Quantitative Planning and Decision-Making Techniques (With Particular Reference to Agricultural Applications)

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Since the birth of civilisation mankind has been experimenting with numbers The study of primes, radicals, and other number forms fascinated the early mathematicians and from their researches there emerged a body of knowledge which is known today as the Theory of Numbers

In similar fashion the representation of physical units (length, mass etc) by the symbols (x, y, z, etc) led the early mathematicians to theories describing the relationships which can exist between these abstractions Their study of equations, complex numbers, series, and other like topics has resulted in the algebras with which we are familiar today

The representation of time as an abstract quantity contributed to the development of the calculus which heralded the development of what has been called "the modern mathematics"

However, up to the end of the nineteenth century mathematics was usually studied for its own sake, or in conjunction with the attempt to understand the physical sciences It seldom occurred to the people of that era to try to apply their mathematics to their economic, agricultural or industrial systems

The Taylor brothers and the Gilbreaths in America, and Bedaux in France, started a revolution in the traditional concepts of the relationships between men, money and materials At the lowest level in the industrial hierarchy, the shop-floor, the man and his job were examined in the cold light of scientific measurement A startling discovery was made – a man was quantifiable Once the full implications of this discovery had been grasped a transformation occurred in industrial planning ideas The catalyst for this revolution was a method of examination and analysis which is now known as Work Study

Work Study can rightly be described as a forerunner of modern planning techniques Work Study ,in spite of its undoubted usefulness, however, had serious limitations in its range of application

During the period 1920-1950, mathematical ideas were being examined and developed, notably in the field of linear and matrix algebra Von Neumann proved the celebrated mini-max theorem and in conjunction with Morgenstern published their important work *The Theory of Games and Economic Behaviour* Wassily Leontief was researching in inter-industry problems which culminated in the publication of *The Structure of the American Economy, 1919-1929* G B Dantzing, the Rand mathematician, published his general method for finding the optimum of a linear function subject to linear constraints – the Simplex Method Erlang, the Swedish engineer, first formulated the mathematical theories of congestion which today have wide application in industry and commerce

These developments, and many others, have ushered in the modern quantitative era which is exemplified by the widespread application of mathematics to planning problems of all descriptions

The majority of mankind's economic and industrial planning ideas are quantifiable and therefore are susceptible to mathematical analysis The rapid development of the electronic computer with its prodigious capacity for calculation has also stimulated the application of mathematics to these problem areas It is undoubtedly true that this trend will continue, and probably accelerate, with the result that techniques which were formerly thought of as pure mathematical abstractions will become the most modern of planning methods This point is perhaps well illustrated by the present application of Markov Chain theory to problems in agriculture and industry

It is intended in this paper to describe the application of

(i) Mathematical Programming, and
(ii) Markov Chain Theory

to some of the problems in agriculture, industry and economics with which the Operational Research scientist is faced today

MATHEMATICAL PROGRAMMING AND PROBLEMS OF ALLOCATION

Allocation problems involve the allocation of resources to sectors of the economy, to geographical areas, to industries or to individual jobs This class of problem occurs when the available resources are in scarce supply Therefore, it is desirable to allot the resources to the activities in such a way as to minimise the total cost, maximise the total return, or attain another such "rational" objective

Many allocation problems can be represented by a matrix as follows

Resources	Activities to be Accomplished				Amount of Resources Available
	A_1	A_2	A_j	A_n	
R_1	c_{11}	c_{12}	c_{1j}	c_{1n}	b_1
R_2	c_{21}	c_{22}	c_{2j}	c_{2n}	b_2
R_p	c_{p1}	c_{p2}	c_{pj}	cp_n	b_p
R_m	c_{m1}	c_{m2}	c_{mj}	c_{mn}	b_m
Amount of Resources required	a_1	a_2	a_j	a_n	

The entries in the cells, c_{ij}, represent the cost (or return) that results from allocating one unit of resource R_i to activity A_j, and if the cost (or return) from allocating an amount x_{ij} of resource i to activity j is equal to x_{ij} c_{ij}, we have a *Linear* allocation problem

