CR Singularities in Codimension 2

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Declaration

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Summary

In this thesis we study the real submanifolds of codimension 2 in a complex manifold near a CR singularity. The thesis has 3 chapters. In Chapter 1 we shall make a small introduction where we will remind some basic notions and known results. The first chapter has 3 parts. In the first part we recall some basic notions. The second part represents an preparation for the second chapter. The third part represent a preparation for the third chapter. The main result of the thesis represent the content of Chaper 2. We generalize to a higher dimensional case Huang-Yin's normal form in $\mathbb{C}^2$. The main tool is given by the Fisher decomposition and our construction is done following the lines of Huang-Yin's normal form construction.

The last Chapter contains some remarks about a family of analytic discs attached to a real submanifold and some applications.

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Chapter 2 represent my first paper [3] and Chapter 3 represent my second paper [4].

Contents

Chapter 1

Introduction

1.1 Real Submanifolds in the Complex Space

In this Section we shall give some basic notions about the real submanifolds in the complex space.

A (smooth) real submanifold of $\mathbb{C}^N$ of codimension d is a subset $M \subset \mathbb{C}^N$ such that for every point $p_0 \in M$ there exists a smooth real vector-valued function $\rho = (\rho_1, \dots, \rho_d)$ defined in U such that

$$M \cap U = \{z \in U;\ \rho(z, \bar{z}) = 0\},$$

with differentials $d\rho_1, \dots, d\rho_d$ linearly independent in U.

When $d = 1$, M is called a hypersurface. If $p \in \mathbb{R}^{2N} \equiv \mathbb{C}^N$, we define the tangent space to $\mathbb{C}^N$ at p as follows

$$T_p\mathbb{C}^N = T_p\mathbb{R}^{2N} := \left\{ X = \sum_{j=1}^{N} \left(a_j \frac{\partial}{\partial x_j} + b_j \frac{\partial}{\partial y_j} \right), \quad a_1, b_1, \dots, a_j, b_j, \dots, a_N, b_N \in \mathbb{R} \right\}.$$

If $p \in M$ and $X \in T_p\mathbb{C}^N$, we say that X is tangent to M at p and we write that $X \in T_pM$ if

$$\sum_{j=1}^{N} \left(a_j \frac{\partial \rho_k}{\partial x_j}(p, \bar{p}) + b_j \frac{\partial \rho_k}{\partial y_j}(p, \bar{p}) \right) = 0, \quad k = 1, \dots, d.$$

Since, for any two local defining equations ρ, ρ' for M there exist $a(z, \bar{z})$ a $d \times d$-matrix such that $\rho(z, \bar{z}) = a(z, \bar{z})\rho(z, \bar{z})$ it follows that the previous definition does not depend on choice of the local defining equation.

Similarly we define the complexified tangent spaces $\mathbb{C}T_p\mathbb{C}^N$ and $\mathbb{C}T_pM$ by allowing the coefficients in the expressions above to be complex numbers. Then $\dim_{\mathbb{R}}(T_pM) = \dim_{\mathbb{C}}(\mathbb{C}T_pM) = 2N - d$. Therefore the mappings $M \ni p \longrightarrow T_pM$ and $M \ni p \longrightarrow \mathbb{C}T_pM$ define real and complex vector bundles over M, denoted by TM and respectively by $\mathbb{C}TM$.

Using the following notations

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right),$$

any $X \in \mathbb{C}T_p\mathbb{C}^N$ can be written uniquely as follows

$$X = \sum_{j=1}^{N} \left(a_j \frac{\partial}{\partial z_j} + b_j \frac{\partial}{\partial \bar{z}_j} \right); \quad a_1, b_1, \dots, a_j, b_j, \dots, a_N, b_N \in \mathbb{C}.$$

A tangent vector X is holomorphic if $a_1 = \dots = a_N = 0$, and antiholomorphic if $b_1 = \dots = b_N = 0$. We denote by $T_p^{0,1}\mathbb{C}^N$ the space of antiholomorphic vectors and respectively by $T_p^{1,0}\mathbb{C}^N$ the space of holomorphic vectors.

For $p \in M$ we define the space of antiholomorphic vectors tangent to M at p as follows

$$T_p^{0,1}M = V_p := T_p^{0,1}\mathbb{C}^N \cap \mathbb{C}T_pM.$$

Here

$$\dim_{\mathbb{C}} V_p = N - \operatorname{rank}_{\mathbb{C}} \left(\frac{\partial \rho_k}{\partial \bar{z}_j}(p,\bar{p}) \right)_{j=1,\dots,N;k=1,\dots,d}.$$

M is called a CR manifold if the map $M \ni q \mapsto \dim_{\mathbb{C}} T_q^{0,1}M$ is constant.

A point $p \in M$ is called a CR singularity if it is a discontinuity point for the map

$$M \ni q \mapsto \dim_{\mathbb{C}} T_q^{0,1}M$$

defined near p.

A point p in a CR manifold N is called a non-minimal point if N contains a proper CR submanifold S containing p such that $T_p^{(1,0)}S = T_p^{(1,0)}N$ (see [23] for more details).

A smooth mapping $T : M \to M$ is called a formal transformation.

1.1.1 CR Singularities and Normal Forms in $\mathbb{C}^2$

The study of real submanifolds in a complex space near an CR singularity goes back to the celebrated paper [2] of Bishop. Bishop considered the case when there exists coordinates (z,w) in $\mathbb{C}^2$ such that near a CR singularity $p = 0$, a real 2-codimensional submanifold $M \subset \mathbb{C}^2$ is defined locally by

$$w = z\bar{z} + \lambda\left(z^2 + \bar{z}^2\right) + \mathrm{O}(3), \tag{1.1}$$

where $\lambda \in [0,\infty]$ is a holomorphic invariant called the Bishop invariant. When $\lambda = \infty$, M is understood to be defined by the equation $w = z^2 + \bar{z}^2 + \mathrm{O}(3)$. If λ is non-exceptional Moser-Webster proved in [28] that there exists a formal transformation that sends M into the normal form

$$w = z\bar{z} + (\lambda + \varepsilon u^q)\left(z^2 + \bar{z}^2\right), \quad \varepsilon \in \{0,-1,+1\}, \quad q \in \mathbb{N}, \tag{1.2}$$

where $w = u + iv$. Here the Bishop invariant λ is called non-exceptional if the following quadratic equation in x has no roots of unity: $\lambda x^2 - x + \lambda = 0$ or if $\lambda \notin \left\{0, \frac{1}{2}, \infty\right\}$.

When $\lambda = 0$ Moser derived in [27] the following partial normal form:

$$w = z\bar{z} + 2\Re\left\{\sum_{j\geq s} a_j z^j\right\}. \tag{1.3}$$

Here $s := \min\left\{j \in \mathbb{N}^\star;\ a_j \neq 0\right\}$ is the simplest higher order invariant, known as Moser's invariant. When $s = \infty$ Moser proved in [27] that (1.3) is holomorphically equivalent to the quadric $\{w = z\bar{z}\}$. Moser's partial normal form is the subject of an action of the infinite dimensional group of formal self-transformations of the quadric $\{w = z\bar{z}\}$ that fix the origin. When $s < \infty$, the problem of reducing the previously mentioned group action was completely solved by Huang and Yin in the recent deep paper [20]. Among other results, they proved that (1.3) can be formally transformed into the following normal form

$$w = z\bar{z} + 2\Re\left\{\sum_{j\geq s} a_j z^j\right\}, \quad a_s = 1, \quad a_j = 0, \quad \text{if} \quad j = 0,1 \bmod s, \quad j > s. \tag{1.4}$$

Further studies concerning the real submanifolds near a CR singularity were done by Ahern-Gong in [1], Coffman in [5], [6], [7], [8], Gong in [12], [13], [14].

1.1.2 CR Singularities and Analytic Discs in $\mathbb{C}^2$

Let Δ be the unit open disc from $\mathbb{C}$ and let S^1 be its boundary. A map $f : \overline{\Delta} \longrightarrow \mathbb{C}^2$ is called an analytic disc if $f|_{\overline{\Delta}}$ is continuous and $f|_{\Delta}$ analytic. We say that f is an analytic disc attached to M if $f\left(S^1\right) \subset M$. In the case when $\lambda \in \left[0, \frac{1}{2}\right)$, Kenig-Webster proved in [24] the existence of an unique family of 1-dimensional analytic disks shrinking to the CR singularity $p = 0$. These discs are mutually disjoint and form a smooth hypersurface $\widetilde{M}$ with boundary M in a neighborhood of the point $p = 0$. In the real-analytic case, Huang-Krantz proved in [17] that $\widetilde{M}$ is a real-analytic hypersurface across the boundary manifold M.

In Chapter 3 we shall study the higher dimensional analog case of Kenig-Webster's Theorem in $\mathbb{C}^2$.

1.1.3 Thesis Organization

In Chapter 2 we shall prove a generalization of Huang-Yin's normal form to a higher dimensional analog case. In Chapter 3 we shall generalize Kenig-Webster's result in $\mathbb{C}^2$ to a higher dimensional analog case in $\mathbb{C}^{N+1}$ and then we shall provide some applications.

Chapter 2

The Normal Form Construction

In this chapter, we construct a higher dimensional analogue of Huang-Yin's normal form in $\mathbb{C}^2$. Let $(z,w) = (z_1,\dots,z_N,w)$ be the coordinates in $\mathbb{C}^{N+1}$ and let $M \subset \mathbb{C}^{N+1}$ be a real submanifold of codimension 2. We consider the case when there exists a holomorphic change of coordinates (see [9], [10] or [19]) such that near $p=0$, M is given by

$$w = z_1\bar{z}_1 + \cdots + z_N\bar{z}_N + \sum_{m+n\geq 3} \varphi_{m,n}(z,\bar{z}), \tag{2.1}$$

where $\varphi_{m,n}(z,\bar{z})$ is a bihomogeneous polynomial of bidegree (m,n) in $(z,\bar{z})$.

Some of our methods extend the construction methods of Huang-Yin's normal form in $\mathbb{C}^2$. First, we give a generalization of Moser's partial normal form, called here the extended Moser lemma (Theorem 2.1.2), which uses the trace operator (see e.g. [33], [34], [35]):

$$\operatorname{tr} := \sum_{k=1}^{N} \frac{\partial^2}{\partial z_k \partial \bar{z}_k}. \tag{2.2}$$

In $\mathbb{C}^2$ Moser's partial normal form eliminates the terms in the local defining equation of M of positive degree in both z and $\bar{z}$. The higher dimensional case considered here brings new difficulties. In $\mathbb{C}^{N+1}$ the extended Moser lemma eliminates only iterated traces of the corresponding terms. However, these terms can still contribute to higher order terms in the construction of the normal form. Recently, similar normal forms were constructed for Levi-nondegenerate hypersurfaces in $\mathbb{C}^{N+1}$ by Zaitsev in [34]. The main instrument is given by the Fisher decomposition.

The condition that (1.3) contains nontrivial higher order terms has the following natural generalization to the higher dimensional case:

$$\Re\left\{\sum_{k\geq 3} \varphi_{k,0}(z)\right\} \not\equiv 0, \tag{2.3}$$

where here and throughout the chapter we use the following abbreviation

$$\varphi_{k,0}(z) := \varphi_{k,0}(z,\bar{z})$$

as the latter polynomials do not depend on $\bar{z}$. As a consequence we obtain that $s := \min\left\{k \in \mathbb{N}^\star;\ \varphi_{k,0}(z) \not\equiv 0\right\} < \infty$. Then s is a biholomorphic invariant and $\varphi_{s,0}(z)$ is invariant (as tensor). We call the integer $s \geq 3$ the generalized Moser invariant. In this chapter we will use the following notations

$$\Delta(z) := \varphi_{s,0}(z), \quad \Delta_k(z) := \partial_{z_k}\left(\varphi_{s,0}(z)\right), \quad k = 1,\dots,N. \tag{2.4}$$

The extended Moser lemma provides us a partial normal form that is not unique, but that is determined up to an action of the infinite dimensional group $\mathsf{Aut}_0(M_\infty)$, the formal self-transformation group of the model $M_\infty := \{w = z_1\bar{z}_1 + \cdots + z_N\bar{z}_N\}$ that fix the origin. The next step is to reduce the action of the above mentioned group on the partial normal form. In order to do this, we use the methods recently developed by Huang and Yin in [20]. In particular, we follow the lines of Huang-Yin's normal form construction [20] considering instead of the model $M_\infty := \{w = z_1\bar{z}_1 + \cdots + z_N\bar{z}_N\}$, the model $w = z_1\bar{z}_1 + \cdots + z_N\bar{z}_N + \Delta(z) + \overline{\Delta(z)}$ and we adapt the powerful Huang-Yin's weights system to our higher dimensional analog case.

Before we will give the statement of our main theorem, we introduce the following definition

Definition 2.0.1. *For a given homogeneous polynomial* $V(z) = \sum_{|I|=k} b_I z^I$ *we consider the associated Fisher differential operator*

$$V^\star = \sum_{|I|=k} \bar{b}_I \frac{\partial^{|I|}}{\partial z^I}. \tag{2.5}$$

We would like to mention that the Fisher decomposition was used also by Ebenfelt in [11].

We consider the class of submanifolds such that in their defining equations, the polynomial $\Delta(z)$ defined in (2.4) satisfies the following nondegeneracy condition:

Definition 2.0.2. *The polynomial* $\Delta(z)$ *is called nondegenerate if for any linear forms* $\mathscr{L}_1(z), \ldots, \mathscr{L}_N(z)$, *one has*

$$\mathscr{L}_1(z)\Delta_1(z) + \cdots + \mathscr{L}_N(z)\Delta_N(z) \equiv 0 \quad \Longrightarrow \quad \mathscr{L}_1(z) \equiv \cdots \equiv \mathscr{L}_N(z) \equiv 0. \tag{2.6}$$

In Section 2.1 we prove that our non-degeneracy condition is invariant under any linear change of coordinates.

We prove the following result:

Theorem 2.0.3. *Let* $M \subset \mathbb{C}^{N+1}$ *be a 2-codimensional real (formal) submanifold given near the point* $0 \in M$ *by the formal power series equation*

$$w = z_1\bar{z}_1 + \cdots + z_N\bar{z}_N + \sum_{m+n\geq 3} \varphi_{m,n}(z,\bar{z}), \tag{2.7}$$

where $\varphi_{m,n}(z,\bar{z})$ *is a bihomogeneous polynomial of bidegree* (m,n) *in* $(z,\bar{z})$ *satisfying (2.3). We assume that the homogeneous polynomial of degree s defined by (2.4) is nondegenerate. Then there exists a unique formal map*

$$(z',w') = (F(z,w), G(z,w)) = (z,w) + O(2), \tag{2.8}$$

that transforms M into the following normal form:

$$w' = z'_1\overline{z'}_1 + \cdots + z'_N\overline{z'}_N + \sum_{\substack{m+n\geq 3\\ m,n\neq 0}} \varphi'_{m,n}\left(z',\overline{z'}\right) + 2\Re\left\{\sum_{k\geq s} \varphi'_{k,0}\left(z'\right)\right\}, \tag{2.9}$$

where $\varphi'_{m,n}\left(z',\overline{z'}\right)$ *is a bihomogeneous polynomial of bidegree* (m,n) *in* $\left(z',\overline{z'}\right)$ *satisfying the following normalization conditions*

$$\begin{cases} \mathrm{tr}^{m-1}\varphi'_{m,n}\left(z',\overline{z'}\right) = 0, & m \leq n-1, \quad m,n \neq 0; \\ \mathrm{tr}^{n}\varphi'_{m,n}\left(z',\overline{z'}\right) = 0, & m \geq n, \quad m,n \neq 0. \end{cases} \tag{2.10}$$

$$\begin{cases} \left(\Delta^t\right)^\star \varphi'_{T,0}(z) = 0, & \text{if } T = ts+1;\ t \geq 1, \\ \left(\Delta_k \Delta^t\right)^\star \left(\varphi'_{T,0}(z)\right) = 0,\ k = 1,\ldots,N, & \text{if } T = (t+1)s;\ t \geq 1. \end{cases} \tag{2.11}$$

A few words about the construction of the normal form. We want to find a formal biholomorphic map sending M into a formal normal form. This leads us to study an infinite system of homogeneous equations by truncating the original equation. As in the paper [20] of Huang-Yin, this system is a semi-non linear system and is very hard to solve. We have then to use the powerful Huang-Yin's strategy defining the weight of z_k to be 1 and the weight of $\bar{z}_k$ to be $s-1$, for all $k = 1\ldots,N$. Since $\mathrm{Aut}_0(M_\infty)$ is infinite-dimensional, it follows that the homogeneous linearized normalization equations (see sections 3 and 4) have nontrivial kernel spaces. By using the preceding system of weights and similar arguments as in the paper [20] of Huang-Yin, we are able to trace precisely how the lower order terms arise in non-linear fashion: The kernel space of degree $2t+1$ is restricted by imposing a normalization condition on $\varphi'_{ts+1,0}(z)$ and the kernel space of degree $2t+2$ by imposing normalization conditions on $\varphi'_{ts,0}(z)$. The non-uniqueness part of the lower degree solutions are uniquely determined in the higher order equations.

We would like to mention here the pseudo-normal form constructed by Huang-Yin in [19] for the real submanifolds defined by (2.7). Our normal form is a natural generalization of Huang-Yin's normal form in $\mathbb{C}^2$. We observe that our normalization conditions are invariant under the linear changes of coordinates that preserves the model $M_\infty := \{w = z_1\bar{z}_1 + \cdots + z_N\bar{z}_N\}$.

A few words about the normal form construction organization: In course of section 2.1 we will give a generalization of Moser's partial normal form and we will make further preparations for our normal form construction. The normal form construction will be presented in the course of sections 2.2 and 2.3. In section 2.4 we will prove the uniqueness of the formal transformation map.

2.1 Preliminaries, Notations and the extended Moser lemma

Let $(z_1,\dots,z_N,w)$ be the coordinates in $\mathbb{C}^{N+1}$. Let $M\subset\mathbb{C}^{N+1}$ be a real submanifold defined near $p=0$ by

$$w = z_1\bar{z}_1 + \cdots + z_N\bar{z}_N + \sum_{m+n\geq 3}\varphi_{m,n}(z,\bar{z}), \tag{2.12}$$

where $\varphi_{m,n}(z,\bar{z})$ is a bihomogeneous polynomial of bidegree (m,n) in $(z,\bar{z})$, for all $m,n\geq 0$.

Let M' be a real submanifold defined by

$$w' = z'_1\overline{z'}_1 + \cdots + z'_N\overline{z'}_N + \sum_{m+n\geq 3}\varphi'_{m,n}\left(z',\overline{z'}\right), \tag{2.13}$$

where $\varphi'_{m,n}\left(z',\overline{z'}\right)$ is a bihomogeneous polynomial of bidegree (m,n) in $\left(z',\overline{z'}\right)$, for all $m,n\geq 0$.

