·9031	·9066	·9034	·9062	·9130
males	·9421	·9401	·9367	·9345	·9348	·9360
females	·9020	·8986	·9069	·9037	·9360	·9161

The Pratschke correction, $(R^{-2})'$, shows the fit of the Cobb-Douglas to be only very slightly inferior to the alternative polynomials. These latter contain parameters whose level of significance is lower than those of the Cobb-Douglas. Should these be eliminated, the modified fit might be less good than that of the Cobb-Douglas. Clearly, additional criteria for selection are required.

(b) Logical Criteria

As mentioned earlier, each function has implications for production logic in terms of the type of returns to be expected, marginal productivity of inputs and rates of substitution between them. To expedite the selection process the characteristics and implications of each function have been tabulated. In addition, Figs. 1, 2 and 3 present the isoquants and isoclines associated with the equations for females (weighted[22] in the case of the quadratic and square root functions.) From the tabulation it is apparent that, in many ways, the square root function is a compromise between the Cobb-Douglas and Quadratic forms. Many features of the Cobb-Douglas, such as constant returns to scale and no defined maximum, conflict with such knowledge as exists on production logic, and warrant its rejection as an unsuitable form. Figure 4 shows that the Cobb-Douglas response curve tends to flatten out as input increases, so that, even where (as in this case)[23] diminishing returns to scale occur, no maximum is defined. Unless an economic optimum is defined for small magnitudes of input, the Cobb-Douglas will overestimate the input requirements which equate marginal revenue and marginal

22. For an explanation of weighting see below.

23. This figure illustrates the shape of the response curve when protein is held constant at 150 units annually and is based on the equation:—L(value of output) = 3·1 $S^{\cdot 3} B^{\cdot 56}$.

TABLE 6: *Tabulation of main features of selected forms of the production function and their implications for the respective isoquant maps.*

Characteristics etc.	*Cobb-Douglas*	*Quadratic*	*Square Root*
1. Returns to scale (in this specific case)	Constant (sum of exponents $=1\cdot0\pm\cdot04$)	Diminishing and negative at high inputs	
2. Marginal productivity of inputs	Declines at declining rate	Decline at constant rate	Decline at declining rate
3(a) Declining total product	Impossible	Yes	
3(b) maximum	none	clearly defined	
4. Elasticity of production	constant	declines at constant rate	declines at declining rate
5. Rate of input substitution	constant	ranges from zero to infinity	
		declines at constant rate	declines at declining rate
6. Optimum mix of inputs at different output levels	constant	changing proportions	
7. Zero inputs yield	zero output	negative output	
8. Production possible on one input	no	yes, on barley alone	
9. Low levels of both inputs yield	small positive output	negative output below threshold input levels	
*Feature of Maps**	*GEOMETRIC PRESENTATION*		
Interval between isoquants (1.)	constant	widening	
Ridgelines (3b, 5)	same as axes	linear converging on maxium	curvilinear converging on origin and maximum
Isoclines (3a, 6, 7, 8)	linear fan out from origin	linear converging on maximum	curvilinear converging on origin and maximum
Isoquants at low levels of output (9)	asymptotic to axes	cut axes	tend to be asymptotic but may cut axes near origin

*Figures in brackets after the feature relate to the corresponding characteristics in the upper part of the tabulation.

Figs. 1 to 4: *Fig 1, 2 and 3. Isoquants, isoclines, etc. derived from various forms of production function based on equations for female pigs only. Fig. 4 Response curve for Cobb-Douglas function holding protein constant at 150 units per annum.*