Allocation problems with linear return or cost functions have been studied the most intensively because of the availability of the powerful techniques of *Linear Programming* and also because of relatively small data requirements

The application of these techniques to the problems of industry and commerce involve the assumption that the amounts of resources available (b_i), the amounts required (a_j) and the cost (c_{ij}) are known exactly This is not always the case and it is sometimes desirable to test the *sensitivity* of the solution to possible errors in these coefficients The technique known as *Parametric Linear programming* allows one to carry out such sensitivity analyses

LINEAR PROGRAMMING MODELS

A linear programming formulation can be written as either

(i) a primal model, or

(ii) a dual model

depending on whether the objective is minimisation (primal) or maximisation (dual) Both types of model are connected by the duality theorems of linear programming whereby the solution of one model contains the solution of the other

The general statement of primal and dual can be expressed in matrix algebra as follows

Min	cX	Max	b^1Z
subject to restrictions	$AX \geqslant b$	subject to	$A^1Z \leqslant c^1$
	$X \geqslant 0$		$Z \geqslant 0$

where

$c = 1 \times n$ row vector
$X = n \times 1$ column vector
$A = m \times n$ matrix
$b = m \times 1$ column vector
$Z = m \times 1$ column vector

Note (i) The objective function and the constraints must obey the linearity rules

(ii) A general method of solution known as the *Simplex Method* was developed in the 1940s by G B Dantzing, late of the Rand Organisation, and now a Professor in the Computer Science Department of Stanford University

(iii) Refinements to the general technique exist whereby it is possible to obtain-

(a) solutions to problems having integer and certain other non-linear constraints or costs, *and*

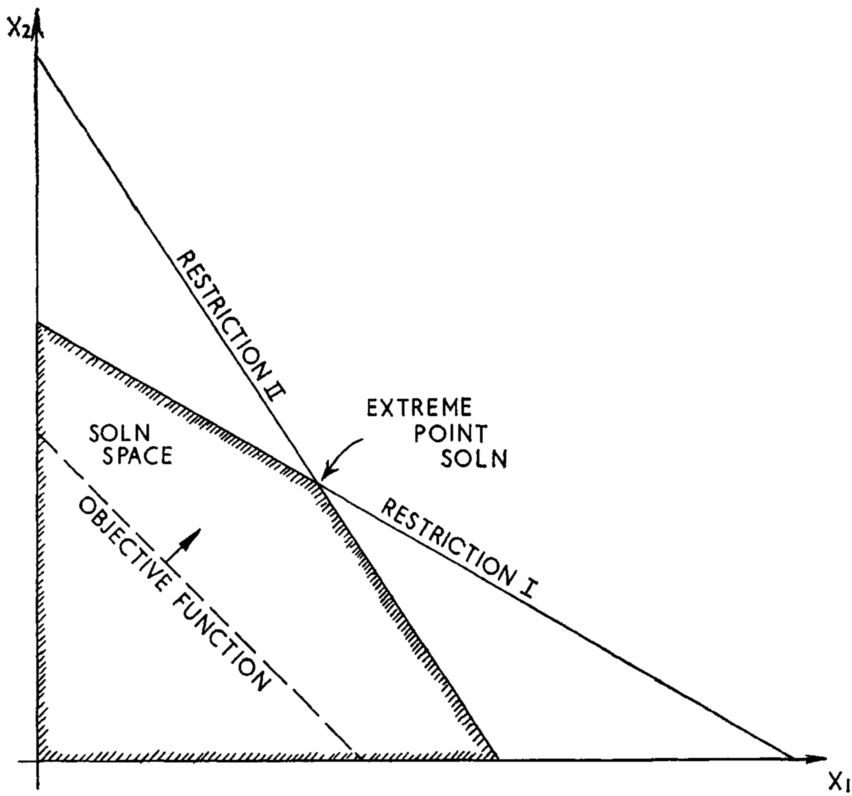
(b) parametric solutions

GEOMETRICAL STRUCTURE

All linear programming problems have basically the same geometrical structure A system of linear constraints forms a solution space (a convex set) within which an objective function must be maximised or minimised

