Discrete exterior calculus with applications to flows and spinors

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by

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B.A. (Mod.) (Hons.)M.Sc.

A Thesis submitted to
The University of Dublin
for the degree of

PhD

Department of Mathematics
University of Dublin
Trinity College

October, 2004

Declaration

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8 October, 2004

Acknowledgements

It is a pleasure to thank my supervisor Professor James C. Sexton who made this work possible by taking me as a research student, obtained funding for me and has been at the origin of the research on spinors. I would like to thank Professor Siddhartha Sen who gave me a lot of guidance and supervised my research on discretising geometry. As this thesis is completed, we have some ongoing research, and I hope to continue to work with them in the future.

In the course of my research I was fortunate to meet two experts in the two areas I approach in this thesis and I am very grateful to them for their remarks and for pointing out many references. Professor Alain Bossavit has gone through what is now Chapter 3 in great detail during a long day of work in Paris. On the topic of spinors I discussed it in some details with Professor Noboru Kawamoto. Thanks to Professor Denjoe O'Connor and Dr. Takuya Tsukioka at DIAS. I am very grateful to Dr. Samik Sen with whom I did what is now Chapter 5 and has been my direct colleague in this area, thanks to Dr. Mike Peardon for giving me programming ideas and to Dr. Sinéad Ryan, my internal examiner.

Thanks to many research students and staff at Trinity and in the Dublin area: to Audrey Crosbie, Karen O'Doherty and to Dr. Xavier Martin. Thanks to other Dubliners: Veronika Achleitner, Liam Greene SJ, and Marton Gyöngyösi.

I dedicate this thesis to my parents (Merci!).

The research has been supported partly by a Trinity postgraduate award and then by a research scholarship from the Higher Education Authority of Ireland as part of the IITAC project.

A discrete exterior calculus, is an analogue of differential forms defined on a discretised space or lattice, together with well-defined operations. What we mean by it depends on how much of the continuum properties one requires to tackle a specific problem. A collection of discretised analogies of the operators $\{d, \wedge, i_v\}$ acting on some discrete analogue of forms is a discrete exterior calculus if it satisfies at least some of the defining algebraic relations of the continuum mappings.

The interest in this field of research, at the interface between numerical analysis, pure mathematics and theoretical physics, is motivated by the need for discrete approaches to concrete problems in enginering through the mathematical modeling of fluids (discussed in Part II); but also in fundamental physics, in models of gravity (we discuss metric and Riemannian geometry in Part II), and in finite models of high energy physics (as we see in Part III) where algebraic and topological issues appear and are crucial. Using the language of algebraic topology, one can show that there is a doubling of fermionic particles on the lattice [1]. In both core areas of fundamental physics, the need for non-perturbative approaches is itself a strong motivation to discretise them with some exact algebraic properties being captured.

The discrete exterior calculus itself is regarded as an outstanding problem (see for example [2]), in fact one finds that using chains and co-chains, respectively discrete objects and continuum analogies of differential forms, and that the discrete wedge is non-associative. It is common to accept this limitation, while it prevents the consideration of modeling discrete analogies of continuous symmetries for example. In terms of algebras, it prevents us from considering associative algebras for which the objects are discrete versions of differential forms.

We put the algebraic properties at the forefront of the discrete theory, and we adopt what is partly an axiomatic approach to its construction. The alternative approach we proposed was to start with a class of discretisation schemes which we can refer to as "geometric discretisation" (described in the introduction: Part I) which gives rise to a triple of operators which we call $\mathcal{T}$ for topological,

$$\mathcal{T} = \{\wedge, d, \star\}. \tag{1}$$

It is based on the simplicial or hyper-cubic homological complex and uses the so-called Whitney map which maps a chain to a co-chain, the latter being effectively a differential form. The construction of this map (denoted W) is guided by the hypothesis of Stokes theorem. Our discussion starts with the hyper-cubic complex since it is the most used in the literature. We found that by going to the product space of chains and introducing a suitable product denoted $\rfloor$, one could define the contraction operation i_v and make it compatible with the other operations in $\mathcal{T}$. Such a modification was not trivial and leads to a new picture which is presented in Part II. We introduce a new framework which we call $\mathcal{L}$ for Lie,

$$\mathcal{L} = \{i_v, \wedge, d, \star\}, \tag{2}$$

and it contains the original theory $\mathcal{T}$ as a subspace.

With these structures in place, we could consider the Lie derivative as given by the Cartan formulae which uses d and i_v. We showed how the non-associativity of the discrete wedge plays no part in the derivation of the Jacobi identities using the new formalism. Then, a definition for the Lie algebra was given, and a Lie group interpretation was found, thus completing the prescription for flows. The conceptual picture is well under control, for example regarding the topology (i.e the de Rham complex for the product space), and some preliminary numerical implementation has been carried out on the torus.

With the discrete exterior calculus in place, we looked at the problem of metric data. The metric of the embedding space is the flat Euclidean metric, which is itself related to the Hodge star operator. So we considered also induced metrics by taking hyper-surfaces in $\mathbb{R}^N$. We did not construct a discrete Hodge star for the induced metric itself.

The aim of that part was to construct a covariant derivative, for which the metric connection was required. This problem, does not a priori call for a unique answer and we consider in turn two different pictures for the connection, one which is based on the Regge approach; and the second one, which is more in line with the discrete exterior calculus, is based on exploiting the Cartan structure equations. It amounts to providing the list of operators constituting $\mathcal{L}$ with a collection of one-forms, the vielbein. We end up with a theory we called $\mathcal{G}$ in a way that suggests metric,

$$\mathcal{G} = \{\{\hat{\theta}^{(a)}\}, i_v, d, \wedge, \star\}. \tag{3}$$

We discuss both approaches and how they lead to curvature, although only the second one leads to a satisfactory covariant derivative.

In Part III, we turn to the problem of discretising fermion fields. The theory naively discretised gives rise to the so-called fermion doubling problem. This problem is not resolved by refining the lattice and is a quite deep mathematical symptom. It can be formulated using algebraic topology (and particularly using the index theorem) ultimately leading to a relation between the number of right-handed and left-handed particles, as was done by Nielsen and Ninomiya [1]. Grosso modo, the problem comes from the fact that the lattice has a periodic structure and counting the fields has something to do with topology.

It was found that the problem could be expressed using a theory based on discretising the operators $\{\wedge, d, \star\}$ where for instance the Hodge star plays the role of γ_5 and hence can be used to construct chirality projectors. The equation that governs the theory is the Dirac-Kähler equation and after discretising the theory [3, 4], spatial shift operators prevent some of the algebraic relations between the various operators present. This discrepency prevents us from extracting a discrete analogue of the Dirac field from the discretised Dirac-Kähler equation.

An alternative formulation using the space $\mathcal{T}$ was constructed by us [16], in which the Whitney elements are the basis set of fields and thus avoids the usual "lattice" hypothesis of the Nielsen-Ninomiya theorem (e.g fermions are defined at vertices), but this description was somewhat incomplete. Recently we found that the Clifford product $\vee$ could only be properly defined in the discrete theory if we used the space $\mathcal{L}$, and we have shown how this leads to a discrete model with $SU(4)$ invariance of the Dirac-Kähler equation exactly realized. This symmetry is exact in the discrete setting because of the way we defined $\mathcal{L}$ in Part II. This constitutes a new result, and we have not fully analyzed its consequences at this point. At the end of Part III, we discuss different ways the theory may accommodate gauge fields.

To guide the reader through this thesis, we list the new results (to our knowledge) that are presented.

- Setup of a contraction operation, by introducing a product on the space of chains. Whitney elements play a central part (Section 3.2).

- We verify that the contraction operation satisfies the defining algebraic properties (Section 3.2). Also, we show that the extension of the original geometric discretisation (i.e the space $\mathcal{T}$) is consistent in that it contains $\mathcal{T}$ and remains topology preserving (Appendix to Chapter 3, Section 3.6.2). So the old operators in $\mathcal{T}$ have a new meaning in $\mathcal{L}$.

- As an application of the new axioms, we construct a Lie derivative and verify that it satisfies the Jacobi identities. A Lie bracket is constructed and a Lie group is defined (Section 3.3).

- We outline the way the model which is discrete can be put in equations to be solved numerically, we used a singular value decomposition algorithm to solve an equation used in fluid dynamics involving the Laplacian and the Lie derivative. We singled out problems which can be solved exactly and those that can be solved with a least square fit while still preserving the algebraic relations which can be enforced in $\mathcal{L}$ (Section 3.4).

- The metric is given in a way analogous to the vielbein approach which is the natural one here. The approximation is presented (Section 4.1).

- The curved geometry is then described following an analogous way to the Regge approach (the connection hops from cell to cell), and secondly using the Cartan structure equations, the neater way (Whitney elements vary in the interior of the cell). The crucial point is that the formulation allows us to solve the Cartan equations for the connection. Effectively we set-up the space $\mathcal{G}$ (Section 4.2).

- A covariant derivative is constructed, and the connection coefficients are worked out for a patch of the two-sphere (Section 4.2).

- Some general issues regarding the scheme are discussed and some potential applications in theoretical physics (Section 4.3).

- For spinors we describe how the space $\mathcal{T}$ is well suited for posing the algebraic conditions that capture fermion doubling (Chapter 5).

- A new discretised Clifford product based on the space $\mathcal{L}$ is defined which complements what the space $\mathcal{T}$ provides us with (Section 6.2). We show how the reduction to the Dirac field can be carried out in the discrete setting (Section 6.3).

- A discussion of various gauging procedures which may be considered, each of which is a possible direction for future work. Special emphasis is placed on the fiber bundle and the cohomology aspects of the theory (Chapter 7).

Before we start, a bibliographic note is in order. Chapter 2 and 3 are taken from the article [82], Chapter 2 being a shorter version of the introduction in [14], Chapter 4 is part of the follow-up article in preparation. Chapter 5, is based on [15, 16]. Chapter 6 is the basis for an article in preparation. Chapter 7 is partly based on discussions with Professor Siddhartha Sen.

Contents

Part I

Introduction

Chapter 1

Basic concepts of algebraic and differential topology

Problems in physics can often be formulated using the coordinate independent language of differential forms [5, 6]. The setup can be characterized in terms of the following objects: a manifold M on which p-forms (anti-symmetric tensor fields) ω_p are defined, the presence of a differential operator d that maps a p-form to a $(p+1)$-form, a wedge product, $\wedge$, which allows a p-form and q-form to be multiplied to produce a $(p+q)$-form in a way that allows a Leibniz-type rule to be valid for the operator d when it acts on the product and the Hodge star operator, $\star$.

The Hodge star operator requires that the manifold M have a metric. It maps a p-form to a $(D-p)$-form where D is the dimension of the manifold M. Using the $\star$ operator a scalar product between p-forms can be introduced. This immediately allows the adjoint $d^\dagger$ of d to be defined. The geometric discretisation scheme [7, 8, 9, 10, 11, 12, 13] constructs an analogous model involving discrete objects for the continuum system described. Each of the elements (ω_p, d, $\wedge$, $\star$, $d^\dagger$) now have discrete analogies.

Two maps allow the approximation scheme to be defined precisely. The de Rham map that maps continuum variables, such as ω_p, to discrete variables σ_p, "chains" and the Whitney map which maps discrete objects σ_p to continuous variables $W[\sigma_p]$ "cochains". Results establishing the nature of the approximation and questions of convergence have been established [12]. One generic feature of the scheme needs to be stressed. The discrete analogue of the Hodge star mapping $\sigma_p \rightarrow \star\sigma_p$ produces a different space of objects. Hence in order to properly work out discrete analogies of $d^\dagger$ it becomes necessary to work in the space which is symbolically $M_K \oplus \star M_K$ where M_K is the discretisation of the manifold M. We will also insist on the fact that $M_K = \star\star\, M_K$. There is a doubling of spaces introduced [7].

The reference for current work in this area is the collection of papers presented at the recent workshop [2]. In previous work it has been shown that the geometric discretisation scheme captured topological features of a system very nicely [14]. Extension of the ideas to accommodate fermions, in a Dirac-Kähler framework, have also been carried out [15, 16].

Let us start with a short review of some concepts of topology.

1.1 Elements of topology

To introduce topology [17, 18], we will see how the natural starting point is that of homotopy, and then that of homology, which leads to the topological invariants in the simplicial theory. Yet, we stress that homotopy is somewhat more fundamental, following [17] closely in the next subsection.

(i) Homotopy: We start by defining what is meant by two maps being *homotopic*. Given two spaces X and Y, let

$$\text{Map } (X,\, Y)\,, \tag{1.1}$$

be the space of continuous maps from X to Y. Then consider two such continuous maps α and β. Then, it is said that α and β are *homotopic*, denoted by

$$\alpha \sim \beta, \tag{1.2}$$

if there is a continuous map such that

$$F : [0,\ 1] \times X \longrightarrow Y, \tag{1.3}$$

$$(0,\ x) \longmapsto \alpha(x), \tag{1.4}$$

$$(1,\ x) \longmapsto \beta(x). \tag{1.5}$$

With this notion, one can divide the space Map $(X,\ Y)$ into equivalence classes of maps written as

$$[X,\ Y]. \tag{1.6}$$

Having alluded to homotopy in relation to maps, we now induce homotopy in relation to spaces. A space X is said to be of the same *homotopy type* as another space Y which we denote symbolically by

$$X \sim Y, \tag{1.7}$$

if there exists continuous maps

$$\alpha : X \longmapsto Y, \tag{1.8}$$

$$\beta : Y \longmapsto X, \tag{1.9}$$

such that

$$\alpha\, o\, \beta \sim I_Y, \tag{1.10}$$

$$\beta\, o\, \alpha \sim I_X. \tag{1.11}$$

where I_X and I_Y are the identity maps on X and Y respectively. Note, that homotopic equivalence between the two maps does not constrain them to be invertible.

Now, if one is to consider the circle S^1 to form the space $[S^1,\ Y]$, it is necessary to introduce a (fixed) base point in order to turn the space into a group under composition of maps. This leads to the space of based maps

$$\mathrm{Map}_0(X,\ Y), \tag{1.12}$$

which maps the base point of X to the base point of Y. As above, we introduce the associated equivalence class

$$[X,\, Y]_0. \tag{1.13}$$

Of special importance is the first homotopy group, $\pi_1(X)$ of a pointed space X which is also known as the fundamental group and is defined by

$$\pi_1(X) = [S^1,\, X]_0. \tag{1.14}$$

Note that the higher homotopy groups are defined accordingly, introducing the spheres $S^n, n = 2, 3, \ldots$, we have

$$\pi_n(X) = [S^n,\, X]_0. \tag{1.15}$$

Also, it is intuitively clear that

$$[X,\, Y]_0 / \pi_1(Y) \sim [X,\, Y]. \tag{1.16}$$

We are now in a position to extract the cohomology groups.

(ii) Cohomology: Let us introduce the *Eilenberg-McLane spaces* denoted by $K(G,\, n)$ where G is a group and n a positive integer. They have the property that

$$\begin{aligned} \pi_m(K(G,\, n)) &= G, \text{ if } m = n; \\ &= 0, \quad \text{otherwise.} \end{aligned} \tag{1.17}$$

In turn, we define the n^{th} *cohomology group* of X with coefficients in the group G by

$$H^n(X;\, G) = [X,\, K(G,\, n)]_0. \tag{1.18}$$

We now turn to what is closer in spirit to our approach, namely using a differential complex as introduced below. It will be seen in due course that this is the formulation which leads to the use of a simplicial complex.

(iii) The de Rham complex: We take a pair $\{E^n(X),\, d_n\}$ consisting of spaces $E^n(X)$

and maps d_n defined on,

$$\ldots \xrightarrow{d_{n-2}} E^{n-1}(X) \xrightarrow{d_{n-1}} E^{n}(X) \xrightarrow{d_{n}} E^{n+1}(X) \xrightarrow{d_{n+1}} \ldots, \tag{1.19}$$

and we require the composition of two maps to be zero:

$$d_n \, o \, d_{n-1} = 0. \tag{1.20}$$

The pair $\{E^n(X), d_n\}$ constitutes a *complex*. The n^{th}-cohomology group of X associated to the complex is defined by

$$H^n_E(X) = \frac{ker\, d_n}{Im\, d_{n-1}}. \tag{1.21}$$

If we take $E_n = \Omega^n(X)$ where $\Omega^n(X)$ is the space of r-forms, and $d_n = d$ is the exterior derivative; we have the *de Rham complex* which is the one we will consider in this thesis. It can be shown that the cohomology groups for this complex are isomorphic to the cohomology groups defined using the Eilenberg-McLane spaces, that is

$$H^n_{\text{de Rham}}(X) \cong H^n(X, \mathbb{R}). \tag{1.22}$$

This takes us now to the simplicial complex, by considering the dual complex which arises through the theorem of de Rham [19]. Before we get to that central link in the theory, we wish to consider another such theorem, that of Stokes, linking the exterior derivative to the boundary operator.

(iv) The Stokes theorem: Let M be an oriented n-dimensional C^∞ manifold, and ω an $(n-1)$-form on M with compact support. Then

$$\int_M d\omega = \int_{\partial M} \omega. \tag{1.23}$$

This theorem is required in order to prove:

(v) The de Rham Theorem: Given M a real C^∞ manifold. A singular p-chain, σ on

M is

$$\sigma = \sum a_i f_i, \tag{1.24}$$

where

$$f_i : \triangle \longrightarrow M, \tag{1.25}$$

is from the standard simplex $\triangle$ to M is said to be *piecewise smooth* if the maps f_i extend to C^∞ maps on a neighborhood of $\triangle$ to M (2D manifolds).
Now, let us consider the space of piecewise smooth integral p-chains $C_p^{ps}(M, \mathbb{Z})$. Considering the space

$$C_\star^{ps}(M, \mathbb{Z}) = \bigoplus_p C_p^{ps}(M, \mathbb{Z}), \tag{1.26}$$

it is clear that under the application of the boundary operator ∂, the space $C_\star^{ps}(M, \mathbb{Z})$ forms a sub-complex of $C_\star(M, \mathbb{Z})$. It then makes sense to consider

$$H_p^{ps}(M, \mathbb{Z}) = \frac{Z_p^{ps}(M, \mathbb{Z})}{\partial C_{p+1}^{ps}(M, \mathbb{Z})}. \tag{1.27}$$

The inclusion map $C_\star^{ps}(M, \mathbb{Z}) \longrightarrow C_\star(M, \mathbb{Z})$ induces the isomorphism

$$H_p^{ps}(M, \mathbb{Z}) \cong H_p(M, \mathbb{Z}). \tag{1.28}$$

The statement of the de Rham theorem is then:
The map

$$H_{DR}^\star(M) \longrightarrow H_{\text{sing}}^\star(M, \mathbb{R}), \tag{1.29}$$

is an isomorphism.
Following [19], we point out that the isomorphism of de Rham is functorial in the following sense: consider a map $f : M \longmapsto N$ which is differentiable on C^∞ manifolds, and let φ be a closed p-form on N which is the image of $[\varphi] \in H_{\text{sing}}^p(N, \mathbb{R})$ under the

de Rham map, if $\sigma = \sum a_i f_i$ is a piecewise smooth p-cycle on M then we have

$$< f^\star \varphi, \sigma > = \sum_i a_i \int_\triangle f_i^\star f^\star \varphi, \tag{1.30}$$

$$=< \varphi, f_\star \sigma > .$$

(vi) The homology complex: The dual complex to (iii) is then given by the pair $\{E_n, \partial_n\}$ giving rise to

$$\ldots \overset{\partial_{n-2}}{\longleftarrow} E_{n-1}(X) \overset{\partial_{n-1}}{\longleftarrow} E_n(X) \overset{\partial_n}{\longleftarrow} E_{n+1}(X) \overset{\partial_{n+1}}{\longleftarrow} \ldots, \tag{1.31}$$

and,

$$\partial_{n-1} \, o \, \partial_n = 0. \tag{1.32}$$

The n-th homology group of X is given by

$$H_n^E(X) = \frac{ker\, \partial_{n-1}}{Im\, \partial_n}. \tag{1.33}$$

(vii) Ring structure of cohomology groups: We introduce the ring $H_E^\star(X)$. The product is called the *cup product* and is denoted by $\cup$. In the case of the de Rham complex, the cup product coincides with the wedge product $\wedge$. Then,

$$H^\star(X;\mathbb{R}) = \bigoplus_{n\geq 0} H^n(X;\mathbb{R}), \tag{1.34}$$

and,

$$\cup : H^m(X;\mathbb{R}) \times H^n(X;\mathbb{R}) \longrightarrow H^{m+n}(X;\mathbb{R}), \tag{1.35}$$

$$([\omega], [\eta]) \longmapsto [\omega] \cup [\eta] = [\omega \wedge \eta]. \tag{1.36}$$

The cup product of two cohomology class is a cohomology class in the higher cohomology group.

(viii) Poincaré duality: If M is a compact m-dimensional manifold then

$$H^r(M) \cong H^{m-r}(M). \tag{1.37}$$

The proof of the theorem as given in [19] is very instructive for the purpose of constructing a discrete analogue of the Hodge star, which is presented in the next introductory chapter.

Finally, it is worth pointing out that homology and cohomology theories have been axiomatized, these axioms define a homology theory as a covariant functor, they are known as the Eilenberg-Steenrod axioms [20].

1.2 Simplicial complexes

We now introduce the basic definitions regarding simplicial complexes [21, 22].

(i) Simplicial complex: Consider the space $\mathbb{R}^N$ for sufficiently large N. Take $l+1$ points $v_0, v_1, \ldots, v_l$ in $\mathbb{R}^N$. These points are chosen in general positions, that is given a point v_k in the list above, one can form l linearly independent vectors with the remainder of the list.

To a set, $\sigma = \{v_0, v_1, \ldots, v_l\}$, we associate the smallest convex set including those points defined as:

$$\|\sigma\| = \{\alpha_0 v_0 + \ldots + a_l v_l;\ a_i \geq 0,\ a_0 + \ldots + a_l = 1\}. \tag{1.38}$$

This defines an *l-simplex*. The points are then referred to as the vertices of the simplex. Now that we have a simplex, we consider a collection of simplexes and introduce a *simplicial complex* denoted K:

1) An arbitrary face of a simplex of K belongs to K, that is, if $\sigma \in K$ and $\sigma' \leq \sigma$ then $\sigma' \in K$.

2) If σ and σ' are two simplexes of K, the intersection $\sigma \cap \sigma'$ is either empty or is a *face* of σ in σ'.

We denote by $|K|$ the union of all simplices belonging to the simplicial complex K. For a topological space X, if we can choose a simplicial complex K and a homeomorphism

$$t : |K| \longrightarrow X, \tag{1.39}$$

then it is called a *triangulation* of X.

(ii) Singular homology: Given any r-simplex $|\sigma|$, we consider the *standard r-simplex* denoted $\triangle^r$ which is

$$\triangle^r = \{(x_1, \ldots x_r) \in \mathbb{R}^k \,|\, x_i \geq 0,\ x_1 + \ldots + x_r \leq 1\}, \tag{1.40}$$

in turn, a *singular r-simplex* is a continuous map

$$\sigma : \triangle^r \longrightarrow X. \tag{1.41}$$

We just mention the singular complex functor [20],

$$S_\star : \{\text{Space, continuous maps}\} \mapsto \{\text{Chain complexes, chain maps}\}, \tag{1.42}$$

since it appears in the Eilenberg-Zilber theorem below.

(iii) C^∞ triangulations of C^∞ manifolds: It is crucial to understand the relation between the simplicial complex and the space from which it is extracted. This will be by means of a *triangulation* of the space which will usually be a manifold, and the relation is settled by the next theorem.

Taking the space X to be an n-dimensional C^∞ manifold M, the map t of Eq. 1.39 is called a C^∞ *triangulation*, if for any given n-simplex $|\sigma|$ of K, the restriction of t to $|\sigma|$ is a C^∞ embedding.

The Whitehead theorem states that any C^∞ manifold has a C^∞ triangulation, and any triangulation of the boundary of a C^∞ manifold can be extended to a triangulation of the whole manifold.

The next section mentions some material which is somewhat outside the scope of the discussion, it is here for completeness and should play a part if one is to express the results of this thesis in the language of categories [24].

1.3 Homological algebras and product structure

In this section, we would like to mention the construction of the cup product, as it is done rigorously in the literature [20, 23].

(i) The Eilenberg-Zilber theorem: Let TOP^2 be the category whose objects are pairs of spaces (X, Y) and whose morphisms are continuous maps. Then, the two functors

$$F : (X, Y) \longmapsto S_\star (X \times Y), \tag{1.43}$$

$$F' : (X, Y) \longmapsto S_\star (X) \otimes S_\star (Y), \tag{1.44}$$

from TOP^2 to the category of chain complexes are naturally equivalent.

The functorial property then implies that there are chain homotopy equivalence maps called Eilenberg-Zilber maps:

$$A : S_\star (X \times Y) \longrightarrow S_\star (X) \otimes S_\star (Y), \tag{1.45}$$

and,

$$B : S_\star (X) \otimes S_\star (Y) \longmapsto S_\star (X \times Y). \tag{1.46}$$

(ii) The cup product: Firstly, introduce a trivial mapping, the so-called diagonal map denoted $\triangle$ such that

$$\triangle : X \longrightarrow X \times X, \tag{1.47}$$

$$x \longmapsto (x, x). \tag{1.48}$$

Now, given $a \in H^p X$ and $b \in H^q X$ then the cup product of a and b is defined by,

$$a \cup b = \triangle^\star (a \times b). \tag{1.49}$$

We thus have a map:

$$H^pX \otimes H^qX \longrightarrow H^{p+q}X, \tag{1.50}$$

$$a \otimes b \longmapsto a \cup b. \tag{1.51}$$

For more details, we send the reader to the literature on algebraic topology and homological algebras.

Let us now turn to the background material which we make direct use of in this work.

Chapter 2

The original scheme: "geometric" discretisation

2.1 Differential forms, manifolds and complexes

The general program of the topological discretisation is described at length in [14]. On a manifold, we have fields $\omega^{(p)}$ which are differential forms (antisymmetric tensor fields), and the various operations of interest are: to consider a theory defined on an arbitrary manifold M of dimension D. Suppose the theory has fields $\omega^p(\vec{x})$, where $\vec{x} \in M$ and $p = 0, 1, \ldots, D$. The theory is constructed using the following objects which are defined on the manifold M: p-forms ω^p which are generalized antisymmetric tensor fields, the exterior derivative $d : \Omega^p(M) \to \Omega^{p+1}(M)$, the Hodge star operator $* : \Omega^p(M) \to \Omega^{D-p}(M)$, which is required to define scalar products, and the wedge operator $\omega^p \wedge \omega^q = \omega^{p+q}$. On a manifold M of dimension D, the operations $(\wedge,\, d,\, \star)$ on p-forms, ω^p where $(p = 0, \ldots, D)$, satisfy the following:

1. $\omega^p \wedge \omega^q = (-1)^{pq} \omega^q \wedge \omega^p$.

2. $d(\omega^p \wedge \omega^q) = d\omega^p \wedge \omega^q + (-1)^p \omega^p \wedge d\omega^q$.

3. $*\omega^p = \omega^{D-p}$.

4. $** = (-1)^{Dp+1}$.

5. $d^2 = 0$, $(d^*)^2 = 0$.

6. $d^\dagger = (-1)^{D(p+1)+1} * d*$. ($d^\dagger$ is the adjoint of d).

The Laplacian on p-forms is

$$\Delta_p = d_{p-1}d_p^* + d_{p+1}^* d_p, \tag{2.1}$$

and the inner product is

$$(\omega^p, \eta^p) = \int_M \omega^p \wedge *\eta^p. \tag{2.2}$$

We would like to discretise the fields of Ω^p, the inner product $(\cdot, \cdot)$ and the operators $(\wedge, d, \star)$ such that discrete analogies of the above interrelationships hold. To do this it is necessary to first introduce a few standard ideas. We first discretise the manifold M by replacing M by a collection of discrete objects, known as simplices, glued together [18]. For $p \geq 0$, a p-simplex $\sigma^{(p)}$ is defined to be the convex hull in some Euclidean space $\mathbb{R}^m$ of a set of $p+1$ points $v_0, v_1, \ldots, v_p \in \mathbb{R}^m$ where the vertices are independent in that

$$\sum_{i=0}^{p} \lambda_i v_i = 0, \tag{2.3}$$

and

$$\sum_{i=0}^{p} \lambda_i = 0, \tag{2.4}$$

implies that $\lambda_i = 0$ for $i = 0, \ldots, p$ where $\lambda_0, \lambda_1, \ldots, \lambda_p$ are real numbers. Geometrically the *barycenter* of a given n-simplex $\sigma^{(n)} = (v_0, v_1, \ldots, v_n)$ is defined by

$$\hat{\sigma}^{(n)} = \frac{1}{n+1}(v_0 + v_1 + \cdots + v_n). \tag{2.5}$$

Thus the barycenter of $\sigma^{(1)} = [v_0, v_1]$ is the midpoint between the vertices v_0 and v_1. We can now describe a particular way that a given manifold M can be discretised. Let S be a collection of simplices $\{\sigma_i^{(n)}\}$, $n = 0, 1, \ldots, D$, with the property that the faces of the simplices which belong to S also belong to it. The elements of S glued together in the following way is known as a simplicial complex [18], [25]:

1. $\sigma_i^{(n)} \cap \sigma_j^{(k)} = 0$ if $\sigma_i^{(n)}$, $\sigma_j^{(k)}$ have no common face.

2. $\sigma_i^{(n)} \cap \sigma_j^{(k)} \neq 0$ if $\sigma_i^{(n)}$, $\sigma_j^{(k)}$ have precisely one face in common, along which they are glued together.

In many cases of interest (including all differentiable manifolds [25]), M can be replaced by a complex K which it is topologically equivalent to. K is then said to be a triangulation of M (it is not uniquely defined). In this way of discretising M, the building blocks are zero, one, ..., D-dimensional objects, all of which are simplices e.g.generalized oriented tetrahedra.

Some manifolds can be discretised using a hyper-cubic complex (e.g the torus, but not the sphere), others can be discretised using a complex in which triangles, squares and other polygons are required (such complexes are called CW complexes, discussed in Section 4.3.2). We will work with a hyper-cubic complex (Section 3.1, and Part III), and it will be used as a discretisation of the torus (Section 3.4) and as a patch of the sphere (Chapter 4) in this thesis.

2.2 The discrete operations

Let us first recall the construction of the discrete Hodge star for the simplicial complex which leads to L. In analogy with the continuum property (3. above), a map from a $(D-p)$-dimensional object to a p-dimensional object was introduced. Now consider the barycentres of the building blocks of the system K. We first construct $(D-p)$-dimensional objects whose vertices are barycentres of a sequence of successively higher dimensional simplices, where each simplex is a face of the following one. In other words, they are $(D-p)$-dimensional objects of the form $\{\hat{\sigma}_p, \hat{\sigma}_{p+1}, \ldots, \hat{\sigma}_D\}$, where σ_n is a face of σ_{n+1}. The orientation of these are set so as to be compatible with the manifold. Joining these objects together gives us the dual of σ_p.

We can now give the general rule for mapping an n-simplex $\sigma_n = [v_0, \ldots, v_n]$ to a $(D-n)$ dimensional object as follows: We think of σ_n as an element of a simplicial complex K. We have

$$*_K : [v_0, \ldots, v_n] \to \cup[\hat{\sigma}_n, \hat{\sigma}_{n+1}, \ldots, \hat{\sigma}_D], \tag{2.6}$$

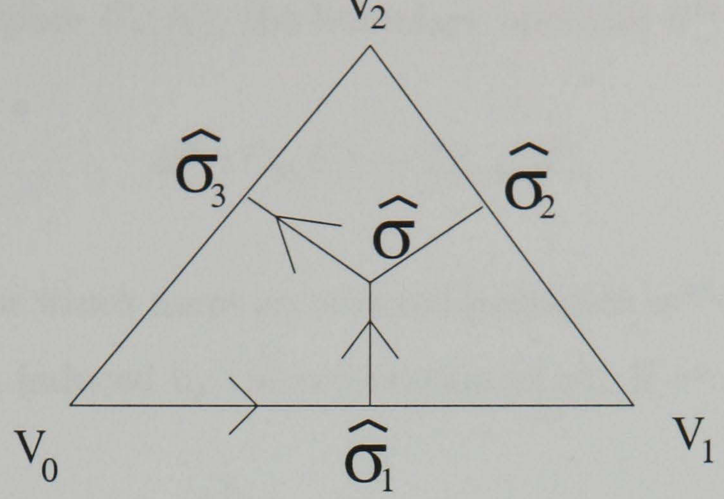

Figure 2-1: Dual complex L.

where $\hat{\sigma}_{n+1}$ is the barycenter of an $(n+1)$-simplex which has σ_n as a face (See Fig. 2-1). $\hat{\sigma}_{n+2}$ is the barycenter of an $(n+2)$-simplex which has σ_{n+1} as a face and so on. These objects have to be coherently oriented with respect to $[v_0, \ldots, v_n]$. The set of these cells constitutes the dual space $\hat{K}$ of K. By this procedure, a discrete version of the Hodge star operation was constructed. Let us explain. The Hodge star operator involves forms. It maps p-forms in D dimensions to a $(D-p)$-form. The $*_K$ map involves not forms but geometrical objects. There is a simple correspondence relation between these two cases. Given a p-form, ω_p, and a p-dimensional geometrical space, Σ_p, the p-form can be integrated over Σ_p to give a number. Thus Σ_p and ω^p are objects that can be paired. We can write this as a pairing

$$(\omega^p, \Sigma_p) = \int_{\Sigma_p} \omega^p.$$

In order to proceed, we need to introduce some more structure. We start by associating with a simplicial complex K, containing $\{\sigma_p^i\}$ $(i = 1, \ldots, K_p; p = 0, \ldots, D)$ a vector space consisting of finite linear combinations over the reals of the p-simplices it contains. This vector space is known as the space of p-chains, $C_p(K)$. For two elements $\sigma_p^i, \sigma_p^j \in C_p(K)$, a scalar product $(\sigma_p^i, \sigma_p^j) = \delta_j^i$ can be introduced. An oriented p-simplex changes sign under a change of orientation i.e. if $\sigma_p = [v_0, \ldots, v_p]$ and τ is a permutation of the indices $[0, \ldots, p]$, then $[v_{\tau(0)}, \ldots, v_{\tau(p)}] = (-1)^\tau [v_0, \ldots, v_p]$, with τ denoting the number of transpositions needed to bring $[v_{\tau(0)}, \ldots, v_{\tau(p)}]$ to the order $[v_0, \ldots, v_p]$.