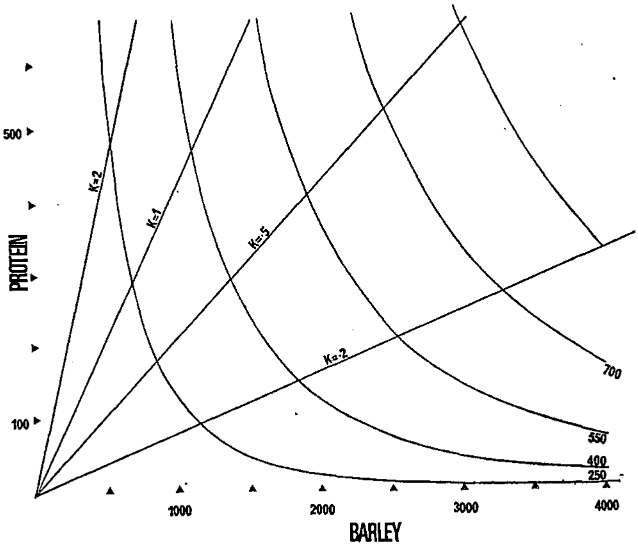


Fig. 1: *Cobb-Douglas unweighted*

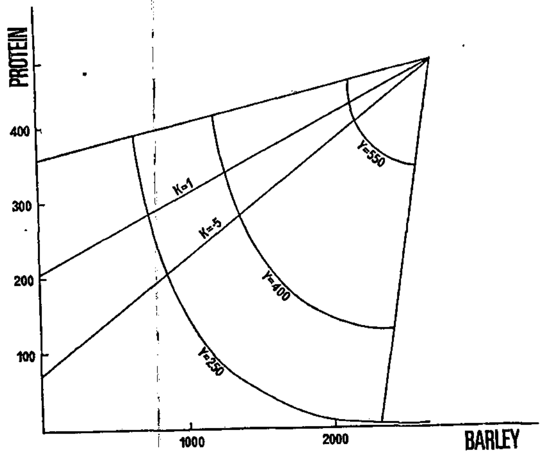


Fig. 2: *Quadratic weighted.*

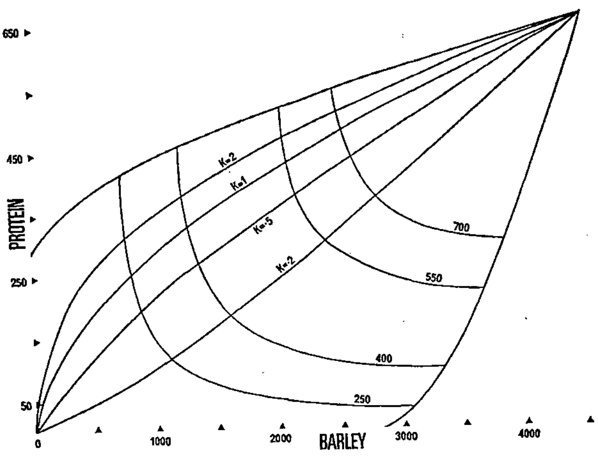


Fig. 3: *Square Root weighted.*

Protein constant at 150 f.u.

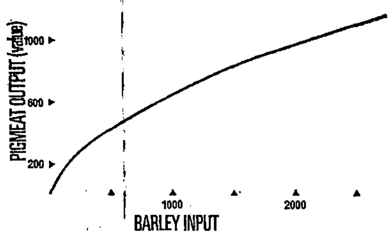
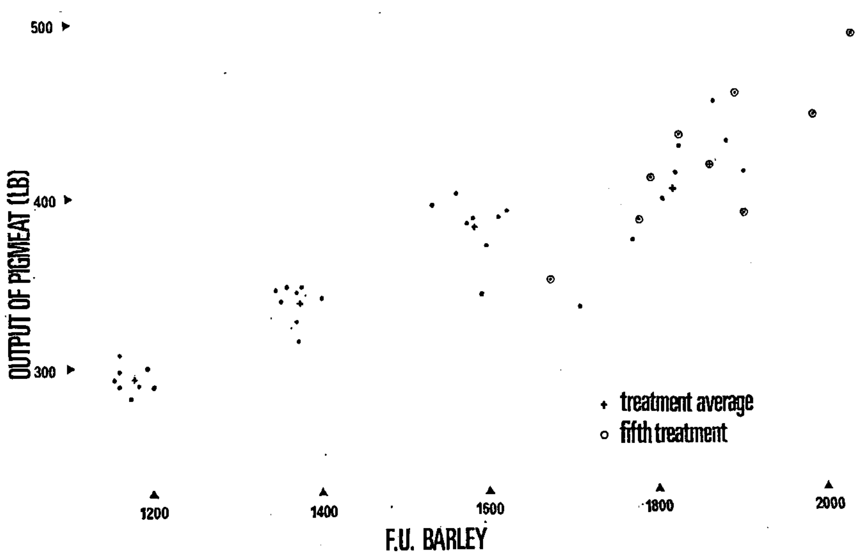


Fig. 4: *Response curve* $L = 3.1\ S^{13}\ B^{56}$

cost. Modifications which would make the Cobb-Douglas more flexible, make it also difficult to compute.

The possibility of output based on one input is a strong point in favour of the quadratic and square root functions. However, Australian research[24] found that, while pigs performed exceptionally well on skim alone, they failed to fatten on a diet exclusively of wheat—the reverse of what is implied by the Danish work. The concept of a threshold quantity, necessary for maintenance, after which the pigs grow, is also reasonable.

Fig. 5: *Scatter diagram showing Annual Output Response (lb. dwt.) from 150 f.u. of Protein and variable amounts of Barley.*



Both the quadratic and square root functions defined maxima which lay far beyond the range of observations.[25] This meant that before a choice could be made between these functions a more fundamental question needed answering: did the data cover the relevant range of inputs? Scatter diagrams, such as Figure 5, were plotted to relate barley intake to pigmeat output at given protein levels. These showed that, at lower levels of energy, the ability of the pigs to consume was fairly constant, though the ability to convert barley to pigmeat varied considerably. Thus, at the 150 protein level, energy intake at barley level 2 ranged from 1,221 to 1,256 units, while output varied from 323 to 391 lb. In this case the "pig" consuming 1,221 units produced the 391 lb. of output. At level 4 of barley, the range in intake was greater, i.e., 1,637 to 1,745 while the variation in output was less, i.e., 374 to 434. At the highest level of barley (level 5), the

24. G. E. Battese, L. H. Duloy, J. M. Holder, and B. R. Wilson, "The determination of optimal rations for pigs fed milk and grain", *Journal of Agricultural Economics*, Vol. XIX, No. 3, 1968.

25. The physical maxima of the quadratic were 687 lb for males and 1514 lb for females.

ability to consume was even more varied, 1,713 to 1,938; and output was again dispersed, 395 to 452.

Now it is a well known phenomenon in studies of animal nutrition that the "component on genetic variability increases with increases in the level of feeding plane".[26] On examination, the variances of treatments were found to vary considerably. At the two lower levels of *protein*, there was no significant difference in the levels of intake or output at the two highest levels of *barley* feeding. At the highest level of protein, there were significant differences but the difference in output was less significant than that for intake. Clearly, the pig's appetite, and its capacity to convert food to meat, were becoming even more significant elements with increasing intensity of feeding. Protein was a limitational factor at higher levels of barley; but increases in protein tended to emphasise the disparity in the performance of the individual pigs. This aspect of the experiment seemed to imply that very little would be gained by extending the levels of treatment in the way economists often suggest. The presence of heteroscedasticity, or unequal variances, would vitiate the results of fitting the equations to the data. To overcome this, a weighted regression was fitted, using as weights the inverse of the within treatment variances in the dependent variable. Since, for a given barley treatment, the actual amount of energy received can vary between animals, the above estimates of error variance may be biased upwards because part of the within treatment variation is caused by differences in the level of energy received by the animals on a particular treatment. However, the amount of bias involved seems reasonably small (roughly about 10 per cent, or less, of within treatment