The geometry of linear programming can best be illustrated by means of the following simple two-variable model

max	$3x_1+4x_2$	Objective Function
subject to	$x_1+2x_2\leqslant 14$	Restriction I
	$4x_1+x_2\leqslant 20$	Restriction II
	$x_1\geqslant 0$ -----	
	$x_2\geqslant 0$ -----	Non negativity conditions



The actual extreme point at which the objective function is maximised depends on the "slope" of the function The extension into "n" variables is obvious and an analytical optimization method is necessary since we are in En space

RANGE OF APPLICATION OF THE LINEAR MODEL

At first sight it may seem an over-simplification to try and represent real-world situations by linear models In actuality the linear model is a sufficiently accurate mathematical representation of many physical, economic and financial situations for it to be acknowledged as a major planning and research tool It has been found particularly suitable for application as a planning and research tool in the following fields

(1) *Agriculture*

E O Heady, W Chandler and others have pioneered, in the United States, the application of linear programming methods to agricultural planning both at the regional and state level They have used this technique to indicate the optimum distribution of commodities subject to such limitations as land, labour, capital etc In addition they have also used the linear programming approach to investigate the effect of variable price, yield and capital quantities on the optimum solution

In general the planning models usually take the form

$$\text{Max } Z = c_1x_1 + c_2x_2 + \quad + c_nx_n \qquad \text{Maximization of "Profits"}$$

$$\begin{aligned} \text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \quad + a_{1n}x_n \leqslant b_1 \\ & a_{21}x_1 + a_{22}x_2 + \quad + a_{2n}x_n \leqslant b_2 \\ & \quad \cdot \qquad\qquad\qquad\qquad \text{Constraints due to land, labour, etc} \\ & \quad \circ \\ & a_{m1}x_1 + a_{m2}x_2 + \quad + a_{mn}x_n \leqslant bm \\ & \qquad x_j \geqslant 0, \ j = 1, 2, \quad n \end{aligned}$$

The central problem in the application of linear models to agricultural planning lies in the accurate estimation of the coefficients The coefficients $c_1, c_2, \quad c_n$ are the unit returns forthcoming from each production process $x_1, x_2, \quad x_n$ The two main methods used to obtain efficient estimates of these coefficients are

(i) production function analysis, and
(ii) gross margin analysis

The use of either method of analysis depends on the planning questions to be answered and the availability of data

The coefficients $b_1, b_2 \quad b_m$ are production restraints These restraints usually refer to

(i) land
(ii) capital
(iii) labour
(iv) equipment
etc

Land, and equipment constraints are usually relatively easy to ascertain However, the problem of labour profiles is not so easy to overcome and requires careful examination of any records or other sources of information that are available A typical labour profile is shown in Fig 1

FIG 1

Month	Total labour supply (hrs)	Total service lab reqrs	Total Labour Available for Specific Enterprises	
			Per month	Per monthly group
December	250	30	220	
January	240	20	220	620
February	200	20	180	
March	230	20	210	405
April	255	60	195	
May	310	10	300	
June	320	20	310	600
July	320	10	310	580
August	300	30	270	
September	240	90	150	
October	260	60	200	560
November	260	50	210	

Hours of labour available in monthly groups usually comprise the labour restrictions used in programming agricultural plans

The estimation of the coefficients a_{11}, a_{12}, a_{mn}, which are usually termed "production" coefficients, is probably the most difficult task in the formulation of the relevant mathematical model Additional effort in this respect is generally more important than in additional refinements in the mathematical approach This additional effort is necessary since if the input coefficients are too low, the subsequent plan will be non-feasible because it will require more resources than are available If, however, the input coefficients are too high, the farm-firm will find that surplus resources exist, and a better plan can be found