Given the vector space $C_p(K)$, the boundary operator ∂^K can be defined as

$$\partial^K : C_p(K) \to C_{p-1}(K). \tag{2.7}$$

It is the linear operator which maps an oriented p-simplex $\sigma^{(p)}$ to the sum of its $(p-1)$ faces with orientation induced by the orientation of σ^p. If $\sigma^p = [v_0, \dots, v_p]$, then

$$\partial \sigma^p = \sum_{i=0}^{p} (-1)^i [v_0, \dots, \hat{v}_i, \dots v_p], \tag{2.8}$$

where $[v_0, \dots, \hat{v}_i, \dots, v_p]$ means that the vertex v_i has been omitted from σ^p to produce the face "opposite" to it.

Given that $C_p(K)$ is a vector space, it is possible to define a dual vector space $C^p(K)$, consisting of dual objects known as cochains; that is we can take an element of $C_p(K)$ and an element $C^p(K)$ to form a real number. Since the space $C_p(K)$ has a scalar product, namely if $\sigma_p^i, \sigma_p^j \in C_p(K)$ then $(\sigma_p^i, \sigma_p^j) = \delta_{ij}$. We can use the scalar product to identify $C_p(K) \equiv C^p(K)$, so that we can consider oriented p-simplices as elements of $C^p(K)$ as well as $C_p(K)$. We can write our boundary operation as

$$([v_0, \dots, \hat{v}_i, \dots, v_p], \partial^K [v_0, \dots, v_p]) = (-1)^i. \tag{2.9}$$

This suggests introducing the adjoint operation d^K defined as

$$(d^K [v_0, \dots, \hat{v}_i, \dots, v_p], [v_0, \dots, v_p]) = ([v_0, \dots, \hat{v}_i, \dots, v_p], \partial^K [v_0, \dots, v_p]). \tag{2.10}$$

This is the coboundary operator which maps $C_p(K) \to C_{p+1}(K)$.

Indeed we have

$$d^K [v_0, \dots, v_p] = \sum_v [v, v_0, \dots, v_p], \tag{2.11}$$

where the sum is over all vertices v such that $[v, v_0, \dots, v_p]$ is a (p+1) simplex.

The boundary operators ∂_K and the coboundary operator d^K have the property

$\partial_K \partial_K = d^K d^K = 0$. Furthermore,

$$d^K : C_p(K) \to C_{p+1}(K), \tag{2.12}$$

$$\partial_K : C_p(K) \to C_{p-1}(K). \tag{2.13}$$

These operators are the discrete analogies of the operators d and

$$d^\dagger = (-1)^{D(p+1)+1} * d* \tag{2.14}$$

which act on forms.

These operators could be defined only when a scalar product was introduced in the vector space $C_p(K)$. At this stage we have a discrete geometrical analogue of d, $d^\dagger$ and $\star$. We have also commented on the fact that the operation $\star$ maps simplices into dual cells i.e. not simplices. If the original simplicial system is described in terms of the union of the vector spaces of all p-chains then the space into which elements of the vector space are mapped by $\star$ is not contained within this space, unlike the situation for the Hodge star operation on forms.

2.3 Relation to the Whitney map

We now need a way to relate a p-chain to a p-form (see for example Harrison [27], and other work by the same author on Chainlets). This together with a construction which linearly maps p-forms to p-simplices allows us to translate expressions in the continuum to a corresponding discrete geometrical objects. We start with the construction of the linear maps from p-chains to p-forms due to Whitney [28]. In order to define this map, we need to introduce barycentric coordinates associated with a given p-simplex σ^p. Regarding σ^p as an element of some $\mathbb{R}^N$, we introduce a set of real numbers $(\mu_0, \ldots, \mu_p)$ with the property

$$\mu_i \geq 0, \tag{2.15}$$

$$\sum_i \mu_i = 1. \tag{2.16}$$

A point $x \in \sigma^p$ can be written in terms of the vertices of σ^p and these real numbers as

$$x = \sum_{i=0}^{p} \mu_i v_i. \tag{2.17}$$

Note if any set of $\mu_i = 0$ then the vector x lies on a face of σ^p. One can think of x as the position of the center of mass of a collection of masses $(\mu_0, \ldots, \mu_p)$ located on the vertices $(v_0 \ldots, v_p)$ respectively. Setting $\mu_i = 0$ for instance means the centre of mass will be in the face opposite the vertex v_i. These ideas suggest that the real numbers μ_i may be replaced by functions satisying $\mu_i([j]) = \delta_{ij}$ (we will return to this construction in Section 3.1 in the context of the hyper-cubic complex). The Whitney map can now be defined. We have

$$W^K : C^p(K) \rightarrow \Omega^p(M), \tag{2.18}$$

where $\Omega^p(M)$ is the space of p-forms. If $\sigma^p \in C^p(K)$ then

$$W[\sigma^p] = p! \sum_{i=0}^{p} (-1)^i \mu_i d\mu_0 \wedge \ldots \hat{d\mu_i} \wedge \ldots d\mu_p, \tag{2.19}$$

where $\hat{d\mu_i}$ means this term is missing, and $(\mu_0, \ldots, \mu_p)$ are the barycentric coordinate functions of σ^p.

We next construct the linear map from p-forms to p-chains. This is known as the de Rham map. We have

$$A^K : \Omega^p(M) \rightarrow C^p(K), \tag{2.20}$$

defined by

$$< A^K(\omega^p), \sigma^p >= \int_{\sigma^p} \omega^p, \tag{2.21}$$

for each oriented p-simplex $\in K$.

A discrete version of the wedge product can also be defined using the Whitney and de Rham maps such that $\wedge^K : C^p(K) \times C^q(K) \to C^{p+q}(K)$ as follows:

$$x \wedge^K y = A^K(W^K(x) \wedge W^K(y)). \tag{2.22}$$

It has many of the properties of the continuous wedge product in that it is skew-symmetric and obeys the Leibniz rule but it is non-associative.

At this stage we have introduced all the building blocks necessary to discretise a system preserving topological structures. We summarize the properties of the maps introduced in the form of a theorem. We have

Theorem (Whitney [28])

1. $A^K W^K =$ Identity.

2. $dW^K = W^K d^K$, where $d : \phi^p \to \phi^{p+1}$.

3. $\int_{|\beta|} W^K(\alpha) =< \alpha, \beta >$, $\alpha, \beta \in K$.

4. $d^K A^K = A^K d$.

This theorem shows how d^K can be considered as the discrete analogue of d. We now show how $*^K$ can be considered as discrete analogue of $*$. For this we need barycentric subdivision.

In order to construct the star map, two geometrically distinct spaces were introduced. The original simplicial decomposition K with its associated set of p-chains $C^p(K)$ and the dual cell decomposition L with its associated set of p-chains $C^p(L)$. These spaces are distinct. However both belong to the first barycentric subdivision of K. This allows the use of the $*^K$ operation if we think of K and L as elements of

BK. The latter being the complex defined by the finer subdivision of the complex K which contains both K and L.

We proceed as follows. Let BK and L denote the barycentric subdivision and dual triangulation, and

$$*^K : C^p(K) \rightarrow C^{n-p}(L). \tag{2.23}$$

However $C^p(K)$ and $C^p(L)$ are both contained in $C^p(BK)$ as we have seen. Let

$$W^{BK} : C^p(BK) \rightarrow \Omega^p(M), \tag{2.24}$$

denote the Whitney map. Then we have [9, 26], for $x \in C^p(K), y \in C^{n-p-1}(L)$,

1. $< *^K x, y >= \int_M W^{BK}(Bx) \wedge W^{BK}(By)$,
   $< *^L y, x >= \int_M W^{BK}(By) \wedge W^{BK}(Bx)$.

2. $\partial^K = (-1)^{np+1} *^L d^L *^K$ on $C^p(K)$,
   $\partial^L = (-1)^{nq+1} *^K d^K *^L$ on $C^q(L)$.

for the hyper-cubic complex. These are the discrete analogies of the interrelationships between $d, d^\dagger, *$ and $< \cdot, \cdot >$ in the continuum.

Note $K \neq L$ and that properties of ∂_K, d_K analogous to those for differential forms only hold if K , L are both regarded as elements of BK. This feature will play a role in the discussion of the metric and of spinors. This concludes the introductory part.

Part II

Geometry

Here we turn to the practical problem of capturing geometrical information present in the original continuum problem in the discrete approximation scheme described without spoiling topological structures that it encodes. The approach we introduce tackles the problem of defining what is known as a discrete exterior calculus and different attempts to address parts of this problem are available; Bossavit [29] has introduced a discrete analogue of contraction by considering the dual mapping "extrusion" in the context of the electromagnetic theory. Hirani and co-workers [30] have developed part of a discrete exterior calculus which is closely linked in spirit to the topological theory (the geometric discretisation described in Part I. being an example), but they do not study the Lie derivative which is of much importance here.

In contrast to other works, in our approach the Whitney map which maps discrete objects to continuum variables plays an essential role. Here, a product space of chains allows the notion of contraction to be defined. This is essential for introducing an analogue of the Lie derivative, an operation that describes the way objects change. The structure constructed gives rise to a consistent discrete exterior calculus, which is to our knowledge a new result. The original geometric discretisation scheme captured topological data. Here we will consider the hyper-cubic version of it [26] as our starting point. An alternative derivation of the Whitney elements is introduced which suits our purpose. First, we point out the important shortcomings of the original geometric discretisation method:

- The discrete wedge is non-associative.
- Operators are only exact as operations on the discrete chains, (i.e there is no direct correspondence or "functor" between the discrete operations and the continuum ones).
- Whitney elements play a spectator role and so the theory is topological only.

Recently, Samik Sen [26] has shown that by replacing a simplicial complex by a hyper-cubic complex, two new features emerge compared to the original proposal: firstly, the Hodge star operator satisfies $\star^K \star^L = Id$ and secondly, the associated dual complex being hyper-cubic as well, means it can be provided with explicit expressions for the Whitney elements.

The second item of the list above is crucial to us, because we are going to take the Whitney forms as the basic building blocks of the theory. Then, we use the product space of chains to generate the appropriate collection of forms which will in turn allow us to consider a discretised system with symplectic geometric data.

In contrast, the original scheme does not utilize the Whitney forms beyond providing a space that gives rise to the de Rham co-homology complex, so in that case we need not know the explicit form of the Whitney elements to proceed topologically.

Here, we put the Whitney elements and the various combinatorial manipulations of them at the forefront of the formulation. A discrete operation is then *exact* if when mapped to the continuum its algebraic properties hold exactly. Then in turn, by accommodating the other operations in the theory ($\wedge$, d, $\star$) we can show that the combinatorics of Whitney elements that we introduce is such that the topological data is still captured. We find that:

- The basis of Whitney forms encodes the topology and is very constraining that way, but allows for the product space of chains.
- Secondly, the problem of the non-associativity of the wedge is by-passed by not relying on the wedge but on a new product in order to define the contraction.

These arguments both provide a strong motivation for considering the product space of chains.

In Chapter 3 we derive a Whitney map in the context of a hyper-cubic complex. Then, the main features of the new scheme are introduced in relation with the definition of a contraction operation; then we proceed with the Lie derivative, the Lie algebra and associated group action on the space of Whitney forms. We discuss the convergence and a numerical implementation. In the Appendix we outline in what sense the theory still captures topology, and display some calculations. In Chapter 4, we tackle the metric along with the covariant derivative and the curvature.

Chapter 3

Setting up the axiomatics of a discrete exterior calculus

3.1 The hyper-cubic complex

In what follows, we will consider a hyper-cubic complex, since it is the most used in the literature, and we have already pointed out its advantages. We consider singular homology rather than homology (see the definition of a simplicial complex [18]). Let us describe how to handle the new setting. Consider the surface of the cube, topologically the two-sphere S^2. Consider the face [0123] in Fig. 3-1, and let us follow the steps of the construction of the Whitney map given in the last section. The zero-form Whitney elements are obtained by application of the boundary conditions and are given by:

$$\{\mu_0 = (1-x)(1-y),\ \mu_1 = x(1-y),\ \mu_2 = xy,\ \mu_3 = y(1-x)\}. \tag{3.1}$$

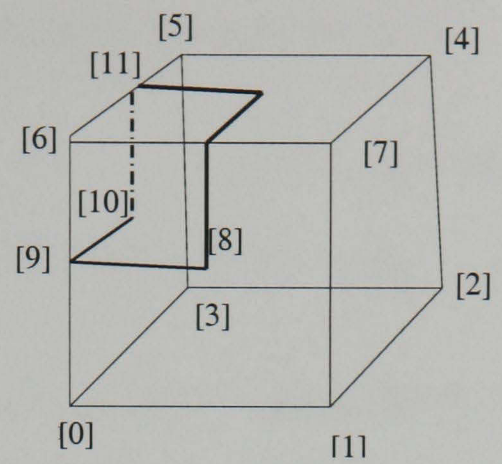

Figure 3-1: The cube with labels used throughout the discussion. The thick loop is a holonomy around the vertex [6], discussed in the next chapter.

The book-keeping for the co-boundary operator is as follows. Consider the simplex [0123]. Then the application of d_K is shown below.

$$d^K[0] = [10] + [30], \tag{3.2}$$

$$d^K[01] = [0123], \tag{3.3}$$

$$d^K[10] = [1023] = -[0123], \tag{3.4}$$

$$d^K[0123] = 0. \tag{3.5}$$

In addition, the orientation is fixed by specifying a two-chain, in this case [0123]. The way to check that the d operation is correct, is by verifying that the Stokes theorem holds. However, by using the formulae of Whitney directly given the cubic barycenter coordinates one finds that the requirement $W^K d^K = dW^K$ is not satisfied. This makes the basis list of forms an incorrect basis of Whitney elements as noted by Samik Sen [26].

To remedy this problem we derive the Whitney map by induction. A Whitney map is defined as follows. Take the barycenter coordinates satisfying

$$\mu_i([j]) = \delta_{ij}, \tag{3.6}$$

and generate the Whitney elements by application of

$$dW^{(r)} = W^{(r+1)}d^K, \tag{3.7}$$

$$< \sigma_i^{(r)}, W^{(r)}(\sigma_j^{(r)}) > = \delta_{ij}. \tag{3.8}$$

which is used as the definition of $W^{(r+1)}$ given $W^{(r)}$. The Whitney elements thus obtained satisfy automatically the Stokes theorem.

The reader may check that the Whitney elements given in the Appendix do indeed satisfy this rule (and may skip the rest of this section).

We will sketch the proof by induction. The first observation is that for zero co-chains, adopting the functional form of the example of the square above, we find that any vertex $[i]$ has within a given top dimensional simplex $\triangle^n$ of which it is a face the following expression:

$$W([i]) = \Pi_k f_k, \tag{3.9}$$

where each factor is either

$$f_i = f_i^+ = (1 - x_i), \tag{3.10}$$

$$or$$

$$= f_i^- = x_i. \tag{3.11}$$

The factors enforce the boundary conditions in Eq. 3.8, and depend on which bounding simplex we are taking for the evaluation of the Whitney map. The claim is then that we can extract the remaining elements, i.e of higher degrees. An immediate observation is that application of d will suppress factors f_i and replace them up to sign by a form dx^i. Moreover, the factor f_i that is suppressed relaxes one of the boundary conditions relating to the simplex containing both the original one and its extension caused by application of d^K. In turn, this leads to a well-defined pull-back map for integration over a given simplex $\sigma_i^{(r)}$ which we denote by

$$< \sigma_i^{(r)}, \; . \; >^{\star} \colon \Omega^r(M, \mathbb{R}) \longleftarrow \mathbb{R}, \tag{3.12}$$

and the space of Whitney elements is generated by the image (denoted $\Im$) of $1 \in \mathbb{R}$ under the pull-back for each and every r-simplex in $\sigma_i^{(r)} \in \Delta^{(r)}(K, \mathbb{R})$,

$$W^{(r)}\Delta^{(r)}(K, \mathbb{R}) = \Im(< \Delta^r(K), \; . \; >^{\star} 1). \tag{3.13}$$

This map, gives us the Whitney elements. Take a $\sigma_i^{(r)}$, then assume $W^{(r)}(\sigma_i^{(r)})$ is given. Then by application of the pull-back map after evaluating $dW(\sigma_i^{(r)})$ using Eq. 3.7 gives $W^{(r+1)}$ for a $(r+1)$-simplex. This process is done until the list of $(r+1)$-simplices is exhausted.

Consider a basic example as given in the Appendix. Apply the exterior derivative to get a linear combination of terms which match $W([01])$, $W([03])$ in the square [0123]. Then select $W([01])$.

A side issue is that of completeness. One might be under the impression that we are calculating W for exact forms in

$$Z^r(K, \mathbb{R}). \tag{3.14}$$

Obviously, this is not true, the pullback allows us to evaluate the Whitney map on the basis set of $C^r(K, \mathbb{R})$ which is the collection of simplices $\Delta^r(K)$. The Whitney elements obtained are the correct ones (see the appendix).

To visualize the Whitney elements, we refer to Fig. 3-2. A one-chain with non-zero coefficients along the edges parallel to the y-axis has the following shape, you see that the form jumps over cells which are parallel to the 1-chain but this does not affect the differentiability of the form since $d = dx \wedge \frac{\partial}{\partial x}$ on a dy-form. In multiple dimensions you get all derivatives apart from $\frac{\partial}{\partial y}$.

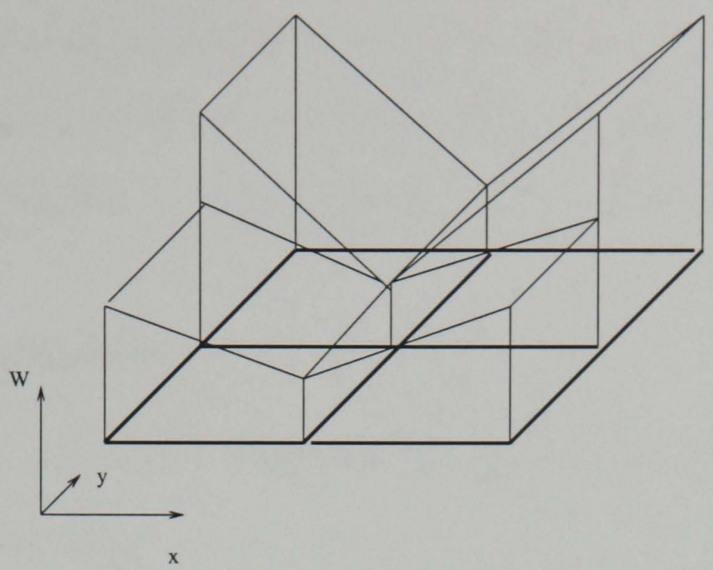

Figure 3-2: The two dimensional plane with a square complex, coordinates (x, y) and a Whitney form with function coefficient sketched in the W-axis.

At this point we take the topological theory above and modify it in order to consider geometry. The key construction that leads us to geometry is the formula given for a hyper-cubic interior product Eq. 3.17 and the numerous properties it possesses as we proceed to describe.

3.2 Setting up the exterior calculus

In order to extend the range of applicability of the method we need to be able to study "flows". This involves introducing the idea of a vector field in the discrete setting and constructing an analogue of the Lie derivative. The vector field enters the discussion in the interior product below as the coefficient function of a form Eq. 3.16, and since we work in the space $\mathbb{R}^N$, we can identify upper and lower indices [1] .

To do this we need to define a discrete analogue of the interior product. Geometrical information in the sense of introducing a metric in this frame corresponds to following the vielbein formalism (Chapter 4). The metric of the embedding space is the Euclidean flat metric (recall Part I, the introduction).

We now begin our construction by noting that the discretisation scheme is too restrictive in order to consider a geometry. What we need is a contraction map i_v in

[1]When one considers the induced metric on a hypersurface $h_{\alpha\beta} = \frac{\partial X^a}{\zeta^\alpha}\frac{\partial X^b}{\zeta^\beta}\delta_{ab}$ [38], one can still integrate vector fields over edges in $\mathbb{R}^N$, so there is no ambiguity here, Chapter 4 addresses the induced metric in more detail.

order to consider vector fields and geometry. With d and i_v we will construct the Lie derivative L_v (and in turn, in the next chapter, with some prescription for the metric and the Lie derivative we will be able to construct a covariant derivative ∇_v).

3.2.1 Interior product

The problem we start with is to construct a hyper-cubic analogue of the interior product (or contraction) map:

$$i_v\omega^{(r)} = \frac{1}{(r-1)!}v^\nu \omega_{\nu\mu_2\ldots\mu_r} dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_r}. \tag{3.15}$$

Our strategy is to apply the de Rham map to the dual one-form of the vector field v represented as a one-chain (i.e a linear combination of edges) and to the degree $(r-1)$-form that results from ω after contraction. Thus providing us with a pair of chains. Let us extract it from the continuum by rewriting Eq. 3.15 as

$$i_v\omega^{(r)} = \frac{1}{(r-1)!}v^\nu \rfloor^K \omega_{\nu\mu_2\ldots\mu_r} dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_r}, \tag{3.16}$$

and then notice that this expression, as a pairing of two forms, can be represented on the chain space of the K complex as:

$$i_{\alpha^{(1)}}\sigma^{(r)} = \alpha^{(1)} \rfloor^K \eta^{(r-1)}(\sigma). \tag{3.17}$$

where α is a one-chain, σ is an r-chain and η an $(r-1)$ chain which results from contracting σ. It is similar to what we have in the continuum with exception that α, a one-chain plays the role of a coefficient function and so should be treated as such. The second remark is that in fact this leads to a well-defined operation on the complex and is exact when the form ω and the dual to the vector field v are in fact Whitney elements. Also, we introduced a generalized product on the space of chains, denoted by $\rfloor^K$ the choice of the symbol is made to stress that it comes from the contraction operation. It clearly distinguishes between the chain on its right and that on its left. Let us postpone the formal construction and consider examples.

Start with a 1D example. Consider the edge [01] on the line. Then, the Whitney elements are

$$\{W([0]) = 1 - x,\, W([1]) = x,\, W([01]) = dx\}, \tag{3.18}$$

and

$$i_{[01]}[01] = [01]\rfloor^{K}([0] + [1]). \tag{3.19}$$

That is, we took the sum of the two vertices that remain when we remove the edge [01]. Also, the construction is guided by the embedding in the continuum:

$$W(i_{[01]}[01]) = W([01]) \times W([0] + [1]) = dx \times 1 = 1. \tag{3.20}$$

Again we said that the one chain α plays the role of a function coefficient so, in this case it is 1 and hence the result. If you contract [0] or [1] with the edge [01], you get zero because the edge [01] is not a face of either vertices. Consider the 2D example of the square (Fig. 3-3).

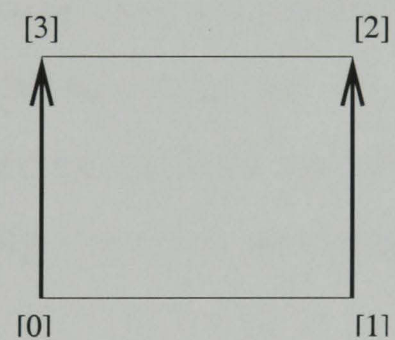

Figure 3-3: Two parallel edges in a given square which arise from contraction of [0123] with one of the edges [01] or [32].

The Whitney elements are:

$$\{W[0] = (1-x)(1-y),\, W[1] = x(1-y),\, W[2] = xy,\, W[3] = (1-x)y, \tag{3.21}$$

$$W([01]) = (1-y)dx, W([12]) = xdy,\, W([32]) = ydx, \tag{3.22}$$

$$W([03]) = (1-x)dy,\, W([0123]) = dx \wedge dy\}. \tag{3.23}$$

This time there are many combinations of two chains which lead to a non-zero contraction, for example the contraction of the square with an edge, say [01]. Let us see two examples which will help us to specify i_v.

$$i_{[01]}[0123] = [01]\rfloor^K ([03] + [12]), \tag{3.24}$$

$$i_{[01]}[32] = [01]\rfloor^K ([3] + [2]). \tag{3.25}$$

And they lead to

$$W(i_{[01]}[0123]) = W([01])\rfloor^K W([03] + [12]) = (1 - y)dx\rfloor^K dy = (1 - y)dy, \tag{3.26}$$

$$W(i_{[01]}[32]) = W([01])\rfloor^K W([3] + [2]) = (1 - y)dx\rfloor^K y = (1 - y)y. \tag{3.27}$$

To comment, the first calculation shows how when contracting a 2-chain, we get a linear combination of the edges perpendicular to the edge which specifies the vector field. This generalizes obviously to higher degree chains: we get a linear combination of the $(r - 1)$-simplices which are orthogonal to the vector field (an edge). The second calculation is there because it shows that although [01] is not a face of [32], the contraction still applies because they are parallel within the same cell of highest degree which plays the role of the local open set.

We formalize a little the interior product we have constructed and in the next sections, we will have to modify the other operators accordingly, notably d^K the co-boundary.

The discretised interior product is a map from the product space of chains to the product space of chains. We have already seen in what sense it is exact by constructing basic examples in this section. We define the map

$$i : C^{(1)}(K, \mathbb{R}) \times C^{(r)}(K, \mathbb{R}) \longrightarrow C^{(1)}(K, \mathbb{R}) \times C^{(r-1)}(K, \mathbb{R}) \tag{3.28}$$

$$(v, \sigma) \longmapsto v\rfloor^K \eta(\sigma) \tag{3.29}$$

where $\eta(\sigma)$ is the $(r - 1)$-chain that results from the contraction by taking a linear combination of faces of $\sigma^{(r)}$. There is also a sense in which the contraction is done in a "local" open set which is determined by the cell (or simplex) of highest dimension which contains both v and σ.

Also, the Whitney map applied to $i_v\sigma$ treats the vector field as a function coefficient i.e:

$$W(v\rfloor^K \eta(\sigma)) = W(v)\rfloor^K W(\eta(\sigma)) = \varphi^0(W(v)) \wedge W(\eta(\sigma)), \tag{3.30}$$

where φ^0 deletes the constant form part of the Whitney element,

$$\varphi^0(\omega dx) = \omega. \tag{3.31}$$

The generalization of this operation when more than one contraction has been applied to a chain, like for the Jacobi identities below, is done in the obvious way.

Having shown that the operation works in the cell, we return briefly to the consideration of chains, we alluded to them in the discussion of FIG. 3.3. There is a pairing of chains in $i_v\sigma$ which makes the map bilinear. It is clear that each simplex gives rise to a co-chain with support over all cells that have that simplex as a face. This is related to the notion of calculating in the "local" open set we alluded to. Otherwise, we are left with the consideration of constant vector fields, we will return to that in the discussion of the Lie derivative. We write the bi-linearity of the interior product as

$$i_v\sigma = \sum_{i,j} i_{v_i}\sigma_j = \sum_j \sum_{\{v_i|\,(v_i,\sigma_j)<\sigma_k^{(n)}\}} i_{v_i}\sigma_j. \tag{3.32}$$

In words, given a one simplex v_i from the one-chain v, and one term σ_j from the chain σ, if they are both faces of a higher dimensional hyper-cube $\sigma_k^{(n)}$, then the terms contribute. It is then straightforward to establish that the discrete interior product squared is zero.

Having introduced the interior product, we allow for geometry, but in doing so, the closure of the operations is on the product space of chains. We will establish that the other operations $\wedge$, d and $\star$ can be modified in a way that makes the theory consistent topologically. Also, we will see that the Jacobi identities work exactly, thus limiting the result of the chain operation to the product of two such spaces, while the intermediate steps involve three. So let us modify the discrete theory to accommodate the new operator i_v.

3.2.2 Exterior derivative

A priori it is already under control, since if ω is a Whitney form, we know how to calculate $d\omega$, it is a linear combination of $(r+1)$-degree Whitney elements using the co-boundary $d^K = \delta^K$ on the complex. However $di_v\sigma = d(v\rfloor^K\eta(\sigma))$ is problematic, we need to make a prescription for it since it is an object in the product space of chains. The outcome is that we can define it in a consistent way which also leaves the cohomology intact. To see this, go back to the 2D example and let

$$d^K(i_{[01]}[0123]) = d^K([01]\rfloor^K([0]+[1])) = 2[01]\rfloor^K([30]+[21]), \qquad (3.33)$$

while mapping both sides to the continuum we get the correct result which is

$$di_v((1-y)dx) = -2(1-y)dy. \qquad (3.34)$$

The operation is well-defined in general, when applying d after i_v, we only take partial derivatives with respect to the orthogonal directions to both $v^{(1)}$ and $\sigma^{(r)}$ and $r \geq 1$. So we can apply the Leibniz rule for functions in a combinatorial way by writing

$$d^K(i_v\sigma) = 2v\rfloor^K d^K\eta(\sigma), \qquad (3.35)$$

where d^K is the original co-boundary map, and thus, the homology of the theory in the product space is the same as that of the original theory and leads to the desired result:

$$W(d^K i_v\sigma) = dW(i_v\eta(\sigma)). \qquad (3.36)$$

To justify Eq. 3.35, we can give a proof, by taking the general Whitney form in a hyper-cube which was discussed above. Let

$$W(\sigma^{(r)}) = \Pi_{i=1}^{p} f_i^- \Pi_{j=p+1}^{n-r} f_j^+ \bigwedge_{k=1}^{r} dx^{\mu_k}. \qquad (3.37)$$

Then, starting with the right hand side of Eq. 3.35, we get

$$W(2v\rfloor^K d^K\eta(\sigma)) = 2(W(v))\rfloor^K(\sum_l \frac{\partial}{\partial x^l}dx^l\wedge)W(\eta(\sigma)), \tag{3.38}$$

the point is that when you move the derivatives $\frac{\partial}{\partial x^l}$ across to the left, you pick up all the terms since the vector has at least every f_i factor that σ has. There is a minor sign issue which as to be watched out for, since as stated above, the contraction operation with edges that are parallel to σ within a given cell has to be included, and $\frac{\partial}{\partial x^l}f_l^{\pm} = \mp 1$.

Returning to the co-homology. The di_v-property in Eq. 3.35 can be used twice to show that

$$(d^K)^2(i_v\sigma) = v\rfloor^K(d^K)^2\eta(\sigma) = 0, \tag{3.39}$$

since $(d^K)^2 = 0$ on the space of chains. We had the property that

$$(d^K)^2 C^r(K, \mathbb{R}) = 0, \tag{3.40}$$

and we have deduced that

$$(d^K)^2 i_v C^r(K, \mathbb{R}) = 0. \tag{3.41}$$

The di_v-property is particularly convenient since it also helps to consider exact chains as well as closed chains. Let us consider an $(r+1)$-chain which is exact,

$$\eta^{(r+1)} = d^K\sigma^{(r)}. \tag{3.42}$$

then, consider a one-chain representing the vector field v and assume that we contracted an $(r+1)$-chain to obtain $v\rfloor^K\sigma$. Then, immediately we get

$$v\rfloor^K\eta^{(r+1)} = \frac{1}{2}d^K(v\rfloor^K\sigma^{(r)}) = v\rfloor^K(d^K\sigma^{(r)}). \tag{3.43}$$

which is a rewriting of Eq. 3.42 leading to the Homology groups

$$H^{(r)}(M, \mathbb{R}) \cong B^{(r)}(M, \mathbb{R})/Z^{(r)}(M, \mathbb{R}). \tag{3.44}$$

The picture is the following: we had the link between the de Rham co-homology and the homology as a basis for discretising differential forms with the operator d. Now, we have extended the theory to the product space of chains in a way that preserves the content of the topological theory, and furthermore gives us access to the geometry. Whitney elements have a central role in the theory.

3.2.3 The wedge product and the Hodge star

The discrete wedge product is a simple generalisation of the original one Eq. 2.22, but it is worth pointing out how it co-exists with i_v since we have to check that it acts as an anti-derivation on the space of chains i.e

$$i_v(\sigma^{(r)} \wedge^K \beta^{(p)}) = (i_v\sigma^{(r)}) \wedge^K \beta^{(p)} + (-1)^r \sigma^{(r)} \wedge^K i_v\beta^{(p)}. \tag{3.45}$$

Again, the criterion is given by the continuum analogue which is

$$W(i_v(\sigma^{(r)} \wedge^K \beta^{(p)}) = W(i_v\sigma^{(r)}) \wedge^K \beta^{(p)}) + (-1)^r W(\sigma^{(r)}) \wedge^K W(i_v\beta^{(p)}). \tag{3.46}$$

But we have already proved that because the interior product is exact on the space of Whitney forms. So by using

$$i_{Wv}W = Wi_v, \tag{3.47}$$

we rely on the continuum identity and so Eq. 3.45 holds. As is known, the wedge gives rise to the cohomology ring, and the discrete wedge (or cup product) is non-associative, or it is only up to a factor. By acting with d on $\sigma \wedge^K \eta$, we invoke the Liebnitz rule which is exact only after integration (application of the de Rham map) in the original geometric discretisation. The point is that Whitney forms were only present as dummy integration variables which are compatible with d, while here we want exactness before integrating thus preserving the topological properties associated to d and to i_v. Then, taking the wedge product of two "contracted" chains is done in the natural way (where $\otimes$ is the tensor product)

$$(\zeta\rfloor^K\eta(\sigma))\wedge^K(\beta\rfloor^K\eta(\alpha))=(\zeta\otimes\beta)(\eta(\sigma)\wedge^K\eta(\alpha)), \tag{3.48}$$

again in a similar fashion as for d. As for the chain space, we then have a discrete analogue of

$$H^{(r)}(M,\mathbb{R})\wedge H^{(p)}(M,\mathbb{R})\cong H^{(r+p)}(M,\mathbb{R}). \tag{3.49}$$

This can be seen by noting that the term under brackets is the ordinary wedge operation in the chain space.

The Hodge star has two representatives in the discrete theory, one in the K subdivision and one in the L subdivision. We write:

$$\star^K(\beta\rfloor^K\eta(\sigma))=\beta\rfloor^K(\star^K\eta(\sigma)). \tag{3.50}$$

Then, the associated properties of $\star^K$, $\star^L$ and $(d^K)^\dagger$ hold, since using Eq. 3.50, we get

$$(d^K)^\dagger=(\star^L d^K\star^K)(\star^L d^L\star^K)=\star^L(d^L)^2\star^K=0, \tag{3.51}$$

where we used the fact that

$$\star^K\star^L=Id_L,\ \star^L\star^K=Id_K. \tag{3.52}$$

and in much the same way as for the wedge, we still have the Poincaré duality

$$H^{(r)}(M,\mathbb{R})\cong H^{(n-r)}(M,\mathbb{R}). \tag{3.53}$$

What is important at this point is that the operations ($\wedge$, d, $\star$) are essentially topological and the function coefficient is passive or gives rise to a factor of two which does not modify the homology, a discussion of the co-homology of the theory in the product space is given in the Appendix. Let us turn to the Lie derivative.