TABLE 7: *Regression Coefficients and related statistics for selected forms of production functions based on weighted data*

Equation Values			*Independent Variable*			
Goodness of fit (1)	*Constant Term*	*Name (2)*	*Value of Coefficient*	*S.E. of Coefficient*	*'t' Value*	*Level of Significance*[3]
			LINEAR			
			both sexes			
·9339	−12·63	*S*	·37569	·01599	23·50	****
(·9316)		*B*	·21521	·00792	27·16	****
			males			
·9372	−33·38	*S*	·34786	·02401	14·49	****
(·9350)		*B*	·22762	·00781	29·13	****
			females			
·8939	8·494	*S*	·42270	·026895	15·72	****
(·8902)		*B*	·18703	·01100	17·01	****

26. R. T. Plank and A. Berg, "The heritance and plane of nutrition in swine. 1, Effect of season plane of nutrition, sex and sire on feed lot performance and carcase characteristics", *Canadian Journal of Animal Science, Vol. 33,* 1963.

TABLE 7—*continued*

Equation Values		*Independent Variable*				
Goodness of fit (1)	*Constant Term*	*Name (2)*	*Value of Coefficient*	*S.E. of Coefficient*	*'t' Value*	*Level of Significance*[3]
			QUADRATIC CROSS PRODUCT			
			both sexes			
·9550	−200·96	*S*	·867294	·117040	7·41	****
(·9530)		B_2	·409115	·082666	4·95	****
		S_2	−·001117	·000190	5·87	****
		B	−·000078	·000028	2·81	***
		SB	·000096	·000059	1·64	*
			males			
·9718	−235·40	*S*	·600495	·150559	3·99	****
(·9692)		B_2	·489223	·081805	5·98	****
		S_2	−·000649	·000254	2·56	**
		B	−·000096	·000027	3·52	****
		SB	·000073	·000059	1·23	
			females			
·9574	−196·30	*S*	1.02539	·147456	6·95	****
(·9534)		B_2	·393022	·102425	3·84	****
		S	−·001438	·000219	6·57	****
		B	−·000086	·000034	2·55	***
		SB	·000138	·000077	1·78	*
			SQUARE ROOT			
			both			
·9553	−807·37	*S*	−·871563	·218924	3·98	****
(·9533)		*B*	−·230916	·158464	1·46	
		S	32·2703	7·7707	4·15	****
		B	29·5870	12·5322	2·36	****
		SB	·22362	·14425	1·55	**
			males			
·9717	−894·76	*S*	−·356201	·295204	1·21	
(·9691)		*B*	−·317652	·158644	2·00	***
		S	17·5068	10·1512	1·72	****
		B	39·2182	12·7126	3·08	****
		SB	·139382	·142163	0·98	
			females			
·9559	−854·10	*S*	1·20424	·250431	4·81	****
(·9518)		*B*	−·290338	·192979	1·50	
		S	40·2370	9·59303	4·19	****
		B	30·2445	15·5239	1·95	****
		SB	·330880	·192844	1·72	**

(1) First value relates to R^2, second value i.e. within brackets relates to adjusted R^2.
(2) *S* means Soya bean meal and other protein rich feed;
B means barley; both measured in fodder units.
(3) **** significant at 0·1 %; *** at 1 %; ** at 5 %; * at 10 %.

variation), as indicated by the scatter diagrams plotted. The results[27], given in Table 7 are based on separate weighting factors calculated for males and females separately and combined.

The choice between the quadratic and square root form was made in the light of the weighted equations (which are those used to plot Figures 2 and 3). The maximum levels of physical output were, in lb. annually:

	Males	*Females*	*Both*
Quadratic	634	576	622
Square Root	817	677	804

The tendency for the square root form to predict higher levels of maximum output was repeated when optimum outputs were calculated over the range of likely movements in price ratios and between the inputs themselves. Some typical results are shown in Table 8.

TABLE 8: *Optimum inputs and output at various price levels is based on the quadratic and square root functions for female pigs only*

Unit prices of			*Optimum Inputs and Outputs*					
			Quadratic			*Square Root*		
Y	*S*	*B*	*S*	*B*	*Y*	*S*	*B*	*Y*
shillings per cwt.								
250	28	24	416	2,055	543	456	2,332	581
250	33	33	399	1,599	480	420	1,921	530
250	48	24	387	2,031	536	403	2,279	567
300	28	24	428	2,156	553	482	2,565	604
300	33	33	413	1,971	534	448	2,164	562
300	48	24	404	2,137	548	434	2,514	594
Physical Maximum			482	2,667	576	670	4,467	677

In the experiment, protein inputs ranged from 150 to 420, barley inputs from 888 units to 2,037 units. The highest level of output was 568 lb. (for females, 543 lb.). Quadratic optima tended to be within the range of observations; those of the square root above the upper limit. The performance of the individual pigs on the high intensities of feeding suggested that an upper limit has been