We define the hermitian product

$$\langle z,t\rangle = z_1\bar{t}_1 + \cdots + z_N\bar{t}_N, \quad z=(z_1,\dots,z_N),\ t=(t_1,\dots,t_N)\in\mathbb{C}^N. \tag{2.14}$$

Let $(z',w') = (F(z,w),G(z,w))$ be a formal map which sends M to M' and fixes the point $0\in\mathbb{C}^{N+1}$. Substituting this map into (2.13), we obtain

$$G(z,w) = \langle F(z,w),F(z,w)\rangle + \sum_{m+n\geq 3}\varphi'_{m,n}\left(F(z,w),\overline{F(z,w)}\right). \tag{2.15}$$

In the course of this chapter, we use the following notations

$$\varphi_{\geq k}(z,\bar{z}) = \sum_{m+n\geq k}\varphi_{m,n}(z,\bar{z}), \quad \varphi_k(z,\bar{z}) = \sum_{m+n=k}\varphi_{m,n}(z,\bar{z}), \quad k\geq 3. \tag{2.16}$$

We write $F(z,w) = \sum_{m,n\geq 0}F_{m,n}(z)w^n$, $G(z,w) = \sum_{m,n\geq 0}G_{m,n}(z)w^n$, where $G_{m,n}(z)$, $F_{m,n}(z)$ are homogeneous polynomials of degree m in z. By using w satisfying (2.12) and the notations (2.16), by (2.15) it follows that

$$\begin{aligned}\sum_{m,n\geq 0}G_{m,n}(z)\left(\langle z,z\rangle + \varphi_{\geq 3}(z,\bar{z})\right)^n &= \left\|\sum_{m_1,n_1\geq 0}F_{m_1,n_1}(z)\left(\langle z,z\rangle + \varphi_{\geq 3}(z,\bar{z})\right)^{n_1}\right\|^2 + \\ \varphi'_{\geq 3}\left(\sum_{m_2,n_2\geq 0}F_{m_2,n_2}(z)\left(\langle z,z\rangle + \varphi_{\geq 3}(z,\bar{z})\right)^{n_2},\right. &\left.\overline{\sum_{m_3,n_3\geq 0}F_{m_3,n_3}(z)\left(\langle z,z\rangle + \varphi_{\geq 3}(z,\bar{z})\right)^{n_3}}\right).\end{aligned} \tag{2.17}$$

Since our map fixes the point $0\in\mathbb{C}^{N+1}$, it follows that $G_{0,0}(z)=0$, $F_{0,0}(z)=0$. Collecting the terms of bidegree

$(1,0)$ in $(z,\bar{z})$ in (2.17), we obtain $G_{1,0}(z)=0$. Collecting the terms of bidegree $(1,1)$ in $(z,\bar{z})$ in (2.17), we obtain

$$G_{0,1}\langle z,z\rangle = \langle F_{1,0}(z),F_{1,0}(z)\rangle . \tag{2.18}$$

Then (2.18) describes all the possible values of $G_{0,1}(z),F_{1,0}(z)$. Therefore $\Im G_{0,1}=0$. By composing with a linear automorphism of $\Re w=\langle z,z\rangle$, we can assume that $G_{0,1}(z)=1$, $F_{1,0}(z)=z$.

By using the same approach as in [34] (this idea was suggested me by Dmitri Zaitsev), the "good" terms that can help us to find the formal change of coordinates under some normalization conditions are the following

$$\varphi_{m,n}(z,\bar{z}),\quad \varphi'_{m,n}(z,\bar{z}),\quad G_{m,n}(z)\langle z,z\rangle^n,\quad \langle F_{m,n}(z),z\rangle\langle z,z\rangle^n,\quad \langle z,F_{m,n}(z)\rangle\langle z,z\rangle^n. \tag{2.19}$$

We recall the trace decomposition (see e.g. [33], [34]):

Lemma 2.1.1. *For every bihomogeneous polynomial $P(z,\bar{z})$ and $n\in\mathbb{N}$ there exist $Q(z,\bar{z})$ and $R(z,\bar{z})$ unique polynomials such that*

$$P(z,\bar{z}) = Q(z,\bar{z})\langle z,z\rangle^n + R(z,\bar{z}),\quad \operatorname{tr}^n R = 0. \tag{2.20}$$

By using Lemma 2.1.1 and the "good" terms defined previously by (2.19), we develop a partial normal form that generalizes Moser's Lemma [27]. We prove the following result:

Theorem 2.1.2 (Extended Moser Lemma). *Let $M\subset\mathbb{C}^{N+1}$ be a 2-codimensional real-formal submanifold. Suppose that $0\in M$ is a CR singularity and the submanifold M is defined by*

$$w=\langle z,z\rangle + \sum_{m+n\geq 3}\varphi_{m,n}(z,\bar{z}), \tag{2.21}$$

where $\varphi_{m,n}(z,\bar{z})$ is bihomogeneous polynomial of bidegree (m,n) in $(z,\bar{z})$, for all $m,n\geq 0$. Then there exists a unique formal map

$$(z',w') = \left(z+\sum_{m+n\geq 2}F_{m,n}(z)w^n,\, w+\sum_{m+n\geq 2}G_{m,n}(z)w^n\right), \tag{2.22}$$

where $F_{m,n}(z)$, $G_{m,n}(z)$ are homogeneous polynomials in z of degree m with the following normalization conditions

$$F_{0,n+1}(z)=0,\quad F_{1,n}(z)=0,\quad \textit{for all } n\geq 1, \tag{2.23}$$

that transforms M into the following partial normal form:

$$w'=\langle z',z'\rangle + \sum_{\substack{m+n\geq 3\\ m,n\neq 0}}\varphi'_{m,n}(z',\overline{z'}) + 2\Re\left\{\sum_{k\geq 3}\varphi'_{k,0}(z')\right\}, \tag{2.24}$$

where $\varphi'_{m,n}(z,\bar{z})$ are bihomogeneous polynomials of bidegree (m,n) in $(z,\bar{z})$, for all $m,n\geq 0$, that satisfy the trace normalization conditions (2.10).

Proof. We construct the polynomials $F_{m',n'}(z)$ with $m'+2n'=T-1$ and $G_{m',n'}(z)$ with $m'+2n'=T$ by induction on $T=m'+2n'$. We assume that we have constructed the polynomials $F_{k,l}(z)$ with $k+2l<T-1$, $G_{k,l}(z)$ with $k+2l<T$.

Collecting the terms of bidegree (m,n) in $(z,\bar{z})$ with $T=m+n$ in (2.17), we obtain

$$\varphi'_{m,n}(z,\bar{z}) = G_{m-n,n}(z)\langle z,z\rangle^n - \langle F_{m-n+1,n-1}(z),z\rangle\langle z,z\rangle^{n-1} - \langle z,F_{n-m+1,m-1}(z)\rangle\langle z,z\rangle^{m-1} + \varphi_{m,n}(z,\bar{z}) + \ldots, \tag{2.25}$$

where "..." represents terms which depend on the polynomials $G_{k,l}(z)$ with $k+2l<T$, $F_{k,l}(z)$ with $k+2l<T-1$ and on $\varphi_{k,l}(z,\bar{z})$, $\varphi'_{k,l}(z,\bar{z})$ with $k+l<T=m+n$.

Collecting the terms of bidegree (m,n) in $(z,\bar{z})$ with $T:=m+n\geq 3$ in (2.25), we have to study the following cases:

(1) **Case $\mathbf{m < n-1,\ m,n \geq 1}$**. Collecting the terms of bidegree (m,n) in $(z,\bar{z})$ in (2.25) with $m < n-1$ and $m,n \geq 1$, we obtain

$$\varphi'_{m,n}(z,\bar{z}) = -\langle z, F_{n-m+1,m-1}(z)\rangle \langle z,z\rangle^{m-1} + \ldots \tag{2.26}$$

We want to use the normalization condition $\mathrm{tr}^{m-1}\varphi'_{m,n}(z,\bar{z}) = 0$. This allows us to find the polynomial $F_{n-m+1,m-1}(z)$. By applying Lemma 2.1.1 to the sum of terms which appear in "...", we obtain

$$\varphi'_{m,n}(z,\bar{z}) = \left(-\langle z, F_{n-m+1,m-1}(z)\rangle + D_{m,n}(z,\bar{z})\right)\langle z,z\rangle^{m-1} + P_1(z,\bar{z}), \tag{2.27}$$

where $D_{m,n}(z,\bar{z})$ is a polynomial of degree $n-m+1$ in $\bar{z}_1,\ldots,\bar{z}_N$ and 1 in $z_1,\ldots,z_N$ with determined coefficients by the induction hypothesis and $\mathrm{tr}^{m-1}(P_1(z,\bar{z})) = 0$. Then, by using the normalization condition $\mathrm{tr}^{m-1}\varphi'_{m,n}(z,\bar{z}) = 0$, by the uniqueness of the trace decomposition we obtain that $\langle z, F_{n-m+1,m-1}(z)\rangle = D_{m,n}(z,\bar{z})$. It follows that

$$F_{k,l}(z) = \overline{\partial_z\left(D_{l+1,k+l}(z,\bar{z})\right)}, \quad \text{for all } k > 2, l \geq 0, \tag{2.28}$$

where $\partial_z := (\partial_{z_1},\ldots,\partial_{z_N})$.

(2) **Case $\mathbf{m > n+1,\ m,n \geq 1}$**. Collecting the terms of bidegree (m,n) in $(z,\bar{z})$ in (2.25) with $m > n+1$ and $m,n \geq 1$, we obtain

$$\varphi'_{m,n}(z,\bar{z}) = \left(G_{m-n,n}(z)\langle z,z\rangle - \langle F_{m-n+1,n-1}(z), z\rangle\right)\langle z,z\rangle^{n-1} + \ldots \tag{2.29}$$

In order to find the polynomial $G_{m-n,n}(z)$ we want to use the normalization condition $\mathrm{tr}^n\varphi'_{m,n}(z,\bar{z}) = 0$. By applying Lemma 2.1.1 to the sum of terms which appear in "..." and to $\langle F_{m-n+1,n-1}(z), z\rangle$, we obtain

$$\varphi'_{m,n}(z,\bar{z}) = \left(G_{m-n,n}(z) - E_{m,n}(z)\right)\langle z,z\rangle^n + P_2(z,\bar{z}), \tag{2.30}$$

where $E_{m,n}(z)$ is a polynomial with determined coefficients by the induction hypothesis and $\mathrm{tr}^n(P_2(z,\bar{z})) = 0$. Then, by using the normalization condition $\mathrm{tr}^n\varphi'_{m,n}(z,\bar{z}) = 0$, by the uniqueness of the trace decomposition we obtain that $G_{m-n,n}(z) = E_{m,n}(z)$. It follows that

$$G_{k,l}(z) = E_{k+l,l}(z), \quad \text{for all } k \geq 2,\ l \geq 0. \tag{2.31}$$

(3) **Case $\mathbf{(n-1,n),\ n \geq 2}$**. Collecting the terms of bidegree $(n-1,n)$ in $(z,\bar{z})$ in (2.25) with $n \geq 2$, we obtain

$$\varphi'_{n-1,n}(z,\bar{z}) = \varphi_{n-1,n}(z,\bar{z}) - \langle F_{0,n-1}(z), z\rangle \langle z,z\rangle^{n-1} - \langle z, F_{2,n-2}(z)\rangle \langle z,z\rangle^{n-2} + \ldots \tag{2.32}$$

In order to find $F_{2,n-2}(z)$ we want to use the normalization condition $\mathrm{tr}^{n-2}\varphi'_{n-1,n}(z,\bar{z}) = 0$. By applying Lemma 2.1.1 to the sum of terms in "...", we obtain

$$\varphi'_{n-1,n}(z,\bar{z}) = -\left(\langle F_{0,n-1}(z), z\rangle \langle z,z\rangle + \langle z, F_{2,n-2}(z)\rangle - C_{n-1,n}(z,\bar{z})\right)\langle z,z\rangle^{n-2} + P_3(z,\bar{z}), \tag{2.33}$$

where $\mathrm{tr}^{n-2}(P_3(z,\bar{z})) = 0$ and $C_{n-1,n}(z,\bar{z})$ is a determined polynomial of degree 1 in $z_1,\ldots,z_N$ and degree 2 in $\bar{z}_1,\ldots,\bar{z}_N$. We take $F_{0,n-1}(z) = 0$ (see (2.23)). Next, by using the normalization condition $\mathrm{tr}^{n-2}\varphi'_{n-1,n}(z,\bar{z}) = 0$ and by the uniqueness of the trace decomposition we obtain that $\langle z, F_{2,n-2}(z)\rangle = C_{n-1,n}(z,\bar{z})$. It follows that

$$F_{2,n-2}(z) = \overline{\partial_z\left(C_{n-1,n}(z,\bar{z})\right)}, \tag{2.34}$$

where $\partial_z := (\partial_{z_1},\ldots,\partial_{z_N})$.

(4) **Case $\mathbf{(n,n-1),\ n \geq 2}$**. Collecting the terms of bidegree $(n,n-1)$ in $(z,\bar{z})$ in (2.25) with $n \geq 2$, we obtain

$$\varphi'_{n,n-1}(z,\bar{z}) = \left(G_{1,n-1}(z)\langle z,z\rangle - \langle F_{2,n-2}(z), z\rangle - \langle z, F_{0,n-1}(z)\rangle \langle z,z\rangle\right)\langle z,z\rangle^{n-2} + \varphi_{n,n-1}(z,\bar{z}) + \ldots \tag{2.35}$$

In order to find $G_{1,n-1}(z)$ we want to use the normalization condition $\mathrm{tr}^{n-1}\varphi'_{n,n-1}(z,\bar{z})=0$. By using (2.23) and by applying Lemma 2.1.1 to $\langle F_{2,n-2}(z),z\rangle$ (see (2.34)) and to the sum of terms in "...", we obtain

$$\varphi'_{n,n-1}(z,\bar{z})=\left(G_{1,n-1}(z)-B_{n,n-1}(z)\right)\langle z,z\rangle^{n-1}+P_4(z,\bar{z}), \tag{2.36}$$

where $\mathrm{tr}^{n-1}\left(P_4(z,\bar{z})\right)=0$ and $B_{n,n-1}(z)$ is a determined polynomial. By the uniqueness of the trace decomposition we obtain that $G_{1,n-1}(z)=B_{n,n-1}(z)$, for all $n\geq 2$.

(5) Case $(\mathbf{n},\mathbf{n})$, $\mathbf{n}\geq\mathbf{2}$. Collecting the terms of bidegree (n,n) in $(z,\bar{z})$ in (2.25) with $n\geq 2$, we obtain

$$\varphi'_{n,n}(z,\bar{z})=G_{0,n}(z)\langle z,z\rangle^{n}-\langle F_{1,n-1}(z),z\rangle\langle z,z\rangle^{n-1}-\langle z,F_{1,n-1}(z)\rangle\langle z,z\rangle^{n-1}+\varphi_{n,n}(z,\bar{z})+\dots \tag{2.37}$$

By taking $F_{1,n-1}(z)=0$ (see (2.23)), we obtain $\varphi'_{n,n}(z,\bar{z})=G_{0,n}(z)\langle z,z\rangle^{n}+\dots$. In order to find $G_{0,n}(z)$ we use the normalization condition $\mathrm{tr}^{n}\varphi'_{n,n}(z,\bar{z})=0$. By applying Lemma 2.1.1 to the sum of terms in "..." we obtain that $\varphi'_{n,n}(z,\bar{z})=\left(G_{0,n}(z)-A_n\right)\langle z,z\rangle^{n}+P_5(z,\bar{z})$, where A_n is a determined constant and $\mathrm{tr}^{n}\left(P_5(z,\bar{z})\right)=0$. By the uniqueness of the trace decomposition we obtain that $G_{0,n}=A_n$, for all $n\geq 3$.

(6) Case $(\mathbf{T},\mathbf{0})$ and $(\mathbf{0},\mathbf{T})$ $\mathbf{T}\geq\mathbf{3}$. Collecting the terms of bidegree $(T,0)$ and $(0,T)$ in $(z,\bar{z})$ in (2.25), we obtain

$$\begin{cases} G_{T,0}(z)+\varphi'_{T,0}(z)=\varphi_{T,0}(z)+a(z) \\ \varphi'_{0,T}(\bar{z})=\varphi_{0,T}(\bar{z})+b(\bar{z}) \end{cases}, \tag{2.38}$$

where $a(z)$, $b(\bar{z})$ are the sums of terms that are determined by the induction hypothesis . By using the normalization condition $\varphi'_{0,T}(\bar{z})=\overline{\varphi'_{T,0}(z)}$ we obtain that $G_{T,0}(z)=\varphi_{T,0}(z)-\overline{\varphi_{0,T}(\bar{z})}+a(z)-\overline{b(\bar{z})}$. □

The extended Moser lemma leaves undetermined an infinite number of parameters (see (2.23)). They act on the higher order terms. In order to determine them and complete our partial normal form we will apply in the course of Sections 3 and 4 the following two lemmas:

Lemma 2.1.3. *Let $P(z)$ be a homogeneous pure polynomial. For every $k\in\mathbb{N}^{\star}$, there exist $Q(z)$, $R(z)$ unique polynomials such that*

$$P(z)=Q(z)\Delta(z)^{k}+R(z),\quad \left(\Delta^{k}\right)^{\star}\left(R(z)\right)=0. \tag{2.39}$$

Lemma 2.1.4. *For every homogeneous polynomial $P(z)$ of degree $(t+1)s$ there exists a unique decomposition*

$$P(z)=L(z)+C(z),\quad \left(\Delta_k\Delta^{t}\right)^{\star}\left(C(z)\right)=0,\quad k=1,\dots,N, \tag{2.40}$$

such that $L(z)=\left(\Delta_1(z)A_1(z)+\dots+\Delta_N(z)A_N(z)\right)\Delta(z)^{t}$, where $A_1(z),\dots,A_N(z)$ are linear forms.

Lemma 2.1.3 and Lemma 2.1.4 are particular cases of the Fisher decomposition (see [33]). The polynomial $L(z)$ defined by Lemma 2.1.4 is uniquely determined, but the linear forms $A_1(z),\dots,A_N(z)$ are not necessarily uniquely determined. In order to make them uniquely determined we consider a nondegenerate polynomial $\Delta(z)$ (see (2.4) and Definition 2.0.2).