To obtain the production coefficients it is necessary to determine the amount of a particular input required to produce an acre of corn, oats and so on A typical matrix of these coefficients including the expected output per acre (in £'s) and the requirements is shown in Table 1

TABLE 1

Varlables (Crops)	Constraints					
	Output/ Acre	Land (Acres)	Capital/ Acre (£'s)	Labour A (M Days)	Labour B (M Days)	Labour C (M Days)
Oats (X^1)	31 54	1	9 8	0 6	0 3	0 6
Wheat (X^2)	30 27	1	12 06	0 8	0 4	0 8
Barley (X^3)	35 45	1	10 2	0 6	0 3	0 6
Corn (X^4)	37 55	1	8 8	0 8	0 4	0 8
Apples (X^5)	10 6	1	10 4	0 3	1 0	2 7
Seed Potatoes (X^6)	208 5	1	61 57	4 0	1 1	10 9
Maln Potatoes (X^7)	140 39	1	90 70	4 0	1 1	10 9
Designations		$\leqslant$	$\leqslant$	$\leqslant$	$\leqslant$	$\leqslant$
Requlre-ments		66,280	1,316,329	92,338	32,856	199,214

Table 1 represents in a simplified manner, the basic information necessary in order to construct a linear model of land utilisation for Co Antrim This model, which in the interests of simplicity ignores livestock, is as follows

$$
\begin{aligned}
\text{Max} \quad & =31\ 54x_1+30\ 27x_2+35\ 45x_3+37\ 55x_4+10\ 5x_5+ \\
& \quad 208\ 5x_6+104\ 39x_7 \\
\text{subject to} \quad & +x_1+x_2+x_3+x_4+x_5+x_6+x_7 \leqslant 66,280 \\
& 9\ 8x_1+12\ 06x_2+10\ 2x_3+8\ 8x_4+10\ 4x_5+61\ 57x_6+ \\
& \quad 90\ 7x_7 \leqslant 1,316,329 \\
& 0\ 6x_1+0\ 8x_2+0\ 6x_3+0\ 8x_4+0\ 3x_5+4\ 0x_6+4\ 0x_7 \\
& \quad \leqslant 92,338 \\
& 0\ 3_1+0\ 4x_2+0\ 3x_3+0\ 4x_4+x_5+1\ 1x_6+1\ 1x_7 \leqslant 32,856 \\
& 0\ 6x_1+0\ 8x_2+0\ 6x_3+0\ 8x_4+2\ 7x_5+10\ 9x_6+10\ 9x_7 \\
& \quad \leqslant 199,214 \\
& \qquad x_j \geqslant 0 \quad j=1, \quad 7
\end{aligned}
$$

The solution to this model yields the following information

Grow (i) 11,277 acres of Barley
(ii) 33,003 acres of Corn
(iii) 14,792 acres of Seed Potatoes

Some of the consequences of this policy are

(i) 7,208 acres of land will be left idle
(ii) The cash output, which is the highest obtainable, will be £4,723,240

It is of course unnecessary to add that this model is both too simple and somewhat naive The next level of sophistication would be to restrict both the acreages of corn, main and seed potatoes in accordance with demand and practicability Analysis of past records indicates that these acreages could be restricted to

$$x_4 \leqslant 1{,}021$$
$$x_6 \leqslant 3{,}722$$
$$x_7 \leqslant 14{,}253$$

The resulting solution to this augmented model yields a solution which is more realistic in that the acreages for corn, seed and main potatoes have been constricted to within acceptable limits of growth

Further levels of sophistication include the addition of constraints for

(i) A minimum level of acceptable income
(ii) The allocation of available "tractor hours" to the various enterprises
(iii) The inclusion of various types of livestock such as cattle, sheep etc (this, of course, requires the addition of new variables) etc

The ultimate aim is to produce a realistic agricultural planning model for the region in order to examine in detail the effect of variations in prices and quantities