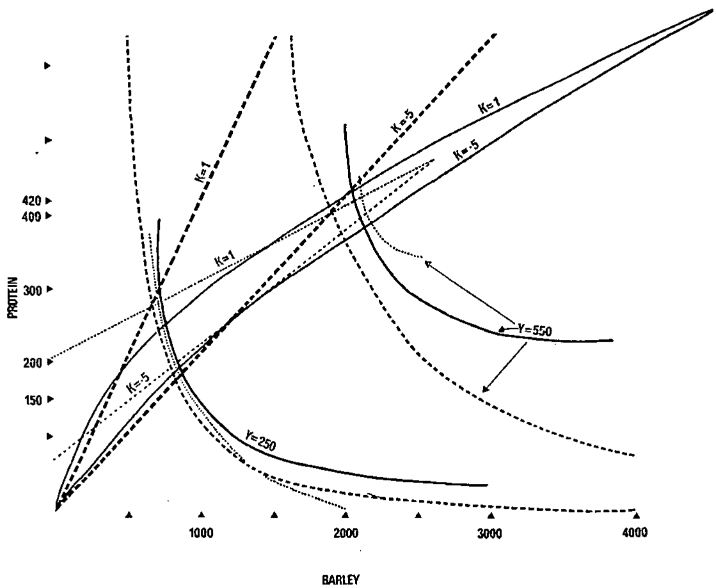
27. Table VII gives no results for the Cobb-Douglas for reasons given earlier. However, if the Cobb-Douglas was satisfactory from a production logic viewpoint, the calculation of weights based on the inverse of the variance would have presented some difficulties.

reached, and that the quadratic, with its lower maximum levels, reflected better the underlying realities.

Isoquant and Isocline Analysis

An interesting sidelight on the selection process is obtained by juxtaposing Figures 2, 3 and 4 and examining the map in the area of economic interest, i.e., between the 250 and 550 lb. isoquants, and between the isoclines appropriate when the unit price of protein is the same as, and when it is twice, that of barley, i.e., $k = 1{\cdot}0$, and $k = 0{\cdot}5$. The 250 isoquants are very similar and the inter-

Fig. 6: *Superimposition of selected isoquants and isoclines from Figs. 1, 2 and 3.*



sections with the $k = 0{\cdot}5$ isoclines are very close together. The intersection of the $k = 1$ isoclines are very similar for the quadratic and Cobb-Douglas. The square root predicted the same barley requirements but less protein. This low intensity of output, and (over the likely range of price relationships) the consequences of using the input combination appropriate to a different function, would involve minimal differences in costs. This also applies at the high levels of 550 lb. annual output in the case of the quadratic and square root functions; but the requirement of the Cobb-Douglas, that the isocline fan out from the origin, resulted in its calling for much larger protein inputs. If the indifference map for the equations for males were juxtaposed, a much closer measure of agreement would be discovered. This means that, in this case study, failure to select the most appropriate form of equation may not lead to greatly inferior managerial decisions.

Regression based on group averages

Geary[28] has shown that, where residuals are regular, grouping of observations leads to very little loss of efficiency in the estimates. In this study the residues were not regular. Nevertheless it seemed worthwhile to experiment with a regression based on the 15 group averages. These averages would replace the scatter of observations at the higher levels of input by an estimation of central tendencies for each treatment. In particular, it would emphasise the flatness of the upper end of the regression line.

TABLE 9: *Regression Coefficients and related statistics for the Quadratic Cross Product form of function based on treatment averages*

Equation Values		*Independent Variable*				
Goodness of fit (1)	*Constant Term*	*Name (2)*	*Value of Coefficient*	*S.E. of Coefficient*	*'t' Value*	*Level of (3) Significance*
			Both Sexes			
·9953	−239·5568	S	·79905	·14503	5·51	****
(·9922)		B	·47714	·08882	5·37	****
		S^2	−·00099	·00018	5·49	****
		B^2	−·00010	·00003	3·86	****
		SB	·00010	·00006	1·69	
			Males			
·9946	−305·3164	S	·62414	·16682	3·74	****
(·9910)		B	·58270	·09871	5·90	****
		S^2	−·00052	·00021	2·49	**
		B^2	−·00012	·00003	4·23	****
		SB	·00001	·00007	0·18	
			Females			
·98999	−163·1838	S	·94533	·20650	4·58	***
(·9832)		B	·36250	·12929	2·80	***
		S^2	−·00143	·00025	5·63	****
		B^2	−·00008	·00004	2·05	*
		SB	·00020	·00008	2·31	****

N.B. The other forms of equation were also fitted to the treatment average data.

(1) First value relates to R^2, second value i.e. within brackets relates to adjusted R^2.
(2) *S* means Soya bean meal and other protein rich feed;
B means barley; both measured in fodder units.
(3) **** significant at 0·1%; *** at 1%; ** at 5%; * at 10%.

28. R. C. Geary, "Effect of grouping on the efficiency of least squares regression—study of a Sample case," Appendix to Pratschke *op. cit.* (1969).

The parameters obtained, and presented in Table 9, were used to predict both the physical maxima and the optima under one set of prices (from Table 7, i.e., 250, 28 and 24) and the results compared with those derived from the equations based on the weighted individual observations.

TABLE 10: *Comparison of predictions derived from equations based on treatment averages and those based on weighted individual observations.*

	Treatment Averages				Weighted Individuals			
	Max.	*Optimum*			*Max.*	*Optimum*		
	Y	*Y*	*S*	*B*	*Y*	*Y*	*S*	*B*
Males	592	566	513	2,013	633	601	503	2,248
Females	599	561	438	2,147	576	543	416	2,055
Both Sexes	586	577	451	2,063	622	586	436	2,270

The weighted regressions predict higher output and higher barley input than do the treatment average equations relating to males and both sexes. All weighted equations indicate lower protein inputs. Compared with the group average equations, the weighted equations displayed a considerable difference in the maximum and optimum output of males *vis-à-vis* females. Nevertheless, being based on all the data, the weighted regression estimates were preferred. The striking feature of the group average regressions was the similarity of their predictions to those of the weighted equations. In this they were clearly superior as predictive tools to the unweighted equations.