The following proposition shows us the nondegeneracy condition on $\Delta(z)$ is invariant under any linear change of coordinates:

Proposition 2.1.5. *If $\Delta(z)$ is nondegenerate and $z\longmapsto Az$ is a linear change of coordinates, then $\Delta(Az)$ is also nondegenerate.*

Proof. Let $\widetilde{\Delta}(z)=\Delta(Az)$, where $A=\left\{a_{jk}\right\}_{1\leq j,k\leq N}$. Therefore $\widetilde{\Delta}_j(z)=\sum_{k=1}^{N}\Delta_k(Az)a_{jk}$, for all $j=1,\dots,N$. We consider $\mathscr{L}_1(z),\dots,\mathscr{L}_N(z)$ linear forms such that $\mathscr{L}_1(z)\widetilde{\Delta}_1(z)+\dots+\mathscr{L}_N(z)\widetilde{\Delta}_N(z)\equiv 0$, or equivalently $\sum_{j,k=1}^{N}\Delta_k(Az)\mathscr{L}_j(z)a_{jk}\equiv 0$. Since $\Delta(z)$ is nondegenerate and $\left\{a_{jk}\right\}_{1\leq j,k\leq N}$ is invertible it follows that $\mathscr{L}_1(z)\equiv\dots\equiv\mathscr{L}_N(z)\equiv 0$. □

The system of weights : Following Huang-Yin's approach in [20], we define the system of weights for $z_1, \bar{z}_1, \ldots, z_N, \bar{z}_N$ as follows. We define $\mathrm{wt}\{z_k\} = 1$ and $\mathrm{wt}\{\bar{z}_k\} = s-1$, for all $k = 1, \ldots, N$. If $A(z,\bar{z})$ is a formal power series we write $\mathrm{wt}\{A(z,\bar{z})\} \geq k$ if $A\left(tz, t^{s-1}\bar{z}\right) = \mathrm{O}\left(t^k\right)$. We also write $\mathrm{Ord}\{A(z,\bar{z})\} = k$ if $A(tz, t\bar{z}) = t^k A(z,\bar{z})$. We denote by $\Theta_m^n(z,\bar{z})$ a series in $(z,\bar{z})$ of weight at least m and order at least n. In the particular case when $\Theta_m^n(z,\bar{z})$ is just a polynomial we use the notation $\mathbb{P}_m^n(z,\bar{z})$. We define the set of the normal weights as follows

$$\mathrm{wt}_{nor}\{w\} = 2, \quad \mathrm{wt}_{nor}\{z_1\} = \cdots = \mathrm{wt}_{nor}\{z_N\} = \mathrm{wt}_{nor}\{\bar{z}_1\} = \cdots = \mathrm{wt}_{nor}\{\bar{z}_N\} = 1.$$

Notations : If $h(z,w)$ is a formal power series with no constant term we introduce the following notations

$$\begin{aligned} h(z,w) &= \sum_{l\geq 1} h_{nor}^{(l)}(z,w), \quad \text{where } h_{nor}^{(l)}\left(tz, t^2w\right) = t^l h_{nor}^{(l)}(z,w), \\ h_{\geq l}(z,w) &= \sum_{k\geq l} h_{nor}^{(k)}(z,w). \end{aligned} \tag{2.41}$$

2.2 Proof of Theorem 2.0.3-Case $T+1 = ts+1, t \geq 1$

By applying the extended Moser lemma we can assume that M is given by the following equation

$$w = \langle z, z\rangle + \sum_{m+n\geq 3}^{T+1} \varphi_{m,n}(z,\bar{z}) + \mathrm{O}(\mathrm{T}+2), \tag{2.42}$$

where $\varphi_{m,n}(z,\bar{z})$ satisfies (2.10), for all $3 \leq m+n \leq T$.

We make induction on $T \geq 3$. Assume that (2.11) holds for $\varphi_{k,0}(z)$, for all $k = s+1, \ldots, T$ with $k = 0, 1 \bmod (s)$. If $T+1 \notin \{ts;\, t \in \mathbb{N}^\star - \{1,2\}\} \cup \{ts+1;\, t \in \mathbb{N}^\star\}$ we apply the extended Moser lemma. In the case when $T+1 \in \{ts;\, t \in \mathbb{N}^\star - \{1\}\} \cup \{ts+1;\, t \in \mathbb{N}^\star\}$, we search a formal map which sends our submanifold M to a new submanifold M' given by

$$w' = \langle z', z'\rangle + \sum_{m+n\geq 3}^{T+1} \varphi'_{m,n}\left(z', \overline{z'}\right) + \mathrm{O}(\mathrm{T}+2), \tag{2.43}$$

where $\varphi'_{m,n}\left(z', \overline{z'}\right)$ satisfies (2.10), for all $3 \leq m+n \leq T$ and $\varphi'_{k,0}\left(z'\right)$ satisfies (2.11), for all $k = s+1, \ldots, T$ with $k = 0, 1 \bmod (s)$. We will obtain that $\varphi'_{k,0}(z) = \varphi_{k,0}(z)$ for all $k = s, \ldots, T$.

In the course of this section we consider the case when $T+1 = ts+1$. We are looking for a biholomorphic transformation of the following type

$$\begin{aligned} \left(z', w'\right) &= (z + F(z,w), w + G(z,w)), \\ F(z,w) &= \sum_{l=0}^{T-2t} F_{nor}^{(2t+l)}(z,w), \quad G(z,w) = \sum_{\tau=0}^{T-2t} G_{nor}^{(2t+1+\tau)}(z,w), \end{aligned} \tag{2.44}$$

that maps M into M' up to the degree $T+1 = ts+1$. In order for the preceding mapping to be uniquely determined we assume that $F_{nor}^{(2t+l)}(z,w)$ is normalized as in the extended Moser lemma, for all $l = 1, \ldots T$. Substituting (2.44) into (2.43) we obtain

$$w + G(z,w) = \langle z + F(z,w), z + F(z,w)\rangle + \sum_{m+n\geq 3}^{T+1} \varphi'_{m,n}\left(z + F(z,w), \overline{z + F(z,w)}\right) + \mathrm{O}(\mathrm{T}+2), \tag{2.45}$$

where w satisfies (2.42). By making some simplifications in (2.45) by using (2.42), we obtain

$$\begin{aligned}\sum_{\tau=0}^{T-2t} G_{nor}^{(2t+1+\tau)}(z,\langle z,z\rangle+\varphi_{\geq 3}(z,\bar z)) &= 2\Re\left\langle z, \sum_{l=0}^{T-2t} F_{nor}^{(2t+l)}(z,\langle z,z\rangle+\varphi_{\geq 3}(z,\bar z))\right\rangle \\ &+\left\|\sum_{l=0}^{T-2t} F_{nor}^{(2t+l)}(z,\langle z,z\rangle+\varphi_{\geq 3}(z,\bar z))\right\|^2 \\ &+\varphi'_{\geq 3}\left(z+\sum_{l=0}^{T-2t} F_{nor}^{(2t+l)}(z,\langle z,z\rangle+\varphi_{\geq 3}(z,\bar z)), \overline{z+\sum_{l=0}^{T-2t} F_{nor}^{(2t+l)}(z,\langle z,z\rangle+\varphi_{\geq 3}(z,\bar z))}\right)-\varphi_{\geq 3}(z,\bar z).\end{aligned} \tag{2.46}$$

Collecting the terms with the same bidegree in (2.46), we find $F(z,w)$ and $G(z,w)$ by applying the extended Moser lemma. Since we don't have components of $F(z,w)$ of normal weight less than $2t$ and $G(z,w)$ with normal weight less than $2t+1$, collecting in (2.46) the terms with the same bidegree (m,n) in $(z,\bar z)$ with $m+n<2t+1$, we obtain that $\varphi'_{m,n}(z,\bar z)=\varphi_{m,n}(z,\bar z)$.

Collecting the terms of bidegree (m,n) in $(z,\bar z)$ with $m+n=2t+1$ (like in the proof of the extended Moser lemma) we find $G_{nor}^{(2t+1)}(z,w)$ and $F_{nor}^{(2t)}(z,w)$ as follows. We prove the following lemma:

Lemma 2.2.1. *$G_{nor}^{(2t+1)}(z,w)=0$, $F_{nor}^{(2t)}(z,w)=aw^t-z\langle z,a\rangle w^{t-1}$, where $a=(a_1,\dots,a_N)\in\mathbb{C}^N$.*

Proof. Collecting the pure terms of degree $2t+1$ in (2.46), we obtain that $\varphi_{2t+1,0}(z)=\varphi'_{2t+1,0}(z)$. Collecting the terms of bidegree (m,n) with $m+n=2t+1$ in $(z,\bar z)$ and $0<m<n-1$ (2.46), we obtain

$$\varphi'_{m,n}(z,\bar z)=-\langle z,F_{n-m+1,m-1}(z)\rangle\langle z,z\rangle^{m-1}+\varphi_{m,n}(z,\bar z). \tag{2.47}$$

Since $\varphi_{m,n}(z,\bar z)$, $\varphi'_{m,n}(z,\bar z)$ satisfy (2.10), by the uniqueness of the trace decomposition, we obtain $F_{n-m+1,m-1}(z)=0$. Collecting the terms of bidegree (m,n) in $(z,\bar z)$ with $m+n=2t+1$ and $m>n+1$ in (2.46), we obtain

$$\varphi'_{m,n}(z,\bar z)=G_{m-n,n}(z)\langle z,z\rangle^n-\langle F_{m-n+1,n-1}(z),z\rangle\langle z,z\rangle^{n-1}+\varphi_{m,n}(z,\bar z). \tag{2.48}$$

Since $F_{m-n+1,n-1}(z)=0$ it follows that $G_{m-n,n}(z)=0$. Collecting the terms of bidegree $(t-1,t)$ and $(t,t-1)$ in $(z,\bar z)$ in (2.46), we obtain the following two equations

$$\begin{aligned}\varphi'_{t-1,t}(z,\bar z)&=-\left(\langle F_{0,t-1}(z),z\rangle\langle z,z\rangle+\langle z,F_{2,t-2}(z)\rangle\right)\langle z,z\rangle^{t-2}+\varphi_{t-1,t}(z,\bar z),\\ \varphi'_{t,t-1}(z,\bar z)&=G_{1,t-1}(z)\langle z,z\rangle^{t-1}-\left(\langle F_{2,t-2}(z),z\rangle+\langle z,F_{0,t-1}(z)\rangle\langle z,z\rangle\right)\langle z,z\rangle^{t-2}+\varphi_{t,t-1}(z,\bar z).\end{aligned} \tag{2.49}$$

By using (2.49) it follows that $G_{1,t-1}(z)=0$. We set $F_{0,t-1}(z)=a=:(a_1,\dots,a_N)$ and we write $F_{2,t-2}(z)=\left(F_{2,t-2}^1(z),\dots,F_{2,t-2}^N(z)\right)$. Since $\varphi_{m,n}(z,\bar z)$, $\varphi'_{m,n}(z,\bar z)$ satisfy (2.10), by the uniqueness of the trace decomposition, by (2.49) we obtain the equation $\langle z,a\rangle\langle z,z\rangle+\langle F_{2,t-2}(z),z\rangle=0$, that can be solved as

$$F_{2,t-2}^k(z)=-\frac{\partial}{\partial\bar z_k}\left(\langle z,a\rangle\langle z,z\rangle\right)=-z_k\langle z,a\rangle,\quad k=1,\dots,N. \tag{2.50}$$

Therefore $F_{nor}^{(2t)}(z,w)=aw^t-z\langle z,a\rangle w^{t-1}$, where $a=(a_1,\dots,a_N)\in\mathbb{C}^N$. □

By Lemma 2.2.1 and by (2.41) we conclude that $F(z,w)=F_{nor}^{(2t)}(z,w)+F_{\geq 2t+1}(z,w)$ and $G(z,w)=G_{\geq 2t+2}(z,w)$. We also have $F_{\geq 2t+1}(z,w)=\sum\limits_{k+2l\geq 2t+1}F_{k,l}(z)w^l$, where $F_{k,l}(z)$ is a homogeneous polynomial of degree k. It follows that

$$\operatorname{wt}\{F_{\geq 2t+1}(z,w)\}\geq\min_{k+2l\geq 2t+1}\{k+ls\}\geq\min_{k+2l\geq 2t+1}\{k+2l\}\geq 2t+1. \tag{2.51}$$

Next, we prove that $\operatorname{wt}\left\{\overline{F_{\geq 2t+1}(z,w)}\right\}\geq ts+s-1$. Since $\operatorname{wt}\left\{\overline{F_{\geq 2t+1}(z,w)}\right\}\geq\min\limits_{k+2l\geq 2t+1}\{k(s-1)+ls\}$, it is enough to prove that $k(s-1)+ls\geq ts+s-1$ for $k+2l\geq 2t+1$. Since we can write the latter inequality as $(k-1)(s-1)+ls\geq ts$, for $(k-1)+2l\geq 2t$, it is enough to prove that $k(s-1)+ls\geq ts$, for $k+2l\geq 2t$. Since

$s \geq 3$ it follows that $ks - 2k \geq 0$. Hence $2k(s-1) + 2ls \geq ks + 2ls$. It follows that $k(s-1) + ls \geq \frac{s}{2}(k+2l) \geq \frac{2ts}{2} = ts$.

Lemma 2.2.2. *By using the previous calculations, we give the following immediate estimates*

$$\begin{gathered}\operatorname{wt}\left\{F_{\geq 2t+1}(z,w)\right\} \geq 2t+1, \quad \operatorname{wt}\left\{\overline{F_{\geq 2t+1}(z,w)}\right\} \geq ts+s-1, \quad \operatorname{wt}\left\{\|F_{\geq 2t+1}(z,w)\|^2\right\} \geq ts+2, \\ \operatorname{wt}\left\{F_{nor}^{(2t)}(z,w)\right\} \geq ts+2-s, \quad \operatorname{wt}\left\{\overline{F_{nor}^{(2t)}(z,w)}\right\} \geq ts, \quad \operatorname{wt}\left\{\left\|F_{nor}^{(2t)}(z,w)\right\|^2\right\} \geq ts+2, \\ \operatorname{wt}\left\{\left\langle F_{nor}^{(2t)}(z,w), F_{\geq 2t+1}(z,w)\right\rangle\right\}, \qquad \operatorname{wt}\left\{\left\langle F_{\geq 2t+1}(z,w), F_{nor}^{(2t)}(z,w)\right\rangle\right\} \geq ts+2,\end{gathered} \tag{2.52}$$

where w satisfies (2.42).

As a consequence of the preceding estimates, we obtain

$$\|F(z,w)\|^2 = \left\|F_{nor}^{(2t)}(z,w)\right\|^2 + 2\Re\left\langle F_{nor}^{(2t)}(z,w), F_{\geq 2t+1}(z,w)\right\rangle + \|F_{\geq 2t+1}(z,w)\|^2 = \Theta_{ts+2}^{2t+2}(z,\bar{z}), \tag{2.53}$$

where w satisfies (2.42). We observe that the preceding power series $\Theta_{ts+2}^{2t+2}(z,\bar{z})$ has the property $\operatorname{wt}\left\{\overline{\Theta_{ts+2}^{2t+2}(z,\bar{z})}\right\} \geq ts+2$.

In order to apply the extended Moser lemma in (2.46) we have to evaluate the weight and the order of the terms which appear and are not "good". Beside the previous weight estimates (see (2.52) and (2.53)) we also need to prove the following lemmas:

Lemma 2.2.3. *For all $m,n \geq 1$ and w satisfying (2.42), we have the following estimate*

$$\varphi'_{m,n}\left(z+F(z,w), \overline{z+F(z,w)}\right) = \varphi'_{m,n}(z,\bar{z}) + 2\Re\left\langle \Theta_s^2(z,\bar{z}), \overline{F_{\geq 2t+1}(z,w)}\right\rangle + \Theta_{ts+2}^{2t+2}(z,\bar{z}), \tag{2.54}$$

where $\operatorname{wt}\left\{\overline{\Theta_{ts+2}^{2t+2}(z,\bar{z})}\right\} \geq ts+2$.

Proof. We make the expansion $\varphi'_{m,n}\left(z+F(z,w), \overline{z+F(z,w)}\right) = \varphi'_{m,n}(z,\bar{z}) + \dots$, where in "$\dots$" we have different types of terms involving $F_{k',l'}(z)$ with $k'+2l' < m+n$ and normalized terms $\varphi_{k,l}(z,\bar{z})$, $\varphi'_{k,l}(z,\bar{z})$ with $k+l < m+n$. In order to study the weight and the order of terms which can appear in "$\dots$" it is enough to study the weight and the order of the following particular terms

$$A_1(z,w) = F_1(z,w)z^I\bar{z}^J, \quad A_2(z,w) = z^{I_1}\bar{z}^{J_1}\overline{F_1(z,w)}, \quad B_1(z,w) = F_2(z,w)z^I\bar{z}^J, \quad B_2(z,w) = \overline{F_2(z,w)}z^{I_1}\bar{z}^{J_1},$$

where $F_1(z,w)$ is the first component of $F_{nor}^{(2t)}(z,w)$ and $F_2(z,w)$ is the first component of $F_{\geq 2t+1}(z,w)$. Here we assume that $|I| = m-1$, $|I_1| = m$, $|J_1| = n-1$, $|J| = n$.

By using (2.52) we obtain $\operatorname{wt}\{A_1(z,w)\} \geq m-1+ts+2-s+n(s-1) \geq ts+2$. It is equivalent to prove that $m-1+s(n-1)-n \geq 0$. This is true because $m-1+s(n-1)-n \geq m-1+3(n-1)-n \geq m+3n-4-n \geq 3+n-4 \geq 0$. On the other hand, we have $\operatorname{Ord}\{A_1(z,w)\} \geq m-1+2t+n \geq 2t+2$.

By using (2.52) we obtain $\operatorname{wt}\{A_2(z,w)\} \geq m+ts+(n-1)(s-1) \geq ts+2 \Longleftrightarrow m+(s-1)(n-1) \geq 2$. We have $m+(n-1)(s-1) \geq m+2(n-1) \geq m+2n-4 \geq 0$, and this is true because $m+n \geq 3$ and $m,n \geq 1$. On the other hand we have $\operatorname{Ord}\{A_2(z,w)\} \geq m+2t+n-1 \geq 2t+2$.

In the same way we obtain that $\operatorname{Ord}\{B_1(z,w)\}$, $\operatorname{Ord}\{B_2(z,w)\} \geq 2t+1$. By using (2.52), every term in "$\dots$" that depends on $F_2(z,w)$ can be written as $\Theta_s^2(z,\bar{z})F_2(z,w)$. From here we obtain our lemma. □

Lemma 2.2.4. *For w satisfying (2.42) and for all $k > s$, we have the following estimation*

$$\varphi'_k(z+F(z,w)) = \varphi'_k(z) + 2\Re\left\langle \Theta_s^2(z,\bar{z}), \overline{F_{\geq 2t+1}(z,w)}\right\rangle + \Theta_{ts+2}^{2t+2}(z,\bar{z}), \tag{2.55}$$

where $\operatorname{wt}\left\{\overline{\Theta_{ts+2}^{2t+2}(z,\bar{z})}\right\} \geq ts+2$.

