A Simple Approach to Production Functions via Factor Costs

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Abstract: Some common ground between input-output accounting and Cobb-Douglas (C.D.) production functions is explored. The problem of how to express value added as a function of labour, capital stock and time, is re-stated. For a single production process the accountant's model of the growth of value added is examined and the surplus shown to mean a saving of costs. The same growth of value added is now explained by a C.D. production function which includes an exponential time trend for technical progress. Approximate identification of C.D. parameters with coefficients of the accountant's model is found. Next the form of measurement used for the capital stock is shown to affect the parameters of the C.D. production function—the smaller the apparent volume of capital stock the greater the volume of value added to be explained by technical progress. Fairly obvious conclusions are drawn.

THE theory of production functions and their numerical investigation have occupied a considerable part of published economic literature since the Cobb-Douglas function appeared in the American Economic Review supplement in March 1928. Two recent textbooks, by Bridge (1971) and by Thirlwall (1972) give interesting summaries of the development of the subject up to about 1971 and describe the mounting dissatisfaction with the numerical results. Bridge (p. 395) comments: "Problems have been handled with considerable ingenuity, but it seems that little trust can be placed in the empirical results obtained. In none of the other fields we have reviewed has so little agreement occurred". "Unfortunately, the empirical results for industry production functions

*I wish to thank the Referee for drawing my attention to obscurities and ambiguities of an earlier draft of this essay. My views are subjective and not to be taken as authoritarian or dogmatic in any sense. I am exploring the common ground between input-output accounting (which deals with gross output expressed as a function of all inputs), and production functions (which usually confine themselves to value added). The views of colleagues on how to bridge the gap between the realities of the accountant and those of the Cobb-Douglas or other production function would be welcomed. What follows is a preliminary attack on the problem.

do not suggest that we have isolated anything even approximating the technical relations that the conceptual production function envisages." (p. 397).

The following quotation, from Thirlwall (p. 49) brings us to the crux of the measurement problem. "It was not until Abromovitz in 1956 and Solow in 1957 showed that between 80 and 90 per cent of the growth of output per head in the American economy over the century could not be accounted for by increases in capital per head that the production function started to be used in earnest as a technique in the applied economics of growth. . . . For the non-farm sector of the American economy for the period 1919–57 . . . approximately 90 per cent of the growth of output per head could not be accounted for by increases in capital per head." The share of output growth not accounted for by the growth of the factors of labour and capital is referred to in Thirlwall as "total productivity or technical progress" (p. 45).

In view of the above adverse comments on the outcome of research extending over some forty years, what is the purpose of the present essay? Its purpose can be summarised under four headings:

(1) To explore the simplest possible production case—that of a single production process—as seen through the eyes of an accountant. We examine the accountant's model of growth of value added, expressed in simple mathematical formulae, and show that the surplus (if there is one) means a saving in costs.

(2) To look at the same growth of value added, measured by a Cobb-Douglas production function which includes an exponential time-trend for technical progress, and to notice how difficulties of measurement start to appear. We find approximate identification of Cobb-Douglas parameters with coefficients of the accountant's model, for linear or quadratic approximations being used for the Cobb-Douglas function. We consider the possibilities and implications of (*a*) constant returns to scale, (*b*) increasing returns to scale.

(3) To show how the form of measurement used for the capital stock affects the parameters of the Cobb-Douglas production function. The smaller the apparent volume of capital stock, the greater the volume of value added to be explained by technical progress.

(4) To draw some fairly obvious conclusions.

An Arithmetical Analysis of Gross Value Added

Let us suppose that a production process has a single homogeneous physical gross output which is sold at a constant price per unit and that all input costs, other than those of labour and capital, are in strict proportion to the value of gross output. Then when all costs per unit, other than those of labour and capital, have been priced at constant prices and deducted from the value of gross output, a constant proportion of that value remains, and this residue will be defined to be gross value added. It is denoted gross because it includes the full cost of the capital stock used in the process. It will be assumed that output, etc., relates to a production period of a year, unless otherwise indicated. This method of getting the value added is none other than the double deflation method pioneered by R. C. Geary and referred to as the Geary-Fabricant method in the literature on volume index of net output or value added.

Notation

V_o = Gross value added in period o, the base period or year.