Selection of Variables

Having selected an equation form, the question remains—how to treat variables whose coefficients are not significant at, say, the 10 per cent level (B^2 in two instances and *SB* in the case of males only)? There is no sure answer to this, since research workers may quite rationally differ in the weight they attach to logical and statistical criteria. Some prefer to start out with an initial hypothesis about the appropriate algebraic form of the equation, and tend to retain all coefficients, even if their standard errors are relatively large. To them, the predictive power of the entire equation is more important than the interpretation of the individual regression coefficients. Others will only retain those coefficients which are significant at the conventional 5 per cent probability level.

In this study all coefficients were retained, partly because this appeared to agree with production logic, and partly because the element to be eliminated differed from equation to equation and it seemed better to retain a standard equation form. To be more specific, B^2 was retained because a curvilinear form for barley was deemed more appropriate than a linear form. *SB* was retained because the positive reaction between barley and protein, which it implies, was found to be appropriate when the data was subjected to an analysis of variance at the outset of the study. It should be noted that the fitting of quadratic equations commonly results in only the linear term having significant coefficients.

Backfat Thickness as an Index of Quality

So far the discussion has assumed that one lb. of pigmeat output is as good as another; and the recommended procedure has been in general to feed high levels of both protein and barley. This assumption can now be relaxed and cognisance taken of the influence of feeding on quality. As this was a Danish experiment, the thickness of backfat was presented as a measure of quality.

Average backfat thickness is not an adequate measure of grade under Irish conditions, so the relationship between it and Irish grading needs to be established. Irish regulations use four grading criteria—length and backfat thickness at shoulder, midback and loin. A report by Lucas, McDonald and Calder[29] stated that a reduction in the level of feeding was unlikely to affect length, but would probably result in reductions in shoulder and backfat thickness, up to a maximum of 10 and 6 mm. respectively.

If length is unaffected by feeding plane, the problem reduces to finding the probability that pigs of a particular backfat thickness will belong to a specified Irish grade based on three fat measurements. This was estimated by studying the grade distribution of 1,082 Irish pigs for which backfat measurements were available.[30] For example, the study reported 128 pigs with backfat thickness in the interval 30·8 to 31·8 mm. Of these, 31 were "A special", 96 "A" and 1 "B".

Application of the appropriate grade prices would give the weighted average price to be expected for this thickness interval. Since the experimental results recorded the backfat thickness of each pig, all the information was now available to calculate the value of output. There is, however, one difficulty. Not only does the absolute level of the pigmeat price fluctuate but the margins between grades widen and narrow almost continuously.[31] The problems this introduces are discussed elsewhere.[32] This paper uses the weighted average of weekly quotations published by major bacon curers in the period July, 1968, to June, 1969. In, shillings per cwt. deadweight these were: 289, 280, 259, 249 and 242 for the grades of A special, A, B1 and C respectively.

The two equation approach

The first approach to introducing quality consideration was to attempt to define

29. I.A.M. Lucas, I. McDonald and A.S.C. Calder, "Some further observations upon the effects of varying the plane of feeding for pigs between weaning and becon weight", *Journal of Agricultural Science*, Vol. 54, p. 81 *et seq.*

30. M. Ross, unpublished study, 1965.

31. See M.Ross, *Economics of Pig Production*, Economic Research Series No. 6, An Foras Talúntais. Dublin, 1962.

32. See above, footnote 2.

unit prices of output, P_y, as a function of protein and barley. This was done in two stages

$$P_y = f(F) \text{ and } F = f(S, B)$$

i.e., backfat thickness (F) is a function of the inputs, and unit price (P_y) is a function of backfat thickness.

Using the above prices for the range 29·3 to 45·3 mm. in the backfat scale gave:—

$$P_y = 385{\cdot}26 - 3{\cdot}087\ F \text{ mm.} \qquad \text{(A)}$$

Starting at 285/- per cwt. for pigmeat with a backfat of 29·3 mm., each additional mm. of fat would reduce the price by 3/- a cwt.

The next step was to relate backfat (F) to level of inputs. The most acceptable forms tested were:—

$$F = 23{\cdot}8 - {\cdot}0058S + {\cdot}0086B \qquad R^2 {\cdot}393\ ({\cdot}386) \qquad \text{(B)}$$
$$[{\cdot}0030] \quad [{\cdot}00112]$$

which shows that protein reduces fat, and therefore improves grade, while barley has the opposite effect. The coefficient of S was significant at the 6·0 per cent level; that for B was significant at 0·1 per cent. The value of R^2 is low; but for 120 observations is significant at the 0·1 per cent level. If protein is not regarded as having a favourable influence on fat formation, then equation B could be formulated in terms of barley only.

$$F \text{ mm.} = 21{\cdot}3 + {\cdot}0092\ B \qquad R^2\ {\cdot}377\ ({\cdot}372) \qquad \text{(C)}$$

$$F \text{ mm.} = 27{\cdot}6 + {\cdot}00000317B^2 \qquad R^2\ {\cdot}382\ ({\cdot}377) \qquad \text{(D)}$$
$$[{\cdot}00000037]$$

B and B^2 were significant at the 0·1 per cent level.

Substituting equation B in equation A gives:

$$P_y = 311{\cdot}8 + {\cdot}0179S - {\cdot}0265B. \qquad \text{(E)}$$

The use of protein raises the unit price while feeding barley lowers it, through their respective influences on fat formation.

