

A Note on the Logarithmic Transformation of the RAS

E. W. HENRY

THE apparently new Least-Squares (LS) minimisation method of R. C. Geary could be applied to the logarithms of the x_{ij} , if the required numerical data were available. For such logarithmic data, the LS λ -solution by Geary has an interesting formal resemblance to the logarithmic transformation of the RAS multipliers r_i and s_j . In the following discussion a comparison will first be made between the form of the logarithmic transform of the RAS and the form of the LS λ -solution. A constructed numerical example will then be given and it will be seen that transforming to logarithms is of no help in calculating the R and S multipliers. Since the logarithm of a negative number is of the form $p + i\pi$, where i is $\sqrt{-1}$, the RAS, for any matrix having one or more negative entries, has its logarithmic transform in the domain of complex variable and is not fully accounted for by the real-variable part alone. Hence a possible explanation of the empirical experience of non-convergence, in attempted iterative solution of problems having one or more negative entries in the inter-industry matrix.

Comparison of Forms

In the Geary notation, let all entries be positive or zero, with ξ_{ij} being the typical original entry, after global scaling. Suppose that the \hat{R} and \hat{S} have been found. Then the RAS matrix is as follows:

$$\begin{bmatrix} r_1 \xi_{11} s_1 & r_1 \xi_{12} s_2 & \cdot & \cdot & \cdot & \cdot & r_1 \xi_{1n} s_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ r_m \xi_{m1} s_1 & \cdot & \cdot & \cdot & \cdot & \cdot & r_m \xi_{mn} s_n \end{bmatrix}$$

There are m rows and n columns.

By taking logarithms, but keeping blanks as blanks and showing separately the matrix of logarithms of the ξ_{ij} , the logarithmic transform is as follows:

$$\begin{bmatrix} \log \xi_{ij} \end{bmatrix} + \begin{bmatrix} \log r_1 + \log s_1, & \log r_1 + \log s_2, & \dots, & \log r_1 + \log s_n \\ \cdot & \cdot & \cdot & \cdot \\ \log r_m + \log s_1, & \dots, & & \log r_m + \log s_n \end{bmatrix}$$

Compare the second matrix with the λ -matrix below, to be super-imposed on some original matrix:

$$\begin{bmatrix} \lambda_1 + \lambda_{.1} + \lambda, & \lambda_1 + \lambda_{.2} + \lambda, & \dots, & \lambda_1 + \lambda_{.n} + \lambda \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_m + \lambda_{.1} + \lambda, & \dots, & & \lambda_m + \lambda_{.n} + \lambda \end{bmatrix}$$

where $\lambda_{.m}$ and $\lambda_{.n}$ are zero.

A correspondence between λ_{ij} and $\log r_i$ can be observed, likewise between $\lambda_{.j}$ and $\log s_j$ and the blank entries in the two matrices coincide. There is, however, an additive constant λ in every non-zero element. It may be pointed out here that λ is the Lagrange multiplier of the linear equation for the overall sum of deviations, in the method of Geary. If instead of this condition, a condition for, say, row m could be substituted, this treatment would remove the λ as such and replace it by a non-zero λ_m , just like any of the other λ_{ij} . The $\lambda_{.n}$ would still be zero, but there would be no constant λ occurring as part of every element.

In numerical solutions having m values r_i and n values s_j only $m+n-1$ of these are independent, corresponding to the $m+n-1$ independent constraints used above in the LS method, and any one of them, such as s_n , taken as unity sets the scale, which determines all the other values of r_i and s_j , with $r_i s_j$ being invariant, regardless of the scale. If $\lambda_{.n}$ which is zero be assigned to $\log s_n$, then $s_n = 1$. Consequently λ must be assigned to $\log r_m$, giving $r_m = e^\lambda$. In the case of the row m condition replacing the overall condition for sum of deviations, r_m would have e^{λ_m} assigned to it. It follows from row m that $s_1 = e^{\lambda_{.1}}$, $s_2 = e^{\lambda_{.2}}$, \dots , $s_{n-1} = e^{\lambda_{.n-1}}$.

Rowwise, there is complete consistency for

$$r_1 = e^{\lambda_{.1} + \lambda}, r_2 = e^{\lambda_{.2} + \lambda}, \dots, r_{m-1} = e^{\lambda_{.m-1} + \lambda}$$

For row m condition being used, λ vanishes. The s_m is the only unit multiplier, all the rest being different from unity.

Since logarithmic transformation must be applied to the ζ_{ij} , as well as to the r_i and s_j in order to permit correspondence of the r_i and s_j with λ_i , λ_j and λ in the comparison of forms shown above, each of these ζ_{ij} must be positive, but their logarithms may be negative. The LS method of Geary accepts negative as well as positive entries, with no confinement to positive entries, for the minimisation procedure.

A Constructed Numerical Example

In the following numerical illustration, the original matrix A has not been globally scaled to have the same grand total as the RAS matrix, in order to avoid the use of rather awkward fractions and logarithms. This global scaling would effectively mean scaling down either each r_i or each s_j , as specified below, by $40/66.75$ and putting a constant additive term ($-\log(40/66.75)$) in each non-blank element of the matrix ($\log r_i + \log s_j$) as given below. The inclusion or omission of this constant term in no way affects the lack of comparability between formulae (A) and (B) below, nor the conclusions obtained.