L_o = Labour supply, in standard man-years, in period o.

K_o = The full cost of capital stock, at constant prices for use in the process in period o. We may regard this as a volume of capital stock, although this interpretation is not necessary for accounting purposes.

w = Constant wage-rate per man-year of labour.

d = Constant cost per annum of the use of the capital stock, a fixed proportion spread evenly over the life of the stock. The accountant views it as depreciation allowance.

i = Constant rate per annum of interest charge, on the capital stock as an investment.

x = A proportionate increase in gross value added.

y = A proportionate increase in the labour supply.

z = A proportionate increase in the full cost of the capital stock valued at base-period prices.

S = A surplus, being part of gross value added after labour and capital costs have been allowed.

Now suppose that for period o the selling price of the gross output is such that the gross value added is just sufficient to cover the costs of labour and capital and the interest payable on the latter, with the surplus S at zero level:

$$V_o = wL_o + (d+i)K_o \tag{1}$$

We may include in the interest any profits which we wish; there is no restriction to any reasonable rate of return on the investment, under the name of interest.

For some later period, value added, labour and capital have changed by proportionate amounts x, y and z respectively. There is no restriction on effects of relative price changes, thus substitution between labour and capital may occur, in order to maximise the surplus S. Neither is there a restriction forbidding economies of scale, for labour and capital. It is assumed, however, that all costs, except the factors included in gross value added, stay in fixed proportion to gross output. Thus gross value added continues to be the same proportion of gross output at constant prices, as for the base year. The analysis of components of gross value added gives

$$(2) \qquad (1+x)V_o = (1+y)wL_o+(1+z)(d+i)K_o+S$$

where S, the surplus, may be positive, negative or zero. Subtraction of (1) from (2) and division by V_o yields

$$(3) \qquad x = y(wL_o/V_o)+z[(d+i)K_o/V_o]+S/V_o$$

Thus, in index-number format, the increase in gross value added is partly accounted for by the growths of the base-year share of labour cost and also of the base-year share of capital costs, but there is a third term, S/V_o, which must be considered, to explain x fully. If S/V_o is, positive, then clearly it arises from some increase in efficiency of use of the factors labour and capital—it is a saving of costs for the later period versus the base year. The surplus emerges in volume index form, with gross value added for period o as the base of the index. It is a function not only of the growth rates of labour and capital costs as defined above, since it requires x as well as these to account for it, hence the measurement problem is apparent. In the usual real-life situation of a positive S, x requires S/V_o as an explanatory variable and vice versa. The cases of zero or negative values of S do not merit much attention as they are untypical of normal growth processes to be analysed in conjunction with the Cobb-Douglas function.

Equation (3) has further interesting possibilities. It follows from (1) that wL_o/V_o and $(d+i)(K_o/V_o)$ sum to unity. If the values for these fractions were 0·75 and 0·25 respectively, substitution in (3) would give:

$$(4) \qquad x = 0{\cdot}75y+0{\cdot}25z+S/V_o$$

The untypical case of y and z each being equal to x gives $S = o$, i.e., there is no surplus because labour and capital costs increase at the same rate as value added. For $x = 0{\cdot}1$, $y = 0{\cdot}05$ and $z = 0{\cdot}1$, $S/V_o = 0{\cdot}0375$, the latter being a 3·75 per cent increase attributable to the surplus, and due to either technical progress or increasing returns to scale or both, if one is seeking the causes of this saving in factor costs.

The Cobb-Douglas (C.D.) Production Function

A widely-used form of this function, shown below in (8) and including the factor e^{Tt} for "technical progress" from period o to period t, will be considered. Some further notation is required. The units of measurement are the same as above.

Further notation

V_t = Gross value added in period t, at constant prices, via double deflation, as explained above.

L_t = Labour supply in period t, in standard man-years, a measure of the volume of labour used.

K_t = The full cost of capital stock, at constant prices, for use in the process during period t. It is a measure of the physical volume of the capital stock.

d_t = The proportion of the original full cost of the capital stock, i.e. of K_t, attributable to depreciation.

K_t^1 = The depreciated value, at constant prices, of the original full cost of the capital stock, thus

$$K_t = K_t^1(1+d_t).$$

A = A constant multiplicative factor.

t = A time trend indicator, having values 0, 1, 2, . . . t for periods 0, 1, 2, . . . t respectively.