2 *Industrial Applications*

The models which have been described in an agricultural context can also be applied to many problems in modern industry The mathematical programming approach has been applied to many diverse problem areas which include

(i) Production scheduling
(ii) Mixing problems in the chemical and feed-stock industries
(iii) Problems of allocation
(iv) Transportation and routing problems

In each case the basic approach is to represent the situation in terms of a mathematical model having an objective function which must be maximised or minimised and subject to various constraints

The following simple illustration typifies the method of approach

TABLE 2

Process	Production time per product			Daily available Production time
	x_1	x_2	x_3	
Turning	6 mins	2 mins	4 mins	200 mins
Boring	2 mins	2 mins	12 mins	160 mins
Profit per unit	£12	£8	£24	

The manufacturer's problem is to plan his daily production schedule so that he will maximise his profit

Expressed as a linear programming problem the mathematical representation is

Max $P=12x_1+8x_2+24x_3$
subject to $6x_1+2x_2\times4x_3\leqslant200$
$2x_1+2x_2\times12x_3\leqslant160$
$x_1\geqslant0$
$x_2\geqslant0$
$x_3\geqslant0$

where x_1, x_2 and x_3 refer to the quantities of X_1, X_2 and X_3 to be produced.

The Simplex technique can be applied to yield the following optimum solution

Manufacture (a) 10 units of X_1
(b) 70 units of X_2
Profit (c) £680 day

3 *Parametric Linear Programming*

Once the solution to a particular linear model has been obtained it is then natural to enquire as to the reliability of this solution in the event of a variation in the parameters This enquiry leads us into the domain of Parametric Linear Programming where the objective is to establish the nature of the effect on the optimal solution of variations in either

(i) the coefficients of the objective function
(ii) the coefficients of the variables in the restriction equations
(iii) the requirements

If we express a linear model in the terms of matrix algebra

Max cX
subject to $AX\leqslant b$
$X\geqslant0$

then the variations referred to above

(i) Variations in the c matrix
(ii) Variations in the A matrix
(iii) Variations in the b matrix

Techniques exist whereby one is able to analyse the effect of these variations and also to establish the limits within which an optimal solution remains stable

This technique has obvious applications in practice

REFERENCES

1 Haley, K B *Mathematical Programming in Business and Industry*, Macmillan 1968
2 *Heady, E O and Chandler, W, *Linear Programming Methods*, Iowa State U Press 1966
3 Heady, E O and Egert, A C, *Mathematical Linear Programming of Regional patterns (Location and Quantities)*, Iowa State U Press
4 *Equilibrium Analysis of Income—Improving Adjustments on Farms in the Lake States Dairy Region*, 1965, University of Minnesota Agricultural Experiment Station

*Many descriptions of the application of L P to Agriculture

MARKOV PROCESSES – THEORY AND APPLICATIONS

Theory

A sequence of experiments that can be subjected to a probalistic analysis is called a stochastic process If the set of possible outcomes is finite then it is said that the process is finite

Stochastic processes can be classified by indicating special properties by the outcome functions of the process

In a Markov Chain process there is a given set of states

$S_1, S_2, \text{———} S_r$

and it is only possible for the process to be in one of these states at a given time The process moves successively from one state to another and each move is called a step The probability that the process moves from S_i to S_j depends only on the state S_i that it occupied before the step The *transition probability* P_{ij}, describing the probability of moving from S_i to S_j, is given for every ordered pair of states In addition an initial *starting state* is specified at which the process is assumed to begin

These characteristics can be readily expressed in a *transition matrix* P

	S_1	S_2	S_k
S_1	P_{11}	P_{12}	P_{1k}
S_2	P_{21}	P_{22}	P_{2k}
S_k	P_{k1}	P_{k2}	P_{kk}

An important characteristic of this type of matrix is that the sum of the components in any given row is 1 A further important characteristic of this type of matrix is that it converges to a *limiting* state provided the markov chain is regular