3.3 Lie derivative and continuous symmetries

With the operations d and i_v in place, it is a matter of consistency that the discrete Lie derivative, as given by the Cartan formulae

$$L_v = i_v d + d i_v, \tag{3.54}$$

should have the following analogue:

$$L_v = i_v^K d^K + d^K i_v^K. \tag{3.55}$$

We will find that we can go quite a long way with it and consider a discrete version of the Lie group. As we saw, we consider the vector field as being represented by a one-cochain which is its dual differential form. The theory is seen as a theory of differential forms and so this choice is imposed to us. If we were to consider the vector fields in the dual complex as $\star^K v$ in the L, we would lose the common support that both the vector field and the form must have in order to carry through the construction of the exterior calculus. We start with examples, since we already have the operators.

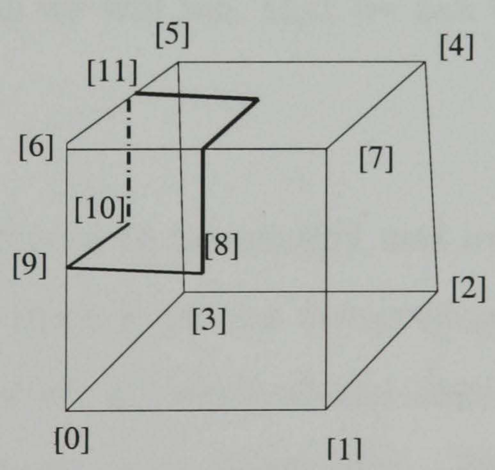

Figure 3-4: The cube with labels. The thick loop is a holonomy around the vertex [6], discussed in the next chapter.

Example 1: (see the Appendix, and Fig. 3-4), to investigate the construction, we start with the vector field (dual to) Eq. 3.157 and the form Eq. 3.158. These are the continuum analogies of Eq. 3.162, Eq. 3.163. The vector and the form will be considered on three faces, [0176], [1247] and [0653] for this purpose. To each of these corresponds a volume element two-form (i.e a constant form such as $dx \wedge dz$). Due to the orthogonality of the form with the vector chosen, we get $i_v v^b = 0$. Next, evaluation of dv^b in the discrete setting Eq. 3.165 is obtained by means of applying the rules for the hyper-cubic complex. After application of the Whitney map the result (e.g Eq. 3.168) is obtained by using the rule we have introduced.

This means that the transcription into an ordinary differential form is obtained by suppressing the form part (like a dx or a $dx \wedge dy$) on the left hand side of the $\rfloor^K$ symbol. So by substituting the list of Whitney elements given in the Appendix, and the rule, we obtain Eq. 3.168 which leads to the identification of the various terms in Eq. 3.161. We have found the following:

$$W(L_{\hat{v}}\hat{v}_b) = L_{W\hat{v}}W(\hat{v}_b). \tag{3.56}$$

We now move to a more delicate case, for which we will need to clarify the notion of the tangent space $T_U M$ of vector fields, where U is an open set taken to be the cell. We wish to point out that we do not think of the tangent space TM as a fiber bundle. The topology of the complex is too restrictive for that, open sets only overlap on a common boundary. However, the Whitney elements, because their support overlaps over boundaries guarantee as we will see, that we can consider properties of vector fields.

Example 2: Take the vector u to be parallel and in fact identical to v^b. In this case, one has to be careful since if we use the proposed formulae for the interior product, we get Eq. 3.175, which is clearly wrong since there should be some terms with coefficient $v_{Z_1}v_{Z_2}$ as shown in the appendix. This is the point at which we introduce the rule above from which we get (3.176). Then,

$$\begin{aligned} di_u v_b &= 2(v_{Z_2}^2 - v_{Z_1}v_{Z_2})[06]\rfloor^K([10]+[76]) + 2(v_{Z_1}^2 - v_{Z_1}v_{Z_2})[17]\rfloor^K([01]+[67]), \\ &= 2(-(v_{Z_2}^2 - v_{Z_1}v_{Z_2})(1-x) + (v_{Z_1}^2 - v_{Z_1}v_{Z_2})x)dx, \\ &= 2(v_{Z_1} - v_{Z_2})(v_{Z_2}(1-x) + v_{Z_1}x)dx, \end{aligned} \tag{3.57}$$

as required to match the continuum expression. For the $i_u d$-term, the problem of support which led us to modify the coefficients does not arise and we get

$$\begin{aligned} i_u dv_b &= (-v_{Z_2}^2[06] - v_{Z_1}v_{Z_2}[17] + v_{Z_1}^2[17] + v_{Z_1}v_{Z_2}[06])\rfloor^K([01] + [67]), \\ &= (-v_{Z_2}^2(1-x) - v_{Z_1}v_{Z_2}x + v_{Z_1}^2 x + v_{Z_1}v_{Z_2}(1-x))dx, \\ &= (v_{Z_1} - v_{Z_2})(v_{Z_2}(1-x) + v_{Z_1}x)dx, \end{aligned} \tag{3.58}$$

which gives the correct matching on [0617], not forgetting the multiplicative factor of two, specified by the rule for "di_v". This gives the correct matching.

The combination of Example 1 and Example 2 cover all cases of a pair of a form and a vector field that can be expressed in the present framework (i.e parallel or orthogonal). They constitute the two parts of the proof that the discretised Lie is the correct one. What we are describing here is a flow from the face [0176] to the face [0653] and check its z-component at the edge [06] which joins the two faces. The tangent vector to the integral curve is thus rotated by a right-angle as it propagates along its flow line. The component v_x lives in [0176], the component v_y lives in [0653] while the component v_z lives in both.

Evaluating components of the discrete Lie derivative as we have done above will be particularly useful when we turn to discuss the covariant derivative in the Chapter 4. But before we address this issue, we ought to check how the Lie algebra property, and notably how the Jacobi identities translate to the lattice. This property gives us a lifting of the Lie algebra of vector fields (which we have yet to define) to the algebra of derivations acting on the space of forms. In the continuum we write:

$$\left[L_v,\, L_u\right] = L_{\left[v,\, u\right]}. \tag{3.59}$$

Although we are in a position to evaluate the left-hand side of Eq. 3.59, we need to make a prescription for the complex based bracket $[v,\, u]$, and in fact we know that in the infinite dimensional representation of vector fields which we adopt, the continuum commutator is clear:

$$\big[u,\, v\big] = (u^i \partial_i v^j - v^i \partial_i u^j)\partial_j. \tag{3.60}$$

Or conversely, we need to show that starting with $\big[L_v,\, L_u\big]$, the vector field which is the argument of the Lie derivative on the right-hand side is indeed the discrete analogue of $[v,\, u]$ given in Eq. 3.60.

Given two one-chains, σ and η, on a 2D complex, we introduce their bracket

$$\big[\,.\,,\,.\,\big]_K : C^1(K) \times C^1(K) \longrightarrow C^2(K) \times C^1(K), \tag{3.61}$$

$$([\sigma],\, [\eta]) \longmapsto \big[\sigma,\, \eta\big]_K, \tag{3.62}$$

where in components, assuming summation over the indices, we are given $\sigma = \sigma_{ij}[ij]$ and $\eta = \eta_{ij}[ij]$, then,

$$\big[v,\, u\big]_K = (\sigma_{ij}\eta_{kl} - \eta_{ij}\sigma_{kl})[ijkl]\rfloor^K[ij]. \tag{3.63}$$

Example: Take the face [0671], and using the previous example, we take:

$$\hat{u} = v_{X_1}[01] + v_{X_2}[67], \tag{3.64}$$

$$\hat{v} = v_{Z_2}[06] + v_{Z_1}[17]. \tag{3.65}$$

The continuum analogies are

$$W\hat{u} = v_{X_1}(1-z)dx + v_{X_2} z dx, \tag{3.66}$$

$$W\hat{v} = v_{Z_2}(1-x)dz + v_{Z_1} x dz. \tag{3.67}$$

Application of Eq. 3.60, leads to

$$[Wu, Wv] = \big(v_{X_1}(1-z) + v_{X_2}z\big)\big(v_{Z_1} - v_{Z_2}\big)dz \\ + \big(v_{Z_2}(1-x) + v_{Z_1}x\big)\big(v_{X_2} - v_{X_1}\big)dx. \tag{3.68}$$

It is straightforward to reproduce this formula on the lattice by using Eq. 3.63.

The construction can be clarified further by considering the example of $so(3)$, the rotations in 3D. We will setup the Lie algebra and show that it is well-defined on the complex. Second, we will explain how, although the Jacobi identities are found to hold, the lifting to the group is only approximate which is because integral curves are approximated by Whitney elements. Consider the $so(3)$ generators and the form ω respectively given by

$$X = y\partial_z - z\partial_y, \tag{3.69}$$

$$Y = z\partial_x - x\partial_z, \tag{3.70}$$

$$Z = y\partial_x - x\partial_y, \tag{3.71}$$

$$\omega = W([65]) = (1-x)zdy. \tag{3.72}$$

A direct calculation in the continuum leads to

$$L_{[X,Y]}W([65]) = [L_X, L_Y]W([65]) = -z(1-x)dx - yzdy, \tag{3.73}$$

which are the continuum Jacobi identities for the Lie. We now proceed to analyse the discrete version of the Jacobi identities by constructing the complex based version of this calculation. We will use the rules we have introduced, notably the di_v-rule. The central feature is that of closure, although we use a product space, we can recover the exact expressions by simply identifying:

$$1 \wedge \phi = \phi, \tag{3.74}$$

after applying the latter rule. In the 3D coordinates available, the generators are exactly represented by:

$$\hat{Y} = ([17] + [24]) - ([67] + [54]),\ \hat{X} = ([35] + [24]) - ([65] + [74]). \tag{3.75}$$

because a look at the table of Whitney elements in the Appendix, shows that these are the dual one forms of the vector fields after mapping the chains to the continuum. The bracket we proposed was guided by requiring 1) the closure of the algebra under the bracket, 2) the functional form of $[X,\, Y]$, 3) the Jacobi identities.

$$[X,\, [Y, Z]] + [Y,\, [Z, X]] + [Z,\, [X, Y]] = 0, \tag{3.76}$$

and at the level of the Lie derivative it reads:

$$L_{[X,Y]} = L_X L_Y - L_Y L_X, \tag{3.77}$$

we will derive this relation below for the complex based theory. This means that we seek a complex based bracket operation that maps to the continuum one by a Lie algebra morphism. With the basis of forms we have, we can already set what the answer is:

$$[\hat{X},\, \hat{Y}]^K = A([W(X),\, W(Y)]). \tag{3.78}$$

and this is satisfactory because it is an exact relation and not an approximation as:

$$W[\hat{X},\, \hat{Y}]^K = Z, \tag{3.79}$$

the continuum Z, which means that W is a Lie algebra morphism. So this would be enough to establish 2), and certainly proves the existence of the discretised bracket.

What we have is a Lie algebra $\mathbf{g}$ in the infinite dimensional representation of vector fields. The map W is a Lie algebra morphism if and only if, the collection of one-forms dual to the generators of $\mathbf{g}$ can be written as a linear combination of Whitney elements.

The bracket in Eq. 3.78 is an implicit definition based on the continuum one. It is satisfactory as we were able to give an exact analogue of the vector fields X, Y, Z. We proceed to the calculation of the Jacobi identities for the Lie, and first of all we need the list of edges that are mutually parallel, so that we will know when to apply the discrete contraction:

$$L_1 = \{[17], [06], [35], [24]\}, \tag{3.80}$$

$$L_2 = \{[67], [54], [32], [01]\}, \tag{3.81}$$

$$L_3 = \{[65], [74], [12], [03]\}. \tag{3.82}$$

The detailed calculation is done in the Appendix, since all the rules we introduced play a part, we display in curly brackets the associated continuum calculation for the purpose of illustration. The picture should be clear, the algebra of vector fields is local and is exactly captured. The Jacobi identities for the algebra

$$ad([x, y]) = [ad(x), ad(y)], \tag{3.83}$$

which are satisfied here, give rise to a local approximation of flows in the Lie group (the notation is the standard one, see [32]), consider the vector fields X_x, X_y, satisfying

$$[X_x, X_y] = X_{[x, y]}, \tag{3.84}$$

and the flow of X_x is given by

$$\Phi_x(t, a) = a \exp(tx). \tag{3.85}$$

This flow is only approximate since it cannot be captured in the space of Whitney elements (for example a vector field rotating in the x, y plane). The limitation is easy to understand, owing to the exactness of the vector fields in the theory, the algebra is exact. However the space of co-chains as given by the Whitney elements does not lead to the correct integral curves and the associated relations, we only obtain an approximation, which is for the flow of a vector field Φ_x. Its discrete counterpart $\hat{\Phi}_x$ is controlled by correction terms of order $\Phi_x - W(\hat{\Phi}_x)$.

We now gather the results regarding continnous symmetries. A Lie group action on the space of chains and associated Lie algebra of vector fields are such that:
1) The Lie algebra is exactly captured if the generators are elements of the space of Whitney elements.
2) The associated Lie group G is exactly captured if the integral curves of the vector fields leave the space of Whitney elements invariant. Otherwise it is approximated by the Whitney forms.

We now address the issue of implementing the technique we have just presented and discuss some numerical results.

3.4 Convergence of the model and numerical implementation

We have seen that the results are exact on the space we have introduced, however the Jacobi identity for the Lie is only satisfied in the local picture, flows in the group are only approximate in general. Overall, this means that we have good control over the convergence.

An issue related to convergence is that of refining the triangulation (which is here made of squares). A discussion of this in the context of simplicial categories [24] is worth considering. In that picture as here, one works with chain complexes $C_.$ and their dual co-chain complexes and it is argued that they are to be preferred to homology. Due to the Whitehead theorem [24] they are invariant for simply connected spaces. That is, taking subdivisions of the original complex and one can relate the associated chain spaces. In turn, given the chain space and the dual cochain space, one can define the various mappings we have introduced above.

In the present perspective, we insist on the fact that refining the triangulation is the only means to obtain a better approximation to forms, which are themselves approximated by Whitney elements.

Next, we discuss the example of a one-form field on the torus, satisfying the equation

$$(L_v - \eta \triangle)\omega = 0, \tag{3.86}$$

which is relevant to fluid dynamics [31], and is a first step toward setting up the Navier-Stokes equation in this formalism. We start by tackling the problem analytically and then numerically. The square "triangulation" of the torus is shown in Fig. 3-5.

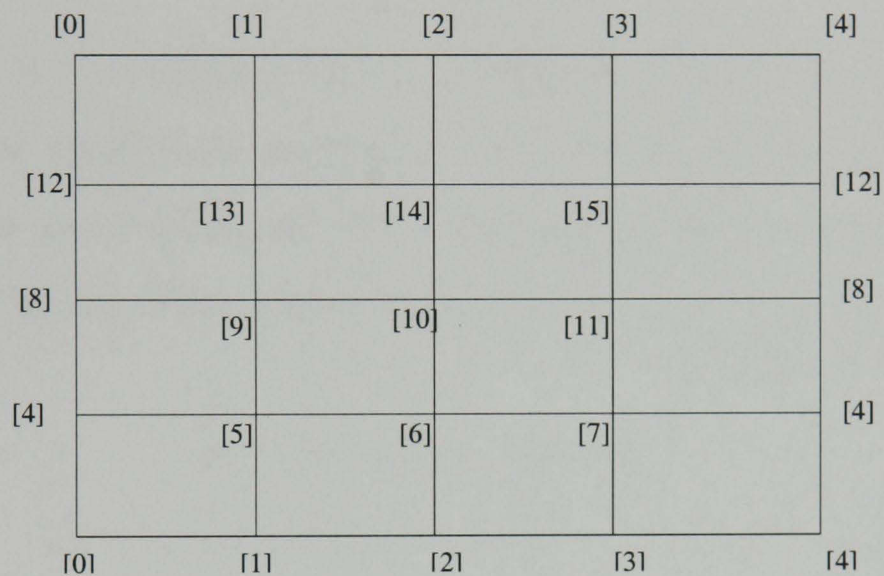

Figure 3-5: Square complex as a discretisation of the torus T^2

The vector field is constant, taken along the x-axis in the plane of the page $v = v_x \partial_x$. We seek a one-form solution as a linear combination of Whitney elements. The Lie derivative is then:

$$L_v\omega = (y_+ - y_-)v_x dy. \tag{3.87}$$

The notation we use for components is such that it can be applied to every 2D cell. The sign and i or j indices denote a specific edge. For example in $[5, 6, 10, 9]$, x_+ is the component on the edge $[9, 10]$ and y_- is the component on the edge $[59]$. Similarly when a component outside the cell is needed, we take for example $y_{+,j}$ to be the $[10, 14]$ component. Next is the Laplacian term:

$$\begin{aligned} dd^\dagger\omega = &(-(x_+ - x_{+,-i} + y_{-,j} - y_-) + (x_{+,i} - x_+ + y_{+,j} - y_+))[x+] \\ &+ ((x_+ - x_{+,-i} + y_{-,j} - y_-) - (x_- - x_{-,-i} + y_- - y_{-,-j}))[y-] \\ &+ ((x_{+,i} - x_+ + y_{+,j} - y_+) - (x_{-,i} - x_- + y_+ - y_{+,-j}))[y+] \\ &+ ((x_{-,i} - x_- + y_+ - y_{+,-j}) - (x_- - x_{-,-i} + y_- - y_{-,-j}))[x-], \end{aligned} \tag{3.88}$$

and

$$
\begin{aligned}
d^\dagger d\omega = &((y_{+,j} - x_{+,j} - y_{-,j} + x_+) - (y_+ - x_+ - y_- + x_-))[x+] \\
&+ ((y_+ - x_+ - y_- + x_-) - (y_- - x_{+,-i} - y_{-,-i} + x_{-,-i}))[y-] \\
&+ ((y_{+,i} - x_{+,i} - y_+ + x_{-,i}) - (y_+ - x_+ - y_- + x_-))[y+] \\
&+ ((y_+ - x_+ - y_- + x_+) - (y_{+,-j} - x_- - y_{-,-j} + x_{-,-i}))[x-]. \qquad (3.89)
\end{aligned}
$$

The first solution is the constant zero-mode solution. Then, L_v is zero because the y-edges have equal coefficients across each square which has the same range of y-coordinates as the given edge, say [48]. This is expressed by Eq. 3.90, we refer to these regions as ribbons. This implies the following:

$$y_{-,-i} = y_- = y_+ = y_{+,i} = Y, \qquad (3.90)$$

$$y_{-,j} = y_{+,j} = Y_+, \qquad (3.91)$$

$$y_{-,-j} = y_{+,-j} = Y_-. \qquad (3.92)$$

In this case, Eq. 3.86 reduces to:

$$\triangle\omega = 0. \qquad (3.93)$$

To get zero we need either each component be zero or each pair of coefficients (e.g the two $[x+]$ components) to cancel out. Setting all x to be zero leaves only:

$$\triangle\omega = (Y_+ - 2Y + Y_-)[y-] + (Y_+ - 2Y + Y_-)[y+]. \qquad (3.94)$$

Our equation is finally

$$Y_+ - 2Y + Y_- = 0. \qquad (3.95)$$

This gives a constant form, $\omega = Cdy$, as an exact solution. Let us think about Eq. 3.95. After all this is an approximation to the second derivative:

$$(\partial_x)^2\omega_y = Y_+ - 2Y + Y_-, \qquad (3.96)$$

using a finite difference rule for the derivative:

$$D_x\omega = \frac{1}{h}(\omega(x+h) - \omega(x)), \tag{3.97}$$

which is correct to order h. Since $[y-]$ has the same coefficient as $[y+]$, the associated Whitney form has constant coefficient function, so having Whitney elements presents no benefit compared to the usual finite difference approach.

In order to move to a more general example, we consider again a solution in y-components, but this time we let y increase as we move from one y-ring to the next one, that is in every cell we let $y_+ - y_- = \theta$. Then,

$$L_v\omega = (y_+ - y_-)v_x dy = \theta v_x dy \tag{3.98}$$

In order to cancel the Lie term, we require as before that all x-coefficients be zero and so

$$y_+ - y_- = y_{+,j} - y_{-,j}, \tag{3.99}$$

$$y_+ - y_- = y_{+,-j} - y_{-,-j}, \tag{3.100}$$

$$y_+ - y_- = y_{+,j} - y_{-,j}, \tag{3.101}$$

$$y_+ - y_- = y_{+,-j} - y_{-,-j}. \tag{3.102}$$

Equate the second line and the third line of Eq. 3.88. We find that they have equal coefficients after noting that using Eq. 3.99, we can rerite the second line of Eq. 3.89 as

$$y_{-,j} - y_- - y_- + y_{-,-j} = y_{+,j} - y_+ - y_- + y_{-,-j}, \tag{3.103}$$

similarly for the third line of Eq. 3.88. Next, equate the second and the third line of Eq. 3.89. Then,

$$y_+ - y_- - (y_- - y_{-,-i}) = y_{+,i} - y_+ - y_+ + y_-, \tag{3.104}$$

$$y_+ - y_- = \frac{1}{3}(y_{+,i} - y_{-,-i}). \tag{3.105}$$

We are now in a position to write down a new solution. Let y be constant along each ring parallel to the y-axis. Also, we have seen that

$$y_+ - y_- = \theta. \tag{3.106}$$

If we enforce this for all squares we have a local (in the sense of a single simplex) matching of Whitney elements. However, if we are to look at boundary conditions, particularly for closed surfaces like the torus, we will need some further prescription to guarantee closure of the field. To this end we consider

$$\phi = ae^{i\frac{x}{b}}dy, \tag{3.107}$$

where a and b are constants to be determined. Then, in the interior of a given square, we have

$$\phi = a(1 + i\frac{x}{b})dy + O(x^2), \tag{3.108}$$

and letting $x = ix$, leads to the usual expression, after we enforce the following ansatz:

$$\hat{\phi} = \phi(l_1)(1 - \frac{x}{b}) + \phi(l_2)(\frac{x}{b}), \tag{3.109}$$

$$\phi(l_1) = \int_{|l_1|} ae^{i\frac{x}{b}}dy = ae^{i\frac{x(l_1)}{b}}. \tag{3.110}$$

Moreover, if $b = h$ the edge length of a small square, then we can match the function with the Whitney map applied to a chain field $\hat{\phi}$. We have the matching

$$\phi = W(\hat{\phi}) + O(x^2). \tag{3.111}$$

To take the analogy one step further, note that one retains a bijection between the function e^{ix} and ix. i.e, the function ϕ oscillates over a large number of squares:

$$||y_+ - y_-|| << 2\pi. \tag{3.112}$$

To guarantee closure, let N_x be the number of edges along a circle of the torus (parallel to the x-axis), then require closure (recall $y_+ - y_- = \theta$ a constant):

$$(y_+ - y_-)N_x = 2k\pi. \qquad \{k = 0, 1, 2, \ldots\} \tag{3.113}$$

Then, $k = 0$ leads to the zero mode we already got, the rest give the k-th mode solution. Let us do the continuum calculation: The change of variable $x \mapsto ix$ leads to $dx \mapsto idx$, $\frac{\partial}{\partial x} \mapsto -i\frac{\partial}{\partial x}$, $v_x \mapsto -iv_x$. Applying this to the solution:

$$L_v\phi = v_x \frac{1}{b}\frac{\partial}{\partial ix} e^{i\frac{x}{b}} dy = v_x \frac{a}{b} e^{i\frac{x}{b}} dy, \tag{3.114}$$

$$\star d \star d\phi = \frac{a}{b^2} e^{i\frac{x}{b}} (dy). \tag{3.115}$$

This solves the equation provided $v_x = \frac{\eta}{b}$. In order to strengthen analogy between the continuum and the complex we evaluate:

$$d \ln \phi \, dy = \frac{\phi'}{\phi} dx \wedge dy, \tag{3.116}$$

$$\star d \ln \phi \, dy = \frac{\phi'}{\phi}, \tag{3.117}$$

$$i_v d \ln \phi \, dy = v_x \frac{\phi'}{\phi} dy. \tag{3.118}$$

Then,

$$d \star d \ln \phi \, dy = \frac{\phi''\phi - \phi'\phi'}{\phi^2} dx, \tag{3.119}$$

$$\star d \star d \ln \phi \, dy = \frac{\phi''\phi - \phi'\phi'}{\phi^2}(-dy) = \frac{\phi''}{\phi}(-dy) + \left(\frac{\phi'}{\phi}\right)^2 dy. \tag{3.120}$$

Note that according to the discrete theory we get:

$$\frac{\phi'}{\phi} = \frac{e^{ix_{k+1}} - e^{ix_k}}{e^{ix_k}} = i(x_{k+1} - x_k) + O((x_{k+1} - x_k)^2). \tag{3.121}$$

Using the modulus,

$$\|\left(\frac{\phi'}{\phi}\right)^2\| \sim \|\frac{\phi'}{\phi}\|^2 \sim \epsilon^2. \tag{3.122}$$

Therefore,

$$\begin{aligned} \triangle \ln\phi - L_v \ln\phi &= -\frac{\phi''}{\phi} - v_x\frac{\phi'}{\phi} + O(\epsilon)^2 = \frac{1}{\phi}\Big[-\phi'' - v_x\phi'\Big] + O(\epsilon^2), \\ &= \frac{1}{\phi}\Big[\triangle\phi - L_v\phi\Big] + O(\epsilon^2). \end{aligned} \tag{3.123}$$

In the discrete approach this is written as

$$A^1(\ln\phi) = \sum_j \int_{[j_1 j_2]} \frac{2ki\pi x_j}{a} dy[j_1 j_2] = \sum_j \frac{2ki\pi x_j}{a}[j_1 j_2]. \tag{3.124}$$

Then apply the discrete version of the operator

$$L_v - \triangle, \tag{3.125}$$

and solve the matrix equation. Next, with a varying vector field, the solution is modified, that is the rate of change of the solution starts to vary. This does not modify the calculation above with the minor exception that now

$$\|\frac{\phi'}{\phi}\| = \|\frac{f(x_{k+1})e^{ix_{k+1}} - f(x_k)e^{ix_k}}{f(x_k)e^{ix_k}}\| < \sup\Big[\|\frac{f(x_{k+1})}{f(x_k)}\|,\ 1\Big]O(\epsilon^2) \sim O(\epsilon^2), \tag{3.126}$$

which leads to the same equation as before with a variable v_x.

Regarding the inclusion of modes, we have studied the function $\ln\phi$ where in principle we could include all the modes (along the x-circles):

$$\phi = \sum_k e^{2\pi ikx} f_k(x, y)dy. \tag{3.127}$$

Let

$$\chi = f_k e^{\kappa x}, \tag{3.128}$$

then we will get

$$(\triangle - L_v)\ln(\chi) = \frac{1}{\chi}(-\chi'' - v_x\chi'), \tag{3.129}$$

$$= \frac{1}{\chi}\Big[-(f_k'' - 2\kappa f_k' - \kappa^2 f_k) - v_x(f_k\kappa + f_k')\Big]e^{\kappa x}. \tag{3.130}$$

Substituting functions, note the x^2 term for f_k.

$$f_k = a + bx + cx^2, \tag{3.131}$$

$$v_x = v_0 + v_x x, \tag{3.132}$$

leads to the following equations for the constant term and the x-term:

$$a\kappa^2 + (2b - v_0 a)\kappa - 2c - v_0 b = 0 \tag{3.133}$$

$$b\kappa^2 + (4c - v_0 b - v_x a)\kappa - v_x b - 2v_0 = 0. \tag{3.134}$$

These equations are to be solved for a, b and c in principle. We can then substitute back the solution into our trial solution:

$$\chi = (a + bx + cx^2)e^{\kappa x}, \tag{3.135}$$

thus

$$\chi' = (\kappa(a + bx + cx^2) + b + 2cx)e^{\kappa x}. \tag{3.136}$$

The x term in this expression is then

$$(b + 2c + a\kappa). \tag{3.137}$$

The approximation is given by:

$$\hat{\chi} = (< f >_0 + < f >_1 x)e^{\kappa x}, \tag{3.138}$$

$$=< f >_0 + (< f >_0 \kappa + < f >_1)x + O(x^2). \tag{3.139}$$

So

$$\chi' = (< f >_0 \kappa + < f >_1)(x_{n+1} - x_n) + O(\epsilon^2), \tag{3.140}$$

which is to be compared with Eq. 3.121. The observation being that the theory captures the linear term. We now leave such considerations of matching the continuum solution.

To gather our findings we note that the key features are the use of the logarithm and the selection of the mode by specifying the relevant boundary condition. The k-th mode is given by specifying the constant C which is a normalization and setting:

$$\phi^+ = C, \tag{3.141}$$

$$\phi^- = C + 2\pi k. \tag{3.142}$$

Numerical tests have been done on the torus we have displayed above (with a few extra squares only), and although we have already discussed the convergence, we would like to explain the numerical procedure. It consists of using a singular value decomposition (SVD) algorithm [33].

Two key advantages to this method are first its well-known stability and second that it solves a matrix equation by minimizing the square norm χ^2 in Eq. 3.145, so in some cases were the present theory cannot be solved exactly as a matrix equation, the algorithm gives us a good approximation. SVD is used to solve

$$Ax = b, \tag{3.143}$$

and leads to the following decomposition (the w_j are the singular values):

$$A^{-1} = V.[diag\,\frac{1}{w_j}]U^T. \tag{3.144}$$

The solution is obtained by minimizing

$$\chi^2 = ||Ax - b||^2. \tag{3.145}$$

We also note that the formal errors due to the χ^2 fit, are given by:

$$\triangle\chi^2 = w_1^2(V_{(1)}.\delta a)^2 + \ldots + w_M^2(V_{(M)}.\delta a)^2, \tag{3.146}$$

where $V_{(1)} \ldots V_{(M)}$ are the columns of the vector V given in Eq. 3.144, then the boundaries of the confidence ellipsoids. Consider the graph in Fig. 3-6, we took the following vector field:

$$\begin{aligned} v_x &= 4, \quad \forall x \in [0,\, 3[,\, \forall y; \\ v_x &= -2,\ \forall x \in [3, 6[, \forall y; \\ v_y &= 0,\ \forall\,(x,\, y). \end{aligned} \tag{3.147}$$

and solved numerically Eq. 3.86 for a one-form field. The matrix equation was written using the one-simplices as the basis vector space (there is an equation for each simplex).

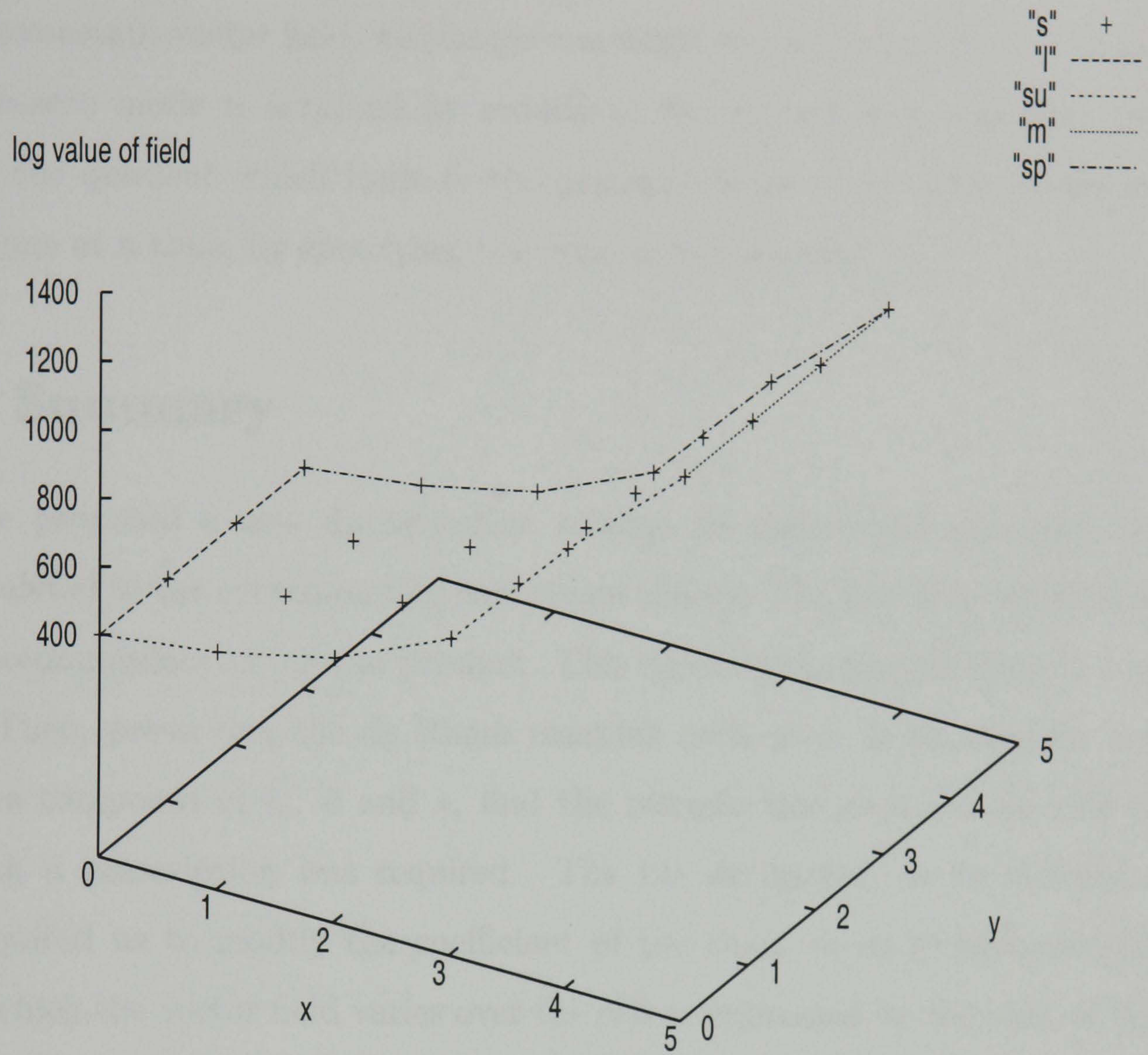

Figure 3-6: One form field solution with values in the range [400, 1400]. Recall that we are plotting the $\log \Phi$ value. The (x, y)-plane contains the torus with vertices at points $a\hat{i} + b\hat{j}$, with $a \in [0, 5]$ and $b \in [0, 3]$. The field is not plotted in the x-interval $[5, 6]$ as the field reaches the value 1400, which is identified with 400.

The boundary conditions are enforced on the solution, by modifying the matrix A in such a way that $C = 400$ (Eq. 3.141) and we identify $\phi^+ = 400$ with $\phi^- = 1400$ (recall that we are considering the logarithm of the solution).

In this example the matrix A is 96×96, because there are $(6 * 4)$ y-edges and the same number of x-edges, and each edge gives rise to two equations, since it has non-zero Whitney form in two cells.