Proof. We make the expansion $\varphi_k'(z+F(z,w)) = \varphi_k'(z)+\dots$. In order to study the weight and the order of terms which can appear in "..." it is enough to study the weight and the order of the following terms

$$A(z,w) = F_1(z,w)z^I, \quad B(z,w) = F_2(z,w)z^I,$$

where $F_1(z,w)$ is the first component of $F_{nor}^{(2t)}(z,w)$ and $F_2(z,w)$ is the first component of $F_{\geq 2t+1}(z,w)$. Here we assume that $|I| = m-1 \geq s$. Then, by (2.52), we obtain that $\mathrm{wt}\{A(z,w)\} \geq s+ts+2-s \geq ts+2$. On the other hand, we have $\mathrm{Ord}\{A(z,w)\} \geq s+2t \geq 2t+2$. By using (2.52), every term in "..." that depends on $F_2(z,w)$ can be written as $\Theta_s^2(z,\bar{z})F_2(z,w)$. From here we obtain our lemma. □

We want to evaluate the weight and the order of the other terms of (2.46). By Lemma 2.3.3 and by Lemma 2.3.4, it remains to evaluate the order and the weight of the terms of the following expression

$$\begin{aligned} S(z,\bar{z}) &= 2\Re\langle F(z,w),z\rangle + 2\Re\left\{\varphi_s'(z+F(z,w))\right\}, \\ &= 2\Re\left\langle F_{nor}^{(2t)}(z,w)+F_{\geq 2t+1}(z,w),z\right\rangle + 2\Re\left\{\Delta\left(z+F_{nor}^{(2t)}(z,w)+F_{\geq 2t+1}(z,w)\right)\right\}, \end{aligned} \tag{2.56}$$

where w satisfies (2.42).

Lemma 2.2.5. *For $F_{nor}^{(2t)}(z,w)$ given by Lemma 2.3.1 and w satisfying (2.42) we have*

$$2\Re\left\langle F_{nor}^{(2t)}(z,w),z\right\rangle = 2\Re\left\{\langle z,a\rangle\Delta(z)w^{t-1}\right\} + \Theta_{ts+2}^{2t+2}(z,\bar{z}), \tag{2.57}$$

where $\mathrm{wt}\left\{\overline{\Theta_{ts+2}^{2t+2}(z,\bar{z})}\right\} \geq ts+2$.

Proof. We compute

$$\begin{aligned} 2\Re\left\langle F_{nor}^{(2t)}(z,w),z\right\rangle &= 2\Re\left\{w^t\langle a,z\rangle\right\} - 2\Re\left\{\langle z,a\rangle\langle z,z\rangle w^{t-1}\right\}, \\ &= 2\Re\left\{\langle z,a\rangle w^t - \langle z,a\rangle\langle z,z\rangle w^{t-1}\right\} + \langle a,z\rangle\left(w^t-\overline{w}^t\right) + \langle z,a\rangle\left(\overline{w}^t-w^t\right), \\ &= 2\Re\left\{\langle z,a\rangle\Delta(z)w^{t-1}\right\} + \Theta_{ts+2}^{2t+2}(z,\bar{z}), \end{aligned} \tag{2.58}$$

where $\mathrm{wt}\left\{\overline{\Theta_{ts+2}^{2t+2}(z,\bar{z})}\right\} \geq ts+2$. □

In the course of our proof we will use the notation $\Delta'(z) = (\Delta_1(z),\dots,\Delta_N(z))$. It remains to prove the following lemma

Lemma 2.2.6. *For w satisfying (2.42) we have the following estimate*

$$\begin{aligned} 2\Re\{\Delta(z+F(z,w))\} =& 2\Re\left\{\Delta(z)-s\langle z,a\rangle\Delta(z)w^{t-1}\right\} \\ &+ 2\Re\left\langle\Delta'(z)+\Theta_s^2(z,\bar{z}),\overline{F_{\geq 2t+1}(z,w)}\right\rangle + \Theta_{ts+2}^{2t+2}(z,\bar{z}), \end{aligned} \tag{2.59}$$

where $\mathrm{wt}\left\{\overline{\Theta_{ts+2}^{2t+2}(z,\bar{z})}\right\} \geq ts+2$.

Proof. By using the Taylor expansion it follows that

$$2\Re\{\Delta(z+F(z,w))\} = 2\Re\left\{\Delta(z)+\sum_{k=1}^{N}\Delta_k(z)F_{\geq 2t}^k(z,w)+L(z,\bar{z})\right\}, \tag{2.60}$$

where $F_{\geq 2t}^k(z,w) = \left(F_{\geq 2t}^1(z,w),\dots,F_{\geq 2t}^N(z,w)\right)$ and $L(z,\bar{z}) = \left\langle\Theta_s^2(z,\bar{z}),\overline{F_{\geq 2t+1}(z,w)}\right\rangle$. We compute

$$\begin{aligned} \sum_{k=1}^{N} 2\Re\left\{\Delta_k(z)F_{\geq 2t}^k(z,w)\right\} &= \sum_{k=1}^{N} 2\Re\left\{\Delta_k(z)\left(a_kw^t - z_k\langle z,a\rangle w^{t-1} + F_{\geq 2t+1}^k(z,w)\right)\right\}, \\ &= \Theta_{ts+2}^{2t+2}(z,\bar{z}) - 2s\Re\left\{\langle z,a\rangle\Delta(z)w^{t-1}\right\} + 2\Re\left\langle\Delta'(z),\overline{F_{\geq 2t+1}(z,w)}\right\rangle, \end{aligned} \tag{2.61}$$

where $\mathrm{wt}\left\{\overline{\Theta^{2t+2}_{ts+2}(z,\bar{z})}\right\} \geq ts+2$. □

For w satisfying (2.42), by Lemma 2.2.5 and by Lemma 2.2.6, we can rewrite (2.56) as follows

$$S(z,\bar{z}) = 2(1-s)\Re\left\{\langle z,a\rangle\Delta(z)w^{t-1}\right\} + 2\Re\left\langle \bar{z}+\Delta'(z)+\Theta^2_s(z,\bar{z}), \overline{F_{\geq 2t+1}(z,w)}\right\rangle + \Theta^{2t+2}_{ts+2}(z,\bar{z}), \tag{2.62}$$

where $\mathrm{wt}\left\{\overline{\Theta^{2t+2}_{ts+2}(z,\bar{z})}\right\} \geq ts+2$. By Lemmas 2.2.1-2.2.6 we obtain

$$\begin{aligned} G_{\geq 2t+2}\left(z,\langle z,z\rangle+\varphi_{\geq 3}(z,\bar{z})\right) =& 2(1-s)\Re\left\{\langle z,a\rangle\Delta(z)\left(\langle z,z\rangle+\varphi_{\geq 3}(z,\bar{z})\right)^{t-1}\right\} \\ &+ 2\Re\left\langle \bar{z}+\Delta'(z)+\Theta^2_s(z,\bar{z}), \overline{F_{\geq 2t+1}\left(z,\langle z,z\rangle+\varphi_{\geq 3}(z,\bar{z})\right)}\right\rangle \\ &+ \varphi_{\geq 2t+2}(z,\bar{z}) - \varphi'_{\geq 2t+2}(z,\bar{z}) + \Theta^{2t+2}_{ts+2}(z,\bar{z}), \end{aligned} \tag{2.63}$$

where $\mathrm{wt}\left\{\overline{\Theta^{2t+2}_{ts+2}(z,\bar{z})}\right\} \geq ts+2$.

Assume that $t=1$. Collecting the terms of total degree $k<s+1$ in $(z,\bar{z})$ in (2.63) we find the polynomials $\left(G^{(k+1)}_{nor}(z,w), F^{(k)}_{nor}(z,w)\right)$ for all $k<s$. Collecting the terms of total degree $m+n=s+1$ in $(z,\bar{z})$ in (2.63), we obtain

$$G^{(s+1)}_{nor}\left(z,\langle z,z\rangle\right) = 2(1-s)\Re\left\{\langle z,a\rangle\Delta(z)\right\} + 2\Re\left\langle z, F^{(s)}_{nor}\left(z,\langle z,z\rangle\right)\right\rangle + \varphi'_{s+1}(z,\bar{z}) - \varphi_{s+1}(z,\bar{z}) + (\Theta_1)^{s+1}_{s+2}(z,\bar{z}). \tag{2.64}$$

By applying the extended Moser lemma we find a solution $\left(G^{(s+1)}_{nor}(z,w), F^{(s)}_{nor}(z,w)\right)$ for the latter equation. We consider the following Fisher decompositions

$$\varphi_{s+1,0}(z) = Q(z)\Delta(z)+R(z), \quad \varphi'_{s+1,0}(z) = Q'(z)\Delta(z)+R'(z), \tag{2.65}$$

where $\Delta^\star\left(R(z)\right) = \Delta^\star\left(R'(z)\right) = 0$. We want to put the normalization condition $\Delta^\star\left(\varphi'_{s+1,0}(z)\right) = 0$. Collecting the pure terms of degree $s+1$ in (2.64), by (2.65) we obtain

$$\varphi'_{s+1,0}(z) = \varphi_{s+1,0}(z) - (1-s)\langle z,a\rangle\Delta(z) = \left(Q(z)-(1-s)\langle z,a\rangle\right)\Delta(z)+R(z), \tag{2.66}$$

where $Q(z)$ is a determined polynomial of degree 1 in $z_1,\ldots,z_N$. It follows that $Q'(z) = Q(z)-(1-s)\langle z,a\rangle$ and $R'(z) = R(z)$. Then the normalization condition $\Delta^\star\left(\varphi'_{s+1,0}(z)\right) = 0$ is equivalent to finding a such that $Q'(z) = Q(z)-(1-s)\langle z,a\rangle = 0$. The last equation provides us the free parameter a.

Assuming that $t\geq 2$, we prove the following lemma (this is the analogue of Lemma 3.3 of Huang-Yin's paper [20]):

Lemma 2.2.7. *Let $N_s := ts+2$. For all $0\leq j\leq t-1$ and $p\in[2t+j(s-2)+2, 2t+(j+1)(s-2)+1]$, we make the following estimate*

$$\begin{aligned} G_{\geq p}(z,w) =& 2(1-s)^{j+1}\Re\left\{\langle z,a\rangle\Delta(z)^{j+1}w^{t-j-1}\right\} + 2\Re\left\langle \bar{z}+\Delta'(z)+\Theta^2_s(z,\bar{z}), \overline{F_{\geq p-1}(z,w)}\right\rangle \\ &+ \varphi'_{\geq p}(z,\bar{z}) - \varphi_{\geq p}(z,\bar{z}) + \Theta^p_{N_s}(z,\bar{z}), \end{aligned} \tag{2.67}$$

where $\mathrm{wt}\left\{\overline{\Theta^{2t+2}_{N_s}(z,\bar{z})}\right\} \geq N_s$ and w satisfies (2.42).

Proof.

Step 1. When $s=3$ this step is obvious. Assume that $s>3$. Let $p_0 = 2t+j(s-2)+2$, where $j\in[0,t-1]$. We make induction on $p\in[2t+j(s-2)+2, 2t+(j+1)(s-2)+1]$. For $j=0$ (therefore $p=2t+2$) the lemma is satisfied (see equation (2.63)). Let $p\geq p_0$ such that $p+1\leq 2t+(j+1)(s-2)+1$. Collecting the terms of bidegree (m,n) in $(z,\bar{z})$ in (2.67) with $m+n=p$, we obtain

$$G^{(p)}_{nor}\left(z,\langle z,z\rangle\right) = 2\Re\left\langle z, F^{(p-1)}_{nor}\left(z,\langle z,z\rangle\right)\right\rangle + \varphi'_p(z,\bar{z}) - \varphi_p(z,\bar{z}) + \mathbb{P}^p_{N_s}(z,\bar{z}). \tag{2.68}$$

By applying the extended Moser lemma we find a solution $\left(F_{nor}^{(p-1)}(z,w), G_{nor}^{(p)}(z,w)\right)$ for (2.68). Assume that p is even. In this case we find $F_{nor}^{(p-1)}(z,w)$ recalling the cases 1 and 3 of the proof of the extended Moser lemma. By using the cases 2 and 4 of the proof of the extended Moser lemma we find $G_{nor}^{(p)}(z,w)$. Since $\mathrm{wt}\left\{\mathbb{P}_{N_s}^{2t+2}(z,\bar{z})\right\} \geq N_s$ we obtain $\mathrm{wt}\left\{\left\langle F_{nor}^{(p-1)}(z,\langle z,z\rangle),z\right\rangle\right\}, \mathrm{wt}\left\{\overline{\left\langle F_{nor}^{(p-1)}(z,\langle z,z\rangle),z\right\rangle}\right\} \geq N_s$. Also $\mathrm{wt}\left\{G_{nor}^{(p)}(z,\langle z,z\rangle)\right\}$, $\mathrm{wt}\left\{\overline{G_{nor}^{(p)}(z,\langle z,z\rangle)}\right\} \geq N_s$. We can bring similar arguments as well when p is even. We obtain the following estimates

$$\begin{gathered}\mathrm{wt}\left\{F_{nor}^{(p-1)}(z,w)\right\} \geq N_s - s + 1,\ \mathrm{wt}\left\{\overline{F_{nor}^{(p-1)}(z,w)}\right\} \geq N_s - 1,\\ \mathrm{wt}\left\{\overline{F_{nor}^{(p-1)}(z,w) - F_{nor}^{(p-1)}(z,\langle z,z\rangle)}\right\} \geq N_s - 1,\quad \mathrm{wt}\left\{F_{nor}^{(p-1)}(z,w) - F_{nor}^{(p-1)}(z,\langle z,z\rangle)\right\} \geq N_s - s + 1,\\ \mathrm{wt}\left\{G_{nor}^{(p)}(z,w)\right\} \geq N_s,\ \mathrm{wt}\left\{G_{nor}^{(p)}(z,w) - G_{nor}^{(p)}(z,\langle z,z\rangle)\right\} \geq N_s,\end{gathered} \tag{2.69}$$

where w satisfies (2.42). As a consequence of (2.68) we obtain

$$\begin{gathered}G_{nor}^{(p)}(z,w) - G_{nor}^{(p)}(z,\langle z,z\rangle) = \Theta_{N_s}^{p+1}(z,\bar{z})',\quad 2\Re\left\langle z, F_{nor}^{(p-1)}(z,w) - F_{nor}^{(p-1)}(z,\langle z,z\rangle)\right\rangle = \Theta_{N_s}^{p+1}(z,\bar{z})',\\ \left\langle \Delta'(z) + \Theta_s^2(z,\bar{z}), \overline{F_{nor}^{(p-1)}(z,w)}\right\rangle + \left\langle \overline{F_{nor}^{(p-1)}(z,w), \Delta'(z) + \Theta_s^2(z,\bar{z})}\right\rangle = \Theta_{N_s}^{p+1}(z,\bar{z})',\end{gathered} \tag{2.70}$$

and each of the preceding formal power series $\Theta_{N_s}^{p+1}(z,\bar{z})'$ has the property $\mathrm{wt}\left\{\overline{\Theta_{N_s}^{p+1}(z,\bar{z})'}\right\} \geq N_s$. Substituting $F_{\geq p-1}(z,w) = F_{nor}^{(p-1)}(z,w) + F_{\geq p}(z,w)$ and $G_{\geq p}(z,w) = G_{nor}^{(p)}(z,w) + G_{\geq p+1}(z,w)$ into (2.67), we obtain

$$\begin{aligned}G_{nor}^{((p)}(z,w) + G_{\geq p+1}(z,w) =& 2(1-s)^{j+1}\Re\left\{\langle z,a\rangle\Delta(z)^{j+1}w^{t-j-1}\right\}\\ &+ 2\Re\left\langle \bar{z} + \Delta'(z) + \Theta_s^2(z,\bar{z}), \overline{F_{nor}^{(p-1)}(z,w) + F_{\geq p}(z,w)}\right\rangle + \varphi_p'(z,\bar{z}) - \varphi_p(z,\bar{z})\\ &+ \mathbb{P}_{N_s}^{p}(z,\bar{z}) + \varphi_{\geq p+1}'(z,\bar{z}) - \varphi_{\geq p+1}(z,\bar{z}) + \Theta_{N_s}^{p+1}(z,\bar{z}).\end{aligned} \tag{2.71}$$

Collecting the pure terms of degree p in (2.68), it follows that $\varphi_{p,0}(z) = \varphi_{p,0}'(z) + \ldots$, where in "..." we have determined terms with the weight less than $p < N_s := ts + 2$. Therefore $\varphi_{p,0}(z) = \varphi_{p,0}'(z)$. We will obtain that $\varphi_{k,0}(z) = \varphi_{k,0}'(z)$, for all $k = 3, \ldots, T$. By making a simplification in (2.71) by using (2.68), it follows that

$$\begin{aligned}G_{\geq p+1}(z,w) =& 2(1-s)^{j+1}\Re\left\{\langle z,a\rangle\Delta(z)^{j+1}w^{t-j-1}\right\} + 2\Re\left\langle \bar{z} + \Delta'(z) + \Theta_s^2(z,\bar{z}), \overline{F_{\geq p}(z,w)}\right\rangle\\ &+ \varphi_{\geq p+1}'(z,\bar{z}) - \varphi_{\geq p+1}(z,\bar{z}) + J(z,\bar{z}) + \Theta_{N_s}^{p+1}(z,\bar{z}),\end{aligned} \tag{2.72}$$

where $\mathrm{wt}\left\{\overline{\Theta_{N_s}^{p+1}(z,\bar{z})}\right\} \geq N_s$ and

$$\begin{aligned}J(z,\bar{z}) =& 2\Re\left\langle z, F_{nor}^{(p-1)}(z,w) - F_{nor}^{(p-1)}(z,\langle z,z\rangle)\right\rangle + 2\Re\left\langle \Delta'(z) + \Theta_s^2(z,\bar{z}), \overline{F_{nor}^{(p-1)}(z,w)}\right\rangle\\ &+ G_{nor}^{(p)}(z,\langle z,z\rangle) - G_{nor}^{(p)}(z,w).\end{aligned} \tag{2.73}$$

By using (2.69) and (3.16) it follows that $J(z,\bar{z}) = \Theta_{N_s}^{p+1}(z,\bar{z})$, where $\mathrm{wt}\left\{\overline{\Theta_{N_s}^{p+1}(z,\bar{z})}\right\} \geq N_s$.