τ = The coefficient of t in the exponential technical progress multiplicative factor.

α = The index of the power of K_t in the production function.

β = The index of the power of L_t in the production function. The parameters α and β are also described as the partial elasticities of output with respect to Capital and Labour, respectively. (Thirlwall (1972), pages 43–44.)

According to the usual conventions,

$$\alpha+\beta = 1 \tag{5}$$

means constant returns to scale;

$$\alpha+\beta < 1 \tag{6}$$

means decreasing returns to scale;

$$\alpha+\beta > 1 \tag{7}$$

means increasing returns to scale.

At this stage we do not specify which of (5) to (7) applies to the function to be used.

The Cobb-Douglas form to be used is specified as follows:

(8) $$V_t = Ae^{\tau t}K_t^{\alpha}L_t^{\beta}$$

For $t = o$, i.e. for period o.

(9) $$V_o = AK_o^{\alpha}L_o^{\beta}$$

Suppose that we had performed standard multiple regression analysis on four time series, namely the logarithm of value added as dependent variable, having $\log L_t$, $\log K_t$ and t as explanatory variables, laying no restrictions on $(\alpha+\beta)$. We then deleted the residuals from the value added time series so that these adjusted values are defined to be $\log V_t$ and fit exactly the regression formula

(10) $$\log V_t = \log A + \alpha \log K_t + \beta \log L_t + \tau t$$

which is (8) above written in logarithmic form.

Approximate Linear Form of C.D. Volume Index

We will suppose that the value added original time series required little or no adjustment (much less than 1 per cent) for year t, in order to get V_t which exactly fits formula (10), and that the same high precision holds for year o. Thus for at least two years, o and t, formula (9) describes with relatively high precision the same data as were used above in the arithmetical accounting analysis. Alternatively, we might suppose that the accountant's series, as used in his analysis, had had V and S simultaneously adjusted by relatively small amounts, so that V fitted formula (10), before using formulae (1) to (3). Their analysis of the growth effects is not impaired by minor prior data adjustments.

Thus we may link up with formulae (1) to (3) above by setting

(11) $$(1+x)V_o = V_t;\ (1+y)L_o = L_t;\ (1+z)K_o = K_t.$$

By substitution of (11) in (8),

(12) $$(1+x)V_o = Ae^{\tau t}K_o^{\alpha}(1+z)^{\alpha}L_o^{\beta}(1+y)^{\beta}$$

Division of (12) by (9) gives

(13) $$1+x = e^{\tau t}(1+z)^{\alpha}(1+y)^{\beta}$$

For τ, y and z fairly small, say $\leqslant 0{\cdot}1$, expansion of the right hand side of (13) gives the approximation

$$(14) \qquad 1+x \simeq (1+\tau t)(1+\alpha z)(1+\beta y)$$

$$\simeq 1+\tau t+\alpha z+\beta y$$

by neglecting second-order small quantities. Thus

$$(15) \qquad x \simeq \tau t+\alpha z+\beta y$$

to the first order of small quantities. Comparison with (3), for coefficients of y and z gives

$$(16) \qquad \beta = wL_o/V_o$$

and

$$(17) \qquad \alpha = (d+i)K_o/V_o$$

Since these cost-shares of base gross value added have been commented on above as adding to unity, it follows that

$$(18) \qquad \alpha+\beta = 1$$

This means constant returns to scale. For the structure being considered such numerical values have frequently been found in empirical investigations, with $\alpha \simeq 0{\cdot}25$ and $\beta \simeq 0{\cdot}75$.

Identification of the third component of x in (3) with that in (15) gives

$$(19) \qquad \tau t = S/V_o$$

which means that the exponential technical progress effect has been identified with the volume of surplus. For the value $0{\cdot}0375$ found above after (4) for S/V_o and for $t = 2$, τ has the value $0{\cdot}01875$, which in the C.D. context is described as an annual growth rate of $1{\cdot}875$ per cent for technical progress.

To summarise results at this point: for growth-rates small enough to permit us to ignore second-order small quantities we find constant returns to scale and technical progress being measured by the volume of surplus, within the Cobb-Douglas framework.