To comment on the result (Fig. 3-6), we note that the change in the vector field (as specified by Eq. 3.147) results in a change of concavity in the graph.

For a constant vector field, we just get a straight line in each section. The inclusion of a non-zero mode is achieved by modifying the matrix in a way that take into account the quotient which leads to the principal value of $i\phi$. The modes are then treated one at a time, by specifying the appropriate matrix.

3.5 Summary

We have proposed a new discretisation scheme for differential geometry which is deeply related to the corresponding continuum theory. The first step which is central, was to accommodate an interior product. This required the introduction of a product space. Then, preserving the de Rham complex motivated us to consider a ring of operators composed of i_v, d and $\star$, and the introduction of a special rule for di_v, for which a prescription was required. The Lie derivative, as an element of the ring, required us to modify the coefficient of the chain so as to accommodate the case in which the vector field varies over the cell as expressed by the sum of Whitney elements. The bracket was also introduced for vector fields in the infinite dimensional representation, for which the geometrical interpretation is crucial. Also, the Jacobi identities for the Lie derivative are exact, but one bears in mind that in general the flows are approximated by Whitney forms. We also have a numerical implementation for the torus and a formal argument for convergence. We are now in a position to introduce metrics in this scheme which we proceed to do in the next chapter.

3.6 Appendix

3.6.1 Hyper-cubic Whitney elements

The list of Whitney elements which are relevant to the discussion are listed in Table. 3.1.

Zero-cochains	One-cochains	Two-cochains
$\mu_0 = (1-x)(1-y)(1-z)$	$W([01]) = dx(1-y)(1-z)$	$W([0123]) = (1-z)dx \wedge dy$
$\mu_1 = x(1-y)(1-x)$	$W([03]) = dy(1-x)(1-z)$	$W([1247]) = xdy \wedge dz$
$\mu_2 = xy(1-z)$	$W([06]) = dz(1-x)(1-y)$	$W([0356]) = (1-x)dx \wedge dz$
$\mu_3 = y(1-x)(1-z)$	$W([17]) = x(1-y)dz$	$W([0167]) = (1-y)dx \wedge dz$
$\mu_4 = xyz$	$W([12]) = x(1-z)dy$	$W([6745]) = zdx \wedge dy$
$\mu_5 = (1-x)yz$	$W([24]) = xydz$	$W([3245]) = ydx \wedge dz$
$\mu_6 = (1-x)(1-y)z$	$W([32]) = y(1-z)dx$	
$\mu_7 = x(1-y)z$	$W([35]) = y(1-x)dz$	
	$W([65]) = z(1-x)dy$	
	$W([67]) = (1-y)zdx$	
	$W([74]) = xzdy$	
	$W([54]) = (1-z)zdx$	

Table 3.1: The Whitney forms for the cube in $\mathbb{R}^3$ Cartesian coordinates

3.6.2 Relation to the de Rham complex

Let us discuss the relation of the setup to the de Rham complex, which is extracted from differential forms under the exterior derivative d. Having shown that on the extended space (after application of i_v), $(d^K)^2 = 0$ is a priori not sufficient. But note that we have apparently lost one important feature, which is the de Rham theorem [22], which states that the map Λ, defined as

$$\Lambda : H_r(M) \times H^r(M) \longrightarrow \mathbb{R}, \tag{3.148}$$

is bilinear and non-degenerate. This map could be expressed as

$$(\sigma, W(\eta)) = \int_\sigma W(\eta). \tag{3.149}$$

However, with the construction introduced above, we seem to have lost the non-degeneracy property of Λ. To see this, let $\zeta^r \in C^r(K, \mathbb{R})$, and let $\eta^{(r+1)} \in C^{r+1}(K, \mathbb{R})$. Then using the rule we saw for mapping to the space of forms, we get

$$\int_{\sigma^r} \varphi^0(W(v)) \wedge W(i_{\hat{v}}\eta), \tag{3.150}$$

to be compared with

$$\int_{\sigma^r} W(\zeta). \tag{3.151}$$

Clearly, the space of forms spanned by products of chains is larger, but we have shown, in the proof of the Hyper-cubic rule that the action of d^K on the larger space has some functorial properties which link it with d, we found that,

$$W d^K i_v = dW i_v, \tag{3.152}$$

and so we may put the various steps of that calculation under the integral symbol. It means that we have a larger space of forms and that the co-homology is "intact". This is another virtue that we extract from the di_v-rule.

$$\begin{array}{ccc} \Omega^r(M,\,\mathbb{R}) & \xrightarrow{id} & \Omega^r(M,\,\mathbb{R}) \\ \uparrow\Lambda & & \uparrow\Lambda \\ C^p\,(K,\,\mathbb{R}) & \xrightarrow{id} & C\,(K,\,\mathbb{R}) \otimes C^p\,(K,\,\mathbb{R}) \end{array} \tag{3.153}$$

A comment on the diagram (Eq. 3.153): we wrote Λ where we could have written W, but we stress that we are using the inner product of a space of chains with the space of differential forms. The vertical arrow on the right relates two larger spaces than that on the left. The diagram being commutative entails the compatibility of the present scheme with the original geometric discretisation. One could say that the introduction of geometry does not modify topology.

In terms of the homology we have,

$$H^r(K \times K,\,\mathbb{R},\,\bar{d}^K) \cong H^r(K,\,\mathbb{R},\,d^K), \tag{3.154}$$

while,

$$B^r(K,\,\mathbb{R},\,d^K) < B^r(K \times K,\,\mathbb{R},\,\bar{d}^K), \tag{3.155}$$

$$Z^r(K,\,\mathbb{R},\,d^K) < Z^r(K \times K,\,\mathbb{R},\,\bar{d}^K). \tag{3.156}$$

Hence, the extended space of chains consisting of elements of the form specified by the rule for immersion (i.e applying $\varphi^{(0)}$) form a larger subspace of the space of forms under the modified Whitney map than does the ordinary chains under the Whitney map. The extended space can be embedded into the product space which has the same topology.

3.6.3 Examples

Example 1: Let

$$v = v_{X_1}(1-z)dx + v_{X_2}zdx + v_{Y_1}(1-z)dy + v_{Y_2}zdy \tag{3.157}$$

$$v^b = v_{Z_2}(1-x)dz + v_{Z_2}(1-y)dz + v_{Z_1}xdz + v_{Z_3}ydz \tag{3.158}$$

Note that

$$i_v v^b = 0. \tag{3.159}$$

Next,

$$dv^b = -v_{Z_2}dx \wedge dz - v_{Z_2}dy \wedge dz + v_{Z_1}dx \wedge dz + v_{Z_3}dy \wedge dz, \tag{3.160}$$

$$\begin{aligned} L_v v^b = i_v dv^b = &-v_{Z_2}(v_{X_1}(1-z) + v_{X_2}z)dz - v_{Z_2}(v_{Y_1}(1-z) + v_{Y_2}z)dz \\ &+ v_{Z_1}(v_{X_1}(1-z) + v_{X_2}z)dz + v_{Z_3}(v_{Y_1}(1-z) + v_{Y_2}z)dz. \end{aligned} \tag{3.161}$$

Lattice calculation:

$$\hat{v} = v_{X_1}[01] + v_{X_2}[67] + v_{Y_1}[03] + v_{Y_2}[65], \tag{3.162}$$

$$\hat{v_b} = v_{Z_2}[06] + v_{Z_1}[17] + v_{Z_3}[35]. \tag{3.163}$$

Again, see the discussion of the interior product,

$$i_{\hat{v}}\hat{v_b} = 0. \tag{3.164}$$

Then,

$$d\hat{v}_b = (v_{Z_1} - v_{Z_2})[6017] + (v_{Z_2} - v_{Z_3})[0653] + v_{Z_1}[1724] + v_{Z_3}[3542], \tag{3.165}$$

and

$$i_{\hat{v}}(v_{Z_1} - v_{Z_2})[6017] = (v_{Z_1} - v_{Z_2})(v_{X_1}([01]\rfloor^K[06] + [01]\rfloor^K[17]) + v_{X_2}([67]\rfloor^K[06] + [67]\rfloor^K[17])), \tag{3.166}$$

$$i_{\hat{v}}(v_{Z_2} - v_{Z_3})[0653] = (v_{Z_2} - v_{Z_3})(v_{Y_1}([03]\rfloor^K[06] + [03]\rfloor^K[35]) + v_{Y_2}([65]\rfloor^K[06] + [65]\rfloor^K[35])), \tag{3.167}$$

the continuum analogue of which is obtained by identifying the sum of Whitney elements with a constant form,

$$Wi_{\hat{v}}(v_{Z_1} - v_{Z_2})[6017] = (v_{Z_1} - v_{Z_2})(v_{X_1}((1-z)dx\rfloor dz) + v_{X_2}(zdx\rfloor dz)), \tag{3.168}$$

$$Wi_{\hat{v}}(v_{Z_2} - v_{Z_3})[0653] = -(v_{Z_2} - v_{Z_3})(v_{Y_1}((1-z)dy\rfloor dz) - v_{Y_2}(zdy\rfloor dz)). \tag{3.169}$$

Which matches the continuum expression after use of the continuum identification, simply by deleting the $\rfloor$ sign and the constant differential which is part of the coefficient form.

<u>Example 2</u>: Next, u is parallel to v_b. Then,

$$i_u v_b = (v_{Z_2}(1-x) + v_{Z_1}x)^2 + (v_{Z_2}(1-y) + v_{Z_3}y)^2, \tag{3.170}$$

and,

$$di_u v_b = 2(v_{Z_1} - v_{Z_2})(v_{Z_2}(1-x) + v_{Z_1}x)dx + 2(v_{Z_3} - v_{Z_2})(v_{Z_2}(1-y) + v_{Z_3}y)dy. \tag{3.171}$$

Next dv_b was given above and,

$$i_u dv_b = (v_{Z_2}(1-x) + v_{Z_1}x)(v_{Z_2} - v_{Z_1})dx + (v_{Z_2}(1-y) + v_{Z_3}y)(v_{Z_2} - v_{Z_3})dy, \tag{3.172}$$

and finally,

$$L_u v_b = (v_{Z_1} - v_{Z_2})(v_{Z_2}(1-x) + v_{Z_1}x)dx + (v_{Z_3} - v_{Z_2})(v_{Z_2}(1-y) + v_{Z_2}y)dy. \quad (3.173)$$

Lattice calculation:

$$\hat{u} = v_{Z_2}[06] + v_{Z_1}[17] + v_{Z_3}[35] = v_b. \quad (3.174)$$

Then,

$$i_{\hat{u}} v_b = v_{Z_2}^2[06]\rfloor^K([0]+[6]) + v_{Z_1}^2[17]\rfloor^K([1]+[7]) + v_{Z_3}^2[35]\rfloor^K([3]+[5]), \quad (3.175)$$

in order to reproduce the cross terms we set:

$$\begin{aligned} i_{\hat{u}} v_b &= (v_{Z_2}^2[06] + v_{Z_1}v_{Z_2}[17])\rfloor^K([0]+[6]) \\ &+ (v_{Z_1}^2[17] + v_{Z_1}v_{Z_2}[06])\rfloor^K([1]+[7]) \\ &+ (v_{Z_3}^2[35] + v_{Z_3}v_{Z_2}[06])\rfloor^K([3]+[5]), \end{aligned} \quad (3.176)$$

and

$$\begin{aligned} d^K i_{\hat{u}} v_b &= (v_{Z_2}^2[06] + v_{Z_1}v_{Z_2}[17])\rfloor^K([10]+[76]) \\ &+ (v_{Z_1}^2[17] + v_{Z_1}v_{Z_2}[06])\rfloor^K([01]+[67]). \end{aligned} \quad (3.177)$$

Next, we already calculated dv_b, so that after retaining the components in [0653] and [0176],

$$\begin{aligned} 2i_{\hat{u}} dv_b &= -v_{Z_2}(v_{Z_2}[06] + v_{Z_1}[17])\rfloor^K([01]+[67]) \\ &+ v_{Z_2}(v_{Z_2}[06] - v_{Z_3}[35])\rfloor^K([03]+[65]) \\ &+ v_{Z_1}(v_{Z_1}[17] + v_{Z_2}[06])\rfloor^K([01]+[67]) \\ &+ v_{Z_3}((v_{Z_3}[35] - v_{Z_2}[06])\rfloor^K([03]+[65]). \end{aligned} \quad (3.178)$$

Gathering the terms, we get

$$i_{\hat{u}} dv_b = (-v_{Z_2}^2[06] - v_{Z_1}v_{Z_2}[17] + v_{Z_1}^2[17] + v_{Z_1}v_{Z_2}[06])\rfloor^K([01] + [67])$$
$$+ (v_{Z_2}^2[06] - v_{Z_2}v_{Z_3}[35] + v_{Z_3}^2[35] - v_{Z_3}v_{Z_2}[06])\rfloor^K([03] + [65]). \quad (3.179)$$

<u>Example 3</u>: The complex based calculation is as follows:

$$\begin{aligned}
i_X[65] &= -([65] + [74])\rfloor([6] + [5]) \\
&\{= -z^2(1-x)\}, \\
i_Y[65] &= 0, \\
d^K i_X[65] &= -2([65] + [74])\rfloor([06] + [76] + [35] + [45]), \\
&\{= -2z(1-x)dz + z^2 dx\}, \\
d^K[65] &= [6574] + [6530], \\
&\{= (1-x)dz \wedge dy - zdx \wedge dy\}, \\
i_X d^K[65] &= -([65] + [74])\rfloor([67] + [54]), \\
&\quad + ([65] + [74])\rfloor([06] + [35]) + ([35] + [24])\rfloor([65] + [03]), \\
&\{= y(1-x)dy + z(1-x)dz - z^2 dx\}, \\
i_Y d^K[65] &= (-[67] - [54])\rfloor([65] + [74]) - ([17] + [24])\rfloor([03] + [65]), \\
&\{= -x(1-x)dy - z^2 dy\}. \qquad (3.180)
\end{aligned}$$

So,

$$\begin{aligned}
L_X[65] &= -([65] + [74])\rfloor([06] + [35]) + ([35] + [24])\rfloor([65] + [03]), \\
&\{= y(1-x)dy - z(1-x)dz,\} \qquad (3.181) \\
L_Y[65] &= i_Y d^K[65]. \qquad (3.182)
\end{aligned}$$

Next,

$$
\begin{aligned}
i_Y L_X[65] &= ([65]+[74])\otimes([17]+[24])\rfloor([0]+[6]+[3]+[5]),\\
&\quad+([65]+[74])\otimes(-[67]-[54])\rfloor(-[7]-[6]-[4]-[5]),\\
&\{=2z(1-x)x-xz(1-x)\},\\
d^K i_Y L_X[65] &= 2([65]+[74])\otimes([17]+[24])\rfloor([10]+[76]+[23]+[45]),\\
&\quad+2([65]+[74])\otimes(-[67]-[54])\rfloor(-[17]-[06]-[24]-[35]),\\
&\{=2z(1-x)dx+3z^2dz-2zxdx+2(1-x)xdz,\\
&-z(1-x)x+xzdx-3z^2dz-x(1-x)dz\}. \qquad (3.183)
\end{aligned}
$$

$$
\begin{aligned}
d^K i_X d^K[65] &= ([65]+[74])\rfloor([6701]+[5423]),\\
&\quad-([65]+[74])\rfloor([0617]+[3542]),\\
&\quad-([35]+[24])\rfloor([6574]+[0312]),\\
&\{=-zdx\wedge dz-2zdz\wedge dx-ydx\wedge dy\},\\
i_Y d^K i_X d^K[65] &= 2([65]+[74])\otimes([17]+[24])\rfloor(-[01]-[67]+[54]+[32]),\\
&\quad+2([65]+[74])\otimes(-[67]-[54])\rfloor(-[35]-[42]-[06]-[17]),\\
&\quad+(-[35]-[24])\otimes(-[67]-[54])\rfloor([65]+[74]+[03]+[12]),\\
&\{=-z^2dz-xzdx+2z^2dz+2xzdx-yzdy\},\\
i_Y i_X d^K[65] &= ([65]+[74])\otimes(-[67]-[45])\rfloor(-[6]-[7]-[5]-[4]),\\
&\quad-([65]+[74])\otimes([17]+[24])\rfloor([0]+[6]+[3]+[5]),\\
&\{=-xz(1-x)-z^3\},\\
d^K i_Y i_X d^K[65] &= ([65]+[74])\otimes(-[67]-[45])\rfloor(-[06]-[17]-[35]-[24]),\\
&\quad-([65]+[74])\otimes([17]+[24])\rfloor([10]+[76]+[23]+[45]),\\
&\{=-z(1-x)dx+xzdx-3z^2dz-x(1-x)dz\}. \qquad (3.184)
\end{aligned}
$$

then

$$
\begin{aligned}
L_Y L_X[65] &= -z(1-x)dx + zxdx - 3z^2dz - x(1-x)dz \\
&\quad + 2z(1-x)dx + 3z^2dz - 2zxdx + 2(1-x)xdz \\
&\quad - z^2dz - xzdx + 2z^2dz + 2xzdx - yzdy \\
&= z(1-x)dx + x(1-x)dz + z^2dz - yzdy \qquad (3.185)
\end{aligned}
$$

Exact matching is also found in the calculation of $L_X L_Y[65]$.

Chapter 4

Capturing metric data

In the last Chapter, we introduced a new discretisation scheme for differential geometry. The starting point was topological and revolved around the homology and co-homology of a manifold captured by the simplicial or hyper-cubic complex. The geometric discretisation scheme was found to be too restrictive to capture geometry, in fact, what we had was a finite list of differential forms (the "cochains") in one to one correspondence with lattice objects, namely, vertices, edges, faces etc. This tight relation between lattice and continuum enforced stringent constraints in the construction of well-defined analogies of the operators present in the continuum ($\wedge$, d, $\star$) with exact algebraic properties, and was enough for the original purpose of the theory (applied to topological theories in physics, Chern-Simons theory).

We now turn to the problem of incorporating realistic metric information in this discretisation scheme. A first step in this direction was taken in Chapter 3. There the notion of the Lie derivative was introduced in the discrete setup. The Lie derivative captures how geometrical objects like tensor fields change. In order to introduce the Lie derivative it was necessary to generalize the procedure by introducing the notion of a product in the space of chains. The construction introduced was not obvious and is motivated and described carefully. An example, fields on the torus, was considered to show how the procedure worked.

Metric aspects can be treated in different ways. The way we proceed is motivated by the question: with the algebraic "kit" which gives access to symmetries

$$\mathcal{L} = \{i_v,\, d,\, \wedge,\, \star\}, \tag{4.1}$$

how can we represent the covariant derivative given by (for example):

$$(\nabla_v v)^b = L_v v^b - \frac{1}{2} dg\,(v,\, v)\,. \tag{4.2}$$

Equivalently we ask: What is the best way (in terms of both algebraic properties and convergence) we can represent the term

$$dg(v, v), \tag{4.3}$$

using the algebraic kit which already provides us with $L_v v^b$?

The starting point will be a discussion of the vielbein metric described in detail below in the case of the two sphere. The idea is that we provide a collection of one-form fields $\{\hat{\theta}^i,\, i = 1, 2\}$ approximated by Whitney forms and then use the kit to extract the metric data built into the embedding of the hyper-surface in $\mathbb{R}^N$.

In fact, we will see that there are two ways to do the connection, one we will refer to as related to the Regge approach [35, 36], which is a way of doing the connection as a matrix relating forms in two cells across a common bounding edge. The other we refer to as based on the continuum formulation of Cartan [22], using the structure equations, which is more in line with the description above. In Fig. 4-1, we have sketched the two pictures of the connection, an important theme here.

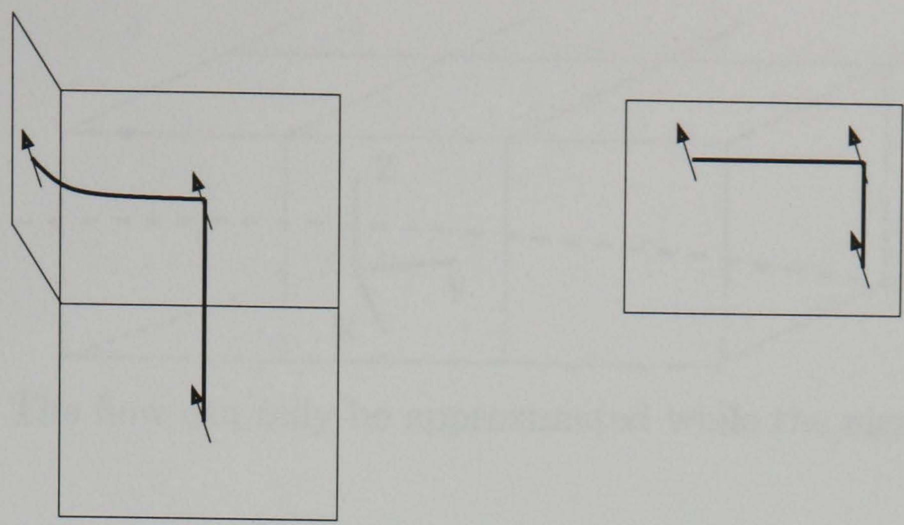

Figure 4-1: Two ways in which the connection (the thick link), may be represented, on the left, a la Regge, on the right a la Cartan. The arrow is a one form, and the two connection components are represented.

The end point is a new kit which gives access to geometry:

$$\mathcal{G} = \{i_v,\ d,\ \wedge,\ \star,\ \{\hat{\theta}^{(a)}\}\}. \tag{4.4}$$

If $\mathcal{T} = (\wedge,\ d,\ \star)$ is the topological kit, we then have $\mathcal{T} \subseteq \mathcal{L} \subseteq \mathcal{G}$.

A brief summary of where we left the abstract construction in Chapter 3 is as follows. Recall that we were able to construct an analogue to the Lie derivative

$$L_v \omega = i_v d + d i_v, \tag{4.5}$$

within our framework.

With these structures under control we thought about the Lie algebra commutator for vector fields $\Xi(M)$ in the infinite dimensional representation, this operation was given a clear finite geometrical meaning. After arguing the special case of $\mathbf{g} = so(3)$ due to its generators matching the Whitney elements we showed how the Lie algebra is "locally" (by which we mean within the local cell) exactly realized (Fig. 4-2).

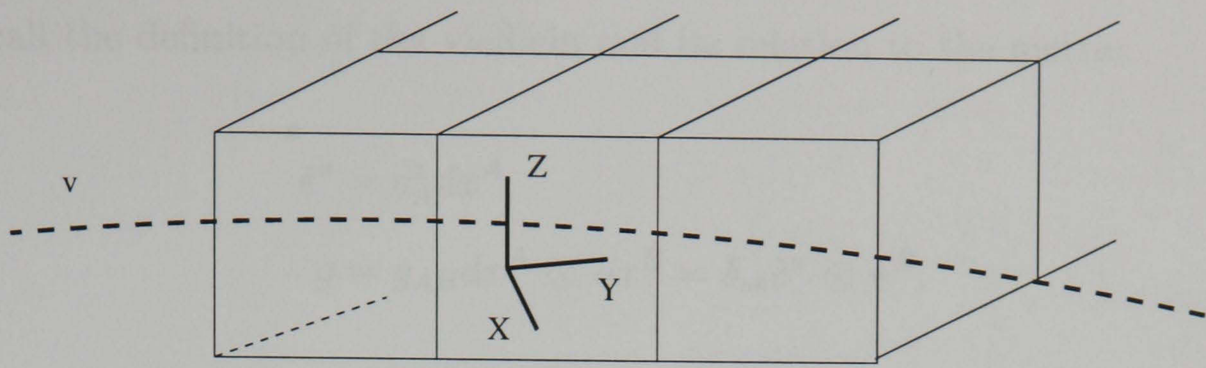

Figure 4-2: The flow can only be approximated while the algebra is exact

One had then control over the lifting of the algebra which is captured in the Jacobi identities, which we made exact on the space of Whitney elements.

$$L_{[u,\, v]} = [L_u,\, L_v]. \tag{4.6}$$

Regarding the Lie group, the flow lines should coincide with the Whitney forms for it to be exact. We have some control up to approximations, but strictly speaking, the theory is exact on flows such as translations which can be captured exactly.

What remains is a discussion of metric aspects of the theory, having a locally Euclidean metric is a priori satisfactory but we need to do more in order to accommodate the vielbein which will relate the theory to the underlying manifold. Then, we consider two different routes to extract metric information, first in a way analogous to the Regge approach, and in the second section, we stay with the vielbein formalism to extract more metric information.

The last part gives an overview of some applications of interest.

4.1 Vielbein formalism and the Whitney map

The vielbein for the two sphere is given by (see [22]):

$$\hat{\theta}^{(1)} = d\theta, \tag{4.7}$$

$$\hat{\theta}^{(2)} = \sin\theta d\phi. \tag{4.8}$$

Also, recall the definition of the vielbein and its relation to the metric:

$$\hat{e}^a = e^a_A dx^A, \tag{4.9}$$

$$g = g_{AB} dx^A \otimes dx^B = \delta_{ab} \hat{e}^a \otimes \hat{e}^B. \tag{4.10}$$

For the two-sphere, we write the matrix of coordinate components of the vielbein:

$$e^1_\theta = 1, \quad e^1_\phi = 0, \tag{4.11}$$

$$e^2_\theta = 0, \quad e^2_\phi = \sin\theta. \tag{4.12}$$

It is natural to respect the symmetry of these functions in the sense that squares correspond in the embedding space $\mathbb{R}^3$ to the regions defined by intersecting parallels and meridians on the sphere. The vielbein is taken to be, in a given square [0123] with [01] along the ϕ-axis and [03] along the θ-axis.

This indicates that as a complex based one-chain, the vielbein may be written approximately as Whitney elements, the coefficients of the simplices have been calculated through integration above.

Then,

$$\hat{\theta}^{(1)} = dy \sim e^\theta, \tag{4.13}$$

$$\hat{\theta}^{(2)} = ((1-y)\sin\theta([01]) + y\sin\theta([32]))dx \sim e^\phi, \tag{4.14}$$

since a unique θ value characterizes edges such as [01] and [32] which are along the parallels. In the square, this is approximated in "local" (x, y)-coordinates of $\mathbb{R}^2$ in a way that approximates the hyper-surface $\mathfrak{H} = (\theta,\, \phi)$ by the plane $\mathfrak{P} = (x,\, y)$ in the embedding space $\mathbb{R}^3$.

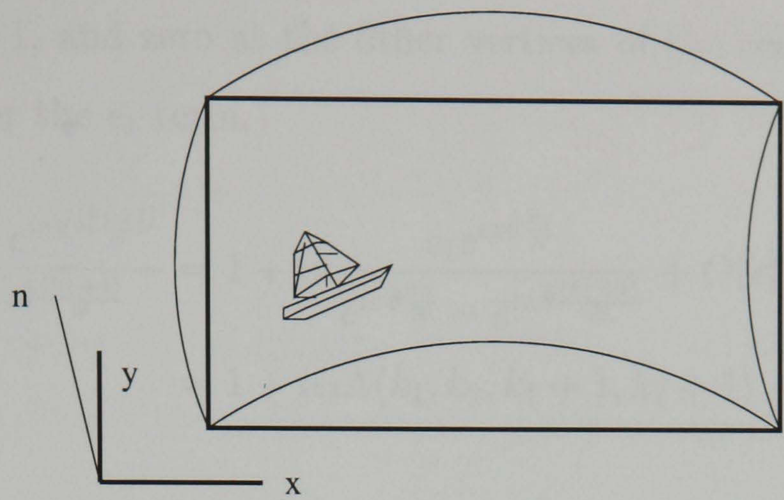

Figure 4-3: A square, as part of the discretisation of the two-sphere. The ship flows over a curvilinear path, which can be projected onto the square. In the case where the flow has a constant direction within a cell, it can be captured exactly (up to a parameterization along the flow line) since it leaves the space of Whitney elements invariant.

We explained in Chapter 3 how the approximation is captured as we extract the Whitney elements directly from the continuum problem. It is appropriate to adopt a similar approach at this point; by using the exponential, we are relating the two sets of coordinates. Start by labeling the vertices by pairs of integers, assuming there are N parallels and meridians, we have the collection of vertices labeled by two integers:

$$(k_1, k_2) = e^{i\pi \frac{k_1}{N}\hat{\theta}} e^{2i\pi \frac{k_2}{N}\hat{\varphi}}, \qquad \{k_1, k_2 \in [0, N-1]\}. \tag{4.15}$$

In order to extract the Whitney elements we consider the small rectangle defined by the four points $(k_1,\, k_2)$, $(k_1{+}1,\, k_2)$, $(k_1{+}1,\, k_2{+}1)$, $(k_1,\, k_2{+}1)$. Then the coordinates on the surface of the sphere $(\theta,\, \varphi)$ locate a given vertex $(k_1,\, k_2)$ and local coordinates $(\epsilon_1,\, \epsilon_2) \in \mathbb{R}^2$ locate points in the square, so a function may be written

$$f(\epsilon_1,\, \epsilon_2) = e^{i\pi\hat{\theta}(\theta+\epsilon_1)} e^{2i\pi\hat{\varphi}(\varphi+\epsilon_2)}. \tag{4.16}$$

Next, consider the function

$$\hat{f}(\epsilon_1,\, \epsilon_2) = \left(\frac{e^{i\pi\hat{\theta}(\frac{k_1}{N}+\epsilon_1)} - e^{i\pi\hat{\theta}\frac{(k_1+1)}{N}}}{e^{i\pi\hat{\theta}\frac{k_1}{N}} - e^{i\pi\hat{\theta}\frac{(k_1+1)}{N}}} \right) \left(\frac{e^{2i\pi\hat{\varphi}(\frac{k_2}{N}+\epsilon_2)} - e^{2i\pi\hat{\varphi}\frac{(k_2+1)}{N}}}{e^{2i\pi\hat{\varphi}\frac{k_2}{N}} - e^{2i\pi\hat{\varphi}\frac{(k_2+1)}{N}}} \right). \tag{4.17}$$

Note that $\hat{f}(0,0) = 1$, and zero at the other vertices of the cell. Expand the terms in ϵ_1, ϵ_2 and consider the ϵ_1 term,

$$\frac{e^{i\pi\hat{\theta}(\frac{k_1}{N}+\epsilon_1)} - e^{i\pi\hat{\theta}\frac{(k_1+1)}{N}}}{e^{i\pi\hat{\theta}\frac{k_1}{N}} - e^{i\pi\hat{\theta}\frac{(k_1+1)}{N}}} = 1 + i\pi\frac{\epsilon_1 e^{i\pi\hat{\theta}\frac{k_1}{N}}}{e^{i\pi\hat{\theta}\frac{k_1}{N}} - e^{i\pi\hat{\theta}\frac{(k_1+1)}{N}}} + O(\epsilon_1^2)$$
$$= 1 + i\epsilon_1\Lambda(k_1, k_2, k_1+1, k_2+1) + O(\epsilon_1^2), \tag{4.18}$$

and substituting Eq. 4.17 into Eq. 4.18 we get

$$\hat{f}(\epsilon_1, \epsilon_2) = (1 + i\epsilon_1\Lambda_1)(1 + i\epsilon_2\Lambda_2) + O(\epsilon_1^2, \epsilon_2^2)$$
$$= (1-x)(1-y) + O(\epsilon_1^2, \epsilon_2^2) \tag{4.19}$$

where we made the following change of variables $x = i\frac{\epsilon_1}{\Lambda}$, $y = i\frac{\epsilon_2}{\Lambda}$. We have extracted a zero-form Whitney element after an approximation. It is easy to get the other Whitney elements. So, in a given cell [0123], the vielbein is

$$\hat{\theta}^{(1)} = [03] + [12], \tag{4.20}$$
$$\hat{\theta}^{(2)} = \sin\theta_{01}[01] + \sin\theta_{32}[32]. \tag{4.21}$$

In using the Whitney elements, new coordinates are taken in $\mathbb{R}^N$ so that

$$d = dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}, \tag{4.22}$$

as before. Note that in spherical coordinates, for the unit sphere S^2, we have the line element

$$(ds)^2 = (d\theta)^2 + \sin^2\theta(d\phi)^2, \tag{4.23}$$

this is normally reflected in d, however, we are using the local orthogonal system of coordinates provided by $(\hat{\theta}, \hat{\varphi})$ and we identify it with a plane in the embedding space with coordinates (x, y, z). The relation between the two sets of coordinates entails the approximation. Let us return to the metric, we now evaluate the matrix O that relates the vielbein in neighboring squares. It is a finite rotation of order $\frac{1}{N}$, because the change in the vectors is abrupt across the boundary. From Eq. 4.20, it is clear that, for this regular "triangulation", the only factor that changes in the vielbein is $\sin\theta$, so that the vielbein changes across the northern and across the southern boundary of a given square. This amounts to placing the connection ω around an edge. This will be discussed in the first part of the next section.

Also, the vielbein varies inside the cell so another viewpoint we will take is that the connection can actually be defined within the cell.

4.2 Covariant derivative and curvature

In the first part, we follow what amounts to a Regge type approach, the connection matrix acts on Whitney elements. We also discuss the curvature. In the second part, we follow the program of the Cartan structure equations which leads to the covariant derivative.

4.2.1 Procedure à la Regge

The connection as a matrix

At this point we can evaluate a dyad $\{e^a_A\}$ using co-chains. This enables one to consider the link vector as we proceed to do. We will in part relate to the formalism of a Hamiltonian lattice gravity model, for which a discrete moving frame formulation has been given (we follow [37]).