This can be illustrated as follows

$$\text{If } P = \begin{bmatrix} 0\ 5 & 0\ 25 & 0\ 25 \\ 0\ 5 & 0 & 0\ 5 \\ 0\ 25 & 0\ 25 & 0\ 5 \end{bmatrix}$$

Then it can be shown that

$$P^8 = \begin{bmatrix} 0\ 4 & 0\ 2 & 0\ 4 \\ 0\ 4 & 0\ 2 & 0\ 4 \\ 0\ 4 & 0\ 2 & 0\ 4 \end{bmatrix}$$

Three important types of Markov chain are

(i) Regular – some power of the transition matrix has only positive elements

(ii) Ergodic – it is possible to go from every state to every other state

(iii) Absorbing – impossible to leave some state of the Markov chain

The theories developed from each type of chain have important implications for applied economics and I would now like to describe their application in the field of agricultural planning

AGRICULTURAL APPLICATIONS

My research interest in this field is to develop a Markovian model that will describe the movements in farm size in Co Antrim as a complement to the work described in the section on linear models Unfortunately the necessary data is not yet to hand so no analysis has yet been carried out However, I would like to describe some other work that has been done using these models to analyse dairy herd size in Co Fermanagh

In the execution of this work I am greatly indebted to Mr D J Alexander of the Ministry of Agriculture (N Ireland) for help and encouragement freely given at all times

DAIRY HERD SIZE – FERMANAGH

An analysis of the size of dairy herds in Fermanagh for the period 1961/67 yields the following information

TABLE 1

Herd size	No of Farms	
	1961/62	1967/68
1-9	3,003	1,126
10-19	635	705
20-29	84	69
30	23	23
Total	3,745	1,923

A more detailed analysis reveals, in the same period, the movements between the various categories

TABLE 2

Herd Size	No of farms 1961/62	1967/68				
		0	1-9	10-19	20-29	30+
0	(2,872)	2,772	100	0	0	0
1-9	3,003	1,700	980	323	0	0
10-19	635	138	46	382	69	0
20-29	84	61	0	0	0	23
30+	23	23	0	0	0	0
Total	3,745	4,694	1,126	705	69	23

Note: (2,872) is the estimated number of potential farms that could enter into the dairying field

From Table 2 it is possible to construct the following transition matrix

TABLE 3

	0 965	0 035	0	0	0
	0 566	0 326	0 108	0	0
P=	0 217	0 072	0 602	0 109	0
	0 726	0	0	0	0 274
	1 000	0	0	0	0

From the theory of Ergodic Marov Chains it is not difficult to demonstrate that P_n tends towards

TABLE 4

0 936	0 050	0 013	0 001	0
0 936	0 050	0 013	0 001	0
0 936	0 050	0 013	0 001	0
0 936	0 050	0 013	0 001	0
0 936	0 050	0 013	0 001	0

if one is content to calculate to three places of decimals

USING THE EQUILIBRIUM VALUES OF THE TRANSITION MATRIX

It is clear that the number of potential entrants assumed affects the rate at which changes in size take place and the rate of exit projected for the industry

The effect of different numbers of potential entrants can be examined through using the transition matrix and a device known as the T-vector

The T-vector indicates the proportion of the original farms plus the number of potential entrants falling into each of the size classes, including O, when sufficient time-periods have passed in order that the exit, entry, growth and decline are in equilibrium

Starting from the recognition that the T-vector for equilibrium is obtained by solving

$$t\,P = t$$

given $\sum_{0}^{n} t_j = 1$

it can be proved that

$$t_0 = \frac{N}{N+C} \quad \text{———(i)}$$

and

$$t_j = \frac{C_j}{N+C}, \quad \text{———(ii)}\ j = 1, 2, \text{———}\ n$$

where n is the number of classes *and* N is the number of potential entrants assumed

C and C_j are some constants which are independent of N and are computed from the original probabilities