Suppose that the row multipliers have been found to be

$$r_1 = 1.5, r_2 = 2.0, r_3 = 2.5,$$

and the column multipliers

$$s_1 = 0.5, s_2 = 1.25, s_3 = 1.0.$$

Let the original matrix be:

	Column 1	Column 2	Column 3	Row sum
Row (1)	6	8	10	24
Row (2)	5		3	8
Row (3)	2	6		8
Column sum	13	14	13	40

After multiplying through each row i by r_i and then each resulting column j by s_j , the resulting RAS matrix is as follows:

B

	Column 1	Column 2	Column 3	Row sum
Row (1)	4.50	15.00	15.00	34.50
Row (2)	5.00		6.00	11.00
Row (3)	2.50	18.75		21.25
Column sum	12.00	33.75	21.00	66.75

A transformation of the original matrix to logarithms gives the following:

	Column 1	Column 2	Column 3	Row sum
Row (1)	log 6	log 8	log 10	log 6 + log 8 + log 10 = log 480
Row (2)	log 5		log 3	log 5 + log 3 = log 15
Row (3)	log 2	log 6		log 2 + log 6 = log 12
Column sum	log 6 + log 5 + log 2 = log 60	log 8 + log 6 = log 48	log 10 + log 3 = log 30	log (480 × 15 × 12) = log (60 × 48 × 30)

The matrix of the logarithmic transforms of the r_i and s_j multipliers is the following:

	Column 1	Column 2	Column 3	Row sum
Row (1)	log 1.5 + log 0.5	log 1.5 + log 1.25	log 1.5 + log 1.0	log (135/64)
Row (2)	log 2.0 + log 0.5		log 2.0 + log 1.0	log 2
Row (3)	log 2.5 + log 0.5	log 2.5 + log 1.25		log (125/32)
Column	log (15/16)	log (375/64)	log 3	log (5 ⁴ × 3 ³ /4 ⁶)

The closest approach of the two problems to each other is denoted by formulae (A) and (B) following, confined to comparison by rows. Since for row i of the RAS, $r_i (\xi_{i1}s_1 + \xi_{i2}s_2 + \dots + \xi_{in}s_n) = x_i^1$, this equality can be written

$$r_i \bar{\xi}_i \bar{s}_i = x_i^1 \quad \text{where } \bar{\xi}_i \text{ is } \xi_{i1} + \xi_{i2} + \dots + \xi_{in}$$

and \bar{s}_i is a weighted average of the s_j , the weights being the $(\xi_{ij}/\bar{\xi}_i)$ for row i . It follows that

$$(A) \quad \log r_i + \log \bar{s}_i = \log (x_i / \xi_i)$$

with the right hand side known, being derived from the original and RAS row sums.

For the logarithmic transform, the row sum for row i of the multipliers is

$$n_i \log r_i + \sum_j \log s_j$$

where n_i is the number of non-blank entries in the row and the summation of s_j corresponds to non-blank entries.

Division of row sum i by n_i gives the average effect per non-blank entry of row i

$$(B) \quad \log r_i + (1/n_i) \sum_j \log s_j$$

The left hand side of (A) is not the same formula as (B) and this fact will appear in the numerical application.

Let us compare the right-hand side of (A) with formula (B) result, for the numerical example above. The former is

$$\log (34.50/24), \log (11.00/8), \log 21.25/8$$

which gives, to base 10,

$$(A) \quad 0.15761, 0.13830, 0.32735,$$

to be compared with

$$(1/3) \log (135/64), (1/2) \log 2, (1/2) \log (125/32),$$

which is

$$(B) \quad 0.10472, 0.15052, 0.29588$$

The numerical values (A) could at best be regarded as rough approximations of the (B) values but have no precise observable relationship to them, i.e. the only way of finding the (B) values is to first find the numerical values of r_i and s_j via iteration, for their values supposedly unknown, and then substitute them in formula (B).

Conclusions

The conclusions to be drawn from the above investigations of form and numerical example are brief. The LS solution of one problem can be consistently assigned to the logarithmic transform of the solution of a related RAS problem.

As such it may possibly explain the fact of empirical convergence via iteration towards exact agreement between the tabular entries and the row and column sums, leading to precise estimation of r_i and s_j , provided no negative entries are included in the original matrix. The numerical results have shown the lack of any usable relation between the known row and column sums before transformation and those of the LS problem which gives as its solution the logarithms of the r_i and the s_j . Thus it is not possible to state the LS related problem in numerical terms, from the data of the original matrix and the row and column sums of the RAS matrix, i.e. the solution via logarithms is not possible from the available data. The author has a definite personal preference for the Geary LS method, to be applied to the untransformed data, as being possibly less artificial than the RAS as a method of distributing change, and as being directly calculable via solution of a set of simultaneous linear equations for the λ_i , λ_j and λ .

*Economic and Social Research Institute,
Dublin.*