Step 2. Assume that we have proved Lemma 2.2.7 for $p \in [2t + j(s-2) + 2, 2t + (j+1)(s-2) + 1]$ for all $j \in [0, t-1]$. We shall prove Lemma 2.2.7 for $p \in [2t + (j+1)(s-2) + 2, 2t + (j+2)(s-2) + 1]$. Collecting the terms of bidegree (m,n) in $(z,\bar{z})$ in (2.67) with $m + n = \Lambda + 1 := 2t + (j+1)(s-2) + 1$, we obtain

$$\begin{aligned}G_{nor}^{(\Lambda+1)}(z,\langle z,z\rangle) =& 2(1-s)^{j+1}\Re\left\{\langle z,a\rangle\Delta(z)^{j+1}\langle z,z\rangle^{t-j-1}\right\} + 2\Re\left\langle z, F_{nor}^{(\Lambda)}(z,\langle z,z\rangle)\right\rangle\\ &+ \varphi_{\Lambda+1}'(z,\bar{z}) - \varphi_{\Lambda+1}(z,\bar{z}) + \mathbb{P}_{N_s}^{\Lambda+1}(z,\bar{z}).\end{aligned} \tag{2.74}$$

Here $\mathrm{wt}\left\{\overline{(\Theta_1)_{N_s}^{\Lambda+1}(z,\bar{z})}\right\} \geq N_s$. We define the following map

$$F_{nor}^{(\Lambda)}(z,w) = F_1^{(\Lambda)}(z,w) + F_2^{(\Lambda)}(z,w), \quad F_1^{(\Lambda)}(z,w) = -(1-s)^{j+1}\langle z,a\rangle\Delta(z)^{j+1}w^{t-j-2}(z_1,\dots,z_N). \tag{2.75}$$

Substituting (2.75) into (2.74), we obtain

$$G_{nor}^{\Lambda+1}(z,\langle z,z\rangle) = 2\Re\left\langle z, F_2^{(\Lambda)}(z,\langle z,z\rangle)\right\rangle + \varphi'_{\Lambda+1}(z,\bar{z}) - \varphi_{\Lambda+1}(z,\bar{z}) + \mathbb{P}_{N_s}^{\Lambda+1}(z,\bar{z}). \tag{2.76}$$

By applying the extended Moser lemma we find a solution $\left(G_{nor}^{(\Lambda+1)}(z,w), F_2^{(\Lambda)}(z,w)\right)$ for (2.76). By using the same arguments as in the Step 1 we obtain the following estimates

$$\begin{gathered}\mathrm{wt}\left\{G_{nor}^{(\Lambda+1)}(z,w) - G_{nor}^{(\Lambda+1)}(z,\langle z,z\rangle)\right\}, \mathrm{wt}\left\{G_{nor}^{(\Lambda+1)}(z,w)\right\},\ \mathrm{wt}\left\{G_{nor}^{(\Lambda+1)}(z,\langle z,z\rangle)\right\} \geq N_s,\\ \mathrm{wt}\left\{F_2^{(\Lambda)}(z,w) - F_2^{(\Lambda)}(z,\langle z,z\rangle)\right\},\ \mathrm{wt}\left\{F_2^{(\Lambda)}(z,w)\right\}, \mathrm{wt}\left\{F_{2,k}^{(\Lambda)}(z,\langle z,z\rangle)\right\} \geq N_s - s + 1,\\ \mathrm{wt}\left\{\overline{F_2^{(\Lambda)}(z,w) - F_2^{(\Lambda)}(z,\langle z,z\rangle)}\right\}, \mathrm{wt}\left\{\overline{F_2^{(\Lambda)}(z,w)}\right\},\ \mathrm{wt}\left\{\overline{F_2^{(\Lambda)}(z,\langle z,z\rangle)}\right\} \geq N_s - 1,\end{gathered} \tag{2.77}$$

where w satisfies (2.42). As a consequence of (2.77) we obtain

$$\begin{gathered}\left\langle \Delta'(z) + \Theta_s^2(z,\bar{z}), \overline{F_2^{(\Lambda)}(z,w)}\right\rangle + \left\langle \overline{F_2^{(\Lambda)}(z,w)}, \Delta'(z) + \Theta_s^2(z,\bar{z})\right\rangle = \Theta_{N_s}^{\Lambda+2}(z,\bar{z})',\\ G_{nor}^{(\Lambda+1)}(z,w) - G_{nor}^{(\Lambda+1)}(z,\langle z,z\rangle) = \Theta_{N_s}^{\Lambda+2}(z,\bar{z})', \quad 2\Re\left\langle F_2^{(\Lambda)}(z,w) - F_2^{(\Lambda)}(z,\langle z,z\rangle), z\right\rangle = \Theta_{N_s}^{\Lambda+2}(z,\bar{z})',\end{gathered} \tag{2.78}$$

where w satisfies (2.42) and each of the preceding formal power series has the property $\mathrm{wt}\left\{\overline{\Theta_{N_s}^{\Lambda+2}(z,\bar{z})}\right\} \geq N_s$. Substituting $F_{\geq\Lambda}(z,w) = F_{nor}^{(\Lambda)}(z,w) + F_{\geq\Lambda+1}(z,w)$ and $G_{\geq\Lambda+1}(z,w) = G_{nor}^{(\Lambda+1)}(z,w) + G_{\geq\Lambda+2}(z,w)$ in (2.67), we obtain

$$\begin{aligned}G_{nor}^{(\Lambda+1)}(z,w) + G_{\geq\Lambda+2}(z,w) = {} & 2(1-s)^{j+1}\Re\left\{\langle z,a\rangle\Delta(z)^{j+1}w^{t-j-1}\right\}\\ & + 2\Re\left\langle \bar{z} + \Delta'(z) + \Theta_s^2(z,\bar{z}), \overline{F_{nor}^{(\Lambda)}(z,w)} + \overline{F_{\geq\Lambda+1}(z,w)}\right\rangle + \varphi_{\Lambda+1}(z,\bar{z}) - \varphi'_{\Lambda+1}(z,\bar{z})\\ & + \varphi'_{\geq\Lambda+2}(z,\bar{z}) - \varphi_{\geq\Lambda+2}(z,\bar{z}) + \mathbb{P}_{N_s}^{\Lambda+1}(z,\bar{z}) + \Theta_{N_s}^{\Lambda+2}(z,\bar{z}).\end{aligned} \tag{2.79}$$

By making a simplification in (2.79) with (2.74), and then by using (2.75), we obtain

$$G_{\geq\Lambda+2}(z,w) = 2\Re\left\langle \bar{z} + \Delta'(z) + \Theta_s^2(z,\bar{z}), \overline{F_{\geq\Lambda+1}(z,w)}\right\rangle + \varphi'_{\geq\Lambda+2}(z,\bar{z}) - \varphi_{\geq\Lambda+2}(z,\bar{z}) + \Theta_{N_s}^{\Lambda+2}(z,\bar{z}) + J(z,\bar{z}), \tag{2.80}$$

where

$$\begin{aligned}J(z,\bar{z}) = {} & 2\Re\left\langle z, F_{nor}^{(\Lambda)}(z,w) - F_{nor}^{(\Lambda)}(z,\langle z,z\rangle)\right\rangle + 2\Re\left\langle \Delta'(z) + \Theta_s^2(z,\bar{z}), \overline{F_{nor}^{(\Lambda)}(z,w)}\right\rangle\\ & + 2(1-s)^{j+1}\Re\left\{\langle z,a\rangle\Delta(z)^{j+1}w^{t-j-1} - \langle z,a\rangle\Delta(z)^{j+1}\langle z,z\rangle^{t-j-1}\right\} + G_{nor}^{(\Lambda)}(z,\langle z,z\rangle) - G_{nor}^{(\Lambda)}(z,w),\\ = {} & 2\Re\left\langle z, F_1^{(\Lambda)}(z,w) - F_1^{(\Lambda)}(z,\langle z,z\rangle) + F_2^{(\Lambda)}(z,w) - F_2^{(\Lambda)}(z,\langle z,z\rangle)\right\rangle\\ & + 2\Re\left\langle \Delta'(z) + \Theta_s^2(z,\bar{z}), \overline{F_1^{(\Lambda)}(z,w)} + \overline{F_2^{(\Lambda)}(z,w)}\right\rangle + G_{nor}^{(\Lambda)}(z,\langle z,z\rangle) - G_{nor}^{(\Lambda)}(z,w)\\ & + 2(1-s)^{j+1}\Re\left\{\langle z,a\rangle\Delta(z)^{j+1}\left(w^{t-j-1} - \langle z,z\rangle^{t-j-1}\right)\right\}.\end{aligned} \tag{2.81}$$

By using (2.77) and (2.78) it follows that

$$\begin{aligned}J(z,\bar{z}) = {} & 2\Re\left\langle z, F_1^{(\Lambda)}(z,w) - F_1^{(\Lambda)}(z,\langle z,z\rangle)\right\rangle + 2\Re\left\langle \Delta'(z) + \Theta_s^2(z,\bar{z}), \overline{F_1^{(\Lambda)}(z,w)}\right\rangle\\ & + 2(1-s)^{j+1}\Re\left\{\langle z,a\rangle\Delta(z)^{j+1}\left(w^{t-j-1} - \langle z,z\rangle^{t-j-1}\right)\right\} + \Theta_{N_s}^{\Lambda+2}(z,\bar{z}),\end{aligned} \tag{2.82}$$

where $\mathrm{wt}\left\{\Theta_{N_s}^{\Lambda+2}(z,\bar{z})\right\} \geq N_s$. We observe that

$$\Re\left\langle z, F_1^{(\Lambda)}\left(z,\langle z,z\rangle\right)\right\rangle = -(1-s)^{j+1}\Re\left\{\langle z,a\rangle\langle z,z\rangle^{t-j-1}\Delta(z)^{j+1}\right\}. \tag{2.83}$$

Since $\mathrm{wt}\left\{F_1^{(\Lambda)}(z,w)\right\} \geq N_s - s$ and $\mathrm{wt}\left\{\overline{F_1^{(\Lambda)}(z,w)}\right\} \geq N_s$, it follows that

$$\Re\left\langle \Theta_s^2(z,\bar{z}), \overline{F_1^{(\Lambda)}(z,w)}\right\rangle = \Theta_{N_s}^{\Lambda+2}(z,\bar{z}), \tag{2.84}$$

where $\mathrm{wt}\left\{\overline{\Theta_{N_s}^{\Lambda+2}(z,\bar{z})}\right\} \geq N_s$. By using (2.83) and (2.84), we can rewrite (2.82) as follows

$$J(z,\bar{z}) = 2\Re\left\langle z, F_1^{(\Lambda)}(z,w)\right\rangle + 2\Re\left\langle \Delta'(z), \overline{F_1^{(\Lambda)}(z,w)}\right\rangle + 2(1-s)^{j+1}\Re\left\{\langle z,a\rangle\Delta(z)^{j+1}w^{t-j-1}\right\} + \Theta_{N_s}^{\Lambda+2}(z,\bar{z}), \tag{2.85}$$

where $\mathrm{wt}\left\{\overline{\Theta_{N_s}^{\Lambda+2}(z,\bar{z})}\right\} \geq N_s$. Substituting the formula of $F_1^{(\Lambda)}(z,w)$ in (2.85), we obtain

$$\begin{aligned} J(z,\bar{z}) &= -2(1-s)^{j+1}\Re\left\{\langle z,a\rangle\Delta(z)^{j+1}w^{t-j-2}\left(\langle z,z\rangle + \left\langle (z_1,\ldots,z_N), \left(\overline{\Delta_1(z)},\ldots,\overline{\Delta_N(z)}\right)\right\rangle\right)\right\} \\ &\quad + 2(1-s)^{j+1}\Re\left\{\langle z,a\rangle\Delta(z)^{j+1}w^{t-j-1}\right\} + \Theta_{N_s}^{\Lambda+2}(z,\bar{z}), \\ &= -2(1-s)^{j+1}\Re\left\{\langle z,a\rangle\Delta(z)^{j+1}w^{t-j-2}\left(\langle z,z\rangle + s\Delta(z) - w\right)\right\} + \Theta_{N_s}^{\Lambda+2}(z,\bar{z}), \\ &= 2(1-s)^{j+2}\Re\left\{\langle z,a\rangle\Delta(z)^{j+2}w^{t-j-2}\right\} + \Theta_{N_s}^{\Lambda+2}(z,\bar{z}), \end{aligned} \tag{2.86}$$

where w satisfies (2.42) and $\mathrm{wt}\left\{\overline{\Theta_{N_s}^{\Lambda+2}(z,\bar{z})}\right\} \geq N_s$.

The proof of our Lemma follows by using (2.86) and (2.80). □

Collecting the terms of bidegree (m,n) in $(z,\bar{z})$ with $m+n = ts+1$ and $t = j-1$ in (2.67), we obtain

$$\begin{aligned} G_{nor}^{(ts+1)}\left(z,\langle z,z\rangle\right) &= 2(1-s)^t\Re\left\{\langle z,a\rangle\Delta(z)^t\right\} + 2\Re\left\langle z, F_{nor}^{(ts)}\left(z,\langle z,z\rangle\right)\right\rangle \\ &\quad + \varphi'_{ts+1,0}(z,\bar{z}) - \varphi_{ts+1,0}(z,\bar{z}) + \left(\Theta_1\right)_{N_s}^{ts+1}(z,\bar{z}). \end{aligned} \tag{2.87}$$

By applying the extended Moser lemma we find a solution $\left(G_{nor}^{(ts+1)}(z,w), F_{nor}^{(ts)}(z,w)\right)$ for (2.87). Collecting the pure terms in (2.87) of degree $ts+1$, it follows that

$$\varphi'_{ts+1,0}(z) - \varphi_{ts+1,0}(z) = (1-s)^t\langle z,a\rangle\Delta(z)^t. \tag{2.88}$$

The parameter a will help us to put the desired normalization condition (see (2.11)). By applying Lemma 2.1.4 to $\varphi'_{ts+1,0}(z)$ and $\varphi_{ts+1,0}(z)$, it follows that

$$\varphi_{ts+1,0}(z) = (1-s)^t Q(z)\Delta(z)^t + R(z), \quad \varphi'_{ts+1,0}(z) = Q'(z)\Delta(z)^t + R'(z), \tag{2.89}$$

where $\left(\Delta^t\right)^\star\left(R(z)\right) = \left(\Delta^t\right)^\star\left(R'(z)\right) = 0$. We impose the normalization condition $\left(\Delta^t\right)^\star\left(\varphi'_{ts+1,0}(z)\right) = 0$. This is equivalent finding a such that $Q'(z) = 0$. Here $Q(z)$ is a determined polynomial. We find a by solving the equation $Q'(z) = (1-s)^t\langle z,a\rangle - Q(z) = 0$.

By composing the map that sends M into (2.42) with the map (2.44) we obtain our formal transformation that sends M into M' up to degree $ts+1$.

2.3 Proof of Theorem 2.0.3-Case $T+1=(t+1)s,\ t\geq 1$

In this case we are looking for a biholomorphic transformation of the following type

$$\begin{gathered}(z',w')=(z+F(z,w),w+G(z,w)),\\ F(z,w)=\sum_{l=0}^{T-2t-1}F_{nor}^{(2t+l+1)}(z,w),\ G(z,w)=\sum_{\tau=0}^{T-2t}G_{nor}^{(2t+2+\tau)}(z,w),\end{gathered}\tag{2.90}$$

that maps M into M' up to the degree $T+1=(t+1)s$. In order to make the mapping (2.90) uniquely determined we assume that $F_{nor}^{(2t+l+1)}(z,w)$ is normalized as in the extended Moser lemma, for all $l=1,\ldots T-2t-1$. Replacing (2.90) in (2.43), and after a simplification with (2.42), we obtain

$$\begin{aligned}&\sum_{\tau=0}^{T-2t-1}G_{nor}^{(2t+2+\tau)}\left(z,\langle z,z\rangle+\varphi_{\geq 3}(z,\bar{z})\right)=2\Re\left\langle\sum_{l=0}^{T-2t-1}F_{nor}^{(2t+l+1)}\left(z,\langle z,z\rangle+\varphi_{\geq 3}(z,\bar{z})\right),z\right\rangle\\ &+\left\|\sum_{l=0}^{T-2t-1}F_{nor}^{(2t+l+1)}\left(z,\langle z,z\rangle+\varphi_{\geq 3}(z,\bar{z})\right)\right\|^2\\ &+\varphi'_{\geq 3}\left(z+\sum_{l=-1}^{T-2t}F_{nor}^{(2t+l+2)}\left(z,\langle z,z\rangle+\varphi_{\geq 3}(z,\bar{z})\right),\overline{z+\sum_{l=-1}^{T-2t}F_{nor}^{(2t+l+2)}\left(z,\langle z,z\rangle+\varphi_{\geq 3}(z,\bar{z})\right)}\right)-\varphi_{\geq 3}(z,\bar{z}).\end{aligned}\tag{2.91}$$

Collecting the terms with the same bidegree in $(z,\bar{z})$ in (2.91) we will find $F(z,w)$ and $G(z,w)$ by applying the extended Moser lemma. Since $F(z,w)$ and $G(z,w)$ don't have components of normal weight less than $2t+2$, collecting in (2.91) the terms of bidegree (m,n) in $(z,\bar{z})$ with $m+n<2t+2$, we obtain $\varphi'_{m,n}(z,\bar{z})=\varphi_{m,n}(z,\bar{z})$.

Collecting the terms of bidegree (m,n) in $(z,\bar{z})$ with $m+n=2t+2$ in (2.91), we prove the following lemma:

Lemma 2.3.1. *$G_{nor}^{(2t+2)}(z,w)=(a+\bar{a})w^{t+1}$, $F_{nor}^{(2t+1)}(z,w)=w^t\begin{pmatrix}a_{11}&\ldots&a_{1N}\\ \vdots&\ddots&\vdots\\ a_{N1}&\ldots&a_{NN}\end{pmatrix}\begin{pmatrix}z_1\\ \vdots\\ z_N\end{pmatrix}$, where Na is the trace of the matrix $(a_{ij})_{1\leq i,j\leq N}$.*

Proof. Collecting the pure terms of degree $2t+2$ in (2.91), we obtain that $\varphi_{2t+2}(z)=\varphi'_{2t+2}(z)$. Collecting the terms of bidegree (m,n) in $(z,\bar{z})$ with $m+n=2t+2$ and $0<m<n-1$ in (2.91), we obtain

$$\varphi'_{m,n}(z,\bar{z})=-\left\langle z,F_{n-m+1,m-1}(z)\right\rangle\langle z,z\rangle^{m-1}+\varphi_{m,n}(z,\bar{z}).\tag{2.92}$$

Since $\varphi_{m,n}(z,\bar{z})$, $\varphi'_{m,n}(z,\bar{z})$ satisfy (2.10), by the uniqueness of the trace decomposition, we obtain $F_{n-m+1,m-1}(z)=0$. Collecting the terms of bidegree (m,n) in $(z,\bar{z})$ with $m+n=2t+2$ and $m>n+1$ in (2.91), we obtain

$$\varphi'_{m,n}(z,\bar{z})=G_{m-n}(z)\langle z,z\rangle^{n}-\left\langle F_{m-n+1,n-1}(z),z\right\rangle\langle z,z\rangle^{n-1}+\varphi_{m,n}(z,\bar{z}).\tag{2.93}$$

Since $F_{n-m+1,m-1}(z)=0$ it follows that $G_{m-n}(z)=0$.