The Lagrangian in $2+1$-dimensions is

$$L = \frac{1}{4}\epsilon^{\mu\nu\lambda}\epsilon_{abc}\left[R^{ab}_{\mu\nu} + \frac{\Lambda}{3}\right]e^c_\lambda. \tag{4.24}$$

We will only use the Regge idea rather than such a Lorentzian model which poses some interesting problems for our Euclidean signature formulation. The link vector is obvious in our context and has components:

$$l^a_{ij} = \int_{[ij]} e^a_A dx^A. \tag{4.25}$$

Clearly there are two parallel edges in each square corresponding to each of the two A in Eq. 4.25. Then, we can think about the link in the two squares of which it is a boundary component. So, introduce a rotation matrix which maps the components of the link vector in the square S_1 to the components of the link vector in the square S_2:

$$l^a_{ij}(S_1) = O^{ab}(S_1,\, S_2) l^b_{ij}(S_2). \tag{4.26}$$

Now let us write Eq. 4.25 for the approximated vielbein Eq. 4.20:

$$l^\theta_{03} = l^\theta_{12} = 1, \tag{4.27}$$

$$l^\phi_{01} = \sin\theta_{01},\; l^\phi_{32} = \sin\theta_{32}, \tag{4.28}$$

the remaining four combinations being zero. We express the matrix O^{ab} in such a way that it contains submatrices which act on the space of Whitney forms. We denote the new matrix by $\mathcal{O}$. It has entries which are 2×2 submatrices relating Whitney elements among themselves. That is, it ought to be considered in a basis of functions $\{1,\, y\}$ or preferably in a basis $\{y,\, (1-y)\}$. Introducing

$$\mathcal{O} = \begin{bmatrix} \frac{\sin(\pi \frac{k_1'}{N})}{\sin(\pi \frac{k_1}{N})} & 0 \\ 0 & \frac{\sin(\pi \frac{(k_1'+1)}{N})}{\sin(\pi \frac{(k_1+1)}{N})} \end{bmatrix}, \tag{4.29}$$

where $k_1' = k_1 \pm 1$ locates the Northern k_1 value of the square S_1, given S_2. We are considering the matrix $\mathcal{O}$ that connects squares from North to South over which the θ value changes. In the matrix above we could have added entries for x and $(1-x)$ with 1 on the diagonal. The relation between $\mathcal{O}$ and the commonly used matrix O is simply:

$$O^{ab} = \int_{[ij]} \mathcal{O}^1 y + \mathcal{O}^2 (1-y). \tag{4.30}$$

Let us return to the discussion in relation to the Regge approach. We have digressed to show the way the veilbein is expressed using the Whitney elements and how the matrix O (which is itself the spin connection) is given by a larger matrix with constant coefficients. Now, following the usual procedure [37], Stokes theorem can be used to show that the integral in the interior of the square and thus the spin connection ω satisfies

$$\partial_A e^b_B - \omega^{bc}_{[A} e^c_{B]} = 0, \tag{4.31}$$

leading to a null spin connection in the interior of the flat region under consideration. Then, assuming that the spin connection has only one non-zero component along the perpendicular direction i.e perpendicular to the edge (denote by B, the coordinate perpendicular to the A coordinate), Stokes theorem (not based on the complex but by introducing a thin rectangle with two sides parallel to A and two infinitesimal ones parallel to B, and using the ordinary Stokes, see [37]) gives,

$$\int_{[ij]} e^a_A dx^A = (\delta^{ab} + \omega^{ab}_B \delta y) \int_{[lk]} e^b_A dx^A, \tag{4.32}$$

where the term in parenthesis corresponds to an infinitesimal rotation, which is done on the rectangle of width δy with the edge $[ij]$ as the side in S_1 and $[kl]$ the side in S_2. Now as is customary in the Regge approach for a simplicial lattice, the curvature (which is related to a product of matrices O as in Eq. 4.26) is concentrated at the vertices and is twice the deficit angle for a simplicial complex is given by

$$2 \sin \epsilon_i = R^{ab}_i \epsilon_{ab}. \tag{4.33}$$

In the hyper-cubic setting, a similar formulae may be written as we proceed to describe. It is clear that the theory remains a linear one. In this way, we are not inclined to make a different prescription for the way the curvature scalar enters in the theory. The angles ϵ_i are evaluated in the embedding $\mathbb{R}^3$. Furthermore, we are led to think of the edges as straight lines in $\mathbb{R}^3$, not approximately as parallels and meridians. The approximation is captured by the use of Whitney forms and what is effectively a change of coordinates from curvilinear (θ, φ) to Cartesian (x, y).

At present, the problem of discrete gravity in Lagrangian or Hamiltonian approaches [38] is the subject of much research. One example is the causal set formulation [39], also, of special interest are the spin foam models [40] which would be closer to what we have here.

It is not the purpose of this Chapter to present a new model of lattice gravity, but rather to propose new tools.

Curvature

Let us now turn to curvature. It enters the discussion in two natural ways. 1) First through the calculation of local holonomies. Consider a closed infinitesimal loop, and look at the local holonomy given using local coordinates $\{\eta^i\}$. Introduce as is customary

$$\triangle S^{lj} = \frac{1}{2}\oint(\eta^l d\eta^j - \eta^j d\eta^l), \tag{4.34}$$

from which it is found that

$$\triangle A^i = \frac{1}{2}\left(R^r_{ilj}A^r\right)_{x0}\triangle S^{lj}. \tag{4.35}$$

We consider the surface of the cube as a geometrical approximation to the 2-sphere. Now, the sphere is a space of constant curvature i.e:

$$R^\theta_{\nu\rho\sigma} = g^{\theta\mu}\frac{R}{D\,(D-1)}\left(g_{\nu\sigma}g_{\mu\rho} - g_{\nu\rho}g_{\mu\sigma}\right). \tag{4.36}$$

So we consider the holonomy shown in Fig. 4-4 where the field A changing abruptly by a right-angle rotation according to $\triangle A^x = -A^x$ and $\triangle A^y = A^y$ through its path across [06], [65], and [67] in that order. In our approach, this is to be expressed as:

$$\triangle A^x = \langle[9,10],\,A^x\rangle - \langle[89],\,A^x\rangle = -\langle[8,9],\,A^x\rangle, \tag{4.37}$$

$$\triangle A^y = \langle[9,10],\,A^y\rangle - \langle[89],\,A^y\rangle = \langle[9,10],\,A^y\rangle; \tag{4.38}$$

where A^x, A^y are respectively the $x-$ and $y-$ components of the Whitney elements obtained with

$$A = A_x([01]+[67]) + A_y([03]+[65]). \tag{4.39}$$

Next, noting that with $\triangle S = 2$, we get

$$\triangle A^x = \frac{1}{2} R^x_{xxz} A_x \triangle S^{xz} + \frac{1}{2} R^z_{xxz} A_z \triangle S^{xz} = R^x_{xxz} A_x, \tag{4.40}$$

$$\triangle A^y = \frac{1}{2} R^y_{yzy} A_y \triangle S^{yz} + \frac{1}{2} R^z_{zyz} A_z \triangle S^{yz} = R^y_{yzy} A_y, \tag{4.41}$$

leads to $R^x_{xxz} = -1$ and $R^y_{yzy} = 1$. We see that curvature is given at the vertices of the cubic complex. That is, if the label i in the summation below labels vertices, then

$$R(x) = \sum_i \int_{S^2_K} d^2x \, \delta^2(x - x_i) R(x_i), \tag{4.42}$$

where $R(x_i) = R_i$. Next, 2) In the global aspects of the theory, we make contact with the topological features and have the Euler class [22] given by

$$e(A)e(A) = p_1(A), \tag{4.43}$$

where in the case of the two sphere, we have ($\mathcal{R} = \mathcal{R}_{\alpha\beta} \theta^\alpha \wedge \theta^\beta$, the curvature two-form)

$$p_1(S^2) = -\frac{1}{8\pi^2} tr \, \mathcal{R} = (\frac{1}{2\pi} \sin\theta d\theta \wedge d\phi)^2, \tag{4.44}$$

and as is well-known,

$$\int_{S^2} e(S^2) = 2. \tag{4.45}$$

The link with the topology is through the Euler characteristic which is itself related to the Laplacian (which is captured in geometric discretisation), acting on r-forms by the important topological identity

$$\chi(M) = \sum_{r=0}^{m} (-1)^r \dim \, Ker \, \triangle_r, \tag{4.46}$$

where

$$\triangle_r = d^{(r-1)} d^{\dagger\,(r)} + d^{\dagger\,(r+1)} d^{(r)}, \tag{4.47}$$

which is clearly related to the harmonic forms and to homology. We now return to Eq. 4.45 and we express its discrete version as

$$\sum_i R_i = 2. \tag{4.48}$$

For the cube, the discrete symmetry implies that $R_i = \frac{R}{8} = \frac{1}{4}$.

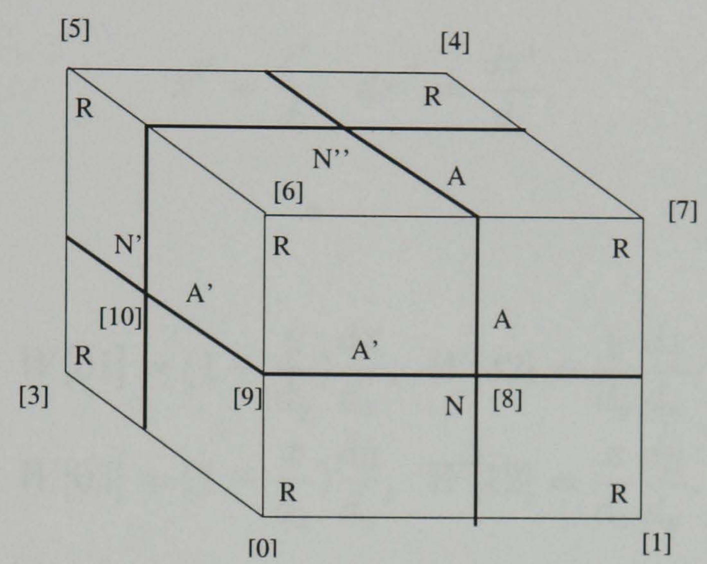

Figure 4-4: On the three faces respectively, there are three norms placed at barycenter N, N', N'' for the vector field under consideration, there are two independent connections placed on the dual L-edges denoted by thick lines, they are A, A' and there is one curvature scalar R placed at vertices of the complex K.

It is worth pausing for a moment to discuss an issue related to the simplicial geometry which forces us to distinguish between shape (rescaling in the embedding space) and metric. The shape of the piecewise square surface is fully determined by one length, the edge of one square, and the way the squares are put together, if we were to take rectangular cells, a finite number of length scales appear in the theory, which is however not detected at the topological level. The metric remains the locally flat metric but the surface areas of cells vary. In such a framework, we would need to modify by some area factors our expressions for curvature.

Let us display how at the level of the Whitney map, the information of link lengths disappears. Set the length of the x-axis edge to be d_x and d_y and d_z for the other edges. This modifies the scalar multiplicative factors in equations which involve the Whitney map. For example we see that we now have the following cubic Whitney elements:

$$\mu_0 = (1 - \frac{x}{d_x})(1 - \frac{y}{d_y}), \;\; \mu_1 = x(1 - \frac{y}{d_y}), \tag{4.49}$$

$$\mu_2 = \frac{xy}{d_x d_y}, \;\; \mu_3 = \frac{y}{d_y}(1 - \frac{x}{d_x}), \tag{4.50}$$

which lead us to make the substitution

$$x^{i'} = \frac{x^i}{d_i}, \;\; dx^{i'} = \frac{dx^i}{d_i}, \tag{4.51}$$

and then

$$W[01] = (1 - \frac{y}{d_y})\frac{dx}{d_x}, \;\; W[32] = \frac{y}{d_y}\frac{dx}{d_x}, \tag{4.52}$$

$$W[03] = (1 - \frac{x}{d_x})\frac{dy}{d_y}, \;\; W[12] = \frac{x}{d_x}\frac{dy}{d_y}. \tag{4.53}$$

So what happens when we deform the unit cube to one with edge length given by d_x, d_y, d_z?

It is essential to note that any 2-simplex is locally flat in the embedding space which is $\mathbb{R}^3$. The metric in the interior of each and every 2-simplex remains the flat metric. It is however clear that the shape of the closed 2-dimensional surface has changed. This feature is captured by the starting data (or embedding in $\mathbb{R}^3$) which is sensitive to the $\mathbb{R}^3$ coordinates of the endpoints of any given edge as

$$v_{ij} = \int_{[ij]} v_{\mathbb{R}^3}. \tag{4.54}$$

While the use of Whitney elements, and possible subdivisions to refine the complex should not affect the metric. It makes some difference since the de Rham map then gives

$$A\left(v^b\right) = \sum_i \int_{\sigma^i} v^b \, [\sigma^i], \tag{4.55}$$

$$\langle \sigma^j, A\left(v^b\right) \rangle = \|vol_{R^3}(\sigma^i)\| v^b_{average(\sigma^i)}, \tag{4.56}$$

where the length are measured in the embedding space $\mathbb{R}^3$. The modification of the Whitney map affects the rescaling which guarantees that $AW = Id$ which is explicitly:

$$\int_{\sigma^i} W\left(\sigma^j\right) = \delta_{ij}. \tag{4.57}$$

In the next section we describe a method that gives access to the covariant derivative.

4.2.2 Procedure à la Cartan

We now return more closely to the vielbein formalism and construct the kit we denoted by $\mathcal{G}$ including a discrete analogue of the covariant derivative.

The structure equations

In the last section, we have seen how one may express the vielbein for the sphere (after a change of coordinates from spherical to Cartesian) as Whitney forms. These have been used as a way to express the vielbein. This suggests in turn that the connection ω is actually defined within the cell, since we can calculate $d\hat{\theta}^{(1)}$, $d\hat{\theta}^{(2)}$ which gives rise to Whitney forms while in the continuum we have the torsion-free Cartan structure equation:

$$d\hat{\theta}^{\alpha} + \omega^{\alpha}_{\beta} \wedge \hat{\theta}^{\beta} = 0, \tag{4.58}$$

which means in the present case

$$d\hat{\theta}^2 = -\cos\theta d\phi \wedge d\theta = -\omega^2_1 \wedge \hat{\theta}^1. \tag{4.59}$$

The approximate version of this equation leads to a matrix with one forms as entries (we denote the one-chain analogue as ω^K, and the matrix in brackets) to get,

$$[\omega] = \begin{bmatrix} 0 & W(\omega^{2K}_1) \\ -W(\omega^{2K}_1) & 0 \end{bmatrix}, \tag{4.60}$$

implying that

$$dW(\hat{\theta}^{2K}) = -W(\omega_1^{2K}) \wedge \begin{bmatrix} W(\hat{\theta}^{1K}) \\ W(\hat{\theta}^{2K}) \end{bmatrix}, \tag{4.61}$$

which is an equation for ω^K. This procedure gives a piecewise constant ω in accordance with the fact that the vielbein varies linearly as Whitney elements in a given cell. Direct evaluation based on the explicit expression for the Whitney form vielbein gives:

$$W(\omega_1^{2K}) = (\sin(\pi\frac{k_1 - 1}{N}) - \sin(\pi\frac{k_1}{N}))dy, \tag{4.62}$$

which is seen as an approximation of $\omega_1^2 = \cos\theta d\phi$. Recall that in the reference square [0123],

$$\hat{\theta}^{1K} = [03] + [12], \tag{4.63}$$

$$\hat{\theta}^{2K} = \sin\theta_{01}[01] + \sin\theta_{32}[32]. \tag{4.64}$$

Thus, we rewrite Eq. 4.62 as

$$\omega_1^{2K} = (\sin\theta_{01} - \sin\theta_{32})([01] + [32]). \tag{4.65}$$

Now, consider the second Cartan structure equation:

$$d\omega_\beta^\alpha + \omega_\gamma^\alpha \wedge \omega_\beta^\gamma = R_\beta^\alpha, \tag{4.66}$$

where as the indices suggest,

$$R_\beta^\alpha = \frac{1}{2}R_{\beta\gamma\delta}^\alpha \hat{\theta}^\gamma \wedge \hat{\theta}^\delta. \tag{4.67}$$

For the two-sphere, we can read off the non-zero components of the Riemann tensor after writing the equation in components leading to $R_{212}^1 = -R_{221}^1 = \sin\theta$ and $R_{112}^2 = -R_{121}^2 = -\sin\theta$. However, in the present model, we do not expect to obtain by direct calculation the curvature since the cell is manifestly flat. Instead, we resort to Stokes theorem after rewriting the second structure equation [41] in the more familiar form

$$d\omega_{12} = -K(p)(\hat{\theta}^1 \wedge \hat{\theta}^2), \tag{4.68}$$

where $K(p)$ is the Gaussian curvature. Then, we can readily integrate this expression, i.e

$$\int_\sigma d\omega_{12} = -\int_\sigma K(p)\hat{\theta}^1 \wedge \hat{\theta}^2, \tag{4.69}$$

and using the Stokes theorem on the left-hand side of the equation, we get

$$\int_{\partial\sigma} \omega_{12} = -\int_\sigma K(x_0)\delta^{(2)}(x - x_0)\hat{\theta}^1 \wedge \hat{\theta}^2. \tag{4.70}$$

where the delta function term $K(x_0)\delta^{(2)}$ places the curvature at a point (as is well known the delta function is a distribution, and so it is well defined under integration). This equation allows a discrete version of the curvature to be defined at vertices. To place the calculation in the context of the discretisation, we select a 2-simplex in the dual complex L, this choice determines a unique vertex in K at which we placed the curvature (denoted x_0). The closed loop $\mathcal{C} = \partial\sigma_i^{(2)}$ is the boundary of the two-simplex. We then have for the left hand side of the previous expression a result as a sum of integrals over L edges:

$$\int_{\mathcal{C}=\partial\sigma_i^{(2)}} \omega_{12} = \sum_i \int_{\sigma_i^{(1)}} \omega_{12}, \tag{4.71}$$

which gives a contribution from the two L-edges parallel to the ϕ axis and the contour $C = \{\sigma_i^{(1)}\}$ listed in a path ordered way by the index i.

Note that we can evaluate the contour integral exactly. Recall that when we introduced the vieblein, we set the exact value of $\sin\theta$ at the parallels, if we also specify the $\sin\theta$ values for the parallels of the dual complex L, we have the exact values we need to do the integral of ω,

$$\int_{\sigma_1^{(1)}} \omega_1^2 = -[\sin\theta]_{\sigma_-^{(0)}}^{\sigma_+^{(0)}}. \tag{4.72}$$

The contour is the same as in the Regge analogue of the previous section. One last point, regarding the Bianchi identities, they are obtained by taking the exterior derivative of the structure equations. In the torsion free case, and using Eq. 4.68 as the second structure equation

$$dd\omega - d(K(p)\hat{\theta}^1 \wedge \hat{\theta}^2) = 0, \tag{4.73}$$

we immediately find in the present case of constant scalar curvature,

$$K(p)d(\hat{\theta}^1 \wedge \hat{\theta}^2) = 0, \tag{4.74}$$

trivially holds.

The covariant derivative

We will consider two different formulae for which we construct an analogue. We have

$$\nabla_X \omega = X^\mu \partial_\mu \omega_\nu dx^\nu + \Gamma^\alpha_{\beta\gamma} X^\alpha \omega_\alpha \hat{\theta}^\beta, \tag{4.75}$$

$$(\nabla_v v)^b = L_v v^b - \frac{1}{2} dg(v, v). \tag{4.76}$$

The second formula has the Lie in it, but the first one has the connection coefficients explicit and is more general as one can pick any pair of form and vector field, not just a vector v and its dual one form v^b. The relation between the two is given by the definition of the connection one-form

$$\omega^\alpha_\beta = \Gamma^\alpha_{\gamma\beta} \hat{\theta}^\gamma. \tag{4.77}$$

In our system, we have a discrete analogue of the vielbein $\hat{\theta}$ and the exterior derivative, which is enough to write the Cartan structure equations which are the key to the Riemannian geometry in the vielbein approach. So we expect the connection to arise from the exterior calculus and the fact that we provided a vielbein. First note that

$$\nabla_X \omega = i_X d\omega + \omega_\alpha X^\beta \omega^\alpha_\beta. \tag{4.78}$$

Then, recall that the covariant derivative is bilinear so taking a vector field with component X^β with $\alpha \neq \beta$,

$$\nabla_{X^\beta}\omega = i_{X^\beta} d\omega + \omega_\alpha i_{X^\beta} d\hat{\theta}^\alpha, \tag{4.79}$$

the case in which $\alpha = \beta$, one gets a minus sign for the second term. The result is now in the convenient notation.

In order to keep track of coordinates, we see that between the $(\theta,\, \phi)$ coordinate system and the non-coordinate basis $(\hat{\theta}^1,\, \hat{\theta}^2)$ we have the following change of variables which is determined by the inner product for one forms and vector fields:

$$X^\theta \mapsto X^1 = X^\theta,\; X^\phi \mapsto X^2 = \frac{X^\phi}{\sin\theta}, \tag{4.80}$$

$$\omega_\theta \mapsto \omega_1 = \omega_\theta,\; \omega_\phi \mapsto \omega_2 = \omega_\phi \sin\theta. \tag{4.81}$$

It is then easy to see that with $\omega = \omega_1 \hat{\theta}^1$,

$$\omega_\alpha i_X d\hat{\theta}^\alpha = \omega_\theta \frac{X^\phi}{\sin\theta} \cos\theta d\phi = \omega_\theta X^\phi \cot\theta d\phi. \tag{4.82}$$

Next, for $\omega = \omega_2 \hat{\theta}^2$,

$$\omega_\alpha i_X d\hat{\theta}^\alpha = \omega_\phi \sin\theta X^\theta (-\cos\theta) d\phi, \tag{4.83}$$

from which we can read off the usual connection coefficients:

$$\Gamma^\phi_{\theta\phi} = \cot\theta, \tag{4.84}$$

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta. \tag{4.85}$$

Let us return to the term $dg(v, v)$.

$$dg(v, v) = \sum_i d < \theta^i,\, v >^2 . \tag{4.86}$$

Since $< \hat{\theta}^{(i)}, v >= \delta^i_j v_j$, it is clear that in the case where we are dealing with a flat space,

$$d < \hat{\theta}, v >^2 = 2v \rfloor dv, \tag{4.87}$$

using the $\rfloor$ symbol. In fact we verify this immediately, using the Euclidean version of the covariant derivative for which:

$$L_v v^b = \left(v^b_\mu \partial_\nu v^\mu + v^\mu \partial_\mu v^b_\nu\right) dx^\nu, \tag{4.88}$$

$$dg\,(v,\, v) = 2v^\mu \partial_\nu v^\mu dx^\nu, \tag{4.89}$$

and we get the familiar expression in the Euclidean setting:

$$\left(\nabla_v v\right)^b = v^\mu \partial_\mu v^b_\nu dx^\nu. \tag{4.90}$$

Substituting the Whitney elements, the same cancellation of terms takes place as in the continuum. Now, in the more general case,

$$\begin{aligned} d\theta(X) &= d(\theta_\alpha X^\alpha) \\ &= X^\alpha d\theta_\alpha + \theta_\alpha dX^\alpha \\ &= X^\alpha \omega^\alpha_\beta \hat{\theta}^\beta + \theta_\alpha \partial_\mu X^\alpha dx^\mu \end{aligned} \tag{4.91}$$

which is the key to

$$dg(X, X) = 2\sum_\alpha \hat{\theta}^\alpha(X) d\hat{\theta}^\alpha(X). \tag{4.92}$$

The relation between equations (4.75) and (4.76) is explicit. Now, is parallel transport well defined ?

Consider the following equation for ω:

$$\nabla_X \omega = 0. \tag{4.93}$$

Let the curve be such that $\{X^\phi = 1, X^\theta = 0\}$. Then, we get

$$i_X d\omega = X^\phi \rfloor^K (\omega_{01} - \omega_{32} + \omega_{12} - \omega_{03})([03] + [12]), \tag{4.94}$$

by direct application of the definition of the discretised interior product and,

$$\begin{aligned} \omega \rfloor i_X d\hat{\theta}^{(\alpha)} &= \omega_1 \rfloor X^\phi \rfloor (\sin\theta_{01} - \sin\theta_{32})([03] + [12]) \\ &= (\omega_{03}[03] + \omega_{12}[12]) \rfloor X^\phi \rfloor (\sin\theta_{01} - \sin\theta_{32})([03] + [12]). \end{aligned} \tag{4.95}$$

The last equation is mapped directly to a differential form as

$$\begin{aligned} W(\omega \rfloor i_X d\hat{\theta}^\alpha) &= \varphi^0(W(X^\phi)) \wedge \varphi^0((\omega_{03}(1-x) + \omega_{12}x)dy) \wedge (\sin\theta_{01} - \sin\theta_{32})dy \\ &= \varphi^0(W(X^\phi))(\omega_{03}(1-x) + \omega_{12}x)(\sin\theta_{01} - \sin\theta_{32})dy \end{aligned} \tag{4.96}$$

while,

$$W(i_X d\omega) = \varphi^0(W(X^\phi)) \wedge (\omega_{01} - \omega_{32} + \omega_{12} - \omega_{03})dy. \tag{4.97}$$

Parallel transport then amounts to the following equations uniquely:

$$\omega_{03} = \omega_{12}, \tag{4.98}$$

$$\omega_{01} - \omega_{32} = \sin\theta_{01} - \sin\theta_{32}, \tag{4.99}$$

since the first one implies that the term $(\omega_{03}(1-x) + \omega_{12}x)$ is equal to a constant which then allows to match the two terms of the covariant derivative after writing the second equation. Our example gave a family of chains obeying the condition for parallel transport.

We end this section with some remarks about coordinates. As in the previous chapter, the vector field is an operator, so in the vielbein picture the object we work with is the one-form and the dual vector field is only defined in relation to it by means of the inner product. For example look at

$$(\frac{X_\phi}{\sin\theta_1}(1-z)+\frac{X_\phi}{\sin\theta_2}z)(\sin\theta_{01}-\sin\theta_{03}) = (\cot\theta_1 - O(z))(1-z)+(\cot\theta_2+O(z))z. \tag{4.100}$$

It is easy to spot that by the mean value theorem and the symmetry of the trigonometric functions, the result is well approximated on the L-edge, since there, the sine values and cos values are approximated at the same angle and give a good approximation for the cotangent. Recall that the relation between contraction and the inner product is

$$i_v\theta = \theta(v) =< \theta,\, v > . \tag{4.101}$$

We have just seen that we only have approximately the basis vectors specified by the vielbein. This does not affect parallel transport and does not prevent the matching with the continuum result in general, because the theory is one in which the objects are differential forms and vector fields are operators.

Consider the one-form we have been dealing with so far v. It is important that we understand the relation between the various coordinate systems we have. There are effectively two, the polar coordinates $(\theta,\, \phi)$ and the Cartesian one $(x,\, y)$. Because differential forms are coordinate independent objects, the choice is subjective. For the basis, we have

$$< \hat{\theta}^i,\, e_j > = \delta^i_j, \tag{4.102}$$

$$< dx^\mu,\, \partial_\nu > = \delta^\mu_\nu, \tag{4.103}$$

but in the model with Whitney elements, the Cartesian coordinates are required.

Remark on the Hodge star

In the construction of metric, we have not placed the Hodge star operator, which for a Riemannian manifold is given by

$$\star(dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_r}) = \frac{\sqrt{g}}{(m-r)!}\epsilon^{\mu_1\ldots\mu_r}_{\nu_{r+1}\ldots\nu_m} dx^{\nu_{r+1}}\wedge\ldots\wedge dx^{\nu_m}, \tag{4.104}$$

at the center of the construction, which is what one might be tempted to do. In fact, in the non-coordinate basis provided by the vielbein, we have the Hodge star in the analogous way to the Euclidean space by replacing dx^{μ_1} by $\hat{\theta}^{\alpha_1}$. That is, when we write $\hat{\theta}^1 \wedge \hat{\theta}^2$, we do have an approximate volume form for the continuum problem. So the formulae

$$\star(\hat{\theta}^{\alpha_1} \wedge \ldots \wedge \hat{\theta}^{\alpha_r}) = \frac{1}{(m-r)!} \epsilon^{\alpha_1 \ldots \alpha_r}_{\beta_{r+1} \ldots \beta_m} \hat{\theta}^{\beta_{r+1}} \wedge \ldots \wedge \hat{\theta}^{\beta_m}, \tag{4.105}$$

can be defined combinatorially using the forms $\hat{\theta}^i$ rather than the basis list of one-cochains with unit coefficient.

4.3 Discussion, applications

The covariant derivative we have evaluated enables us to consider the second term in the Euler equation

$$\frac{\partial v}{\partial t} = -\nabla_v v - \nabla P. \tag{4.106}$$

It is important to have a sense of how the axiomatics of the covariant derivative apply to the map we have constructed. We discuss it very briefly. Again, the exactness of the operations on certain forms and the linearity of the mappings we have is of much help.

4.3.1 Axiomatics

They are: 1) Viewed as a map with two entries, $\nabla_v w$ is bilinear, and,

$$2)\ \nabla_v f w = (L_v f) w + f(\nabla_v w), \tag{4.107}$$

$$3)\ L_v \langle w,\, z\rangle = \langle \nabla_v w,\, z\rangle + \langle w,\, \nabla_v z\rangle, \tag{4.108}$$

$$4)\ \nabla_v w - \nabla_w v = [v,\, w]. \tag{4.109}$$

Briefly, we relate to these properties in the following way: 1) holds trivially; We can also impose 2) which then takes the obvious form

$$\nabla_v f w = (L_v f) \times w + f \times (\nabla_v w). \tag{4.110}$$

This equation supposes that f and w can be expressed as Whitney elements written as $f \rfloor w$, so that we just need to apply the Liebnitz rule. Next, for 3), recall that the inner product is given by integration, and so the left hand side is taken as a zero form field which can be represented by $A < w, z >$ on the chain space and for the right hand side it is required that both $\nabla_v w$ and $\nabla_v z$ are exact. Finally, the last axiom is satisfied due to the exactness.

So in summary, the approximation is at the level of the vielbein itself, once it is included in the theory, the algebraic relations are exact.

4.3.2 Remark on 2D CW complexes

In practical applications, it might be of interest to consider a CW complex. Namely, the generalization of the simplicial complex made of different gones, for the present matter we will consider three gones (triangles) and four gones (squares). In this case, the Euler characteristic

$$\chi(g) = 2 - 2g = V - E + F, \tag{4.111}$$

gives us a constraint on the number of edges, vertices and faces that are acceptable.

We discuss briefly a proposal for the Whitney map on a CW complex we are currently investigating.

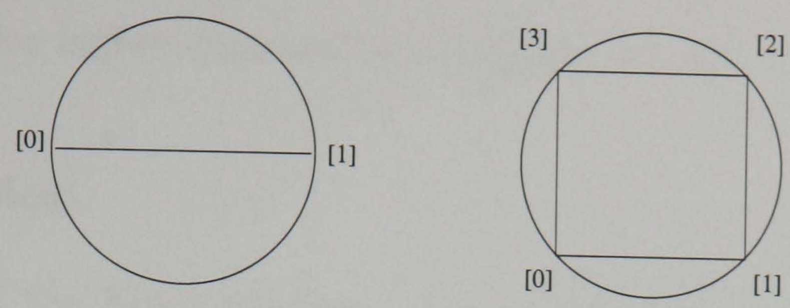

Figure 4-5: To express the Whitney map for a CW complex in 2D, the left figure shows the construction for an edge, and the right side figure shown the setting for a square.

Consider the edge [01] in polar coordinates (Fig. 4-5) and let:

$$W([1]) = \frac{1}{2}r(e^{i\theta} + 1), \tag{4.112}$$

$$W([0]) = 1 - W([1]) = \frac{1}{2}r(1 - e^{i\theta}). \tag{4.113}$$

Then apply d, strictly speaking we should split the interval [01] into two, since one segment has $\theta = 0$ and the other $\theta = \pi$, but it is trivial so we write:

$$\int_{[01]} dW([1]) = \int_{[01]} dr\frac{1}{2}(e^{i\theta} + 1) = \frac{1}{2}\int_1^0 dr - \frac{1}{2}\int_0^1 dr = 1. \tag{4.114}$$

The Whitney map for the CW complex seems not trivial, by taking N-gones as the basic objects, what is sought is a multilinear map which satisfies the boundary conditions as previously. In the case of the square, both coordinates r, θ play a role, so the analysis is a bit more complicated. Another issue is that there is no algorithm such as the iterative process we described in the context of the hyper-cubic complex in order to generate higher order elements. That is, when calculating $d\mu$, we get a linear combination of the Whitney elements of the edges which have the corresponding vertex, but it is impossible to select the individual Whitney elements by pull-back of integration.

Another method due to Bossavit [42, 43] makes use of the elegant construction known as the Poincaré map. This relies on the topological retract as a means to obtain the Whitney map for any 2D CW complex. The Poincaré map is a technique that goes through the exact sequence of homology by generating the basis of forms.

The Whitney map for higher dimensional complexes remains an open problem.

4.3.3 Discussion

We have mentioned the Euler equation. Let us describe in a bit more detail the basic application to fluids. First, it is customary to think of discrete flows, which is described in the book by Arnold and Khesin [31]. The discrete flow in this context is a composition of elementary hops between neighboring cells, using the notation of [31]

$$\sigma_k \, o \, \sigma_{k-1} \, o \, \ldots \, o \, \sigma = \sigma'. \tag{4.115}$$

Next, we obtained the term

$$\nabla_v v. \tag{4.116}$$

We also have the topologist's Laplacian (which differs from the analyst's Laplacian by a sign and by the Ricci curvature):

$$\triangle = dd^{\dagger} + d^{\dagger}d, \tag{4.117}$$

if one is to consider the more general Navier-Stokes equation. In this picture, one is interested primarily in the idealized situation of the steady, incompressible, constant density flow in an embedding three dimensional Euclidean space,

$$(u \,.\, \nabla)u = \nabla_u u = -\nabla P, \tag{4.118}$$

for which the vorticity field

$$\omega = \nabla \times u, \tag{4.119}$$

is frozen in the fluid. This condition can be expressed [44] very simply as

$$L_u \omega = 0. \tag{4.120}$$

In general fields that are frozen in the fluid satisfy

$$\frac{\partial v_t}{\partial t} + L_{u_t} v_t = 0. \tag{4.121}$$

Now we would like to mention conservation laws (see [45] and the references therein). These can be expressed using differential forms, thus generalizing the last equation. Any local invariant such as conserved quantities, Lagrangian invariants, frozen-in vector fields and Frobenius invariants can be recast by specifying a differential form ω^p of appropriate degree satisfying

$$\frac{\partial \omega^p}{\partial t} + L_u \omega^p = 0. \tag{4.122}$$

For example, $\omega^2 = S$ the momentum of a vortex ring.

In the article by Ricca [45], a classification is given of invariants which is thought provoking:

(i) local ; (ii) global; (iii) topological,

which are then put respectively in relation with:

(i) pointwise, (ii) integral and (iii) algebraic,

In our picture, topology is captured, and so we ought to put respectively:

(i) cell-wise, (ii) integral , (iii) combinatorial.

This brings forward the aim of combining the purely topological virtues of geometric discretisation with the present geometric insight in applications to the often highly non-trivial problems of the mathematical description of fluid flow.