Approximate Quadratic Form of C.D. Volume Index

Now, however, let us include the second-order terms of (14), getting

(20) $$x \doteq \alpha z(1+\tau t)+\beta y(1+\tau t)+\alpha z \beta y+\tau t$$

In order to identify (20) with (3) we have to take S as a function of y and z as well as of τ. In this way we get exactly the same results for α and β as are given by (16), (17) and (18). Thus here again we have constant returns to scale, since α and β add to unity.

The volume of surplus, however, S/V_o, is now only partly explained by τt, being in full:

(21) $$S/V_o \doteq \tau t(\alpha z+\beta y+1)+\alpha z \beta y$$

Thus the growth-rates of labour and capital contribute to S/V_o, as well as technical progress supposedly described by growth-rate τ.

The Question of the Scale of Production

We may now consider the possibility of making $(\alpha+\beta)$ differ from unity particularly $(\alpha+\beta)>1$ since this, being interpreted as increasing returns to scale, is nearer to our experience in the normal event of economic growth. If we look back over the discussion from (11) above down to this point, it appears that the unit value for $(\alpha+\beta)$ seems to be a fairly natural outcome of the comparisons described. In other words, for the time series and formulae and smallish growth rates being considered, the Cobb-Douglas produces constant returns to scale, if $(\alpha+\beta)$ equal to unity defines such constant returns. The surplus S/V_o absorbs, under the name of technical progress, various kinds of savings. We may therefore question the correctness of defining constant returns to scale as the unit sum of α and β. Possibly we should compare the volume index S/V_o with some weighted average or function of y, z and τ to decide on the extent to which "constant returns to scale" is relevant, if we are using the Cobb-Douglas form given above in (8). It would at all events appear that the Cobb-Douglas mathematical form will throw up $(\alpha+\beta)$ of approximate unit value, in many applications. Does this matter terribly, one may ask, if the function is useful as an econometric tool?

How the Unit used to Measure the Volume of Capital Stock affects the Exponential Factor

In the C.D. formula used above, the capital stock K_t has been entered at its full original purchase cost valued at constant prices, as a measure of the volume. If a large number of small homogeneous machines form the capital stock, or if a unique combination of certain kinds of plant, machines, vehicles and buildings is used for a single process, then their aggregate value at constant prices is indeed a measure of their physical volume. One is inclined to think of the man-hour of labour as on average a fairly definite physical amount of a certain grade of skill or handicraft or know-how. Thus the improvement in the design and quality of machines creates a problem of measurement, if one wishes to measure the equivalent volume of homogeneous (less efficient) machines.

Scaling up the constant-price cost of the new machines or making other adjustments is mentioned in Thirlwall, who cites several investigations into adjustment of volume of capital. All that is intended here is to point out that there is a measurement problem.

If we want physical volumes of capital and labour for our production function, then we want each factor to be as homogeneous as possible. If, however, we become accountants, then all that matters is the cost of the capital stock being used (at constant prices) so that we can measure the interest and depreciation included in value added. The accountant's system is easier to apply, in practice.

We find in the literature many examples of depreciated capital stock being used as K_t. That this highly probable underestimation of the PHYSICAL VOLUME of stock being used has the effect of inflating the C.D. exponential index τ can be shown as follows:

We suppose that, for period t, K_t is replaced by $(1+d_t)K^1_t$, where K^1_t is some depreciated value and d_t is a parameter taking acount of the depreciation effect. Let K^1_o be equal to K_o, meaning that $d_t = o$ for $t = o$. Since $K^1_t < K_t$, in place of $K_t = (1+z)K_o$ there will be $K^1_t = (1+z^1)K^1_o$ with $z^1 < z$. Then formulae (12) and (13) become respectively:

$$(12') \qquad (1+x)V_o = Ae^{\tau t}(K^1_o)^{\alpha}(1+d_t)^{\alpha}(1+z^1)^{\alpha}L^{\beta}_o(1+y)^{\beta}$$

$$(13') \qquad 1+x = e^{\tau t}(1+d_t)^{\alpha}(1+z^1)^{\alpha}(1+y)^{\beta}$$

which leads to

$$(15') \qquad x \simeq \tau t+\alpha(d_t+z^1)+\beta y$$

for τ, d_t, y and z^1 small.