Collecting the terms of bidegree $(t+1,t+1)$ in $(z,\bar{z})$ in (2.91), we obtain

$$\varphi'_{t+1,t+1}(z,\bar{z})=\left(G_{0,t+1}(z)\langle z,z\rangle-\langle F_{1,t}(z),z\rangle-\langle z,F_{1,t}(z)\rangle\right)\langle z,z\rangle^{t}+\varphi_{t+1,t+1}(z,\bar{z}).\tag{2.94}$$

Then (2.94) can not provide us $F_{1,t}(z)$. Therefore $F_{1,t}(z)$ is undetermined. We obtain

$$F_{nor}^{(2t+1)}(z,w)=w^t\begin{pmatrix}a_{11}&\ldots&a_{1N}\\ \vdots&\ddots&\vdots\\ a_{N1}&\ldots&a_{NN}\end{pmatrix}\begin{pmatrix}z_1\\ \vdots\\ z_N\end{pmatrix},\quad a_{ij}\in\mathbb{C},\ 1\leq i,j\leq N.\tag{2.95}$$

We write $a_{11}=a+b_{11},\ldots,a_{NN}=a+b_{NN}$ and we use the notations $b_{k,j}=a_{k,j}$, for all $k\neq j$. Then the matrix $(b_{k,j})_{1\leq k,j\leq N}$ represents the traceless part of the matrix $(a_{k,j})_{1\leq k,j\leq N}$. By applying Lemma 2.1.1 to the polynomial $\langle F_{1,t}(z), z\rangle$, we obtain $\langle F_{1,t}(z), z\rangle = a\langle z, z\rangle + P(z,\bar{z})$ with $\operatorname{tr}(P(z,\bar{z})) = 0$, where $P(z,\bar{z}) = \sum_{i,j=1}^{N} b_{i,j} z_i \bar{z}_j$. By using the preceding decomposition we obtain

$$\varphi'_{t+1,t+1}(z,\bar{z}) = (G_{0,t+1}(z) - a - \bar{a})\langle z, z\rangle^{t+1} + \varphi_{t+1,t+1}(z,\bar{z}) - 2\Re\left(P(z,\bar{z})\langle z, z\rangle^t\right). \tag{2.96}$$

Since $\operatorname{tr}(P(z,\bar{z})) = 0$ it follows that $\operatorname{tr}^{t+1}(\Re(P(z,\bar{z})\langle z,z\rangle^t)) = 0$ (see Lemma 6.6 in [34]). □

We can write $F(z,w) = F_{nor}^{(2t+2)}(z,w) + F_{\geq 2t+3}(z,w)$ and $G(z,w) = G_{\geq 2t+2}(z,w)$ (see (2.41)). We have $F_{\geq 2t+2}(z,w) = \sum_{k+2l\geq 2t+2} F_{k,l}(z) w^l$, where $F_{k,l}(z)$ is a homogeneous polynomial of degree k. Therefore $\operatorname{wt}\{F_{\geq 2t+2}(z$ $\min_{k+2l\geq 2t+2}\{k+ls\} \geq \min_{k+2l\geq 2t+2}\{k+2l\} \geq 2t+2$. Next, we show that $\operatorname{wt}\left\{\overline{F_{\geq 2t+2}(z,w)}\right\} \geq ts+s-1$. Since $\operatorname{wt}\left\{\overline{F_{\geq 2t+2}(z,w)}\right\} \geq \min_{k+2l\geq 2t+2}\{k(s-1)+ls\}$, it is enough to prove that $k(s-1)+ls \geq ts+s-1$ for $k+2l \geq 2t+2$. Since we can write the latter inequality as $(k-1)(s-1)+ls \geq ts$ for $(k-1)+2l \geq 2t+1$, it is enough to prove that $k(s-1)+ls \geq ts$ for $k+2l \geq 2t+1 > 2t$. Continuing the calculations like in the previous case we obtain the desired result.

Lemma 2.3.2. *For w satisfying (2.42), we make the following immediate estimates*

$$\begin{gathered}\operatorname{wt}\left\{F_{nor}^{(2t+1)}(z,w)\right\} \geq ts+1, \quad \operatorname{wt}\left\{\overline{F_{nor}^{(2t+1)}(z,w)}\right\} \geq ts+s-1, \quad \operatorname{wt}\left\{\left\|F_{nor}^{(2t+1)}(z,w)\right\|^2\right\} \geq ts+s+1,\\ \operatorname{wt}\{F_{\geq 2t+2}(z,w)\} \geq 2t+2, \quad \operatorname{wt}\left\{\overline{F_{\geq 2t+2}(z,w)}\right\} \geq ts+s-1, \quad \operatorname{wt}\left\{\|F_{\geq 2t+2}(z,w)\|^2\right\} \geq ts+s+1,\\ \operatorname{wt}\left\{\left\langle F_{nor}^{(2t+1)}(z,w), F_{\geq 2t+2}(z,w)\right\rangle\right\}, \ \operatorname{wt}\left\{\left\langle F_{\geq 2t+2}(z,w), F_{nor}^{(2t+1)}(z,w)\right\rangle\right\} \geq ts+s+1.\end{gathered} \tag{2.97}$$

As a consequence of the estimates (2.97) we obtain

$$\|F(z,w)\|^2 = \left\|F_{nor}^{(2t+1)}(z,w)\right\|^2 + 2\Re\left\langle F_{nor}^{(2t+1)}(z,w), F_{\geq 2t+2}(z,w)\right\rangle + \|F_{\geq 2t+2}(z,w)\|^2 = \Theta_{ts+s+1}^{2t+3}(z,\bar{z}), \tag{2.98}$$

where $\operatorname{wt}\left\{\overline{\Theta_{ts+s+1}^{2t+3}(z,\bar{z})}\right\} \geq ts+s+1$.

In order to apply the extended Moser lemma in (2.91) we have to identify and weight and order evaluate the terms which are not "good". We prove the following lemmas:

Lemma 2.3.3. *For all $m,n \geq 1$ and w satisfying (2.42), we make the following estimate*

$$\varphi'_{m,n}\left(z+F(z,w), \overline{z+F(z,w)}\right) = \varphi'_{m,n}(z,\bar{z}) + 2\Re\left\langle \Theta_s^2(z,\bar{z}), \overline{F_{\geq 2t+2}(z,w)}\right\rangle + \Theta_{ts+s+1}^{2t+3}(z,\bar{z}), \tag{2.99}$$

where $\operatorname{wt}\left\{\overline{\Theta_{ts+s+1}^{2t+3}(z,\bar{z})}\right\} \geq ts+s+1$.

Proof. We have the expansion $\varphi'_{m,n}\left(z+F(z,w), \overline{z+F(z,w)}\right) = \varphi'_{m,n}(z,\bar{z}) + \ldots$ (see the proof of Lemma 2.2.3). In order to prove (2.99), it is enough to study the weight and the order of the following particular terms

$$A_1(z,w) = F_1(z,w) z^I \bar{z}^J, \quad A_2(z,w) = z^{I_1}\bar{z}^{J_1}\overline{F_1(z,w)}, \quad B_1(z,w) = F_2(z,w) z^I \bar{z}^J, \quad B_2(z,w) = z^{I_1}\bar{z}^{J_1}\overline{F_2(z,w)},$$

where $F_1(z,w)$ is the first component of $F_{nor}^{(2t+1)}(z,w)$ and $F_2(z,w)$ is the first component of $F_{\geq 2t+2}(z,w)$. Here we assume that $|I| = m-1$, $|J| = n$, $|I_1| = m$, $|J_1| = n-1$.

By using (2.97) we obtain $\operatorname{wt}\{A_1(z,w)\} \geq m-1+ts+1+n(s-1) \geq ts+s+1 \Longleftrightarrow m+ns-n \geq s+1 \Longleftrightarrow m+s(n-1) \geq n+1$ and the latter inequality is true since $m+s(n-1) \geq m+3(n-1) \geq n+1$. On the other hand $\operatorname{Ord}\{A_1(z,w)\} \geq m-1+2t+1+n \geq 2t+3$.

By using (2.97) we obtain $\operatorname{wt}\{A_2(z,w)\} \geq m+(n-1)(s-1)+ts+s-1 \geq ts+s+1$ and the last inequality is equivalent with $m+(n-1)(s-1) \geq 2$. The latter inequality can be proved with the same calculations like in the proof of Lemma 2.2.3. On the other hand, we observe that $\operatorname{Ord}\{A_1(z,w)\} \geq m+2t+1+n-1 \geq 2t+3$.

In the same way we obtain $\text{Ord}\{B_1(z,w)\}$, $\text{Ord}\{B_2(z,w)\} \geq 2t+2$. By using (2.97), every term in "..." that depends on $F_2(z,w)$ can be written as $\Theta_s^2(z,\bar{z})F_2(z,w)$. This proves our lemma. □

Lemma 2.3.4. *For all $k > s$ and w satisfying (2.42), we make the following estimate*

$$\varphi'_{k,0}(z+F(z,w)) = \varphi'_{k,0}(z) + 2\Re\left\langle \Theta_s^2(z,\bar{z}), \overline{F_{\geq 2t+2}(z,w)} \right\rangle + \Theta_{ts+s+1}^{2t+3}(z,\bar{z}), \tag{2.100}$$

where $\text{wt}\left\{\overline{\Theta_{ts+s+1}^{2t+3}(z,\bar{z})}\right\} \geq ts+s+1$.

Proof. We make the expansion $\varphi'_{k,0}(z+F(z,w)) = \varphi'_{k,0}(z) + \ldots$. To study the weight and the order of terms which can appear in "..." it is enough to study the weight and order of the following terms

$$A(z,w) = F_1(z,w)z^I, \quad B(z,w) = F_2 z^I,$$

where $F_1(z,w)$ is the first component of $F_{nor}^{(2t+1)}(z,w)$ and $F_2(z,w)$ is the first component of $F_{\geq 2t+3}(z,w)$. Here we assume that $|I| = m-1 \geq s$. From (2.97) we obtain $\text{wt}\{A(z,w)\} \geq s+ts+1 = ts+s+1$. On the other hand, we have $\text{Ord}\{A(z,w)\} \geq 2t+s+1 \geq 2t+3$. By using (2.97) each term in "..." that depends on $F_2(z,w)$ can be written as $\Theta_s^2(z,\bar{z})F_2(z,w)$. This proves our lemma. □

Lemma 2.3.5. *For w satisfying (2.42) we have the following estimate*

$$\begin{aligned} 2\Re\{\Delta(z+F(z,w))\} =& 2\Re\left\{\Delta(z) + \sum_{k=1}^{N} \Delta_k(z)\left(a_{k1}z_1 + \cdots + a_{kN}z_N\right)w^t\right\} \\ &+ 2\Re\left\langle \Delta'(z) + \Theta_s^2(z,\bar{z}), \overline{F_{\geq 2t+2}(z,w)}\right\rangle + \Theta_{ts+s+1}^{2t+3}(z,\bar{z}), \end{aligned} \tag{2.101}$$

where $\text{wt}\left\{\overline{\Theta_{ts+s+1}^{2t+3}(z,\bar{z})}\right\} \geq ts+s+1$.

Proof. For w satisfying (2.42), we have the expansion

$$2\Re\{\Delta(z+F(z,w))\} = 2\Re\left\{\Delta(z) + \sum_{k=1}^{N} \Delta_k(z)F_{\geq 2t+1}^k(z,w) + L(z,\bar{z})\right\} + \Theta_{ts+s+1}^{2t+3}(z,\bar{z}), \tag{2.102}$$

where $F_{\geq 2t+1}(z,w) = \left(F_{\geq 2t+1}^1(z,w), \ldots, F_{\geq 2t+1}^N(z,w)\right)$ and $L(z,\bar{z}) = \left\langle \Theta_s^2(z,\bar{z}), \overline{F_{\geq 2t+2}(z,w)}\right\rangle$. We compute

$$\begin{aligned} \sum_{k=1}^{N} 2\Re\left\{\Delta_k(z)F_{\geq 2t+1}^k(z,w)\right\} &= \sum_{k=1}^{N} 2\Re\left\{\Delta_k(z)\left(w^t\sum_{j=1}^{N} a_{kj}z_j + F_{\geq 2t+2}^k(z,w)\right)\right\} \\ &= 2\Re\left\{w^t\sum_{k=1}^{N}\Delta_k(z)\left(a_{k1}z_1 + \cdots + a_{kN}z_N\right)\right\} + 2\Re\left\langle \Delta'(z), \overline{F_{\geq 2t+2}(z,w)}\right\rangle. \end{aligned} \tag{2.103}$$

□

Lemma 2.3.6. *For w satisfying (2.42), we have the following estimate*

$$G_{nor}^{(2t+2)}(z,w) - 2\Re\left\langle F_{nor}^{(2t+1)}(z,w), z\right\rangle = 2(a+\bar{a})\Re\{\Delta(z)w^t\} + 2\Re\{P(z,\bar{z})w^t\} + \Theta_{ts+s+1}^{2t+3}(z,\bar{z}), \tag{2.104}$$

where $P(z,\bar{z}) = \sum_{k,j=1}^{N} b_{k,j}z_k\bar{z}_j$ *and* $\text{wt}\left\{\overline{\Theta_{ts+s+1}^{2t+3}(z,\bar{z})}\right\} \geq ts+s+1$.

Proof. For w satisfying (2.42), by Lemma 2.3.1 it follows that

$$
\begin{aligned}
G_{nor}^{(2t+2)}(z,w) - 2\Re\left\langle F_{nor}^{(2t+1)}(z,w), z\right\rangle &= (a+\overline{a})w^{t+1} - 2\Re\left\langle w^t \begin{pmatrix} b_{11}+a & \dots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \dots & b_{NN}+a \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix}, z\right\rangle, \\
&= 2\Re\left\{aw^{t+1}\right\} - 2\Re\left\{aw^t\langle z,z\rangle + P(z,\overline{z})w^t\right\} + \overline{a}\left(w^{t+1} - \overline{w}^{t+1}\right), \\
&= 2\Re\left\{aw^t\left(w - \langle z,z\rangle\right)\right\} - 2\Re\left\{P(z,\overline{z})w^t\right\} + \Theta_{ts+s+1}^{2t+3}(z,\overline{z}), \\
&= 2\Re\left\{aw^t\left(\Delta(z) + \overline{\Delta(z)}\right)\right\} - 2\Re\left\{P(z,\overline{z})w^t\right\} + \Theta_{ts+s+1}^{2t+3}(z,\overline{z}), \\
&= 2(a+\overline{a})\Re\left\{\Delta(z)w^t\right\} - 2\Re\left\{P(z,\overline{z})w^t\right\} + \Theta_{ts+s+1}^{2t+3}(z,\overline{z}),
\end{aligned}
\tag{2.105}
$$

where $\mathrm{wt}\left\{\overline{\Theta_{\mathrm{ts+s+1}}^{\mathrm{2t+3}}(\mathrm{z},\overline{\mathrm{z}})}\right\} \geq \mathrm{ts+s+1}$. □

Substituting $F(z,w) = F_{nor}^{(2t+1)}(z,w) + F_{\geq 2t+2}(z,w)$ and $G(z,w) = G_{nor}^{(2t+2)}(z,w) + G_{\geq 2t+3}(z,w)$ (see (2.41)) into (2.91) and by Lemmas 2.3.2-2.3.6 , we obtain

$$
\begin{aligned}
G_{\geq 2t+3}(z,w) = {} & 2\Re\left\{\left(\sum_{k=1}^{N}\Delta_k(z)\left(a_{k1}z_1 + \dots + a_{kN}z_N\right) - (a+\overline{a})\Delta(z)\right)w^t\right\} + 2\Re\left\{P(z,\overline{z})\left(w^t - \langle z,z\rangle^t\right)\right\} \\
& + 2\Re\left\langle \overline{z} + \Delta'(z) + \Theta_s^2(z,\overline{z}), \overline{F_{\geq 2t+2}(z,w)}\right\rangle + \varphi'_{\geq 2t+3}(z,\overline{z}) - \varphi_{\geq 2t+3}(z,\overline{z}) + \Theta_{ts+s+1}^{2t+3}(z,\overline{z}),
\end{aligned}
\tag{2.106}
$$

where w satisfies (2.42) and $\mathrm{wt}\left\{\overline{\Theta_{\mathrm{ts+s+1}}^{\mathrm{2t+3}}(\mathrm{z},\overline{\mathrm{z}})}\right\} \geq \mathrm{ts+s+1}$. It remains to study the following expression

$$
E(z,\overline{z}) = 2\Re\left\{P(z,\overline{z})\left(w^t - \langle z,z\rangle^t\right)\right\}. \tag{2.107}
$$

Lemma 2.3.7. *For w satisfying (2.42) we make the following estimate*

$$
E(z,\overline{z}) = 2\Re\left\{\left(P(z,\overline{z}) + \overline{P(z,\overline{z})}\right)\Delta(z)\sum_{k+l=t-1} w^k\langle z,z\rangle^l\right\} + \Theta_{ts+s+1}^{2t+3}(z,\overline{z}), \tag{2.108}
$$

where $P(z,\overline{z}) = \sum\limits_{k,j=1}^{N} b_{k,j}z_k\overline{z}_j$ *and* $\mathrm{wt}\left\{\overline{\Theta_{\mathrm{ts+s+1}}^{\mathrm{2t+3}}(\mathrm{z},\overline{\mathrm{z}})}\right\} \geq \mathrm{ts+s+1}$.

Proof. We compute

$$
\begin{aligned}
E(z,\overline{z}) &= 2\Re\left\{P(z,\overline{z})\left(\Delta(z) + \overline{\Delta(z)}\right)\sum_{k+l=t-1} w^k\langle z,z\rangle^l\right\} + \Theta_{ts+s+1}^{2t+3}(z,\overline{z}), \\
&= 2\Re\left\{\left(P(z,\overline{z}) + \overline{P(z,\overline{z})}\right)\Delta(z)\sum_{k+l=t-1} w^k\langle z,z\rangle^l\right\} + \Theta_{ts+s+1}^{2t+3}(z,\overline{z}),
\end{aligned}
\tag{2.109}
$$

where $\mathrm{wt}\left\{\overline{\Theta_{\mathrm{ts+s+1}}^{\mathrm{2t+3}}(\mathrm{z},\overline{\mathrm{z}})}\right\} \geq \mathrm{ts+s+1}$. □

We consider the following notations

$$
\begin{aligned}
&\mathscr{L}(z,\overline{z}) = P(z,\overline{z}) + \overline{P(z,\overline{z})} = \sum_{k,j=1}^{N}\left(b_{k,j} + \overline{b}_{j,k}\right)z_k\overline{z}_j, \\
&Q(z) = \sum_{k=1}^{N}\Delta_k(z)\left(a_{k1}z_1 + \dots + a_{kN}z_N\right) - (a+\overline{a})\Delta(z), \quad Q_1(z) = \sum_{k,j=1}^{N}\left(b_{k,j} + \overline{b}_{j,k}\right)z_k\Delta_k(z).
\end{aligned}
\tag{2.110}
$$

Then, for w satisfying (2.42), by Lemma 2.3.7 and the notations (2.110), we can rewrite (2.106) as follows

$$G_{\geq 2t+3}(z,w) = 2\Re\left\{Q(z)w^t\right\} + 2\Re\left\{\mathscr{L}(z,\bar z)\Delta(z)E_{t-1}(w,\langle z,z\rangle)\right\} + 2\Re\left\langle \bar z + \Delta'(z) + \Theta_s^2(z,\bar z), \overline{F_{\geq 2t+2}(z,w)}\right\rangle$$
$$+ \varphi'_{\geq 2t+3}(z,\bar z) - \varphi_{\geq 2t+3}(z,\bar z) + \Theta^{2t+3}_{ts+s+1}(z,\bar z), \tag{2.111}$$

where $\mathrm{wt}\left\{\overline{\Theta^{2t+3}_{ts+s+1}(z,\bar z)}\right\} \geq ts+s+1$. Here $E_{t-1}(w,\langle z,z\rangle) = \sum\limits_{k+l=t-1} w^k\langle z,z\rangle^l$. For $p \geq 2t+3$ we prove the following lemma (the analogue of Lemma 3.4 of Huang-Yin's paper [20]):

Lemma 2.3.8. *We define $\varepsilon(p) = 0$ if $p < 2t+s$ and $\varepsilon(p) = 1$ if $p \geq 2t+s$, $\gamma(p) = 1$ if $p < ts+2$ and $\gamma(p) = 0$ if $p = ts+2$. Let $N'_s := ts+s+1$. For all $0 \leq j \leq t$ and $p \in [2t+j(s-2)+3, 2t+(j+1)(s-2)+2]$, we have the following estimate*

$$G_{\geq p}(z,w) = 2(1-s)^j\Re\left\{Q(z)\Delta(z)^j w^{t-j}\right\} + 2\gamma(p)(-1)^j\Re\left\{\mathscr{L}(z,\bar z)\Delta(z)^{j+1}\sum_{l_1+l_2=t-j-1} E^{t-j}_{l_1,l_2} w^{l_1}\langle z,z\rangle^{l_2}\right\}$$
$$+2\varepsilon(p)\Re\left\{Q_1(z)\Delta(z)^j w^{t-j}\sum_{l=0}^{j-1}(-1)^{\beta_l}(1-s)^l F_l^{t-j}\right\} + 2\Re\left\langle \bar z + \Delta'(z) + \Theta_s^2(z,\bar z), \overline{F_{\geq p-1}(z,w)}\right\rangle$$
$$+\varphi'_{\geq p}(z,\bar z) - \varphi_{\geq p}(z,\bar z) + \Theta^p_{N'_s}(z,\bar z), \tag{2.112}$$

where $\mathrm{wt}\left\{\overline{\Theta^{\mathrm{p}}_{\mathrm{N}'_s}(z,\bar z)}\right\} \geq \mathrm{N}'_s$ and w satisfies (2.42). Here $E^{t-j}_{l_1,l_2}$ with $l_1+l_2 = t-j-1$ and F_p^{t-j} with $l = 1,\ldots,j-1$ are natural numbers satisfying the following recurrence relations

$$F^{t-j-1}_{l+1} = F^{t-j}_l, \quad F^{t-j-1}_0 = \sum_{l_1+l_2=t-j-1} E^{t-j}_{l_1,l_2}, \quad E^{t-j-1}_{l,t-j-1-l} = \sum_{l'=l}^{t-j-l} E^{t-j}_{t-j-l',l'}.$$

Also $\beta_l \in \mathbb{N}$, for all $l = 1,\ldots,j-1$.