Also regarding dynamics, one is in a position to think of Hamiltonian vector fields,

$$i_v \omega = -dH. \tag{4.123}$$

It is tempting to write a discrete analogue of Noether theorem, and even the more topological moment map of Sourieau (see [46] for the symplectic formulation) in this language and we outline the proposed route to the moment map. The latter relates [47] the action of a Lie group G on a manifold M to the co-adjoint action of G on the dual $\mathbf{g}^\star$. It satisfies the generalization of (4.123)

$$d\mu^X = i_{X^\sharp}\omega, \tag{4.124}$$

where $X^\sharp = \{\exp tX \mid t \in \mathbb{R}\}$, and the equivariance property

$$\langle \mu \, o \, \psi_g, \, X \rangle = \langle Ad_{g^{-1}} \, o \, \mu, \, X \rangle, \tag{4.125}$$

where $\mu^X = \langle \mu, \, . \, \rangle$ is the moment map. One then has to define the flow ψ_t in the discrete context, and clearly this would be a discrete flow through the various cells. Associated to the flow is the map $Ad_g : G \longrightarrow GL(\mathbf{g})$ as

$$\frac{d}{dt} Ad_{exp\, tX} Y|_{t=0} = [X, \, Y], \; \forall X, \, Y \in \mathbf{g}; \tag{4.126}$$

the right-hand side being within our control in the discrete theory. Obtaining the momentum P seems somewhat trivial in out approach. However, the moment map is a very powerful tool in differential geometry and even in gauge theory [48]. We would like to return to it in future work.

Another topic is an approach to the Poisson bracket based on the coherent state approach [49] (we will follow section III of [50]). Briefly, one is given a Lie group, $SU(2)$ for example since it leads to the two-sphere as explained below. Let $\lambda = x + iy$, then the symplectic structure is,

$$\omega_{\lambda\bar{\lambda}} = \omega_{\lambda\bar{\lambda}} d\lambda \wedge d\bar{\lambda} + \omega_{\bar{\lambda}\lambda} d\bar{\lambda} \wedge d\lambda, \tag{4.127}$$

where the two non-zero elements of the matrix are:

$$\omega_{\lambda\bar{\lambda}} = \frac{1}{(1+\lambda\bar{\lambda})^2}, \tag{4.128}$$

$$\omega_{\bar{\lambda}\lambda} = -\omega_{\lambda\bar{\lambda}}. \tag{4.129}$$

Then, we introduce three functions as

$$X_+ = \frac{2\lambda}{1+\lambda\bar{\lambda}}, \tag{4.130}$$

$$X_- = \frac{2\bar{\lambda}}{1+\lambda\bar{\lambda}}, \tag{4.131}$$

$$X_0 = \frac{1-\lambda\bar{\lambda}}{1+\lambda\bar{\lambda}}, \tag{4.132}$$

which describe the two sphere

$$X_+X_- + X_0^2 = 1. \tag{4.133}$$

The Poisson bracket follows,

$$\{X_+, X_-\} = (\omega^{-1})_{\lambda\bar{\lambda}}\partial_\lambda X_+ \partial_{\bar{\lambda}} X_- + (\omega^{-1})_{\bar{\lambda}\lambda}\partial_{\bar{\lambda}} X_+ \partial_\lambda X_- = 2iX_0, \tag{4.134}$$

$$\{X_0, X_\pm\} = \pm X_\pm. \tag{4.135}$$

Again, because we are dealing with the two-sphere,

$$\int \omega d\lambda \wedge d\bar{\lambda} = 4\pi. \tag{4.136}$$

The analogous version of this would be made following the linearization

$$\frac{1}{1+\lambda\bar{\lambda}} = 1 - \lambda\bar{\lambda} + O(\lambda\bar{\lambda})^2, \tag{4.137}$$

from which one still gets the equation of the two-sphere and the algebra with the modified

$$\omega_{\lambda\bar{\lambda}} = 1 - 2\lambda\bar{\lambda} + O(\lambda\bar{\lambda})^2. \tag{4.138}$$

Finally a short note on non-commutative geometry. In that field, much use is made of the so-called Moyal product [55], and one may be tempted to turn the contraction $\rfloor^K$ into some non-commutative product. In [51], a non-commutative product for multiplication of forms with some scalar function was introduced, and was applied to the problem of fermion doubling. A (commutative) algebraic approach to the problem is given in the next two chapters. Closer to the present work and non-commutative geometry is the idea of a non-commutative differential calculus, in which one may introduce contraction and a Lie derivative [52, 53]. It is worth noting the formula for the interior product given in [54],

$$\begin{aligned} i_V(\omega^{i_1} \wedge \dots \omega^{i_n}) &= i_V(\omega^{i_1} \wedge \dots \omega^{i_s}) \wedge (\omega^{i_(s+1)} \dots \omega^{i_n}) \\ &(-1)^s b^i f_i^j \star (\omega^{i_1} \wedge \dots \omega^{i_s}) \wedge i_{t_j}(\omega^{i_{s+1}} \wedge \dots \omega^{i_n} a_{i_1 \dots i_n}), \end{aligned} \tag{4.139}$$

where the $\star$ is the Moyal product which encodes the non-commutativity [55].

4.4 Summary

In this chapter, we have laid out a prescription for capturing metric data. We followed two different roots after making a prescription for the vielbein. First by looking at a connection matrix which hops accross cells in what constitutes an analogue of the Regge approach, and second by considering the differential geometrical formulation which is encoded in the Cartan structure equations. The latter was best because its formalism is close to ours (differential forms) but particularly because the conceptual picture was clear: in the torsionless case, we extract the connection one-form ω by applying the exterior derivative to a collection of N (in $N-$dimensions) specific forms which encode the metric data. This construction in turn gave us access to curvature via the second structure equation, and gave us the covariant derivative.

We have completed the space $\mathcal{L} = \{i_v,\, d,\, \wedge,\, \star\}$ by providing it with a discrete vielbein, which in tun gives access to induced metric data. This has led to the new space $\mathcal{G} = \{i_v,\, d,\, \wedge,\, \star,\, \{\hat{\theta}^a\}\}$. We have commented briefly on how the inducd metric relates to the Hodge star operator in the discrete theory. The latter was defined for the embedding space $\mathbb{R}^N$. Chapter 3 was primarily about setting up the algebraic relations for the objects, so in a way the most problematic operator was i_v but accommodating it has determined the construction to a large extent.

This part serves as background theory for applications, we singled out the concrete problem of fluid dynamics, for which the naturally present topological features of the method, combined with the present geometrical complement should prove useful.

Let us now go back a few steps and start again with the space $\mathcal{T}$ in relation to the problem of discretising fermions in theoretical physics. We will see that eventually, we have to invoke the space $\mathcal{L}$ to progress further with the model.

Part III

Spinors

Spinors can be formulated using the language of differential forms [58]. In order to extract the Clifford algebra, it is necessary to consider the space of inhomogeneous differential forms (i.e Eq. 5.5) under the Clifford product [59] from which a representation of the Clifford algebra can be extracted (Eq. 5.6). This procedure, when the fibration is trivial, is a special case of the Atiyah-Kähler bundle construction.

That such an abstract construction may be involved in the discretisation of fermionic theories is, as we pointed out already, to be linked with the importance of topological techniques as a means to count particles in the theory. A fact which is reflected in algebraic relations which themselves encode algebraic topology (e.g Poincaré duality). In Chapter 5, we are concerned with approaching the problem of discretising fermionic fields in relation to the so-called fermion doubling problem, in a way that is analogous to that advocated independently by Rabin [3] and by Becher and Joos [4]. A must is to avoid falling into the quite constraining hypothesis of the Nielsen-Ninomiya theorem [1], and we do just that by considering fields as Whitney elements, and within the original geometric discretisation which we called $\mathcal{T}$, we can enforce the algebraic relations alluded to above (done in Chapter 5).

A hint that the argument given in Chapter 5 is somewhat incomplete appeared when considering the Clifford product in detail. As the picture of the theory we called $\mathcal{L}$ was clarified we recognized that i_v was formally what was needed in order to introduce a discretised Clifford product that is satisfactory. If $\vee$ was to be associated to a Clifford algebra, it was clear that a spinorial Lie derivative [60] was acting in the background, and so it became a matter of consistency that we introduce the Clifford product in the space $\mathcal{L}$ rather than in the space $\mathcal{T}$. So with the help of the space $\mathcal{L}$ as the discretisation scheme, Chapter 6 contains a more rigorous account of the proposed discretisation of the Dirac-Kähler equation.

In the final Chapter, issues regarding gauging are discussed. It is such an interesting problem that although the picture is not yet satisfactory for the minimal coupling, other gauging procedures like the gluon-gluon term $F \wedge \star F$ can be considered via the non-abelian Stokes theorem. A discussion of the fiber bundle is also included.

Chapter 5

Presentation of the problem in the space $\mathcal{T}$

The lattice provides a regularization of continuum quantum chromodynamics [56] which has played a central role in the study of non-perturbative effects such as quark confinement. However, problems arose which seem to be inherent to the lattice.

Putting fermions on the lattice naively leads to degeneracy, as can be seen from the dispersion relations obtained. When Wilson removed this degeneracy [56], it may have seemed that the loss of chirality was perhaps coincidental but subsequent work [1, 57], based on topological arguments, found that this was not the case.

The argument, proposed by Nielsen and Ninomiya (NN), shows that there are, under reasonable assumptions such as exact conservation of discrete valued quantum numbers (lepton number Q) and locality, an equal number of species of left and right-handed neutrinos which means that you cannot have single fermions since they occur in pairs. Handedness is then taken into account by considering the homotopy class of maps from closed surfaces, enclosing the degeneracy points ω_{deg} embedded in the Brillouin zone into the space of rays formed by the N fundamental fermion components $\mathbb{CP}^{N-1}$. The degenerate fermionic states, $\mid \omega >$, live in this space and satisfy

$$H \mid \omega >= \omega_{deg} \mid \omega > . \tag{5.1}$$

The Dirac-Kähler (DK) formulation [59], which is differential geometric in nature, is advantageous since it gives rise to a non-degenerate energy spectrum in the discrete formulation, which means that the NN theorem is not directly applicable [61]. With this in mind a discrete DK theory [62, 4] can be expressed, using ideas from algebraic topology to provide a natural discrete analogue to the necessary differential geometry. Becher and Joos [4] showed how doubling still arises in this theory after reduction, even though topology is captured and NN is not applicable [1], as the discrete operations used do not satisfy the desired properties.

The algebraic properties of the discrete analogies of the operators $\{\wedge,\ d,\ \star\}$ are used to describe spinors and the action of the Clifford algebra on them. The latter has a differential geometric analogue through the introduction of the Clifford product (CP), both in the continuum and on the lattice. Its action, denoted by $\vee$, is a combination of the operations in the triple $\{\wedge,\ d,\ \star\}$ and plays an important role in the construction.

We will discuss how the problem of discretising fermion fields is usually formulated in the DK approach. We will then give a first part of the analogue that may be constructed in our approach. We will use the space $\mathcal{T}$, which is itself suited to describe the chirality transformations (multiplication by γ_5 can be represented by application of the Hodge star operator).

The discrete operations and the chain spaces K and L have an interesting interpretation in such a framework.

We begin with a brief review of the DK equation in the next section. We then look at how this is dealt with discretely; first, using standard methods where the wedge fails, and then using the space $\mathcal{T}$, we propose an alternative, particularly the aspects related to chirality and the action functional are discussed; the reduction being postponed to the next chapter.

[1]The no-go theorem of Nielsen-Ninomiya can also be avoided using Ginsparg-Wilson [63] methods; namely domain wall and overlap fermions [64].

5.1 The Dirac-Kähler formulation of spinors

In the language of differential forms the Laplacian is given by

$$\triangle = d\delta + \delta d. \tag{5.2}$$

Thus we note that

$$-(d-\delta)^2 = \triangle, \tag{5.3}$$

from which we obtain the Dirac-Kähler equation (DKE) [59, 95]

$$i(d - \delta + m)\psi = 0, \tag{5.4}$$

the solution of which has the following functional form

$$\psi = 1 + f_\mu dx^\mu + \frac{1}{2!} f_{\mu\nu} dx^\mu \wedge dx^\nu + \frac{1}{3!} f_{\mu\nu\lambda} dx^\mu \wedge dx^\nu \wedge dx^\lambda + f_{0123} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3. \tag{5.5}$$

In order to represent spinors in this language, we introduce the Clifford product, which acts on the space of inhomogeneous differential forms and satisfies the following relations:

$$\begin{aligned} 1 \vee 1 &= 1, && (5.6)\\ 1 \vee dx^\mu &= dx^\mu \vee 1 = dx^\mu, && (5.7)\\ dx^\mu \vee dx^\nu &= g^{\mu\nu} \cdot 1 + dx^\mu \wedge dx^\nu, && (5.8) \end{aligned}$$

where $g^{\mu\nu}$ is the Euclidean metric. Through the identification

$$dx^\mu \vee \mapsto \gamma^\mu, \tag{5.9}$$

which arises from representation theory, we relate the differential forms under the CP to the algebra of gamma matrices. It is now immediate that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{5.10}$$

The Dirac field then belongs to a 16 dimensional representation of the algebra of gamma matrices. Furthermore all the representations of the complex Clifford algebra of the 4-dimensional Euclidean space can be decomposed into 4-dimensional irreducible representations [67]; these being equivalent to those generated by the standard gamma matrices.

As we have introduced the Clifford product, an important observation due to Susskind is now in order, namely, that the DKE is invariant under the group $SU(4)$, referred to as a "global flavour symmetry" [4], which acts by right action of the Clifford product with a constant differential U. That is,

$$(d - \delta + m)\,(\Phi \vee U) = \{(d - \delta + m)\,\Phi\} \vee U = 0, \tag{5.11}$$

this being established through the identification $d - \delta \mapsto dx^\mu \vee \partial_\mu$ (which can be derived using (5.6)) and use of the associativity of $\vee$ which is known to hold in the continuum.

We can thus decompose the 16D space of differential forms $\mathcal{D} = \{\Phi\}$ into 4-dimensional invariant subspaces

$$\mathcal{D} = \bigoplus_{b=1}^{4} \mathcal{D}^{(b)}, \tag{5.12}$$

on which the DKE implies the DE.

One can construct a collection of projection operators $P^{(b)}$ mapping forms onto the irreducible subspaces $\mathcal{D}^{(b)}$, where

$$\Phi \vee P^{(b)} = \Phi, \tag{5.13}$$

if $\Phi \in \mathcal{D}^{(b)}$. We also know that $\Phi \vee P^{(b)}$ is a solution of the DKE from the argument given above regarding the global $SU(4)$ symmetry. With this subsidiary condition Eq. 5.13, the DKE for fixed b is equivalent to the DE.

Considering

$$\Phi^{(b)} = \Phi \vee P^{(b)}, \tag{5.14}$$

where Φ is a solution of the DKE, we get

$$(d - \delta + m)\,\Phi^{(b)} \mapsto (\gamma^{\mu}\partial_{\mu} + m)\,\Phi^{(b)}, \tag{5.15}$$

leading to the Dirac equation.

We note that the DKE can also be expressed using the set of equations

$$i(d\omega^{(p-1)} - \delta\omega^{(p+1)}) = -m\omega^{(p)}, \quad p = \{0, 1, \ldots, 4\}. \tag{5.16}$$

These are related. For instance, the $p = 0$ and $p = 4$ equations are equivalent; as can be shown using $\star\star = I$ and $\delta = \star d \star$. A solution to the $p = 0$ equation living in one of the $\mathcal{D}^{(b)}$ is then a solution of the $p = 4$ equation (in the same $\mathcal{D}^{b}$). If we did not have $\star\star = I$ and $\delta = \star d \star$ then the relations between the equations would not be captured and degeneracy would arise.

In this language, taking $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ and with the identification $dx^u \vee \mapsto \gamma^{\mu}$ it is found that γ^5 corresponds, up to a sign, to $\star$; $(\gamma^5)^2 = 1$ while $\star\star = (-1)^p$. Thus the Hodge star operator plays an important part in formulating chiral fermions [3].

Before we address the discretisation of this theory, it is worth looking at the definition of the Clifford product in the continuum, which we express in the form

$$\Phi \vee \Psi = \sum_{p \geq 0} \frac{\text{sign}\,(p)}{p!}\left(\eta^p(e_{\mu_1}\rfloor \ldots e_{\mu_p}\rfloor\Phi)\right) \wedge \left(e^{\mu_1}\rfloor \ldots e^{\mu_p}\rfloor\Psi\right), \tag{5.17}$$

where $\text{sign}\,(p) = (-1)^{\frac{p(p-1)}{2}}$ and $\eta^p\omega^r = (-1)^r\omega^r$. We note that the operations of contraction $\rfloor$ and that of wedging are both used.

5.2 The Dirac-Kähler equation in the space $\mathcal{T}$

After having stressed the importance of the subsidiary conditions above, we now investigate their discrete counterpart [2]. We will first explain the nature of the problem by describing the usual method and then propose our alternative in relation to

[2]Other approaches of interest are given in [66, 51].

it. Again, this will turn out to be linked to the algebraic relations of the discrete operators; particularly of the wedge $\wedge$.

At this point we note that the (continuum) subsidiary conditions can be given the form of a group property known as the reduction group denoted $\mathcal{R}$ and generated by $\tau = idx^1 \wedge dx^2$ and $\epsilon = dx^1 \vee dx^2 \vee dx^3 \vee dx^4$ under $\vee$-multiplication.

On the lattice, we represent p-forms as p-cochains. Thus the Dirac fields are represented by inhomogeneous cochains. We now review the construction of the discrete analogue of the reduction group.

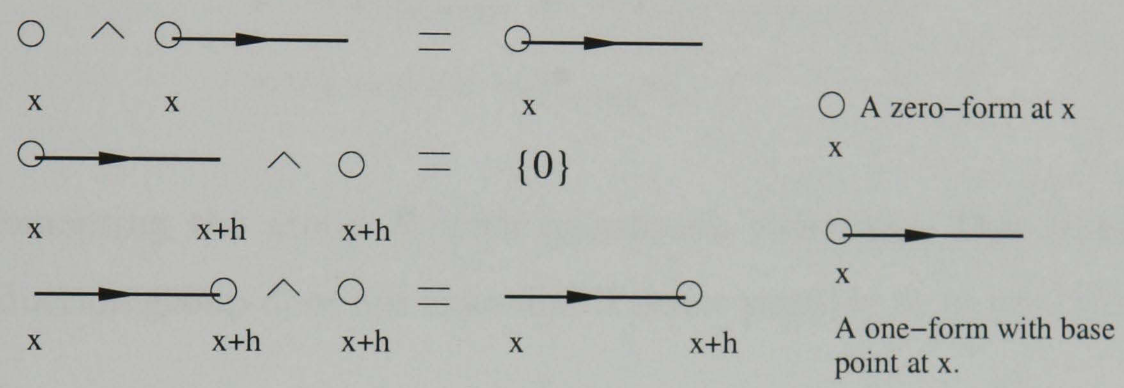

Figure 5-1: Role of the base point in the definition of the discrete wedge

In the approach [4] where the cup product is taken as the discrete wedge, the operation is performed at a base point x. The left and right cochain arguments of the wedge must have the same base point, as illustrated by the above Fig. 5-1.

In the continuum $x \wedge y$ is equal to $y \wedge x$ up to sign [3] which is no longer the case on the lattice; unless we translate the right cochain in such a way that its base point coincides with the other, enabling us to wedge cochains which do not have the same base point.

In the continuum we have

$$dx^1 \vee (dx^1 \wedge dx^2) = 1 \wedge dx^2 = dx^2, \tag{5.18}$$

because we remove any dx^i's which two arguments have in common before wedging them, when taking their Clifford product.

So,

$$(dx \wedge dy) \vee (dx \wedge dy) = 1 \wedge 1 = I, \tag{5.19}$$

[3] The discrete wedges of Birmingham-Rakowski [8] and Albeverio-Schäfer [13] satisfy this property also.

whose discrete analogue is

$$(dx \wedge dy) \vee T(dx \wedge dy) = T_e I, \tag{5.20}$$

where translation operation T_e shifts simplices σ and η so that $\sigma \wedge \eta$ and $\eta \wedge \sigma$ have common support.

It is then found that [4]:

$$\tau^2 = T_{-(e_1+e_2)}, \quad \epsilon^2 = T_{-(e_1+e_2+e_3+e_4)}; \tag{5.21}$$

$$\tau \vee \epsilon = \epsilon \vee \tau = T_{e_1+e_2}\tau\epsilon, \tag{5.22}$$

thus supplementing the group $\mathcal{R}$ with translation elements. This means that the discrete reduction group does not close and it is not possible to impose the subsidiary conditions.

Note that the presence of translation operators in the discrete analogue of the reduction group is unavoidable in this formalism..

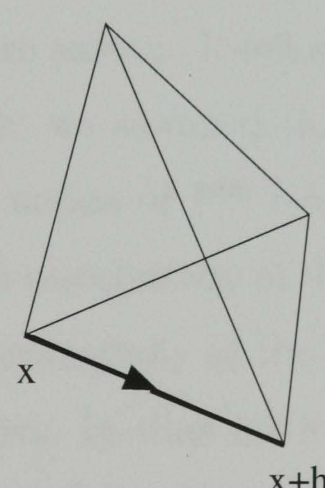

Figure 5-2: Discrete wedge in GD: the higher dimensional region, here a tetrahedron gives common support to the vertex at x, the vertex at $x+h$ and to the edge $[x,\ x+h]$.

In geometric discretisation (GD), or more specifically in the space $\mathcal{T}$, as we have seen, we map any p-chains, σ^p, to p-forms, ω^p, using the Whitney map W and wedge these, before mapping the result back to the lattice, using the de Rham map A.

It is intuitively clear that using the maps A and W, the discrete wedge satisfies $x \wedge^K y = \pm y \wedge^K x$ with no translation element required. Thus $\tau^2 = I$ and the reduction group closes (provided the discrete Clifford product is well-defined, in Chapter 6). We note that the wedge product in the space $\mathcal{T}$ is only associative up to a multiplicative factor which is a function of the degree of the cochains being wedged. Given three cochains σ, η, ζ in K, of degrees p, q and r respectively, the rule is

$$\sigma \wedge^K (\eta \wedge^K \zeta) = \left(\frac{p+q+1}{r+q+1}\right) (\sigma \wedge^K \eta) \wedge^K \zeta. \tag{5.23}$$

The problem with the discrete Clifford product is dealt with in the next Chapter.

The emerging picture for the DK equation in the present framework is that one has a hyper-cubic complex with fields represented by inhomogeneous cochains with constant coefficients in front of simplices such as $[i]$, $[ij]$, $[ijkl]$ and so on. The fact that one has Whitney elements is of importance here in two related ways. First of all, they provide support to the associated form and also are used in the wedge formulae as we have seen.

To compare this with the method described previously, we point out that the vertices at x and at $x+h$ have overlapping support on the edge $[x,\ x+h]$ (Fig. 5-2). If one was to specify a base point to be x or $x+h$, as is done in turn in Fig. 5-1, one would still get the correct non-zero answer. It follows that no translation is necessary.

In the argument given above, we assumed that the inhomogeneous differential forms projected under the right action of $P^{(b)}$ remain solutions of the DKE. In the continuum, this step involved the associativity of the continuum Clifford product. As is apparent from Eq. 5.11 the associativity of the discrete Clifford is conditional on the discrete wedge being associative. In other methods such as [4] the discrete Clifford product is non-associative in general but does not affect the derivation of Eq. 5.11. In the present framework we point out that we are given the exact expression for the $P^{(b)}$ differentials in the discrete theory. Therefore, owing to the fact that the Clifford product gives rise to the algebra displayed in Eq. 5.6, we can assert that

$$\Phi' = \Phi \vee P^{(b)}, \tag{5.24}$$

is also a solution of the DKE.

More explicitly, the $P^{(b)}$ are constant differentials. We will postpone to the next

Chapter further consideration of the technical details related to $\vee$. Suppose the lattice field Φ^K satisfies the discrete DK equation on the space $C(K)$:

$$(d^K - \delta^K + m)\Phi^K = 0, \tag{5.25}$$

then the equations

$$(d^K - \delta^K + m)(\Phi^K \vee^K P^{(b)}) = 0, \tag{5.26}$$

follow without requiring associativity of the discrete wedge, as we will see in the next Chapter.

On the issue of chirality, our discrete Hodge star maps one space into its dual, so we bear in mind that there is a doubling of the space of chains. As is customary in lattice theories we then deduce the relation:

$$\{\gamma_5, \not{D}\} = 0. \tag{5.27}$$

where

$$\not{D} = d - \delta + m. \tag{5.28}$$

In the present DK formalism this is expressed (acting on fields in K, but one can equally swap K and L superscripts) as

$$\{\star, d - \delta\}^K \equiv \star^K(d^K - \delta^K) + (d^L - \delta^L)\star^K, \tag{5.29}$$

since $\star$ plays the role of γ_5 and $\not{D} = d - \delta$. Using $\star^L\star^K = I$ and $\delta^K = \star^L d^L \star^K$, which are both satisfied in GD, we find

$$\begin{aligned}
&\star^K (d^K - \star^L d^L \star^K) + (d^L - \star^K d^K \star^L)\star^K \\
&= (\star^K d^K - \star^K \star^L d^L \star^K) + (d^L \star^K - \star^K d^K \star^L \star^K) \\
&= (\star^K d^K - d^L \star^K) + (d^L \star^K - \star^K d^K) = 0.
\end{aligned} \tag{5.30}$$

Note that γ_5 is identified with $\star^K$ and $\star^L$, up to some sign factor, since $(\gamma_5)^2 = 1$ whereas $\star^L\star^K = \pm 1$. So we always have $\star\star = 1$ provided we insert the correct sign factor, leading to the desired result. Again, there are two spaces involved here, since $\star^K \not{D}^K$ and $\not{D}^L \star^K$ both map objects to their dual space, which is also the case in the Ginsparg-Wilson [63] formalism.

The presence of the two spaces, K and L, is not necessarily a problem in the discretised chiral fermion theory, at least as far as producing an analogue of the action functional for the free fermion theory. In order to check this, we note that the subsidiary conditions carry through under application of $\star$. Namely, if $\omega \in D_K^{(b)}$ then $\star\, \omega \in D_L^{(b)}$. Since the theory is well defined as we interchange K and L, a space such as $D_L^{(b)}$ is well-defined. That is, in self-explanatory notation we can write an L-DKE as we were able to write a K-DKE in Eq. 5.25 (see Chapter 6).

Given the inhomogeneous fields Φ_K, represented by inhomogeneous chains in the complex K, we can now consider the associated action functional S_K:

$$S_K =< \bar{\Phi}_K, (d - \delta + m)\Phi_K >, \tag{5.31}$$

where (denoting complex conjugation by c)

$$\bar{\Phi}_K = \gamma_5 \Phi_K^c = \star^K \Phi_K^c, \tag{5.32}$$

and so Φ_K belongs to the dual space L. It is worth pausing for a moment to appreciate the role played by $\star$ both as γ^5 and as providing the volume form for integration.

The inner product $< \cdot, \cdot >: C^p(L) \times C^{n-p}(K) \longrightarrow \mathbb{C}$ is given by

$$< \sigma^L, \eta^K >= (\sigma^L, \star^K \eta^K) = \frac{(n+1)!}{p!(n-p)!} \int_M W^B(B\sigma^L) \wedge W^B(B\eta^K),$$

where B is the subdivided space containing both K and L and $B\sigma^L$ is the projection of σ^L onto B.

The action is of the form

$$S_K \equiv S\left(\Phi_K,\ \star^K \Phi'_K\right), \tag{5.33}$$

where there is a subtlety since the field $\star^K \Phi_K$ lives in L. Although there is a field in the original and dual complex, one is the image of the other. We can repeat the argument starting with the L complex, and define $S_L(\Phi_L)$. This is in contrast to methods such as [4] in which the volume element is induced by right Clifford multiplication with ϵ. Again, this is due to the close link between the lattice and differential geometry in our method.

Finally we look at the partition function for which the action is a direct sum of its components in the invariant spaces $\mathcal{D}^{(b)}$.

$$S = \sum_{b=1}^{4} S^{(b)}, \tag{5.34}$$

which according to the argument above can be readily rewritten as

$$S = \sum_{b=1}^{4} S_K^{(b)} \oplus S_L^{(b)}. \tag{5.35}$$

Furthermore, $S_K^{(b)}$ and $S_L^{(b)}$ are functionals of the fields Φ_K and Φ_L respectively; their associated representative in the dual spaces L and K. It follows then that the counting of the fields is correct, while we obtain the partition function squared given below Eq. 5.37. To see this, we introduce a matrix M determined by taking

$$\Phi_K^T M^K \Phi_K' = S_K(\Phi_K,\, \Phi_K'). \tag{5.36}$$

The entries of M^K are fixed, up to normalization (volume factors), by

$$M_{ij}^K = S_K(\sigma_i, \sigma_j).$$

We can fix this by normalizing the parameters in the general formula for the discretised fields,

$$\Phi_K = \sum_i \lambda_i [\sigma_i],$$

The space of parameters λ_i determines the partition function measure:

$$Z = \int [d\Phi_K][d\Phi_K^T][d\Phi_L][d\Phi_L^T]e^{\,\Phi_K^T M \Phi_K + \Phi_L^T M \Phi_L}$$
$$= \int [d\Phi_K][d\Phi_K^T]e^{\,\Phi_K^T M \Phi_K} \int [d\Phi_L][d\Phi_L^T]e^{\,\Phi_L^T M \Phi_L} \qquad (5.37)$$

which splits into two independent parts, one for fields defined on K and the other for fields defined on L. So, there is no mixing between the two lattices K and L.

5.3 Summary

In this Chapter, we have pointed out the limitations of discrete models as a means to reduce the Dirac-Kähler equation to the Dirac equation. We saw that the doubling of spaces is related to chirality and that displacement operators made the reduction problematic. Also, following the argument of Rabin, we pointed out how the algebraic relations of the operators d and $\star$ were crucial to the reduction. In that aspect, our approach already differs from [3, 4] since we do have operators $\star$ and d satisfying the desirable algebraic relations and the action functional is well defined. We need now to introduce a deeper approach to the problem in which the Whitney elements play a crucial role.

In the next Chapter we include more technical details of our method which is better formulated in the space $\mathcal{L}$.

Chapter 6

Reduction with a new discrete Clifford product

In the previous chapter, we have undertaken the description of Dirac-Kähler fermions using the space $\mathcal{T}$. It has proved useful, particularly in the way one handles the Hodge star which acts as γ_5. We also argued how the action functional for the free fermion field may be written. The previous Chapter raises some technical and conceptual issues that we now turn to, by building up the model from the start.

The idea is that the description of fermions using the Dirac-Kähler equation puts the topological aspects of the doubling problem in the foreground. As is known, the naive discretisation of the derivative using finite difference leads to doubling, and this problem has a topological manifestation using the index theorem which gives a simple relation between the number of right and left handed particles:

$$N_+ - N_- = 0, \tag{6.1}$$

in contradiction with the chiral anomaly postulated by the standard model. One way out, is to change the topology of the total space in order to change the index [88, 89].

The line of reflection that is the DK one, has been presented in the last chapter. By producing analogies of d, $\wedge$ and $\star$ which give all the operators in the DK equation, the discrete reduction does not lead to the Dirac equation. The Hodge star $\star$ (which plays the role of γ_5) leads one to introduce a dual lattice, a real nightmare for solving doubling because then you have the fermions on one lattice K and the doublers on the other, on L. This feature is unavoidable because the $\star$ maps a p dimensional region to a $(4-p)$ dimensional region. The property $\star\star = 1$ imposes one-to-one for the discrete version and hence the dual complex. However, we have seen how the K and L fields separate cleanly. To advance further, we need to analyse the tools at our disposal. This will lead us to consider the space $\mathcal{L}$ of Chapter 3.

As we pointed out, the lack of associativity of the wedge is a problem since it prevents the associativity of the Clifford product $\vee$ which plays the role of the multiplication operation in the associated Clifford algebra. That property is required in order to isolate one Dirac equation via a projection [4]. We will discuss the proposed solution to this problem.

Again, on the issue of a discretised Clifford product, the recent developments of non-commutative geometry and the setup of a discrete differential geometry [90, 91] provide some clue [66]. A model for fermions has been given along these lines [51]. However, since we just want fermions on a Euclidean lattice, we choose to stick to the original description of DK fermions.

In Chapter 3, a discretisation of differential geometry was carried out, and it motivated the introduction of a product which allowed us to describe the contraction operation. We were then able to verify that by replacing the usual collection of operations $(\wedge, d, \star)$ acting on $C(K)$ with the new "kit" $(i_v, \wedge, d, \star)$ acting on the product space of $C(K)$, we found that all the algebraic requirements were satisfied. Since we have a discretised theory with a Lie derivative analogue, it is not surprising that it plays a role in capturing the global $SU(4)$ symmetry of the DK equation.

We propose to replace a discretisation of *space* out of which fields are specified at vertices of a regular lattice by a discretisation of *fields* for which the underlying discretised space gives support to a finite collection of continuum fields. We should clarify these ideas below. We clarify the starting point of the program by introducing the scheme concretely, covering the $SU(4)$ algebra and in turn the reduction of the Dirac-Kähler equation to the Dirac equation.

Before we proceed, we ask: what can we aim for ?

1) A finite theory with a finite subspace of the continuum fermion fields, with exact algebraic properties that can be embedded in the continuum: Impossible, because the theory is not fundamentally a discrete one (the non-commutative geometry presents an alternative).

2) A finite theory with a finite subspace of approximated fermionic fields with exact algebraic properties: the present picture.

6.1 Three levels, one theory

Let us think in terms of the following three levels; they are:

I) The continuum theory of fermionic modes ψ_k which is not within reach of the discrete theory.

II) The continuum analogue of the discrete theory with fermion fields represented approximately by a finite basis list of differential forms known as Whitney elements.

III) The discrete theory of chains $\hat{\psi}_k$.

$$\hat{\psi}_k = \sum_i [\sigma_i^{(r)}] \int_{\sigma_i^{(r)}|} \psi_k, \tag{6.2}$$

i.e the assignment of coefficients to hyper-cubes.

The second level is really an approximation to the fermionic modes. The convergence is good because of the piecewise linear coefficients of the differential forms extracted using the Whitney map. Convergence may be defined by the L^2 norm:

$$\lim_{a\to 0} ||W\hat{\psi}_k - \psi_k|| \to 0. \tag{6.3}$$

It is not the best way to present the convergence since it looks like an order a (i.e edge lenght) convergence but we can also compare directly $W(\hat{\psi}_k)$ and ψ_k before integrating. Considering

$$e^{ikx}e^{iky} = (1 + ikx)(1 + iky) + O(x^2, y^2), \tag{6.4}$$

with

$$x \mapsto i\frac{x}{k}, \tag{6.5}$$

and similarly for y, then, it is easy to obtain a Whitney form. We see that the first part is in the range of the Whitney map for a hyper-cubic complex after complexifying the fields. This was shown in Chapter 4. So there is a discretisation of the fields and it is at that level that we need a certain number of exact properties which occupy the rest of the discussion. There are two limitations of the original scheme which pose a challenge for our purpose regarding the discretised wedge and consequently the Clifford product. They are:

1) The discrete wedge is non-associative. There is a leading scalar factor which prevents the property.

2) The discrete wedge is approximate. From the point of view of the Whitney elements, the function coefficients after wedging do not match.