(I) N=2,875

Now $t = [0\ 936\ 0\ 050\ 0\ 013\ 0\ 001\ 0]$

thus $t_0 = 0\ 936 = \frac{2{,}872}{2{,}872+C}$

whence $C = 196\ 38$

$t_1 = 0\ 05 = \frac{C_1}{2{,}872+196\ 38} = \frac{C_1}{3068\ 38}$

whence $C_1 = 153\ 42$

$t_2 = 0\ 013 = \frac{C_2}{3{,}068\ 34}$

whence $C_2 = 39\ 88$

and $C_3 = 3\ 07$

If M is the number of active farms in the initial time period, the number of farms F_j in each category in equilibrium will be

$$\text{(iii)}\ F_j = \frac{C_j}{N+C}(M+N)_j\ \ j = 1, 2 \text{———}\ n$$

In addition if E represents the net change in the number of active farms in the dairying industry, then the net number of entrants or exits from this industry is

$$E=F_0-N=\frac{N\ (M+N)}{N+C}-N$$

whence $$E=\frac{N(M-C)}{N+C} \qquad \text{———(iv)}$$

Thus using formula (iii) we can compute the number of farms in each size class

$$F_1=\frac{153\ 42\ [6{,}617]}{3{,}068\ 38}=\frac{C_1\ [M+N]}{N+C}$$

$=330\ 85$ say 331 0

Similar calculations give

$F_2=86\ 00$

$F_3=\ 6\ 62$ say 7 0

$F_4=\ 0$ 19

Using the formula (iv) we can compute the number of farms that have left the industry viz

$$E=\frac{N(M-C)}{N+C}=\frac{2{,}872\times 3{,}548\ 62}{3{,}068\ 38}$$

3,321 5 say 3,321 0

Thus assuming a potential entry of 2,872 and also a continuation of the trends indicated in the period 1961/62 – 1967/68, the following conclusions can be reached at equilibrium

TABLE 5

Herd Size	Number of farms		Reductions	
	1961/62	Equilibrium	No	%
1-9	3,003	331	2,672	88 97
10-19	635	86	549	86 45
20-29	84	7	77	91 66
30+	23	0	23	100 00
Exits	0	3,321	—	—
Total	3,745	3,745	3,321	—

The following table indicates the position at equilibrium when the number of potential entrants is varied

TABLE 6

Herd Size	N=Number of potential entrants assumed		
	2,000	2,872	4,000
1-9	402	331	283
10-19	104	86	74
20-29	8	7	6
30+	0	0	0
Exits	3,231	3,321	3,382
Total	3,745	3,745	3,745

CONCLUSION

The example chosen to represent the application of Markov Chain theory was directly due to the author's interest in the application of mathematical models to the agricultural industry In this respect it is recognised that the model is inherently too simple However, the analyst working in this and other fields must use all the means at his disposal in order to describe the economic forces at work Markov Chain Theory represents a promising line of enquiry – not the least in that it provides a means of directly estimating equilibrium conditions

It should be noted that the number of potential entrants has a definite effect on projections and care should be taken in estimating this statistic

It is of course possible to apply this theory to areas other than that which has been described The pioneering application is generally recognised to be due to I G Adelman who in December, 1958 contributed a paper entitled "A Stochastic Analysis of the Size Distribution of Firms" to the Journal of the American Statistical Association Since then the theory has been applied to many other problem areas and is an expression of man's desire to explore and understand more fully those forces which control his destiny

REFERENCES

1 Judge, G G and Swanson, E R, *Markov Chains – Basic Concepts and Suggested Uses in Agricultural Economics*, Department of Agr Econ AERR-49, University of Illinois

2 Howard, R A, *Dynamic Programming and Markov Processes*, M I T

3 Padberg, D I, *The use of Markov Processes in Measuring Changes in Market Structure*, J Farm Econ 44 189-199, Feb 1962

4 Kemeny, J G and Snell, J L, *Finite Markov Chains*, Van Nostrand, Princeton N Jersey, 1959