Whereas the coefficient of α in (15) was z, here it is d_t+z^1, so that for z^1 being taken as the growth of the capital stock, instead of z, there is a loss of explanation of x, the loss being αd_t.

It is clear that corresponding to τt in (19) there must now be $\tau t+\alpha d_t$ to explain the residue of x not attributable to either L_t or K^1_t.

Suppose that

$$(22) \qquad \tau^1 t = \tau t+\alpha d_t$$

Then $\tau^1 > \tau$ and the apparent increase in the "technical progress" effect is due to the decreased explanatory value of the capital stock $K^1{}_t$ compared with K_t. For $t = 2, \tau = 0{\cdot}01875, \alpha = 0{\cdot}25$ and $d_t = 0{\cdot}1$ (meaning a 5 per cent annual depreciation rate), $\tau^1 = 0{\cdot}03175$, which is 1·3 per cent per annum high than τ.

Conclusions and a Deduction

(1) For growth rates of labour, capital and technical progress small enough to validate the linear approximate form of the C.D. volume index used above at (13), the parameters β and α of that function are identified with base-period shares of labour and capital costs in the gross value added, and the technical progress effect has been identified with the growth of the surplus, which is a saving on labour and/or capital factor costs. Since α and β add to unity, there are constant returns to scale, as defined for C.D. functions.

(2) For valid use of the quadratic approximation to the C.D. volume index, the parameters β and α are again identified with base-period shares of labour and capital costs. Now the volume of surplus, S/V_0, is a function of y and z, as well as of τ. So here again α and β add to unity and there are constant returns to scale, if we accept this property as the necessary condition.

(3) Our method of comparing the C.D. expansion with $(1+x)$ seems to make $(\alpha+\beta=1)$ an inevitable outcome, for identical volumes of both labour and capital being inherent in the two functions which are compared.

(4) It is clear from the above investigations that labour and capital growths cannot fully account for the growth of gross value added, unless they are adjusted to absorb the exponential τt effect, in the typical situation of a positive surplus S in year t.

(5) The surplus S is the outcome of more efficient use of the factors L_t and K_t in year t than in year o, and in the accountant's view might be partly due to economies of scale as well as technical progress. Is the unit sum of α and β a sufficient indicator of constant returns to scale, for the models compared above?

(6) A discounting or depreciation of the original cost of the capital stock has been shown to effectively increase the apparent rate of technical progress. The latter is kept at a minimum level by putting in the stock at full cost at constant prices.

(7) *Deduction*

We consider expanding the right-hand side of (13) so as to show more powers of τ, z and y:

$$(23) \qquad 1+x \simeq \left(1+\tau t+\frac{\tau^2 t^2}{\lfloor 2}+\ldots\right)\left[1+\alpha z+\alpha\frac{(\alpha-1)}{\lfloor 2}z^2+\ldots\right]$$

$$\left[1+\beta y+\beta\frac{(\beta-1)}{\lfloor 2}y^2+\ldots\right]$$

Since α and β are each less than unity, the power-series expansion converges absolutely for

$$(24) \qquad \begin{array}{l} |\tau t| \text{ having any finite value,} \\ |z| \text{ and } |y| \text{ each} < \text{unity.} \end{array}$$

See for instance Allen (1962, page 317 *et seq.*).

Because each series converges absolutely so does their product. Thus, for the stated conditions (24),

$$(25) \qquad 1+x = 1+\alpha z+\beta y+\tau t[1+\phi(\tau, z, y)]$$

where ϕ is a polynomial function of τ, z and y. Thus for any value of τt and for up to 100 per cent increases in labour (y) and capital stock (z), we have an equality connecting the right-hand side of (3) increased by 1·0 with that of (25). By comparing coefficients of z and y we get once more the same cost-shares of base gross value added, for α and β. And again α and β sum to unity. It follows that

$$(26) \qquad S/V_o = \tau t(1+\phi)$$

(8) The process of comparing an accounting model, essentially monetary, with the C.D. production function, which is basically a relation between physical units of factors and output, may be open to question. On this account, if on no other, the above treatment is tentative. The author is not aware of any comparison of forms along the lines explored above and would appreciate the comments of readers.

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