Proof. For $j=0$ and $k=0$ we obtain $p = 2t+3$. Therefore (2.112) becomes (2.111).

Step 1. It follows by a similar approach as in the Step 1 of Lemma 2.2.7.

Step 2. Assume that we proved Lemma 2.3.8 for $m \in [2t+j(s-2)+3, 2t+(j+1)(s-2)+2]$, for $j \in [0,t-1]$. We want to prove that (2.112) holds for $m \in [2t+(j+1)(s-2)+3, 2t+(j+2)(s-2)+2]$. Collecting in (2.112) the terms of bidegree (m,n) in $(z,\bar z)$ with $m+n = \Lambda+1 := 2t+(j+1)(s-2)+2$, we obtain

$$G^{(\Lambda+1)}_{nor}(z,\langle z,z\rangle) = 2\Re\left\langle z, F^{(\Lambda)}_{nor}(z,\langle z,z\rangle)\right\rangle + 2\gamma(p)(-1)^j\Re\left\{\mathscr{L}(z,\bar z)\Delta(z)^{j+1}\langle z,z\rangle^{t-j-1}\sum_{l_1+l_2=t-j-1} E^{t-j}_{l_1,l_2}\right\}$$
$$+2\varepsilon(p)\Re\left\{Q_1(z)\Delta(z)^j\langle z,z\rangle^{t-j}\sum_{l=0}^{j-1}(-1)^{\beta_l}(1-s)^l F^{t-j}_{t-j}\right\} + 2(1-s)^j\Re\left\{Q(z)\Delta(z)^j\langle z,z\rangle^{t-j}\right\}$$
$$+\varphi'_{\Lambda+1}(z,\bar z) - \varphi_{\Lambda+1}(z,\bar z) + \mathbb{P}^{\Lambda+1}_{N'_s}(z,\bar z), \tag{2.113}$$

where $\mathrm{wt}\left\{\mathbb{P}^{\Lambda+1}_{N'_s}(z,\bar z)\right\} \geq N'_s$. We define the following mappings

$$F^{(\Lambda)}_1(z,w) = -(1-s)^j Q(z)\Delta(z)^j w^{t-j-1}(z_1,\ldots,z_N),$$
$$F^{(\Lambda)}_2(z,w) = -\varepsilon(p)Q_1(z)\Delta(z)^j w^{t-j-1}\left(\sum_{l=0}^{j-1}(-1)^{\beta_l}(1-s)^l F^{t-j}_l\right)(z_1,\ldots,z_N),$$
$$F^{(\Lambda)}_3(z,w) = -\gamma(p)(-1)^j\Delta(z)^{j+1}\sum_{l_1+l_2=t-j-1} E^{t-j}_{l_1,l_2} w^{t-j-1}\left(\sum_{l=1}^{N}\left(b_{l,1}+\bar b_{1,l}\right)z_l,\ldots,\sum_{l=1}^{N}\left(b_{l,N}+\bar b_{N,l}\right)z_l\right),$$
$$F^{(\Lambda)}_{nor}(z,w) = F^{(\Lambda)}_1(z,w) + F^{(\Lambda)}_2(z,w) + F^{(\Lambda)}_3(z,w) + F^{(\Lambda)}_4(z,w), \tag{2.114}$$

where $F^{(\Lambda)}_4(z,w)$ will be determined later (see 2.115).

Substituting (2.114) into (2.113), by making some simplifications it follows that

$$G_{nor}^{(\Lambda+1)}(z,\langle z,z\rangle) = 2\Re\left\langle z, F_4^{(\Lambda)}(z,\langle z,z\rangle)\right\rangle + \varphi'_{\Lambda+1}(z,\bar{z}) - \varphi_{\Lambda+1}(z,\bar{z}) + \mathbb{P}_{N'_s}^{\Lambda+1}(z,\bar{z}). \tag{2.115}$$

By applying the extended Moser lemma we find a solution $\left(G_{nor}^{(\Lambda+1)}(z,w), F_4^{(\Lambda)}(z,w)\right)$ for (2.115). By repeating the procedure from the first case of the normal form construction, we obtain the following estimates

$$\begin{gathered}
\mathrm{wt}\left\{G_{nor}^{(\Lambda+1)}(z,w) - G_{nor}^{(\Lambda+1)}(z,\langle z,z\rangle)\right\},\ \mathrm{wt}\left\{G_{nor}^{(\Lambda+1)}(z,w)\right\},\ \mathrm{wt}\left\{G_{nor}^{(\Lambda+1)}(z,\langle z,z\rangle)\right\} \geq N'_s,\\
\mathrm{wt}\left\{F_4^{(\Lambda)}(z,w) - F_4^{(\Lambda)}(z,\langle z,z\rangle)\right\},\ \mathrm{wt}\left\{F_4^{(\Lambda)}(z,w)\right\},\ \mathrm{wt}\left\{F_4^{(\Lambda)}(z,\langle z,z\rangle)\right\} \geq N'_s - s + 1,\\
\mathrm{wt}\left\{\overline{F_4^{(\Lambda)}(z,w)}\right\},\ \mathrm{wt}\left\{\overline{F_4^{(\Lambda)}(z,\langle z,z\rangle)}\right\},\ \mathrm{wt}\left\{\overline{F_4^{(\Lambda)}(z,w) - F_4^{(\Lambda)}(z,\bar{z})}\right\} \geq N'_s - 1,
\end{gathered} \tag{2.116}$$

where w satisfies (2.42). As a consequence of (2.116) we obtain

$$\begin{gathered}
\left\langle \Delta'(z) + \Theta_s^2(z,\bar{z}), \overline{F_4^{(\Lambda)}(z,w)}\right\rangle + \left\langle \overline{F_4^{(\Lambda)}(z,w)}, \Delta'(z) + \Theta_s^2(z,\bar{z})\right\rangle = \Theta_{N'_s}^{\Lambda+2}(z,\bar{z})',\\
\Re\left\langle F_4^{(\Lambda)}(z,w) - F_4^{(\Lambda)}(z,\langle z,z\rangle), z\right\rangle = \Theta_{N'_s}^{\Lambda+2}(z,\bar{z})',
\end{gathered} \tag{2.117}$$

where w satisfies (2.42) and each of $\Theta_{N'_s}^{\Lambda+2}(z,\bar{z})'$ has the property $\mathrm{wt}\left\{\Theta_{N'_s}^{2t+3}(z,\bar{z})\right\} \geq N'_s$. Substituting $F_{\geq\Lambda}(z,w) = F_{nor}^{(\Lambda)}(z,w) + F_{\geq\Lambda+1}(z,w)$ and $G_{\geq\Lambda+1}(z,w) = G_{nor}^{(\Lambda+1)}(z,w) + G_{\geq\Lambda+2}(z,w)$ in (2.112), it follows that

$$\begin{aligned}
G_{nor}^{(\Lambda+1)}(z,w) + G_{\geq\Lambda+2}(z,w) = {} & 2\Re\left\langle \bar{z} + \Delta'(z) + \Theta_s^2(z,\bar{z}), \overline{F_{nor}^{(\Lambda)}(z,w)} + \overline{F_{\geq\Lambda+1}(z,w)}\right\rangle + \varphi'_{\geq\Lambda+1}(z,\bar{z}) - \varphi_{\geq\Lambda+1}(z,\bar{z})\\
& + \mathbb{P}_{N'_s}^{\Lambda+1}(z,\bar{z}) + \Theta_{N'_s}^{\Lambda+2}(z,\bar{z}) + 2(1-s)^j\Re\left\{Q(z)\Delta(z)^j w^{t-j}\right\}\\
& + 2\gamma(p)\Re\left\{(-1)^j\mathscr{L}(z,\bar{z})\Delta(z)^{j+1}\sum_{l_1+l_2=t-j-1} E_{l_1,l_2}^{t-j} w^{l_1}\langle z,z\rangle^{l_2}\right\}\\
& + 2\varepsilon(p)\Re\left\{Q_1(z)\Delta(z)^j w^{t-j}\sum_{l=0}^{j-1}(-1)^{\beta_l}(1-s)^l F_l^{t-j}\right\},
\end{aligned} \tag{2.118}$$

where w satisfies (2.42). After a simplification in the preceding equation by using (2.113), it follows that

$$G_{\geq\Lambda+2}(z,w) = 2\Re\left\langle \bar{z} + \Delta'(z) + \Theta_s^2(z,\bar{z}), \overline{F_{\geq\Lambda+1}(z,w)}\right\rangle + \varphi_{\geq\Lambda+2}(z,\bar{z}) - \varphi'_{\geq\Lambda+2}(z,\bar{z}) + \Theta_{N'_s}^{\Lambda+2}(z,\bar{z}) + J(z,\bar{z}), \tag{2.119}$$

where we have used the following notation

$$\begin{aligned}
J(z,\bar{z}) = {} & 2\Re\left\langle z, F_{nor}^{(\Lambda)}(z,w) - F_{nor}^{(\Lambda)}(z,\langle z,z\rangle)\right\rangle + 2\Re\left\langle \Delta'(z) + \Theta_s^2(z,\bar{z}), \overline{F_{nor}^{(\Lambda)}(z,w)}\right\rangle\\
& + 2(1-s)^j\Re\left\{Q(z)\Delta(z)^j w^{t-j} - Q(z)\Delta(z)^j\langle z,z\rangle^{t-j}\right\} + G_{nor}^{(\Lambda+2)}(z,\langle z,z\rangle) - G_{nor}^{(\Lambda+2)}(z,w)\\
& + 2\gamma(p)(-1)^j\Re\left\{\mathscr{L}(z,\bar{z})\Delta(z)^{j+1}\left(\sum_{l_1+l_2=t-j-1} E_{l_1,l_2}^{t-j} w^{l_1}\langle z,z\rangle^{l_2} - \langle z,z\rangle^{t-j-1}\sum_{l_1+l_2=t-j-1} E_{l_1,l_2}^{t-j}\right)\right\}\\
& + 2\varepsilon(p)\Re\left\{Q_1(z)\Delta(z)^j\sum_{l=0}^{j-1}(-1)^{\beta_l}(1-s)^l\left(F_l^{t-j}w^{t-j} - F_l^{t-j}\langle z,z\rangle^{t-j}\right)\right\},
\end{aligned} \tag{2.120}$$

By using (2.114) the precedent identity becomes

$$
\begin{aligned}
J(z,\bar{z}) = & 2\Re\left\langle z, \sum_{k=1}^{3}\left(F_k^{(\Lambda)}(z,w) - F_k^{(\Lambda)}(z,\langle z,z\rangle)\right)\right\rangle + 2\Re\left\langle \Delta'(z) + \Theta_s^2(z,\bar{z}), \sum_{k=1}^{3}\overline{F_k^{(\Lambda)}(z,w)}\right\rangle \\
& + 2(1-s)^j\Re\left\{Q(z)\Delta(z)^j\left(w^{t-j} - \langle z,z\rangle^{t-j}\right)\right\} + G_{nor}^{(\Lambda+2)}(z,\langle z,z\rangle) - G_{nor}^{(\Lambda+2)}(z,w) \\
& + 2\gamma(p)(-1)^j\Re\left\{\mathscr{L}(z,\bar{z})\Delta(z)^{j+1}\left(\sum_{l_1+l_2=t-j-1} E_{l_1,l_2}^{t-j} w^{l_1}\langle z,z\rangle^{l_2} - \sum_{l_1+l_2=t-j-1} E_{l_1,l_2}^{t-j}\langle z,z\rangle^{t-j-1}\right)\right\} \\
& + 2\varepsilon(p)\Re\left\{Q_1(z)\Delta(z)^j\sum_{l=0}^{j-1}(-1)^{\beta_l}(1-s)^l F_l^{t-j}\left(w^{t-j} - \langle z,z\rangle^{t-j}\right)\right\}.
\end{aligned}
\tag{2.121}
$$

We observe that

$$
\begin{aligned}
\Re\left\langle F_1^{(\Lambda)}(z,\langle z,z\rangle), z\right\rangle &= -(1-s)^j\Re\left\{Q(z)\Delta(z)^j\langle z,z\rangle^{t-j}\right\}, \\
\Re\left\langle F_2^{(\Lambda)}(z,\langle z,z\rangle), z\right\rangle &= -\varepsilon(p)\Re\left\{Q_1(z)\Delta(z)^j\langle z,z\rangle^{t-j}\sum_{l=0}^{j-1}(-1)^{\beta_l}(1-s)^l F_l^{t-j}\right\}, \\
\Re\left\langle F_3^{(\Lambda)}(z,\langle z,z\rangle), z\right\rangle &= -(-1)^j\gamma(p)\Re\left\{\mathscr{L}(z,\bar{z})\Delta(z)^{j+1}\langle z,z\rangle^{t-j-1}\sum_{l_1+l_2=t-j-1} E_{l_1,l_2}^{t-j}\right\}.
\end{aligned}
\tag{2.122}
$$

Since $\mathrm{wt}\left\{F_k^{(\Lambda)}(z,w)\right\} \geq ts+1$ and $\mathrm{wt}\left\{\overline{F_k^{(\Lambda)}(z,w)}\right\} \geq ts+s-1$ for all $k \in \{1,2,3\}$, it follows that

$$
2\Re\left\langle \Theta_s^2(z,\bar{z}), \sum_{k=1}^{3}\overline{F_k^{(\Lambda)}(z,w)}\right\rangle = \Theta_{N_s'}^{\Lambda+2}(z,\bar{z}),
\tag{2.123}
$$

where $\mathrm{wt}\left\{\overline{\Theta_{N_s'}^{\Lambda+2}(z,\bar{z})}\right\} \geq N_s'$. By using (2.116), (2.117), (2.122), (2.123) we can rewrite (2.121) as follows

$$
\begin{aligned}
J(z,\bar{z}) = & 2\Re\left\langle z, \sum_{k=1}^{3}F_k^{(\Lambda)}(z,w)\right\rangle + 2\Re\left\langle \Delta'(z), \sum_{k=1}^{3}\overline{F_k^{(\Lambda)}(z,w)}\right\rangle \\
& + 2(1-s)^j\Re\left\{Q(z)\Delta(z)^j w^{t-j}\right\} + 2(-1)^j\gamma(p)\Re\left\{\mathscr{L}(z,\bar{z})\Delta(z)^{j+1}\langle z,z\rangle^{t-j-1}\sum_{l_1+l_2=t-j-1} E_{l_1,l_2}^{t-j}\right\} \\
& + 2\varepsilon(p)\Re\left\{Q_1(z)\Delta(z)^j w^{t-j}\sum_{l=0}^{j-1}(-1)^{\beta_l}(1-s)^l F_l^{t-j}\right\}.
\end{aligned}
\tag{2.124}
$$

Substituting the formulas of $F_1^{(\Lambda)}(z,w)$, $F_2^{(\Lambda)}(z,w)$ and $F_3^{(\Lambda)}(z,w)$ in (2.124) and using w satisfying (2.42), we obtain

$$
\begin{aligned}
J(z,\bar{z}) = & -2(1-s)^j\Re\left\{Q(z)\Delta(z)^j w^{t-j-1}\left(\langle z,z\rangle + s\Delta(z)\right) - Q(z)\Delta(z)^j w^{t-j}\right\} \\
& - 2(-1)^j\gamma(p)\Re\left\{\mathscr{L}(z,\bar{z})\Delta(z)^{j+1}\sum_{l_1+l_2=t-j-1} E_{l_1,l_2}^{t-j} w^{l_2}\left(w^{l_1} - \langle z,z\rangle^{l_1}\right)\right\} \\
& - 2(-1)^j\gamma(p)\Re\left\{Q_1(z)\Delta(z)^{j+1}\sum_{l_1+l_2=t-j-1} E_{l_1,l_2}^{t-j} w^{t-j-1}\right\} \\
& - 2\varepsilon(p)\Re\left\{Q_1(z)\Delta(z)^j\sum_{l=0}^{j-1}(-1)^{\beta_l}(1-s)^l F_l^{t-j} w^{t-j-1}\left(\langle z,z\rangle + s\Delta(z) - w\right)\right\},
\end{aligned}
\tag{2.125}
$$

By (2.125) and by the next identity (2.126) we obtain the recurrence relations given by the statement of Lemma 23.

$$J(z,\overline{z}) = 2(1-s)^{j+1}\Re\left\{Q(z)\Delta(z)^{j+1}w^{t-j-1}\right\} + 2\gamma(p)(-1)^{j+1}\Re\left\{\mathscr{L}(z,\overline{z})\Delta(z)^{j+2}\sum_{l_1+l_2=t-j-2} E^{t-j-1}_{l_1,l_2} w^{l_2}\langle z,z\rangle^{l_1}\right.$$
$$+2(-1)^{j+1}\Re\left\{Q_1(z)\Delta(z)^{j+1}\sum_{l_1+l_2=t-j-1} E^{t-j}_{l_1,l_2} w^{t-j-1}\right\}$$
$$+2\varepsilon(p)\Re\left\{Q_1(z)\Delta(z)^{j+1}\sum_{l=0}^{j-1}(-1)^{\beta_l+1}(1-s)^{l+1}F^{t-j}_{l}w^{t-j-1}\right\} + \Theta^{\Lambda+2}_{N'_s}(z,\overline{z}), \tag{2.126}$$

where $\operatorname{wt}\left\{\overline{\Theta^{\Lambda+2}_{N'_s}(z,\overline{z})}\right\} \geq N'_s$.