Optimising the Product of Two Functions

The next step is to find the optimising strategy. Given as before that profit is the margin between revenue and cost, i.e.

$$\pi = P_y Y - (P_s S + P_b B) - \Sigma_1^n P_j X_j\ (j = 1, n)$$

where π is profit and the summation sign represents all other costs which are more or less held constant, their optimising conditions are:—

$$\frac{\partial \pi}{\partial S} = P_y \frac{\partial Y}{\partial S} + Y \frac{\partial P_y}{\partial S} - P_s = 0$$

$$\frac{\partial \pi}{\partial B} = P_y \frac{\partial Y}{\partial B} + Y \frac{\partial P_y}{\partial B} - P_b = 0$$

Where P_y is a constant (i.e., grading is not considered) the middle term disappears, and we are left with the partial derivatives of Y equal to the inverse of the prices. When P_y is itself a function of S and B, the equations become more complex. Substitution of the values for Y and P_y, and collecting the terms, results in a pair of simultaneous quadratic equations in S and B.

$$266{\cdot}83 - {\cdot}665S + {\cdot}014B - {\cdot}00006S^2 - {\cdot}000004B^2 + {\cdot}00006SB = P_s \quad \text{(F)}$$
$$132{\cdot}98 + {\cdot}014S - {\cdot}070B + {\cdot}00003S^2 - {\cdot}000006B^2 - {\cdot}000008SB = P_b \quad \text{(G)}$$

Given the values of P_b and P_s, these equations can be solved by an iterative process.

The alternative iterative process assumption—price unrelated to protein

An analysis of variation indicated that there was a significant positive relationship between the level of protein consumption and backfat. This relationship was not strongly confirmed by equation B above, where the coefficient of S was only significant at the 6 per cent level.[33] If protein is not regarded as influencing backfat, equation E must be recalculated to incorporate equation C:—

$$P_y = 319{\cdot}52 - {\cdot}0284B \quad \text{E(a)}$$

which in turn would yield two further quadratic equations F(a) and G(a) for solution by the iterative procedure.

The above presentation assumes a linear relationship between price and backfat. A regression based on 1962 prices[34] took the form:—

$$P_y = -652{\cdot}7 + 82{\cdot}8F - 2{\cdot}45F^2 + {\cdot}02F^3 + {\cdot}000003F^4.$$

If equation B was substituted for F in each term the technical difficulties in obtaining an optimum would prove insoluble.

33. In the co-ordinated trials in Britain, which covered many different locations, breeds of pigs, housing and management conditions but a uniform feeding regime protein, the level of protein was not found to be related to quality, but the experiment was somewhat different, see R. Braude, M. Townsend, G. Harrington and J. G. Rowell, "Effects of different protein contents in the rations of growing fattening pigs", *Journal of Agricultural Science*, 55, 2, 1960.

34. See below, p. 249.

The Quasi-Production Function Approach

An alternative approach to incorporating quality would be to value output at the unit price appropriate to its associated backfat, and to calculate a new set of regressions with "value of output" as the dependent variable, in preference to "volume of output". Since quality is inversely related to the level of barley fed, the lower unit prices of higher outputs would have the effect of dampening down

TABLE 11: *Some derived data relating to various equations forms fitted to data giving in input physical terms and output in monetary terms (prices used for optimising were $P_s=28$, $P_b=22$)*

Sex	Maximum Output*		Optimum Output		Optimum Inputs		Coefficients of Determination†
	Financial	Physical	Financial	Physical	Protein	Barley	
			QUADRATIC FORMS				
			group averages				
Both	1,260	559	1,225	520	442	1,773	·9860 (·9789)
Male	1,145	522	1,125	486	382	1,786	·9724 (·9571)
Female	1,266	557	1,228	512	418	1,771	·9833 (1·9740)
			Unweighted individual observations				
Both	1,312	639	1,266	556	440	1,922	·8309 (·8234)
Male	1,298	796	1,258	553	508	1,837	·8487 (·8347)
Female	1,310	624	1,263	539	421	1,883	·8626 (·8499)
			Weighted individual observations				
Both	1,316	586	1,270	541	466	1,899	·8925 (·8878)
Male	1,234	547	1,210	506	370	1,735	·9258 (·9189)
Female	1,280	543	1,244	509	440	1,767	·9294 (·9229)
			SQUARE ROOT FORMS				
			group averages				
Both	1,329	610	1,263	537	523	1,815	·9877 (·9809)
Female	1,153	540	1,133	497	458	1,588	·9756 (·9620)
Male	1,334	622	1,260	530	462	1,854	·9840 (·9751)
			Unweighted Individual Observations				
Male	1,481	696	1,363	591	812	1,796	·8518 (·8381)
Female	1,397	748	1,300	564	470	1,984	·8637 (·8511)
			Weighted individual observations				
Both	1,440	656	1,337	567	579	2,002	·8935 (·8888)
Male	1,371	632	1,287	549	295	2,159	·9294 (·9229)
Female	1,360	588	1,286	526	504	1,852	·9300 (·9235)

*Obtained by inserting the *S* and *B* values derived from the quasi production function into the corresponding physical production function.

†Figures in brackets relate to $\bar{R}^2$.

the upper levels of the response curve. It might also increase the variance. The degree of "flattening" would be most marked when price differentials between grades are widest.

All the equation forms examined in defining the production function proper were fitted once again to this quasi-production function—quasi in the sense that output was not in physical, (Y), but in monetary, ($P_y.Y$), terms. It would be tedious to give all the details for each equation fitted. Instead, some derived information about each form is presented in Table XI. The linear and Cobb-Douglas forms are omitted since neither produced meaningful optima.