These issues have been addressed in Chapter 3, the resolution is given by the space $\mathcal{L}$ which we are now apply to the problem of spinors. We now turn to the discretisation scheme we propose.

6.2 The Clifford product

To carry out the reduction, we require the following: 1) The Dirac Kähler equation is invariant under application of right multiplication with a constant differential. 2) Given the projectors $P^{(b)}$ and a solution ψ to the Dirac Kähler equation, the field $\psi \vee P^{(b)}$ satisfies the Dirac equation.

What we will find is that after the appropriate definition is made, the Clifford product we introduce does lead to 1) and 2) however we require some exactness and closure properties for this product. The closure property is important to formulate the $SU(4)$ symmetry, and is required to leave invariant the space of chains (not the product space) when Clifford multiplication with a projector $P^{(b)}$ is involved. We will derive this property. Another remark we already alluded to, is that the Clifford product is (after applying the technique of representation theory) the product in

$SU(4)$, and we have argued that the geometric dicretisation needs to be supplemented with the contraction operation in order to capture symmetries. The specific form of the Whitney elements played a role in establishing the rules that lead us to a well-defined discrete version of contraction

$$i_\alpha \sigma = \alpha \rfloor^K \eta(\sigma), \tag{6.6}$$

with the correct properties that lead us in turn to a discrete version of

$$L_\alpha \sigma = (i_\alpha d + d i_\alpha)\sigma. \tag{6.7}$$

In the spinorial context, provided σ is a Killing vector, which is trivial here, one can introduce a Lie for spinors, and write

$$L^f_\sigma = [i_\sigma, d^K]. \tag{6.8}$$

This indicate that the extra structure associated with the contraction operation is to play a role when we consider the Clifford product.

Let us turn to constant differentials. The general expression that we found for the Whitney elements is such that a constant differential can be expressed as a sum of Whitney elements by taking the sum of all the ones which have the same form part within a given cell. So let

$$P^{(b)}_K = \sum_{k,k_r} W(P^{(b)}_{k,\,k_r}) = W \sum_\theta P^{(b)}_{\theta,K}, \tag{6.9}$$

where $P^{(b)}_{k,\,k_r} \in C^{(k_r)}(K)$. The last expression does not make the various chain degrees explicit and is used below. Now we express the Clifford product in the following way:

$$\Phi \vee \Psi = \sum_p (-1)^{p(p-1)/2}((-1)^p (i_\partial)^p \Phi) \wedge (i_\partial)^p \Psi. \tag{6.10}$$

The interior product is defined by

$$i_X\omega = \sum_r (-1)^r X^{\mu_r}\omega_{\mu_1\ldots\mu_p} dx^{\mu_1}\wedge\ldots \hat{dx^{\mu_r}}\wedge\ldots dx^{\mu_p}. \tag{6.11}$$

The interior product maps a p-form to a $(p-1)$-form so using the notation $(i_\partial)^p$ to indicate p-products of the interior product, we obtained the formula above. For example

$$(i_\partial)^{(3)} = i_{\partial_{\mu_1}} i_{\partial_{\mu_2}} i_{\partial_{\mu_3}}. \tag{6.12}$$

In the summation, we specify that one should take an ordered r-tuple of coefficients $\mu_1 < \ldots < \mu_r$. We show some examples:

$$dx^2 \vee dx^3 = dx^2 \wedge dx^3 = -dx^3 \vee dx^2, \tag{6.13}$$

$$dx^4 \vee dx^4 = (i_{\partial_4} dx^4) \wedge (i_{\partial_4} dx^4) = 1 \vee 1 = 1, \tag{6.14}$$

such manipulations give

$$dx^\mu \vee dx^\nu = \delta^{\mu\nu} + dx^\mu \wedge dx^\nu. \tag{6.15}$$

The book-keeping is easy and leads to an associative and distributive product. This provides us with a representation of the Clifford algebra. The role played by the wedge product is explicit in the continuum equation Eq. 6.10 and leads provisionally to the following definition of the discrete Clifford product:

$$A(W(\sigma)\vee W(\eta)) = \sigma \vee^K \eta. \tag{6.16}$$

The approximate nature of the wedge which we have mentioned is a challenge which limits the range of applicability of the formula. To remedy this problem, we proceed in a way similar to the construction of the discrete interior product, by considering the product space of chains.

Definition : Given two chains σ and η, the discretised Clifford product is defined by

$$\zeta = \sigma \vee^K \eta, \tag{6.17}$$

where $\zeta \in C^\star(K) \times C^\star(K)$ and

$$W(\zeta) = W(\sigma) \vee W(\eta). \tag{6.18}$$

Note that the Whitney map as applied to a product element $\sigma\rfloor^K\eta$, is such that

$$W(\sigma\rfloor^K\eta) = (\varphi^{(0)}\, W(\sigma)) \wedge W(\eta). \tag{6.19}$$

The map $\varphi^{(0)}$ extracts the function coefficient of the form $W(\sigma)$ thus extending the space of forms within reach of the discretised theory (see Chapter 3). For the present purpose we consider an example. Take the Whitney element

$$W([01]) = (1-y)dx, \tag{6.20}$$

in two dimensions for the square [0123]. We are using the Euclidean δ_{ij} metric and after contraction with the vector

$$v = (1-y)\frac{\partial}{\partial x}, \tag{6.21}$$

which is represented as a one-chain we obtain,

$$i_{[01]}[01] = [01]\rfloor^K([0]+[1]), \tag{6.22}$$

and under the modified Whitney map we obtain,

$$i_v W([01]) = (1-y)^2, \tag{6.23}$$

which is the correct result. In fact, on the complex, our knowledge of these mappings forces us to suggest the discrete analogue to be $[0]+[1]$ as

$$W([0]+[1]) = (1-x)(1-y) + x(1-y) = (1-y). \tag{6.24}$$

When calculating $\sigma^{(p)} \vee^K \sigma^{(q)}$, we find that in all cases in which at least one of the two chains is mapped to a constant form the result can be reproduced exactly on the a single chain space (i.e not on the product space) by choosing the appropriate linear combination of chains as the result[1]. However, in general, one needs the product space of chains, as defined in Chapter 3 to express the result. We have for example

$$W([01]\rfloor^K([03]+[12])) = (1-y)dy, \tag{6.25}$$
$$W([01]\rfloor^K([0]+[1])) = (1-y)^2, \tag{6.26}$$

which in turn means that with the definition of the discretised Clifford product:

$$[01] \vee^K [0123] = [01]\rfloor^K([03]+[12]), \tag{6.27}$$
$$[01] \vee^K [01] = [01]\rfloor^K([0]+[1]), \tag{6.28}$$

which are correct. What we also need, is the Clifford product of two constant differentials, namely of two one-forms in order to capture the Clifford algebra. Suppose we picked $[0\mu]$ to be the edge parallel to the x^μ-axis. Then, we write

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \tag{6.29}$$

in the specific representation we get the following version of the algebra

$$\{dx^\mu, dx^\nu\}_\vee = dx^\mu \vee dx^\nu + dx^\nu \vee dx^\mu. \tag{6.30}$$

So the key is to consider symbolically within the four dimensional hyper-cube $\sigma_k^{(4)}$

[1]Look at the table of Whitney forms in the Appendix to Chapter 3.

the following chains:

$$\sigma_\mu = \sum_{\{i<j\,|\,[ij]<\sigma_k^{(4)}\}} [ij], \tag{6.31}$$

leading to

$$W\sigma_\mu = dx^\mu. \tag{6.32}$$

Then by application of the definition, we get

$$\sigma_\mu \vee \sigma_\nu = \delta_{\mu,\nu} \sum_{\{[i]<\sigma_k^{(4)}\}} [i], \tag{6.33}$$

since it is the unique linear combination of chains that when mapped to the space of forms gives unity. We now have the Clifford algebra exactly.

6.3 The Dirac-Kähler equation in the space $\mathcal{L}$

We have seen that there are three levels, the continuum DK equation is one, next, a second level is that of the DK equation on the space of Whitney forms (which we refer to as the W-DK equation):

$$(d - \delta + m)W(\sigma) = 0, \tag{6.34}$$

where σ is an inhomogeneous chain. At the third level there is a complex based equation

$$(d^K - \delta^K + m)\sigma = 0. \tag{6.35}$$

The relation between the solutions of the three equations, which are the DK equation itself and the two versions based on the discretised theory, Eq. 6.34 and Eq. 6.35 is crucial. The last two are related up to an approximation which comes from the co-boundary for which

$$\delta W \neq W\delta^K, \tag{6.36}$$

and leads to an approximation for the derivative which makes the result not exact, in contrast to d^K. The obvious example is in 2D where two squares σ_1 and σ_2 have the common edge boundary σ_0 then, starting with the continuum field Ψ taken to be a two-form, we obtain

$$[\psi] = A(\Psi) = [\sigma_1] \int_{\sigma_1} \Psi + [\sigma_2] \int_{\sigma_2} \Psi. \tag{6.37}$$

and

$$\delta^K[\Psi] = [\sigma_0](\int_{\sigma_1} \Psi - \int_{\sigma_2} \Psi). \tag{6.38}$$

which is a finite difference approximation (It is easy to visualize this problem in Fig. 6-1).

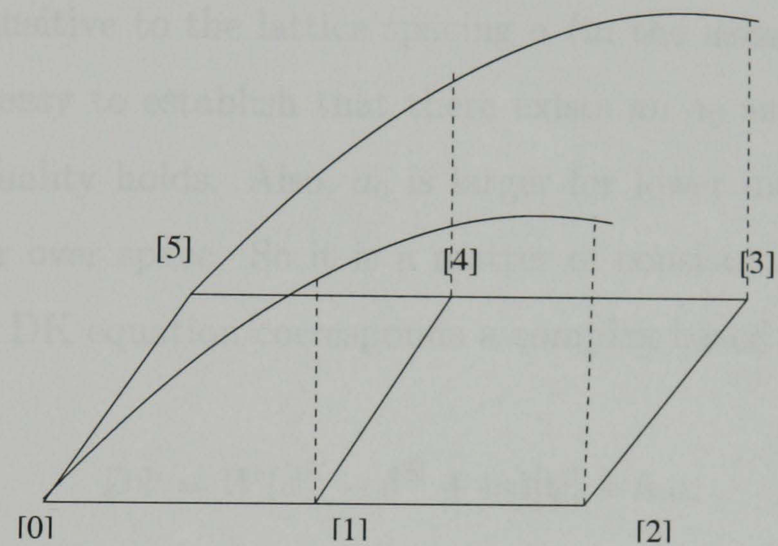

Figure 6-1: 2-form field over a square lattice.

We can write symbolically that

$$\lim_{a \to 0} W(d^K - \delta^K + m) = (d - \delta + m). \tag{6.39}$$

In the limit, looking at the equation and the figure, the derivative δ converges to the continuum limit on the edge σ_0 by virtue of the mean value theorem. Convergence is necessary but our requirements are stronger and we insist on having exact properties in the approximate theory.
So what do we mean by solving the equation ?
Start with the zero-form equation and its analogue zero-chain equation. The given fermionic mode has a representative as a chain

$$[\psi] = A\Psi. \tag{6.40}$$

To proceed, we start with the K-DK equation, the discrete DK equation on the chain space:

Definition : The complex based Dirac field is defined by

$$(d^K - \delta^K + m)[\psi] = 0. \tag{6.41}$$

This is satisfactory if after the approximation is taken into account, we have

$$||\not{D}W([\psi])||_{L_2} = \min_{[\psi']} ||\not{D}W([\psi'])||_{L_2}. \tag{6.42}$$

This condition is sensitive to the lattice spacing a (in the usual way, where a is the edge length). It is easy to establish that there exists an a_0 such that $\forall a$ satsifying $0 < a < a_0$, the equality holds. Also, a_0 is larger for lower modes as the variation of the field is slower over space. So it is a matter of consistency that to a solution Ψ of the continuum DK equation corresponds a complex based solution $[\psi]$ given by Eq. 6.40. Then

$$\not{D}\Psi = W(d^K - \delta^K + m)[\psi] + h.o. \tag{6.43}$$

where $h.o$ refers to higher order terms in a. To summarize, given a solution to the discretised equation on the chain space Eq. 6.41 it satisfies the condition Eq. 6.42. The point is that the failure of Eq. 6.36 to hold is controlled by the following limit

$$\lim_{a\to 0} WA = Id. \tag{6.44}$$

6.3.1 SU(4) invariance I: the proof

Let us reformulate the $\vee P^{(b)}$ invariance by means of a commutative diagram:

$$\begin{array}{ccc} \Psi & \xrightarrow{\vee P^{(b)}} & \Psi' \\ \downarrow A & & \downarrow A \\ \psi & \xrightarrow{\vee^K P^{(b)}} & \psi' \end{array} \tag{6.45}$$

If these transformations fit in this diagram then we can assert that the invariance has an exact discrete counterpart. We have already pointed out how the action of $\vee P^{(b)}$ is exact on Whitney elements because the terms in $P^{(b)}$ are constant differentials. By virtue of the continuum invariance, we find that the invariance of the discrete DK equation holds.

The global $SU(4)$ invariance is embodied in the condition

$$[\not{D}](\Psi \vee P^{(b)}) = ([\not{D}]\Psi) \vee P^{(b)} = 0. \tag{6.46}$$

Now, using the chain expression for $P^{(b)}$ given by Eq. 6.9 and the approximated fermionic field $\Psi = W[\psi]$, and the definition of the discrete Clifford product,

$$\Psi \vee P^{(b)} = (W[\psi]) \vee (W \sum_{\theta} P^{(b)}_{\theta,K}), = W\big([\psi] \vee^K \sum_{\theta} P^{(b)}_{\theta,K}\big), \tag{6.47}$$

we unwind the definitions to find

$$\begin{aligned}
&(\not{D}(W[\psi])) \vee W[\sum_{\theta} P^{(b)}_{\theta,K}] \\
=&\not{D}(W[\psi] \vee W[\sum_{\theta} P^{(b)}_{\theta,K}]), \\
=&\not{D} \sum_{p \geq 0} \frac{\text{sign }(p)}{p!} \left(\eta^p(e_{\mu_1}\rfloor \dots e_{\mu_p}\rfloor [W[\psi]])\right) \wedge \left(e^{\mu_1}\rfloor \dots e^{\mu_p}\rfloor [W \sum_{\theta} P^{(b)}_{\theta,K}]\right), \\
=&\not{D} W\left[[\psi] \vee^K \sum_{\theta} P^{(b)}_{\theta,K}\right],
\end{aligned} \tag{6.48}$$

by application of the continuum invariance. The key step is on the second line where we move $\not{D}$ to the left. The discrete transformation $\vee P^{(b)}$ is the exact analogue of the associated continuum transformation.

The DK-equation (in components in the Appendix) leads to four independent equations out of the sixteen and this in turn gives the Dirac equation for a four component spinor.

$$(\not{\partial} + m)\psi = 0. \tag{6.49}$$

This procedure is called reduction and we now turn to its discrete analogue.

6.3.2 Reduction to the Dirac fields

Symbolically, the left-hand side of the DK equation on the space of chains is:

$$(d^K - \delta^K + m)[\psi] = \sum_\sigma f(\sigma)[\sigma], \tag{6.50}$$

where $f(\sigma)$ are the coefficients, and the sum runs over all chains of all degrees. The projection operators are given by

$$P^{(b)} = \frac{1}{4}(1 + i\text{sign}_1 dx^1 \wedge dx^2) \vee (1 + \text{sign}_2.\epsilon), \tag{6.51}$$

where the pair of signs (sign_1, sign_2) are given by $(-,-)$, $(+,-)$, $(-,+)$ and $(+,+)$ respectively for $b = 1, 2, 3, 4$.

Let us display how we do the continuum reduction. We start with the most general inhomogeneous differential form in 4D.

$$\Psi = f^0 + f^1_\mu dx^\mu + f^2_{\mu\nu} dx^\mu \wedge dx^\nu + f^3_{\mu\nu\sigma} dx^\mu \wedge dx^\nu \wedge dx^\sigma + f^4_{1234} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \tag{6.52}$$

with the appropriate summation. Next, we choose one of the projectors, Eq. 6.51, i.e we specify a pair of signs; take $P^{(0)}$ and solve the equation

$$\Psi \vee P^{(0)} = \Psi. \tag{6.53}$$

There are sixteen coupled equations, one for each basis elements starting with 1 and ending with $dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$. However we notice that they can be grouped in four independent subsystems, for example one of them is

$$f^0 - f^4_{1234} + if^2_{12} + if^2_{34} = 4f^0, \tag{6.54}$$

$$f^1_1 - f^3_{234} + if^1_2 + if^3_{341} = 4f^1_1, \tag{6.55}$$

$$f^3_{134} + f^1_2 + if^3_{234} - if^1_1 = 4f^3_{134}, \tag{6.56}$$

$$f^4_{1234} - f^0 - if^2_{34} - if^2_{12} = 4f^4_{1234}. \tag{6.57}$$

These four equations are actually the first and the last two of the list of sixteen. They can be solved by specifying one parameter $f^0 = \psi_1$ and determining all the others accordingly. The reason for such a choice of solution is simply that since we have four independent subsystems, we have at least one free parameter in each of them, so if one of them has two free parameters, this will lead to at least five independent parameters. This is too many components for a spinor in 4 dimensions.

After considering the other three subsystems of equations, we find in total four independent parameters (ψ_1, ψ_2, ψ_3, ψ_4) which give us a $P^{(0)}$-invariant field, namely

$$\begin{aligned}\Psi = &\psi_1(1 - dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 - idx^1 \wedge dx^2 - idx^3 \wedge dx^4) \\ &+ \psi_2(dx^1 - idx^2 - dx^2 \wedge dx^3 \wedge dx^4 - idx^3 \wedge dx^4 \wedge dx^1) \\ &+ \psi_3(dx^3 - idx^4 - idx^1 \wedge dx^2 \wedge dx^3 - dx^1 \wedge dx^2 \wedge dx^4) \\ &+ \psi_4(dx^2 \wedge dx^4 - dx^1 \wedge dx^3 + idx^1 \wedge dx^4 - idx^3 \wedge dx^2). \end{aligned} \tag{6.58}$$

As we pointed out above, in the continuum formulation, a solution of the Dirac-Kähler equation with this ansatz can be found, which gives a four component Dirac field. In turn, the identification of the Dirac equation can be made after noting that it is a question of rewriting to find:

$$d - \delta = dx^\mu \vee \partial_\mu, \tag{6.59}$$

together with the representation which sets $dx^\mu \vee = \gamma^\mu$. Let us return to our task. Recall that we were searching for a relation between levels II and III of the description of spinors. To make this relation explicit we proceed as follows: consider the functions ψ_k, that is scalar functions which are expressed approximately as zero chains

$$\psi_k = \sum_i \psi_k(i)[i]. \tag{6.60}$$

They are the most general function we can write within the framework. Then we see that due to the use of Whitney elements, the function ψ_k is modified, according to the formulae for the Whitney elements. So with

$$(d^{(r-1)} - \delta^{(r+1)} + m)W(\Psi) = 0, \tag{6.61}$$

you see immediately that with the Ψ given in the chiral representation, we obtain various Whitney forms for different degrees. This is fortunately not a problem since one can multiply the equation by the missing multiplicative function factor and get

$$c^{(r)}(\sigma)(d^{(r-1)} - \delta^{(r+1)} + m)W(\Psi) = 0, \tag{6.62}$$

and so we do have the original function ψ_k in the equation (i.e with all its factors). It is superfluous to include this function since the solutions of the Dirac-Kähler equation are not modified by it as $c(r)$ is not zero. Let us gather our findings.

Dirac-Kähler equation : Given the field Ψ in the chiral representation, it is approximated by a zero-form Whitney element and the associated equation

$$(d - \delta + m)\Psi = 0, \tag{6.63}$$

is equivalent up to higher order term, to the equation

$$(d - \delta + m)W(\psi) = 0. \tag{6.64}$$

The solutions of the discretised equation are in one to one relation with the various modes of the Dirac equation and the invariance of the Dirac-Kähler equation under the $SU(4)$ action is exact at the level of the discretised theory. We are able to express the field in the chiral representation in such a way that the field Ψ can be recast as a four function components spinor as given in the Appendix.

6.3.3 SU(4) invariance II: Exactness of the group action

We would like to insist on the closure of the operation $P^{(b)}\vee$ on the inhomogenous chain: 1) Fermion fields are spanned by a finite set of continuous fields (the Whitney elements) over a hyper-cubic complex. It follows then that the Hilbert space of fields is generated by a finite vector space. The norm is defined to be the L^2 norm given by

$$\|\omega\|^2 = (\omega, \omega) = \int_M \omega \wedge \star\omega. \tag{6.65}$$

The basis set is

$$\mathcal{H}_P = \{\psi \in \bigoplus_r Span_{\mathbb{C}} W^{(r)}\}. \tag{6.66}$$

Next, the "transformed set":

$$\mathcal{H}_T = \{\psi \in \bigoplus_r Span_{\mathbb{C}} W^{(r)} \times Span_{\mathbb{C}} W^{(r)}\}. \tag{6.67}$$

But it turned out that in the case of the projectors, which have the virtue of being constant differentials, we can identify:

$$\mathcal{H}_{T(P^{(b)})} \overset{Id}{\longmapsto} \mathcal{H}_P. \tag{6.68}$$

This is the more stringent closure condition that we required. We have encountered this map as we noted that a constant form extracted from a chain adjoined as we did above amounts to multiplying the various by the appropriate constant coefficient to the various terms after the Clifford product has been done (this is reminiscent of the discussion of the Jacobi identities in Chapter 3). So the group action on the complex is given by

$$SU(4) \times C^{\star}(K) \longrightarrow C^{\star}(K), \tag{6.69}$$

$$(P^{(b)}, \sigma) \longmapsto \sigma'. \tag{6.70}$$

The closure of the group action leaving the finite space $\mathcal{H}_P$ invariant guarantees that it is indeed a group, for instance that each element has an inverse.

2) We maintain exact properties, notably the $SU(4)$ invariance.

3) Exactness of the properties is equivalent to closure of the theory. Although we adjoin various chains in the operations, we maintain the invariance which is what

matters.

6.4 Free propagator: outline.

The functional form of the propagator was given in Chapter 5, We are now in a position to write it more explicitly, starting with

$$Z = \int [d\bar{\phi}][d\phi] e^{-\int_M \bar{\phi} \vee P^{(b)} \wedge \star (d-\delta+m)\phi \vee P^{(b)}}. \tag{6.71}$$

If $\lambda(\sigma)$ is the coefficient for the cell σ, we actually have a simple Lebesgue integral where D is the dimensionality of the highest degree simplex, and N_r the number of hypercubes of that type,

$$\int [d\bar{\phi}][d\phi] = \Pi_{r=0}^{D} \Pi_{i_r=0}^{N_r} \int_{\mathbb{C}} \int_{\mathbb{C}} d\lambda_i d\bar{\lambda}_i. \tag{6.72}$$

Now let us turn to the main issue which is the calculation of the action. We will do it for $P^{(0)}$. Let

$$\sigma^{(b)} = \sigma \vee P^{(b)} = \sum_{r=0}^{D} \sum_{i_r=0}^{N_r} \lambda_{i_r} [\sigma_i^{(r)}]. \tag{6.73}$$

Recalling the discussion of the projectors and the fact that the only pair of chains that give a non-zero value under the bilinear form are identical ones:

$$\int_M W\sigma_i \wedge W \star \sigma_j = \delta_{ij}. \tag{6.74}$$

Then,

$$\int_M W(\sigma^{(b)}) \wedge W \star [(d-\delta+m)\sigma^{(b)}] = \lambda_i^{(r)} M_{r,p}^{i,j} \lambda_j^{(p)} \tag{6.75}$$

The matrix M remains to be calculated, but all the work is already done. The calculation of the Dirac-Kähler term of the action is simple after recalling that δ acts like the boundary operator at the complex level. The matrix is therefore sparse. So much for the theoretical model, a computational implementation, time permiting, would start at this point.

6.5 Summary

It was our intention here to propose new technical tools to think about discretised fermion theories. We found that capturing the Clifford algebra comes at the cost of introducing extra structure in a similar way to what has been done in Chapter 3 for the discretisation of differential geometry. Our approach to the Clifford product is new, and has the merit of being faithful to the continuum while being defined on a finite space. This means that there is closure in the cases of interest.

The fact that we can carry out the reduction is a new result. It does not mean however that we solve the fermion doubling problem, this would be a dangerous claim. It would require more work to settle how we stand in relation to the problem. We have more algebraic properties but the full QCD should be considered to answer the question. Chirality still implies a doubling of spaces and so one might still say a doubling of fields. We will address these issues in future work.

Other approaches based on the non-commutative geometry have been developed regarding treatment of the fermion problem [92].

6.6 Appendix

The explicit form of the projectors which used (which lead to a different version of Ψ:

$$P^{(1)} = \frac{1}{4} - \frac{1}{4}dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 - \frac{1}{4}idx^1 \wedge dx^2 - \frac{1}{4}idx^3 \wedge dx^4, \tag{6.76}$$

$$P^{(2)} = \frac{1}{4} - \frac{1}{4}dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + \frac{1}{4}idx^1 \wedge dx^2 + \frac{1}{4}idx^3 \wedge dx^4, \tag{6.77}$$

$$P^{(3)} = \frac{1}{4} + \frac{1}{4}dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 - \frac{1}{4}idx^1 \wedge dx^2 + \frac{1}{4}idx^3 \wedge dx^4, \tag{6.78}$$

$$P^{(4)} = \frac{1}{4} + \frac{1}{4}dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + \frac{1}{4}idx^1 \wedge dx^2 - \frac{1}{4}idx^3 \wedge dx^4. \tag{6.79}$$

The general Ψ considered by Rabin is:

$$\begin{aligned}\Psi = &\frac{1}{4}\psi_1(1 + idx^2 \wedge dx^3 - idx^1 \wedge dx^4 + dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4)\\ &+ \frac{1}{4}\psi_2(idx^3 \wedge dx^4 - idx^1 \wedge dx^2 - dx^2 \wedge dx^4 - dx^1 \wedge dx^3)\\ &+ \frac{1}{4}\psi_3(idx^1 - dx^4 - idx^2 \wedge dx^3 \wedge dx^4 - dx^1 \wedge dx^2 \wedge dx^3)\\ &+ \frac{1}{4}\psi_4(-dx^2 + idx^3 - dx^1 \wedge dx^3 \wedge dx^4 - idx^1 \wedge dx^2 \wedge dx^4) \qquad (6.80)\end{aligned}$$

Let us express the Dirac-Kähler equation in components where the input is the Ψ given by Rabin. This calculation is not given in his article so it is interesting to have it here for the purpose of the discussion.

$$
\begin{aligned}
&(d-\delta+m)\Psi \\
&= \left[-\frac{1}{4}(\partial_0\psi_3 i - \partial_3\psi_3 + \partial_1\psi_4 + i\partial_2\psi_4) + \frac{1}{4}m\psi_1\right] \\
&+ \left[\frac{1}{4}\partial_0\psi_1 - \frac{1}{4}(i\partial_3\psi_1 + i\partial_1\psi_2 + \partial_2\psi_2) + \frac{1}{4}m\psi_3 i\right] dx^0 \\
&+ \left[\frac{1}{4}\partial_1\psi_1 - \frac{1}{4}(-\partial_2\psi_2 i - i\partial_0\psi_2 - \partial_3\psi_2) + \frac{1}{4}m(-\psi_4)\right] dx^1 \\
&+ \left[\frac{1}{4}\partial_2\psi_1 - \frac{1}{4}(\partial_1\psi_1 i + \partial_3\psi_2 i - \partial_0\psi_2) + \frac{1}{4}mi\psi_4\right] dx^2 \\
&+ \left[\frac{1}{4}\partial_3\psi_1 - \frac{1}{4}(-i\partial_0\psi_1 + \partial_2\psi_2 i - \partial_1\psi_2) + \frac{1}{4}m(-\psi_3)\right] dx^3 \\
&+ \left[\frac{1}{4}(-i\partial_1\psi_3 - \partial_0\psi_4) - \frac{1}{4}(-\partial_2\psi_3 - i\partial_3\psi_4) + \frac{1}{4}m(-i\psi_2)\right] dx^0 \wedge dx^1 \\
&+ \left[\frac{1}{4}(-i\partial_2\psi_3 + i\partial_0\psi_4) - \frac{1}{4}(-\partial_1\psi_3 - \partial_3\psi_4) + \frac{1}{4}m(-\psi_2)\right] dx^0 \wedge dx^2 \\
&+ \left[\frac{1}{4}(-i\partial_3\psi_3 - \partial_0\psi_3) - \frac{1}{4}(+\partial_2\psi_4 + i\partial_1\psi_4) + \frac{1}{4}m(-i\psi_1)\right] dx^0 \wedge dx^3 \\
&+ \left[\frac{1}{4}(\partial_2\psi_4 + i\partial_1\psi_4) - \frac{1}{4}(-i\partial_3\psi_3 - \partial_0\psi_3) + \frac{1}{4}m(i\psi_1)\right] dx^1 \wedge dx^2 \\
&+ \left[\frac{1}{4}(-\partial_1\psi_3 + \partial_3\psi_4) - \frac{1}{4}(i\partial_2\psi_3 - i\partial_0\psi_4) + \frac{1}{4}m(-\psi_2)\right] dx^1 \wedge dx^3 \\
&+ \left[\frac{1}{4}(-\partial_2\psi_3 - i\partial_3\psi_4 - \frac{1}{4}(-i\partial_1\psi_3 - \partial_0\psi_4) + \frac{1}{4}m(\psi_2 i)\right] dx^2 \wedge dx^3 \\
&+ \left[\frac{1}{4}(i\partial_0\psi_1 - i\partial_2\psi_2 + \partial_1\psi_2) - \frac{1}{4}(-\partial_3\psi_1) + \frac{1}{4}m(-\psi_3)\right] dx^0 \wedge dx^1 \wedge dx^2 \\
&+ \left[\frac{1}{4}(i\partial_1\psi_1 - i\partial_3\psi_2 - \partial_0\psi_2) - \frac{1}{4}(\partial_2\psi_2) + \frac{1}{4}m(-i\psi_4)\right] dx^0 \wedge dx^1 \wedge dx^3 \\
&+ \left[\frac{1}{4}(-\partial_2\psi_3 - i\partial_3\psi_4 - \frac{1}{4}(-i\partial_1\psi_3 - \partial_0\psi_4) + \frac{1}{4}m(\psi_2 i)\right] dx^0 \wedge dx^2 \wedge dx^3 \\
&+ \left[\frac{1}{4}(i\partial_3\psi_1 + i\partial_1\psi_2 + \partial_2\psi_2) - \frac{1}{4}(\partial_0\psi_1) + \frac{1}{4}m(-i\psi_3)\right] dx^1 \wedge dx^2 \wedge dx^3 \\
&+ \left[\frac{1}{4}(-i\partial_0\psi_3 + \partial_3\psi_3 + \partial_1\psi_4 - i\partial_2\psi_4) + \frac{1}{4}m\psi_1\right] dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \qquad (6.81)
\end{aligned}
$$

Chapter 7

Discussion of gauging

7.1 Minimal coupling of gauge fields to the fermionic fields

The standard gauge transformations (see [51] for a discussion of gauging the DK-theory) for fermions are

$$\psi \stackrel{G}{\longmapsto} G\psi, \tag{7.1}$$

$$\bar{\psi} \stackrel{G}{\longmapsto} \bar{\psi}G^{-1}, \tag{7.2}$$

where G is a Lie group valued zero-form. and for the connection:

$$A \stackrel{G}{\longmapsto} GAG^{-1} + GdG^{-1}, \tag{7.3}$$

which has a local version of particular interest written as

$$\delta_v = d + [\,.\,, v], \tag{7.4}$$

where v is in the Lie algebra, tangent to the identity of the gauge group, and by expansion, we have

$$G = 1 - v. \tag{7.5}$$

The parallel transporter U or link variable, which is inserted in the lattice action to gauge it, transforms as

$$U \mapsto GUG^{-1}, \tag{7.6}$$

and lives in the groupoid formed by parallel transport along the links over all possible paths [87], it is based on the group G, and the infinitesimal or local gauge transformation are Lie algebra valued. Let us turn to the discussion of gauging in relation to the techniques of lattice QCD. In the DK picture the global $SU(4)$ symmetry is linked to the specific representation of fermions using the Clifford Product, and it has been argued that one could attempt to make it into a local "flavour" gauge symmetry [4].

However, we do not view the Dirac-Kähler equation as the fundamental equation in the theory, but rather as a way to extract the Dirac equation. So, we add gauge labels to the coefficients of the inhomogeneous differential forms, to obtain a Lie algebra $\mathbf{g}$-valued form.

The first observation is that the group action on the gauge labels should leave the reduced spaces invariant. That is if Ψ_{DK} is a solution of the Dirac Kähler equation, then it is a solution of the Dirac equation and satisfies

$$\Psi_{\mathrm{DK}} = P^{(b)}\Psi_{\mathrm{DK}}, \tag{7.7}$$

for a single value of b. Then in turn we require (G as in Eq. 7.1) that

$$G\Psi_{\mathrm{DK}} = P^{(b)}G\Psi_{\mathrm{DK}}. \tag{7.8}$$

In the continuum language one would say that reduction (which is an action of $SU(4)$ on Ω) commutes with gauging the free fermions. Actually, this is trivial to establish after noting that G is a Lie group valued zero form and the $P^{(b)}$ are a sum of constant differentials. So it is obvious that the two actions commute, G only acts on the gauge labels and does not affect the degree of the forms.

The well-known prescription that leads to the minimal coupling of the fermion fields to gauge is to introduce the gauge fields $A_\mu(x)$ and consider the parallel transport operator from the vertex at x to the vertex at $x + \hat{\mu}$:

$$U_\mu(x) = \exp[ieaA_\mu(x)]. \tag{7.9}$$

The operator U is then inserted in the fermion bilinear term of the free Lagrangian. The invariance of the Lagrangian under gauge transformation follows by the usual manipulation of matrices and traces. In effect, we have coupled a zero-form (in the spinor representation of DK only) gauge field to an inhomogeneous form field. Furthermore, this guarantees that the $\mathcal{D}^{(b)}$ are left invariant as we have just seen.