TABLE 12: *Comparison of optima obtained output is paid for on a graded or ungraded basis*

	Without Quality Payments						With Quality Payments		
	P_y=250			P_y=300					
	Y	S	B	Y	S	B	Y	S	B
Both	586	436	2,270	597	449	2,380	536	464	1,867
Males	601	503	2,248	611	522	2,330	503	373	1,716
Females	543	416	2,055	553	428	2,156	506	438	1,744

The introduction of quality acts as a correction for unequal variance, since both are greatest at the higher levels of barley intake. Thus, the optima obtained using weighted and unweighted monetary data are not as disparate as was the case with the physical output. On the contrary, the percentage difference is so small, less than 4 per cent in the most extreme case, as to be insignificant. It is difficult to compare the results from the quasi function with the results obtained where price is not influenced by grade, since it is not clear which average price would provide a fair comparison with the built-in prices in the quasi production function. For that reason, the optimum was based on average prices of both 250 and 300 shillings per cwt. In all cases the weighted forms of the equations have been used. The main result of introducing grade is to cut back barley feeding. The more conservative reduction, where $P_y = 250$, ranged, nonetheless, from 15 to 25 per cent and was due to the negative correlation between barley inputs and quality. Protein, which tends to be positively correlated with quality, increased, or in the case of males fell, by an amount similar to that of barley.

The greater simplicity of the quasi product approach to introducing quality means that it might be preferred even if it meant new calculations every time the margin between the various grades altered. In practice, sensitivity analysis might reveal that the optimum would not be seriously altered over a considerable range of variations in the price relationship between the individual grades.

The consequence of the alternative approaches to quality

To test the power of these alternative approaches to quality the unit prices of protein and barley of 28 and 24 shillings per cwt. were assumed, with the following results:

TABLE 13: *Comparison of optima obtained under various assumptions about quality (calculated for both sexes only)*

	S	*B*	*Y*
1. No quality payments, fixed output price 250/-	436	2,270	586
2. No quality payments, fixed output price 300/-	449	2,380	597
3. Quality payments—quasi production function	464	1,867	536
4. Two equation approach: protein improves quality[35]	443	1,965	550
5. Protein does not improve	424	1,924	542

The optimal strategy of the "two equation" approach called for less protein, more barley, and higher output than the quasi production function. On the other hand, their optimal barley levels were far below those recommended where grading was not commercially practised. It is for the analyst to decide whether the differences in the last three methods are significant in relation to other uncertainties in the production and sales processes.

In the discussion to date we have attempted to meet the conditions laid down above for selecting a production function which would be an error free guide to decision-making—all inputs to be included, the observations to cover the relevant range and the parameters to be specified without error. The form we have selected, if correctly specified, is nonetheless a statistical, rather than a logical function. This means that, strictly speaking, it only applies to the population to which it refers.

The purpose of specifying a function is to provide a guide to producers. Correct specification is half the battle. There still remain the problems of predicting prices and yields. The short lifespan of pigs, and the system of guaranteed prices, mean that reasonably accurate estimates of the general price level can be made. Predicting intergrade margins, while less easy, may still be adequate for management purposes.

Greater uncertainty will normally apply to predictions of yield. Conditions facing producers can never be exactly duplicated in experiments, so there is always some error in transferring experimental data to commercial situations.[36]

35. An interesting academic sidelight on the results is that given a constant unit price of output the marginal rate of substitution between barley and protein at the optimum which would be 0·786 for a price relationship of 28:22 (i.e. 22÷28) was, in fact, 0·436 when price was influenced by the inputs of protein and barley.

36. If the data are gathered by surveying actual producing units, fundamentally the same problem will exist. There will always be some more or less relevant discrepancies between conditions faced by sample producers at the times they are surveyed and those faced by producers who ultimately use the results.

In addition to the problem of transfer, in this experiment the practice of individual feeding of pigs is unlikely to be commercially viable. Normal group feeding will mean delays in replacing pigs (until most of the pen reach bacon weight), greater feed intake and greater variability of performance (intake of individual pigs will vary with their aggressiveness in the group).[37] The possibility of delays in refilling pens, from these or other causes, can be allowed for by altering the assumption of a continuous throughput which underlies the construction of the dependent variable on page 229. A constant delay of, say, 14 days in replacement would have its greatest repercussions on the annual output of fast growing pigs and would depress the higher levels of output more than the lower. Optimum levels would tend to be lower as a result.

The unmeasurable genetic characteristics of the pigs in the experiment introduce other difficulties. Experimental error due to variations between the animals may lead to false estimates of the "true" production function. More important, the decision-maker will not possess ex ante information of the genetic potential of his pigs, though he may have probability estimates of the range. Clearly, an attempt to transfer the results to Irish conditions would also be fraught with hazards. The presence of risk and uncertainty might indicate decision theory as a more appropriate method of analysis than the neoclassical function.

"Mongrel" curves

Two further sources of potential error need to be considered. At the outset the possibility of relating cumulative liveweight gain to cumulative inputs was ruled

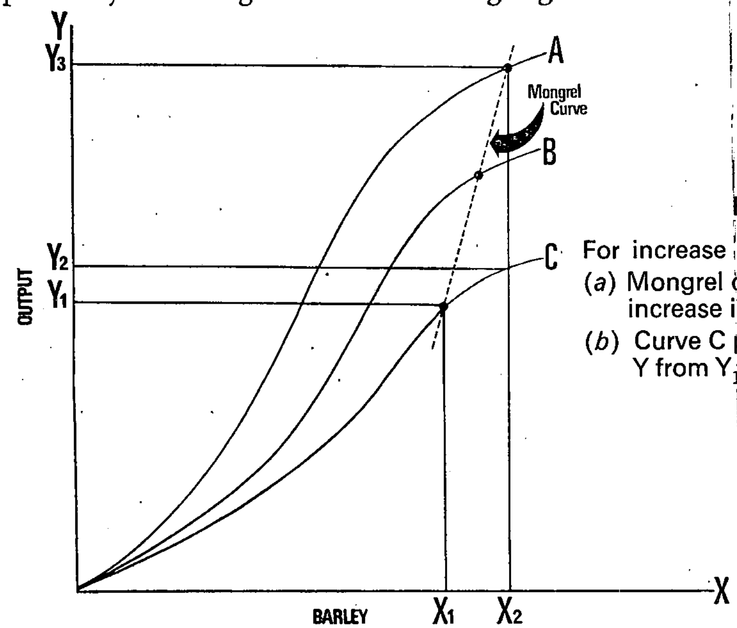


Fig. 7.
For increase from X_1 to X_2
(*a*) Mongrel curve predicts increase in Y from Y_1 to Y_3
(*b*) Curve C predicts increase in Y from Y_1 to Y_2

37. See Per Jonsson, "Investigation of group feeding versus individual feeding and on the interaction between genotype and environment," *Acta Scandinavica*, Vol. 9, 2, p. 208, 1959.

out for lack of data. However, such a function may be a better predictor of the optimal strategy than a function based on one point on each pig production function. This point was first emphasized in the classical study by Jensen,[38] et al. of dairy production functions. In Figure 7 the Y axis might represent output and the X axis input of barley at a constant level of protein. A, B and C are three pigs with different genetic capacities. If the only observation available for each curve was respectively, a, b and c, which some optimal points on their respective curves it might be assumed that they all lie on one production function, such as $R^1 - R^{11}$. It is clear that the marginal productivities calculated from this "mongrel" function would be gross overestimates, e.g., an increase of barley from X_1 to X_2 would appear to increase output from Y_1 to Y_3, whereas for each curve the increase in Y would be much less (from Y_1 to Y_2 in the case of curve C).

In the classical presentation by Jensen, cows were grouped on their own dairy merit, and also on the managerial ability of those running the various experimental stations. Even within the station, dairy merit would differ from cow to cow, and lead to a similar situation. The data in Figure 7 could just as easily be the response to fertiliser, of three different varieties of barley, the response of the same variety on the same soil in different years. Thus, care must be taken to ensure that the points all lie on the same production function if biased results are to be avoided. The group averages, therefore, may have provided a less biased estimate of the function than the use of each observation treated separately. In an analogous sense, failure to recognize what the observations represented led to much misdirected study of supply and demand functions in the 'twenties and 'thirties of this century.

Parameters as Random Variables

The Hicksian model also assumes that the parameters cannot be random variables. Battese, *et al.*,[39] observed that the qualities of feed consumed were a function of both of the experimental diets and the decisions of the animals. This would make them (grain and separated milk) endogenous variables, the predetermined variables being the plane of feeding. To avoid the simultaneous-equations bias in the parameter estimates, Battese, *et al*, treated the quantities consumed as functions of the diets offered the pigs. Even so, they did not ensure full independence between the quadratic equations they employed to explain the quantities of the two feeds consumed.

Concluding Remarks

Where pig production is combined with other farming activities, it might be felt that more satisfactory results would be obtained by taking an overall farm approach, which takes account of the dynamics of feed availability, fixed resources

38. E. Jensen, J. W. Klein, E. Rauchenstein, T. E. Woodward and R. H. Smith, "Input-output relationships in Milk Production" *USDA Technical Bulletin 815*, Washington, 1942.

39. Battese, *op. cit.*

and market conditions. Or it might be felt that the production function which holds other things constant is less useful if the decision relates to changes in the controlled factors. On occasion a programming approach is adopted, which brings with it its own set of statistical and data problems. Generally it is possible to use the production function approach, and optimise so as to ensure equi-marginal returns with other enterprises, provided some estimate exists of what returns are at the margin.

Production function analysis has been subject to considerable criticism. Some of the points at issue have been mentioned above. Others, such as the appropriate goals of the firm (e.g. profit maximization or profit sufficing), and the knowledge of input-output relationships may have more force in studies of industrial firms. If the functions are linear or not continuous, if the objective function and all the constraints are linear, then mathematical programming may be a more appropriate tool of analysis.

Faced with the immense complexity of economic life economists in the Marshallian tradition have become notorious for the qualifications they add to almost every statement. To them the production function was an abstraction which might help clarify their presentation in certain instances. Although often brilliant mathematicians they hesitated to transform the subtle art of decision-making into a science of quantification. Later economists often did not share their timorousness. In the welter of mathematical complexities that followed, the danger loomed large that the production function might become the master of the analyst, not the reverse; and that what was essentially an abstraction might be applied where it did not mirror reality to any great extent.

From the presentation in this paper, it is clear that successful quantification must be based on informed judgement. The selection of a function is often more of an art than science; and interpretation is an exercise in experience and discretion. Though the area of application may change, there can be no substitute for these qualities. If these conditions are met, production function will provide a reasonable guide to efficient resource allocation. In the case study presented above, the optimum feeding for pigs was one which recommended a high level and fast turn over of animals. The optimal diets were remarkably stable, in spite of considerable variations in relative prices, indicating a low degree of substitutability between protein and energy. Identical conclusions were drawn from a study of Australian pigs that had been fed wheat and skim milk, even though genotype management, feed and climate were very different from those of the Danish experiment. Such confirmation is impressive and added to the insights provided by production function analysis ensure it a permanent role in the interpretation, of economic data.

Economic and Social Research Institute, Dublin.