In the usual Dirac-Kähler methods, one diagonalises the free Lagrangian leading to

$$L = a^d \sum_{r,\mu} \bar{\phi}(r) \frac{1}{2a} \frac{1}{2^{\frac{d}{2}}} [Tr\left[\left(\Gamma^{\frac{r}{a}}\right)^\dagger \gamma_\mu \Gamma^{\frac{r}{a}+\hat{\mu}}\right] \phi(r + a\hat{\mu}) - Tr\left[\left(\Gamma^{\frac{r}{a}}\right)^\dagger \gamma_\mu \Gamma^{\frac{r}{a}-\hat{\mu}}\phi(r - a\hat{\mu})\right]], \tag{7.10}$$

where r is the vector that locates vertices, $\hat{\mu}$ is the unit vector in the μ-direction and the Γ matrices are products of the ordinary γ matrices [68]. In this form Eq. 7.10, after taking the trace, has one component of ϕ (this procedure is called "thining"). At this point, one has the option of blocking the fields [69, 70]; this is done by introducing the so-called block coordinates and by means of a unitary transformation. One obtains a Dirac $\otimes$ flavour representation. The gauging is then done by inserting link variables in the Lagrangian Eq. 7.10.

The DK fermions, after thinning, may be identified with "staggered fermions" [70]. In which case, the gauging is done after the thinning. One obtains directly:

$$L^S = a^d \sum_{r,\mu} \bar{\phi}(r) \frac{1}{2^{\frac{d}{2}}} Tr\left[\left(\Gamma^{\frac{r}{a}}\right)^\dagger \gamma_\mu \Gamma^{\frac{r}{a}+\hat{\mu}}\right] \frac{1}{2a} [U_\mu(r)\phi(r + a\hat{\mu}) - U_\mu^\dagger(r - a\hat{\mu})\phi(r - a\hat{\mu})]. \tag{7.11}$$

The reduction we have described can be interpreted as the analogue of diagonalising and thinning, we reduce the degrees of freedom to obtain one fermion with $2^{\frac{d}{2}}$ spinorial components described in any given invariant subspace $\mathcal{D}^{(b)}$ by an action based on a

Lagrangian such as Eq. 7.11. However we can also be more general and consider

$$S_K = \langle \bar{\Phi}_K^{(b)}, \mathbf{U}\left[\left(d^K - \delta^K + m\right) \Phi_K^{(b)}\right] \rangle, \tag{7.12}$$

where $\mathbf{U} = \sum_l U_l$ is the displacement operator labeled by the degree of the cochain it acts on.

We are then tempted to follow the lattice QCD type of approach, i.e by inserting the parallel transport operator U in the action. The fact that the fields are chains of various degree does not a priori pose a problem.

There is however an issue of interpretation: are the inhomogenous form fields to be viewed as the actual physical fields, in the sense that they should be coupled to gauge fields of various dimensions?

In this case, one does not end up with the theory of fermions coupled to gluon fields, but rather with a generalisation.

Then, keeping the inhomogenous chains and coupling them by inserting the appropriate U between them by calculation the various $(p,\, q)$-cases of

$$(\sigma^{(r)},\, (d - \delta + m)\sigma^{(p)}). \tag{7.13}$$

In Fig. 7-1, we give a 2D example, where one might couple fields of various dimensions, so the way the link variables may be inserted in the action Eq. 7.14 is not obvious. Before we consider zero-form fermion fields, one can consider the parallel transport operator connecting simplices. However, the chains of degree higher than a vertex cannot be used as the basic fields in the theory (see [71]), since the gauge invariance of the resulting extended objects is not well-defined [1], as illustrated by the no-go theorem of Teitelboim [73].

Now let us take the connection as a one-form (gluon field), the interpretation is that the Whitney elements which are used before integration in the expression for

[1]The theory of higher dimensional connections opens new possibilities, see the discussion below and [72]

the action

$$S = \int_{|\sigma_i^N|} W(\sigma^{(r)}) \wedge W \star ((d - \delta + m)\sigma^{(p)}), \tag{7.14}$$

play a dummy role as functions, the constant differential form part being used to obtain the volume form (this is the way adopted in [4]) and is the one closest to the current lattice models.

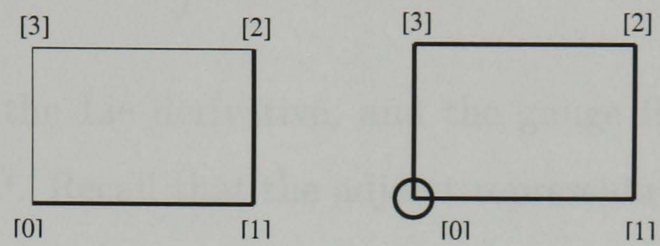

Figure 7-1: Two types of couplings: two edges with a vertex in common and the entire square with one vertex.

Differential geometry enables us to extract a zero form fermion field which can in turn be coupled to a one form gauge field. The simple formulae that enables us to change the degree of forms is

$$dx^\mu \vee \Phi^\mu = dx^\mu \wedge \Phi + i_{\partial_\mu}\Phi. \tag{7.15}$$

The discrete Clifford product (Chapter 6) enables us to handle exactly the Clifford product of any Whitney form with a constant form in a way that is exact. So in the space $\mathcal{L}$ the operation given by Eq. 7.15 is well-defined so we can rewrite any field Φ as a zero form; not in the space of chains, but in the product space of chains (as we have seen). It is then possible to consider an action of the form

$$(\Phi^\dagger, \Phi)_0 = 4 \sum_{a,\, b} \psi_a^{(b)}(x)^\star \psi_a^{(b)}(x)\, \epsilon, \tag{7.16}$$

where ϵ is the volume element $dx^1 \wedge \ldots \wedge dx^4$ which can be gauged by inserting the link variables.

Such models are somewhat disappointing because one ends up with the discrete version of the naive action for which doubling is well known. After a Fourier transform to momentum space, one finds spectrum doubling.

After discussing these partly satisfactory treatments to the minimal coupling we are tempted to propose the most natural one that follows after constructing the space $\mathcal{L}$. Namely, we do not attempt to consider the theory by inserting the link variable as a brute force approach. Instead, we consider the action

$$S = \int \bar{\Phi} \wedge \star(d - \delta + L_A)\Phi, \tag{7.17}$$

where L_A is obviously the Lie derivative, and the gauge fields A are taken in the appropriate Lie algebra[2]. Recall that the adjoint representation is effectively

$$L_A = [A, .]. \tag{7.18}$$

The observation that comes to mind is that this is the correct analogue of the continuum minimally coupled action with fields approximated by the Whitney map. So, we put to use the discreteness associated to the Lie algebra $\mathfrak{g}$. This approach is appealing, essentially because it is well-defined and exploits the construction of the discrete exterior calculus, the space $\mathcal{L}$, in the context of gauge theory. Applying the gauge transformations (see start of the chapter), is done in the usual way by multiplication in the algebra. Regarding the discrete symmetries of the action, the obvious one such as a discrete rotational symmetry are present.

An alternative description for the minimal coupling, which appears in the literature is via the Clifford product. Consider the gauge field

$$A = g\frac{\lambda}{2}A^a_\mu dx^\mu. \tag{7.19}$$

The inhomogeneous form field then is coupled to gauge as

$$(d - \delta)\Phi = iA \vee \Phi. \tag{7.20}$$

This is an interesting construction, as it identifies the Clifford product as multipli-

[2]This requires to separate the gauge part from the spinor part by taking the tensor product, we will return to this.

cation of fields, the latter live in the product of the Atiyah-Kähler bundle with the vector bundle of local color spaces.

Recent developments in Dirac-Kähler formulations of continuum QFT can be found in the context of the Weinberg-Salam model [74] and supersymmmetry [75].

Having the Lie derivative at our disposal, we are tempted to use a similar approach to the problem of the quadratic term of the action. In lattice approaches, it is not problematic and is generally done via the Wilson loop or holonomy as we proceed to discuss.

7.2 Fiber bundle

In order to consider gauge theory in the way that is mathematically well-defined, one needs to consider a principal gauge bundle.

$$\mathrm{Ad}\, P = P \times_{Ad} G. \tag{7.21}$$

We discuss the bundle construction, the common wisdom is that it is not possible to construct a discrete analogue of such a space (new approaches are being developed [76] and [2]), but there seem to be no reasons why one may not construct the relatively special case of a bundle besides the globally trivial bundle which is the product of two spaces. The first problem one encounters is that the base space has a topology which is special: if one associates open sets to simplices or hyper-cubes of all dimensions, their overlap $U^{(r)} \cap V^{(r)}$ is over a common face $T^{(r-1)}$, which is limited. For example, the theory has no well-defined analogue of $dT^{(r-1)}$ in the overlap open set which is of one degree less. A better approach is to set-up the gauge formalism with the tools we have.

7.2.1 The complex as the base space X

Importance of the local description

Recall that the fundamental objects here are differential forms. Moreover, one can consider the Lie algebra of the group of gauge transformations (which is the space of sections of AdP, see [17]), that is turning to the study of

$$\mathrm{ad}P = P \times_{\mathrm{Ad}G} \mathfrak{g}, \tag{7.22}$$

which is the local description of the group of gauge transformations. So we ought to consider the local section of the gauge bundle Eq. 7.21. Locality is defined as within the local cell of highest degree which is associated with the volume form in the theory. The way forward is to consider the local sections of the bundle. Every principal fiber bundle is locally trivial allowing us to choose a local section and in turn to write the connection ω, not as a form on P but rather as a form on the base space M. This leads to the gauge potential A which can be written as

$$A = A^{\alpha}_{\mu} X_{\alpha} dx^{\mu}. \tag{7.23}$$

so that A is a one form with value in the Lie algebra $\mathfrak{g}$. The physical interpretation is that the choice of local section that leads to A is a choice of gauge. So the complex constitutes a choice of gauge.

This picture is sufficient in order to consider the holonomy as we discuss shortly.

7.2.2 The complex as the bundle P

Primality of the Cartan formulation

In principle, the fiber bundle construction or more specifically the fibration of a space, consists in identifying forms according to the fact that they are invariant, horizontal or basic. Each of these notions can be phrased in the Cartan formalism using i_v and L_v by satisfying conditions such as $i_X\omega = 0$ or $L_v\omega = 0$ or both [77]. This encourages us to look into the possibilities of this formalism. The setup is a principal bundle

P and the Lie algebra $\mathfrak{g}$ on the space $\Omega(P)$ of differential forms on P. Precisely the setup of the space $\mathcal{L}$. The space with work in is then the convenient space

$$\mathfrak{g} \otimes \Omega(P). \tag{7.24}$$

The operations $(i_X,\, d,\, L_X)$ act in the following way

$$i_X(Y \otimes \alpha) = Y \otimes i_X\alpha, \tag{7.25}$$

$$L_X(Y \otimes \alpha) = Y \otimes L_X\alpha, \tag{7.26}$$

$$d(Y \otimes \alpha) = Y \otimes d\alpha. \tag{7.27}$$

Then, one recovers the Cartan formalism, reminiscent of the discussion of the theory $\mathcal{G}$ which is the topic of Chapter 4 where it was adopted. The various equations that capture the fiber bundle are:

$$i_X\omega = X, \tag{7.28}$$

$$L_X\omega = [\omega,\, X], \tag{7.29}$$

which are satisfied $\forall x \in \mathfrak{g}$. They express the condition of verticality and equvariance respectively. Finally, the curvature of the algebraic connection ω is

$$\Omega = d\omega + \omega \wedge \omega. \tag{7.30}$$

Which satisfies in turn the properties of horizontality

$$i_X\Omega = 0, \tag{7.31}$$

$\forall X \in \mathfrak{g}$, and equivariance.

This is what is best from our point of view, instead of constructing a fiber bundle from the start, we extract it in the natural way through the space of forms by putting conditions on them. This approach, which we leave here deserves further consideration.

7.3 Holonomy

7.3.1 Integration and cohomology of gauge fields

The holonomy:

$$\exp i \oint_{C=\partial S} A, \tag{7.32}$$

is the classical object which is the key to the quantum gauge theory if one adopts the Feynman viewpoint. It is used in gauge theory extensively and in loop quantum gravity as well. It also plays a central role in further mathematical developments in gauge theory. Before we turn to it, we need to be able to integrate the gauge field over a closed loop. We already have a notion of integration on the base space. The connection A satisfies the Ad-equivariance condition

$$\rho_g^\star A = \mathrm{Ad}(g^{-1})A, \tag{7.33}$$

and its local Lie algebra version ($\rho_\star$ is the push-forward associated to the group action $\rho : G \otimes P \longmapsto P$), which a version of Eq. 7.29 is

$$L_{\rho_\star(x)}\omega = -[x, \omega]. \tag{7.34}$$

The condition Eq. 7.33 enables one to show that if a one-form β is Ad-equivariant, then $d\beta + [A, \beta]$ is also Ad-equivariant. This leads to

$$d_A = d + A, \tag{7.35}$$

$$F_A = dA + A \wedge A. \tag{7.36}$$

A is a $\mathfrak{g}$-valued 1-form, so can extract its Lie algebra part $A = A_a T^a$. We can view the object $A \wedge A$ as

$$A \wedge A = A_\alpha \wedge A_\beta \otimes T^\alpha T^\beta. \tag{7.37}$$

At this point, it is legitimate to ask what happened to the de Rham cohomology and to Stokes theorem. The answer is well known, the Lie algebra has its own cohomology

(See e.g [83, 55]), or d_A-cohomology. We see for example that if the curvature F vanishes, then $d_A^2 = 0$. When the space can be identified with a $\mathfrak{g}$-module V, the cohomology associated to the Lie algebra $\mathfrak{g}$ brings us back to the de Rham we already have, ie, restricting the de Rham cohomology to left invariant differential forms, we get

$$H^l_{\text{de Rham}}(G) = H^l(\mathfrak{g}, k). \tag{7.38}$$

relating the de Rham cohomology of a space, namely G, to the algebra cohomology. In the general case, it is found that A is flat (i.e $\mathrm{d}_A^2 = F = 0$), if and only if the bundle with connection (P, A) is localy diffeomorphic to the trivial bundle with connection $(M \times G, \omega_G)$, where ω_G is the canonical Lie aglebra valued one-from which is left invariant on G.

After these abstract considerations, we see that it is appropriate not to specialise to the flat connections. We start then by replacing the inner product

$$\int_\sigma \omega \in \mathbb{R}, \tag{7.39}$$

by

$$\int_\sigma \omega \in \mathbb{R}\mathfrak{g}. \tag{7.40}$$

It is not obvious (in the case of the G-bundle) how to relate this object to the integration of $d\omega$ since dA is not Ad-equivariant. With such an object, it is clear that no Stokes theorem can be written. A solution is given by the non-abelian Stokes theorem which gives a formula for the holonomy Eq. 7.32 and is a way to study the cohomology of the loop group [78, 72] in simplicial language. The key observation is that the loop is taken on the manifold M so one considers the gauge field A on M (this is motivated by the discussion of the fiber bundle). Let,

$$A^1 \longmapsto A_\sigma[\sigma] \otimes [\sigma]_{\mathfrak{g}}, \tag{7.41}$$

$$A_\sigma = \int_\sigma A. \tag{7.42}$$

We take A in Eq. 7.41 to be an element of $C^{(1)}(K) \otimes C^1(K)$. This choice enables us to obtain a p-form part and a Lie algebra part (i.e in $\mathbf{g}$). So for the purpose below, we will separate the form and Lie algebra part locally,

$$< \sigma,\ A_\sigma[\sigma] \otimes [\sigma]_{\mathbf{g}} >= A_\sigma \sigma_{\mathbf{g}}. \tag{7.43}$$

Let us discuss the theorem that gives us the $F \wedge \star F$ term in the non-abelian case. Its analogue in the present context is easy to construct as we will see.

7.3.2 An adapted non-abelian Stokes theorem

In order to consider gauge theories on the lattice, various techniques have been developed, the mainstream one is to consider the parallel transporter U to be associated to a link on a regular lattice. This technique avoids the construction of a principal G-bundle as is done in continuum Yang-Mills theory. Another technique [59] is to consider inhomogeneous differential forms as providing a representation of spinors by means of the Clifford product. One thus describes fermions by means of the Dirac-Kähler equation, such a representation leads to gauge fields as being composed of not just a 1-form but of all other dimensions as well [95].

A systematic study of gauge theories for any dimensions is given by Alvarez et al [?], they consider models with a flat connection, which gives access to various topological field theories such as the Chern-Simons theory. In a more general context of a non-flat connection, the Wilson loop is

$$W_R(C) = \mathrm{Tr}_R P \exp \oint_{\partial S} A, \tag{7.44}$$

where R denotes an irreducible representation of the Lie algebra $\mathbf{g}$. The expression Eq. 7.44 is given by the non-abelian Stokes theorem [87, 93, 96] which reads

$$Tr\ P\ \exp \oint_S g^{-1} F g = Tr\ \mathcal{P}\ \exp \int_{\partial S} \mathcal{A}. \tag{7.45}$$

The matching between the two sides can be done by Taylor expansion and taking the trace in the algebra, one obtains, using the Euclidean metric, the action

$$S_{YM} = \int_S F \wedge \star F, \tag{7.46}$$

leading to the Yang-Mills equations

$$F = \star F. \tag{7.47}$$

A simple proof is given in [87]. Again, one uses the property that the parallel transporters are elements of a groupoid satisfying by definition

$$(U_1 U_2) U_3 = U_1 (U_2 U_3), \tag{7.48}$$

$$IU = UI = U, \tag{7.49}$$

$$U^{-1} U = U U^{-1} = I. \tag{7.50}$$

More precisely, as a group action, elements U of the groupoid satisfy:

$$I = U(x,\, x), \tag{7.51}$$

$$U^{-1}(x_1,\, x_2) = U(x_2,\, x_1), \tag{7.52}$$

$$U(x_1,\, x)\, U(x,\, x_2) = U(x_1,\, x_2). \tag{7.53}$$

Briefly, the path ordering (the operator $P_{s,t}$ below) is the trivial one. The key intermediate step is to construct the series made of parallel transporters around small rectangles labeled by integers $(m,\, n)$. One is then evaluating

$$\mathrm{P}\exp\left(i \oint A\right) = \lim_{N\to\infty} (P_{s,t})\, \Pi^N_{m,n=1} U^{-1}_{m,\,n} W_{m,n} U_{m,n}. \tag{7.54}$$

It is then found that:

$$W_{m,n} = 1 + \frac{i}{N}\left(A^y_{m,\,n} - A^y_{m-1,\,n}\right) - \frac{i^2}{N^2}\left(A^x_{m,n}A^y_{m,n} - A^y_{m,n}A^x_{m,n}\right), \tag{7.55}$$

$$= 1 + \frac{i}{N^2}\left(\partial_x A^y - \partial_y A^x\right) - \frac{i^2}{N^2}\left(A^x A^y - A^y A^x\right), \tag{7.56}$$

$$= 1 + \frac{i}{N^2} F_{m,n}. \tag{7.57}$$

More details are given in the literature, what we are interested in for our discussion here is the calculation done in Eq. 7.55, because that is how the matching is done perturbatively. So our prescription is to use both Eq. 7.41 and Eq. 7.43. The integration being over a closed loop.

Calculate the path-ordered exponential in the algebra by setting:

$$A \in \Lambda^{\star}(\mathbf{g},\ M) \longmapsto L_A \in Aut\ \mathbf{g}. \tag{7.58}$$

So we change the representation: $A \longmapsto L_A$, and we rely on the Jacobi identities, known to hold in the discrete exterior calculus, as discussed in Part II,

$$\begin{aligned} W &= P\,exp\left(<\sigma,\ A>[\sigma]\right), \\ &= P\left(\mathrm{Id}+<\sigma_1,\ A>[\sigma_1]\right)\ldots P\left(\mathrm{Id}+<\sigma_N,\ A>[\sigma_N]\right), \\ &= \left(\mathrm{Id}+<\sigma_1,\ A>L_{[\sigma_1]}\right)\ldots\left(\mathrm{Id}+<\sigma_N,\ A>L_{[\sigma_N]}\right), \end{aligned} \tag{7.59}$$

and so, using the same (hypercubic) subdivision of the area S, we find

$$P\,exp\left(<\sigma,\ A>[\sigma]\right) = ad\ F_A. \tag{7.60}$$

Returning to the loop, what we have done is the following:

$$Ad\ \exp\left(i\oint A\right) = \exp\ ad\left(i\oint A\right) \tag{7.61}$$

$$= \left(1 + A_{\sigma_1}\sigma^1_{\mathbf{g}}\right)\circ\left(1 + A_{\sigma_2}\sigma^2_{\mathbf{g}}\right)\circ\ldots\left(1 + A_{\sigma_4}\sigma^4_{\mathbf{g}}\right), \tag{7.62}$$

where $\circ$ is the Lie algebra operation in the adjoint representation,

$$\sigma^1_{\mathbf{g}} \circ \sigma^2_{\mathbf{g}} = L_{\sigma^1} \sigma^2_{\mathbf{g}}. \tag{7.63}$$

One is in a position to recover Chern-Simon theory and possible variations on the theme of topological quantum field theory. Not surprisingly, using the Non-abelian Stokes theorem, one can calculate the Monodromy matrix, which is the key to calculate the partition function [94].

Part IV

Conclusions

In this thesis, the central theme has been the development of a new discretisation scheme which is suited for approaching conceptual issues in the discretisation of systems in theoretical physics. We adopted a partly axiomatic approach to establish a discrete exterior calculus, a construction that was not previously available, and showed its consistency in relation to the de Rham cohomology. A fully axiomatic approach would require using the language of functors and categories which is conceptually what we constructed. It was found that we could exploit the discreteness in the algebraic relations and the algebras we require, we were then able to capture them even though the fields themselves were approximated by Whitney forms.

We proposed a novel approach to constructing a discrete analogue of continuum symmetries. This was particularly well suited to describe flows as we saw in Chapter 3, where the Lie derivative is given a rigorous discrete analogue. But also for the formulation of spinors in Chapter 6, where we introduce a satisfactory discretised Clifford product. The new discretised theory for fermions leads to the reduction to the Dirac equation which was previously impossible (we described other methods in Chapter 5) using discrete versions of differential geometry in the context of the Dirac-Kähler formalism.

In Chapter 4, we used the discrete exterior calculus in order to obtain the covariant derivative. We also found that the Cartan formalism for describing the induced metric (by taking a hyper-surface in $\mathbb{R}^N$) had a well-defined discrete analogue.

The Chapter on gauging proposes a discussion of various interesting possibilities, we do not exclude at this point that we might end up with a "no-go theorem" for the minimal coupling of fermions to gauge. Our formulation relies on Whitney forms and we wish to address this question in the future. We worked in the product space of chains, but when it was formally needed, closure of the operations on the space of chains was established. This was done for the Jacobi identities and for the Clifford product with a constant differential which was of central importance in the discussion of the reduction of the Dirac-Kähler field. We also benefited from the elegant approach provided by the use of differential forms in constructing a discrete analogue of the vielbein approach.

Having moved away from the natural habitat of geometric discretisation which was the Abelian Chern-Simons theory, in future work, we would like to address issues in QFT related to theories for which the partition function cannot be evaluated exactly. Some work on actions needs to be done, the partition function needs to be expressed and studied more explicitly. Also, it would be very tempting to construct a discertisation of models based on string theory in which much non-perturbative work is needed and we think that both the work on spinors and on geometry we have done would provide the necessary formalism.

We finish with two tables. The first three columns of Table 7.1 give the list of operators defined, and the space on which they act. We denote by $\times C$ the product space of chains. The last column gives some interesting operators that can be deduced. These operators are well-defined in the discrete theory.

Theory	Operators	Space	Other operators
$\mathcal{T}$	$(\wedge,\, d,\, \star)$	$C(K)$, $C(L)$	δ
$\subseteq \mathcal{L}$	$(i_v,\, \wedge,\, d,\, \star)$	$\times C(K)$, $\times C(L)$	L_v, $[u,v]$, $\vee$
$\subseteq \mathcal{G}$	$(\{\theta^{(a)}\},\, i_v,\, \wedge,\, d\star)$	$\times C(L)$, $\times C(K)$	∇_v

Table 7.1: The discretisation schemes are specified by a space and a collection of operators.

In Table 7.2, we point out the physics that we expect to find.

Theory	Physical content
$\mathcal{T}$	topological abelian gauge theories, topological invariants "Chirality" or γ_5
$\mathcal{L}$	Lie derivative, conservation laws, geometry of flows Dirac-Kähler reduction with Clifford product
$\mathcal{G}$	Euclidean models for gravity

Table 7.2: The schemes and their counterpart in theoretical physics.

Bibliography

[1] H. B. Nielsen and M. Ninomiya, *Phys. Lett.* **B105** 219 (1981).

[2] IMA Workshop: "Compatible Spatial Discretizations for Partial Differential Equations" proceedings. [http://www.ima.umn.edu/talks/workshops/5-11-15.2004/] (2004)

[3] J. M. Rabin, *Nuclear Physics* **B201**, 315 (1982).

[4] P. Becher and H. Joos, *Z. Phys.* **C15**, 343 (1982).

[5] H. Flanders, *Differential Forms with Applications to the Physical Sciences* , Dover Publications (1989).

[6] S. Lang, *Differential and Riemannian Manifolds*, Springer-Verlag (1995).

[7] D. H. Adams, *Phys. Rev. Lett.* **78**, 4155-4158, (1997).

[8] D. Birmingham, M. Rakowski, *Phys. Letters* **B299**, 299 (1993).

[9] D. H. Adams, *R-Torsion and Linking Numbers from Simplicial Abelian Gauge Theories*, T.C.D. Preprint hep-th/9612009.

[10] N. Warner, *Proc. Roy. Soc. London Ser.* **A 383** (1982) 359-377.

[11] R. Hiptmair, *PIER, EMW Publishing, Cambridge, MA*, pp247-269. (2001).

[12] J. Dodzuik, *American Journal of Mathematics*, **Vol. 98, No. 1**, pp79-104. (1976).

[13] S. Albevario and J. Schäfer, *J. Math. Phys.* **36**, 2157 (1995).

[14] Samik Sen, S. Sen, J. C. Sexton and D. H. Adams, *Phys. Rev.* **E61**, 3174 (2000); hep-th/0001030.

[15] V. de Beaucé, S. Sen and J. C. Sexton, Nucl. Phys. Proc. Suppl. **129**, 468 (2004) [arXiv:hep-lat/0309167].

[16] V. de Beaucé and S. Sen, arXiv:hep-th/0305125.

[17] C. Nash, *Differential Topology and Quantum Field Theory*, Academic Press, London (1991).

[18] C. Nash and S. Sen, *Topology and Geometry for Physicists*, Academic Press, London (1983).

[19] P. Griffiths and J. Harris *Principles of Algebraic Geometry*, John Wiley and Sons, Inc, (1994).

[20] J. F. Davis and P. Kirk *Lecture Notes in Algebraic Topology*, AMS, (2001).

[21] S. Morita *Geometry of Differential Forms*, AMS, (2001).

[22] M. Nakahara *Geometry, Topology and Physics*, IOP, (1998).

[23] S.I Gelfand and Yu.I Manin *Methods of Homological Algebras*, Springer, (1996)

[24] R.P. Thomas, *Derived categories for the working mathematician*, Proceedings of the Winter School on Mirror Symmetry, AMS/IP Studies in Advanced Mathematics (2001).

[25] J. G. Hocking and G. S. Young, "Topology", Dover Publications, NY, 1961.

[26] Samik Sen, hep-th/0307166.

[27] J. Harrison, Journal of Geometric Analysis, 8 (1998), no. 5, 797–807.

[28] H. Whitney, "Geometric Integration Theory", Princeton University Press, Princeton, NJ, 1957.

[29] A. Bossavit *COMPEL, 22,*, 3, pp. 470-80 (2003).

[30] A. Hirani, *PhD Thesis* , CALTECH, (2003).

[31] V. I Arnold and B. A Khesin, Topological Methods in Hydrodynamics, *Applied Mathematical Sciences, Springer* **126**, (1998).

[32] D. S. Freed, K. K Uhlenbeck Editors, "Geometry and Quantum Field Theory" *IAS/Park City mathematical series*, AMS. (1994)

[33] W. H. Press et. al. "Numerical recipes in C". CUP. (1992).

[34] T. K Kambe, Fluid Dynamics Research, (2002).

[35] T. Regge, *Nuovo Cimento* **19**, 558 (1961),

[36] R. Sorkin, Phys. Rev. D **12**, 385 (1975).

[37] M. Bander *Phys. Letters* **D38** , 1056 (1988).

[38] J. Ambjorn and H. Dhurrus, CUP monographs (1998)

[39] R. D. Sorkin, *gr-qc/0309009.* (2003).

[40] J. C. Baez, Lect. Notes Phys. **543**, 25 (2000) [arXiv:gr-qc/9905087].

[41] M. do Carmo, "Differential forms and applications", *Universitext* Springer-Verlag (1994).

[42] A. Bossavit, J. Japan Soc. Appl. Electromagn. Mech., a series of six articles, (2000)

[43] A. Bossavit, *"Discretisation of Electromagnetic Problems"*, Lectures at the CMI, [www.icm.edu.pl/edukacja/mat.php] (2001)

[44] P. Boyland,in An introduction to the Geometry and Topology of Fluid Flows *NATO Science series, II*, Vol.47 (2001).

[45] R. L. Ricca in op. cit. *NATO Science series, II*, Vol.47 (2001).

[46] A. Cannas da Silva, Lectures on Symplectic Geometry *Sringer Lecture Notes in Mathematics* **1764** (2001).

[47] V. I. Arnold, Mathematical Methods of Classical Mechanics, *Springer* **60**. (1980).

[48] S. K Donaldson, The moment map in Differential Geometry, Brower lectures, MSRI, video, (1997).

[49] A. Perelomov, Generalised Coherent States and Their Applications *Springer* (1986).

[50] S. K. Paul and S. Sen, J. Math. Phys. **45**, 2124 (2004) [arXiv:hep-th/0211159].

[51] I. Kanamori and N. Kawamoto, Int. J. Mod. Phys. A **19**, 695 (2004) [arXiv:hep-th/0305094].

[52] C. Chryssomalakos, P. Schupp and B. Zumino, Alg. Anal. **6**, 252 (1994) [arXiv:hep-th/9401141].

[53] P. Watts, arXiv:hep-th/9412153.

[54] P. Ascheri and P. Schupp, q-alg/9505023.

[55] P. Etingov and O. Schiffman, Lectures on Quantum Groups, *Lecture in Mathematical Physics Series* International Press (1998).

[56] K. Wilson, *Phys. Rev.* **D10**, 2445 (1974).

[57] L. H. Karsten and J. Smit, *Nuclear Physics* **B183**, 103-140 (1981).

[58] H. Blaine and M-L. Michelsohn *Spin Geometry*, Princeton Mathematical Series, 38. PUP, (1989).

[59] E. Kähler, *Rendiconti de Matematica* (3-4) 21 425 (1962).
W. Graf, *Ann. Inst. Henri Poincaré* **A29** 85 (1978).

[60] See the newsgroup "sci.physics.research". *http://math.ucr.edu/home/baez/spr.html*

[61] P. Becher, *Phys. Lett.* **104B**, 221(1981).

[62] J. Kogut and L. Susskind, *Phys. Rev.* **D11** 395-408 (1975).

[63] P. H. Ginsparg and K. G. Wilson, *Phys. Rev.* **D25**, 2649 (1982).

[64] M. Creutz, *Rev. Mod. Phys* **73**, 119 (2001).

[65] M. B Green, A. Schwarz, E. Witten, *Superstring Theory*, CUP (1987)

[66] J. Vaz, Advances in Applied Clifford algebras, **7** (1997) 37, hep-th/9706189.

[67] E. P. Wigner, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra*, Expanded and improved edition p210 ff, Academic Press, New York, London (1959)

[68] N. Kawamoto and J. Smit, *Nuclear Physics* **B192**, 100 (1981).

[69] H. Kluberg-Stern, A. Morel, O. Napoly and B. Petersson *Nuclear Physics* **B220**, 447 (1983).

[70] G. T Bodwin, E. V Kovács, *Phys. Rev.* **D38**, 1206 (1988).

[71] J. M. Drouffe, K. J. M. Moriarty and C. N. Mouhas, Comput. Phys. Commun. **30**, 249 (1983).

[72] C. Hofman, arXiv:hep-th/0207017.

[73] C. Teitelboim, *Phys. Letters.* **167B**, 63 (1986).

[74] N. Kawamoto, T. Tsukioka and H. Umetsu, Int. J. Mod. Phys. A **16**, 3867 (2001) [arXiv:hep-th/0010254].

[75] N. Kawamoto and T. Tsukioka, Phys. Rev. D **61**, 105009 (2000)

[76] D. Oriti, C. Rovelli and S. Speziale, arXiv:gr-qc/0406063.

[77] R. Coquereaux, "Fiber bundles and connections," PRINT-97-211, in french.

[78] G. Segal, "Loop Groups", *Oxford Science Monographs*, OSP, (1986).

[79] P. Mitra and P. Weisz, *Phys. Letters* **B126**, 355 (1983).

[80] H. Sharatchandra, H.J Thun and P. Weisz, *Nuclear Physics* **B192**, 205 (1981).

[81] M. Golterman and J. Smit *Nuclear Physics* **B245**, 61 (1984).

[82] V. de Beaucé and S. Sen, arXiv:hep-th/0403206. Submitted to the Journal of Mathematical Physics.

[83] J. A. de Azcarraga, J. M. Izquierdo and J. C. Perez Bueno, Rev. R. Acad. Cien. Exactas Fis. Nat. Ser. A Mat. **95**, 225 (2001) [arXiv:physics/9803046].

[84] J. C. Baez, arXiv:hep-th/0206130.

[85] N. Kawamoto, H. B. Nielsen and N. Sato, Nucl. Phys. B **555**, 629 (1999) [arXiv:hep-th/9902165].

[86] N. Kawamoto, N. Sato and Y. Uchida, Nucl. Phys. B **574**, 809 (2000) [arXiv:hep-th/9911228].

[87] B. Broda, arXiv:math-ph/0012035.

[88] L. O'Raifeartaigh, Nucl. Phys. Proc. Suppl. **101**, 107 (2001).

[89] O. Jahn and J. M. Pawlowski, Nucl. Phys. B **642**, 357 (2002) [arXiv:hep-lat/0205005].

[90] A. Dimakis and F. Mueller-Hoissen, J. Math. Phys. **35**, 6703 (1994).

[91] A. Dimakis and F. Mueller-Hoissen, J. Math. Phys. **44**, 1781 (2003)

[92] A. P. Balachandran, X. Martin and D. O'Connor, Int. J. Mod. Phys. A **16**, 2577 (2001) [arXiv:hep-th/0007030].

[93] R. L. Karp, F. Mansouri and J. S. Rno, J. Math. Phys. **40**, 6033 (1999) [arXiv:hep-th/9910173].

[94] E. Witten, Nucl. Phys. B **330**, 285 (1990). Phys. Rev. D **62**, 094504 (2000) [arXiv:hep-lat/0004016].

[95] S. I. Kruglov, arXiv:hep-th/0110251.

[96] D. Diakonov and V. Y. Petrov, Phys. Lett. B **224**, 131 (1989).