The proof of our Lemma follows by using (2.126) and (2.119). □

Collecting the terms of bidegree (m,n) in $(z,\overline{z})$ in (2.112) with $m+n = ts+s$ and $t = j$, we obtain

$$\begin{aligned} G^{(ts+s)}_{nor}(z,\langle z,z\rangle) =& 2(1-s)^t\Re\left\{Q(z)\Delta(z)^t\right\} + 2K\Re\left\{Q_1(z)\Delta(z)^t\right\} + 2\Re\left\langle z, F^{(ts+s-1)}_{nor}(z,w)\right\rangle \\ &+ \varphi'_{ts+s,0}(z,\overline{z}) - \varphi_{ts+s,0}(z,\overline{z}) + (\Theta_1)^{ts+s}_{N'_s}(z,\overline{z}). \end{aligned} \tag{2.127}$$

By applying the extended Moser lemma we find a solution $\left(G^{(ts+s)}_{nor}(z,w), F^{(ts+s-1)}_{nor}(z,w)\right)$ for (2.127). Collecting the pure terms of degree $ts+s$ in (2.127), it follows that

$$\varphi'_{ts+s,0}(z) - \varphi_{ts+s,0}(z) = (1-s)^t Q(z)\Delta(z)^t + KQ_1(z)\Delta(z)^t, \tag{2.128}$$

where $K = (-1)^{\beta_1}k_1(1-s)^{t-1} + \cdots + (-1)^{\beta_{t-1}}k_{t-1}(1-s) + (-1)^{\beta_t}k_t$, with $k_1,\ldots,k_t \in \mathbb{N}$. By the proof of Lemma 2.3.8 (see (2.125) and (2.126)) we observe that $\beta_1 = 1,\ldots,\beta_t = t$. Next, by applying Lemma 2.1.4 to $\varphi_{ts+s,0}(z)$ and $\varphi'_{ts+s,0}(z)$, it follows that

$$\begin{aligned} \varphi_{ts+s,0}(z) &= \left(A_1(z)\Delta_1(z) + \cdots + A_N(z)\Delta_N(z)\right)\Delta(z)^t + C(z), \\ \varphi'_{ts+s,0}(z) &= \left(A'_1(z)\Delta_1(z) + \cdots + A'_N(z)\Delta_N(z)\right)\Delta(z)^t + C'(z), \end{aligned} \tag{2.129}$$

where $(\Delta_k\Delta^t)^\star(C(z)) = (\Delta_k\Delta^t)^\star(C'(z)) = 0$, for all $k = 1,\ldots,N$. We have

$$\begin{aligned} Q(z) &= \sum_{k=1}^{N}\Delta_k(z)\left(a_{k1}z_1 + \cdots + \left(a_{kk} - \frac{a+\overline{a}}{s}\right)z_k + \cdots + a_{kN}z_N\right), \\ Q_1(z) &= \sum_{k=1}^{N}\Delta_k(z)\left((a_{k1}+\overline{a}_{1k})z_1 + \cdots + (a_{kk}+\overline{a}_{kk}-(a+\overline{a}))z_k + \cdots + (a_{kN}+\overline{a}_{Nk})z_N\right). \end{aligned} \tag{2.130}$$

We impose the normalization condition $(\Delta_k\Delta^t)^\star\left(\varphi'_{ts+s,0}(z)\right) = 0$, for all $k = 1,\ldots,N$. By Lemma 2.1.4 this is equivalent to finding $(a_{ij})_{1\leq i,j\leq N}$ such that $A'_1(z) = \cdots = A'_N(z) = 0$. It follows that

$$\begin{aligned} (1-s)^t a_{kj} + K\left(a_{kj}+\overline{a}_{jk}\right) &= c_{kj}, \quad \text{for all } k,j = 1,\ldots,N,\ k\neq j, \\ (1-s)^t\left(a_{kk} - \frac{a+\overline{a}}{s}\right) + K\left(a_{kk}+\overline{a}_{kk}-(a+\overline{a})\right) &= c_{kk}, \quad \text{for all } k = 1,\ldots,N, \end{aligned} \tag{2.131}$$

where c_{kj} is determined, for all $k,j = 1,\ldots,N$. Here $Na = \sum_{k=1}^{N} a_{kk}$. By using the second equation in (2.131) we find $\Im a_{kk}$, for all $k = 1,\ldots,N$. By taking the real part in the second equation in (2.131), we obtain

$$\left(Ns(1-s)^t + 2NKs\right)\Re a_{kk} - \left(2(1-s)^t + 2Ks\right)\sum_{l=1}^{N}\Re a_{ll} = \Re c_{k,k}, \quad k = 1,\ldots,N. \tag{2.132}$$

By summing all the identities in (2.132), it follows that $(1-s)^t N(s-2)\sum_{l=1}^{N} \Re a_{ll} = \sum_{k=1}^{N} \Re c_{k,k}$. Next, going back to (2.132) we find $\Re a_{ll}$, for all $l = 1,\dots,N$. Now, let $k \neq j$ and $k,j \in \{1,\dots,N\}$. By taking the real and the imaginary part in first equation in (2.131), we obtain

$$\begin{aligned} &\left((1-s)^t+K\right)\Re a_{kj} + K\Re a_{jk} = \Re c_{k,j}, \quad K\Re a_{kj} + \left((1-s)^t+K\right)\Re a_{jk} = \Re c_{j,k}, \\ &\left((1-s)^t+K\right)\Im a_{kj} - K\Im a_{jk} = \Im c_{k,j}, \quad -K\Im a_{kj} + \left((1-s)^t+K\right)\Im a_{jk} = \Im c_{j,k}, \end{aligned} \tag{2.133}$$

where $c_{k,j}$ is determined, for all $k,j = 1,\dots,N$ and $k \neq j$. In order to solve the preceding system of equations it is enough to observe that $(1-s)^t\left((1-s)^t+2K\right) \neq 0$. It is equivalent to observe that

$$\begin{aligned} (1-s)^t + 2\left((-1)k_1(1-s)^{t-1} + \cdots + (-1)^t k_t\right) \neq 0, \\ (-1)^t\left((s-1)^t + 2\left(k_1(s-1)^{t-1} + k_2(s-1)^{t-2} + \cdots + k_t\right)\right) \neq 0. \end{aligned} \tag{2.134}$$

By composing the map that sends M into (2.42) with the map (2.90) we obtain our formal transformation that sends M into M' up to degree $ts+s+1$.

2.4 Proof of Theorem 2.0.3-Uniqueness of the formal transformation map

In order to prove the uniqueness of the map (2.8) it is enough to prove that the following map is the identity

$$M' \ni (z,w) \longrightarrow \left(z + \sum_{k\geq 2} F_{nor}^{(k)}(z,w), w + \sum_{k\geq 2} G_{nor}^{(k+1)}(z,w)\right) \in M'. \tag{2.135}$$

Here M' is a manifold defined by the normal form in Theorem 2.0.3. We have used the notations (2.41). We make the proof by induction induction on $k \geq 2$.

Definition 2.4.1. *The undetermined homogeneous parts of the map (2.135) by applying the extended Moser lemma are called the free parameters.*

We prove that $F_{nor}^{(2)}(z,w) = 0$. Here we recall the first case of the normal form construction. We assume that $t = 1$. By repeating the normalization procedures from the first case of the normal form construction, we find that all of the homogeneous components of $F_{nor}^{(2)}(z,w)$ except the free parameter are 0 and that $G_{nor}^{(3)}(z,w) = 0$. By using the same approach as in the first case of the normal form construction (see (2.66)), it follows that

$$\varphi'_{s+1,0}(z) - \varphi_{s+1,0}(z) = (1-s)\langle z,a\rangle\Delta(z) = 0. \tag{2.136}$$

Here a is the free parameter of $F_{nor}^{(2)}(z,w)$. It follows that $a = 0$. Therefore $F_{nor}^{(2)}(z,w) = 0$.

We assume that $F_{nor}^{(2)}(z,w) = \cdots = F_{nor}^{(k-2)}(z,w) = 0$, $G_{nor}^{(3)}(z,w) = \cdots = G_{nor}^{(k-1)}(z,w) = 0$. We want to prove that $F_{nor}^{(k-1)}(z,w) = 0$, $G_{nor}^{(k)}(z,w) = 0$. First, we consider the case when $k = 2t$, with $t \geq 2$. Let $a \in \mathbb{C}^N$ be the free parameter of the polynomial $F_{nor}^{(2t)}(z,w)$. By repeating all the normalization procedures from the first case of the normal form construction it follows that all of the homogeneous components of $F_{nor}^{(2t)}(z,w)$ except the free parameters are 0 and that $G_{nor}^{(2t+1)}(z,w) = 0$. We are interested in the image of the manifold M through the map (2.135) to M up to the degree $ts+1$. We repeat the normalization procedure done during the proof of Lemma 2.2.7 In that case we have considered a particular mapping (see (2.44)). Here we have a general polynomial map with other free parameters. They generate terms of weight at least $ts+2$ that do not change their weight under the conjugation:

$$\begin{aligned} &\mathrm{wt}\{\langle F_{1,m}(z)w^m, z\rangle\}, \quad \mathrm{wt}\{\langle z, F_{1,m}(z)w^m\rangle\} \geq ts+2, \text{ for all } m > t; \\ &\mathrm{wt}\{\langle F_{0,r}(z)w^r, z\rangle\}, \quad \mathrm{wt}\{\langle z, F_{0,r}(z)w^r\rangle\} \geq ts+2, \text{ for all } r \geq t+2. \end{aligned} \tag{2.137}$$

Here $F_{1,m}(z)w^m$, $F_{0,r}(z)w^r$ are the free parameters of $F_{nor}^{(2m+1)}(z,w)$ and $F_{nor}^{(2r)}(z,w)$, for all $m>t$ and $r\geq t+2$. Therefore they cannot interact with the pure terms of degree $ts+1$ (because of the higher weight). Therefore all Lemmas 2.2.1-2.2.6 remain the same in this general case.

By using the same approach as in the first case of the normal form construction (see (2.88)), it follows that

$$\varphi'_{ts+1,0}(z)-\varphi_{ts+1,0}(z)=(1-s)^t\langle z,a\rangle\Delta^t(z)=0. \tag{2.138}$$

It follows that $a=0$. Therefore $F_{nor}^{(2t)}(z,w)=0$.

We assume that $k=2t+1$, with $t\geq 2$. Let $(a_{i,j})_{1\leq i,j\leq N}$ be the free parameter of $F_{nor}^{(2t+1)}(z,w)$. By repeating all the normalization procedures from the first case of the normal form construction, it follows that all of the homogeneous components of $F_{nor}^{(2t+1)}(z,w)$ except the free parameters are 0 and that $G_{nor}^{(2t+2)}(z,w)=0$.

We are interested of the image of the manifold M' through the map (2.44) to M' up to the degree $ts+s+1$. The other free parameters of the map (2.135) generate terms of weight at least $ts+s+1$ that do not change their weight under the conjugation:

$$\begin{aligned} \mathrm{wt}\{\langle F_{1,m}(z)w^m,z\rangle\},\quad \mathrm{wt}\{\langle z,F_{1,m}(z)w^m\rangle\}\geq ts+s+1,\ \text{for all } m>t+1;\\ \mathrm{wt}\{\langle F_{0,r}(z)w^r,z\rangle\},\quad \mathrm{wt}\{\langle z,F_{0,r}(z)w^r\rangle\}\geq ts+s+1,\ \text{for all } r\geq t+3. \end{aligned} \tag{2.139}$$

Therefore all Lemmas 2.3.1-2.3.7 remain true in this general case.

By using the same approach as in the second case of the normal form construction (see (2.128)), it follows that

$$\varphi'_{ts+s,0}(z)-\varphi_{ts+s,0}(z)=(1-s)^tQ(z)\Delta^t(z)=0. \tag{2.140}$$

It follows that $(a_{i,j})_{1\leq i,j\leq N}=0$. Therefore $F_{nor}^{(2t+1)}(z,w)=0$, $G_{nor}^{(2t+2)}(z,w)=0$. This proves that (2.135) is the identity mapping. From here we conclude the uniqueness of the formal transformation (2.8).

Chapter 3

A family of analytic discs

Let $(z_1,\ldots,z_N,w)$ be the coordinates from $\mathbb{C}^{N+1}$. In this chaper, we consider the higher dimensional analog case of (1.1) when the submanifold $M\subset\mathbb{C}^{N+1}$ is defined near $p=0$ by

$$w=z_1\bar{z}_1+\lambda\left(z_1^2+\bar{z}_1^2\right)+Q\left(z_1,\bar{z}_1,z_2,\bar{z}_2,\ldots,z_N,\bar{z}_N\right)+\mathrm{O}(3), \tag{3.1}$$

where $Q\left(z_1,\bar{z}_1,z_2,\bar{z}_2,\ldots,z_N,\bar{z}_N\right)$ is a quadratic form depending on $z_2,\bar{z}_2,\ldots,z_N,\bar{z}_N$ and combinations between $z_2,\bar{z}_2,\ldots,z_N,\bar{z}_N$ and $z_1,\bar{z}_1$. We say that λ is elliptic if $\lambda\in\left[0,\frac{1}{2}\right)$.

In this chapter, we extend Kenig-Webster's Theorem [24]. We prove the following result:

Theorem 3.0.2. *Let $M\subset\mathbb{C}^{N+1}$ be a smooth submanifold defined locally near $p=0$ by (3.1) such that λ is elliptic. Then there exists a family of regularly embedded analytic discs with boundaries on M that are mutually disjoint whose union forms a smooth hypersurface $\widetilde{M}$ with boundary M in a neighborhood of the CR singularity $p=0$.*

The manifold $\widetilde{M}$ given by Theorem 3.0.2 is not necessary a Levi-flat hypersurface as in Kenig-Webster's case from [24] in $\mathbb{C}^2$. For the definition and properties of the Levi-form of a hypersurface we mention here the book [22], page 49.

The existence problem of a Levi-flat hypersurface with prescribed boundary S in $\mathbb{C}^{N+1}$ with $N\geq 2$, was studied by Dolbeault-Tomassini-Zaitsev in [9] under the following natural assumptions on S:

(i) S is compact, connected and nowhere minimal at its CR points;

(ii) S does not contain a complex submanifold of dimension $(n-2)$;

(iii) S contains a finite number of flat elliptic CR singularities.

We would like to mention that properties of nowhere minimal CR submanifolds were studied by Lebl in [30].

The CR singularity $p=0$ is called elliptic if the quadratic part from (3.1) is positive definite. We say that $p=0$ is a "flat" if Definition 2.1 from [9] is satisfied. Under the preceding natural assumptions, Dolbeault-Tomassini-Zaitsev proved the existence of a (possibly singular) Levi-flat hypersurface which bounds S in the sense of currents (see Theorem 1.3, [9]).

The graph case was studied by Dolbeault-Tomassini-Zaitsev in [10]: Let $\mathbb{C}^{N+1}=\left(\mathbb{C}_z^N\times\mathbb{R}_u\right)\times\mathbb{R}_v$, where $w=u+iv$, and let Ω be a bounded strongly convex domain of $\mathbb{C}_z^N\times\mathbb{R}_u$ with smooth boundary $b\Omega$. Let $S\subset\mathbb{C}^{N+1}$, $n\geq 3$, be the graph of a function $g:b\Omega\longrightarrow\mathbb{R}_v$ such that S satisfies the natural assumptions (i), (ii), (iii). Under these assumptions Dolbeault-Tomassini-Zaitsev proved the following result

Theorem 3.0.3. *Let $q_1,q_2\in b\Omega$ be the projections of the complex points p_1,p_2 of S, respectively. Then, there exists a Lipschitz function $f:\overline{\Omega}\longrightarrow\mathbb{R}_v$ which is smooth on $\overline{\Omega}-\{q_1,q_2\}$ and such that $f|_{b\Omega}=g$ and $N=graph(f)-S$ is a Levi-flat hypersurface of $\mathbb{C}^{N+1}$. Moreover, each complex leaf of N is the graph of a holomorphic function $\phi:\Omega'\longrightarrow\mathbb{C}$ where $\Omega'\subset\mathbb{C}^{n-1}$ is a domain with smooth boundary (that depends on the leaf) and ϕ is smooth on Ω'.*

As an application of Theorem 3.0.2 , we solve an open problem regarding the regularity of f given by Theorem 3.0.3 at q_1, q_2, proposed by Dolbeault-Tomassini-Zaitsev in [10].

By combining Theorem 3.0.2 and Theorem 3.0.3 we obtain the following result

Theorem 3.0.4. *Let $M \subset \mathbb{C}^{N+1}$ be a smooth submanifold as in Theorem 3.0.3. Suppose p is a point in M such that M is defined near $p = 0$ by (3.1) satisfying the condition that (i) $p = 0$ is a flat-elliptic CR singularity (ii) any CR point of M near $p = 0$ is non-minimal, and (iii) M does not contain a complex submanifold of dimension $n-2$. Then $\widetilde{M}$ constructed by Theorem 3.0.2 is a smooth Levi-flat hypersurface with boundary M in a neighborhood of $p = 0$.*

In the real analytic case our smoothness result combined with an similar argument as in the paper [23] of Huang-Yin concerning the analyticity of the local hull of holomorphy, gives the following result:

Theorem 3.0.5. *Let $M \subset \mathbb{C}^{N+1}$ be a real analytic submanifold defined near $p = 0$ by (3.1) and that satisfies the assumptions of Theorem 3.0.4. Then $\widetilde{M}$ is a Levi-flat hypersurface real-analytic across the boundary manifold M.*

We prove our results by following the lines of developed by Huang in [17], Kenig-Webster in [24], [25] and in particulary the construction of holomorphic discs developed by Huang-Krantz in [16]. First, we make a perturbation along the CR singularity and then we find a holomorphic change of coordinates depending smoothly on a parameter. Then, we will adapt the methods used in $\mathbb{C}^2$ by Huang-Krantz and Kenig-Webster in our case.

We would like to mention that versions of our result were obtained in a higher codimensional case by Huang in [17] and Kenig-Webster in [25].

3.1 Preliminaries

3.1.1 A Perturbation Along the CR singularity

We construct analytic discs attached to M depending smoothly on

$$X = (z_2, \ldots, z_N) = (x_2 + iy_2, \ldots, x_N + iy_N) \approx 0 \in \mathbb{C}^{N-2}. \tag{3.2}$$

By using the notation $z = z_1$, our manifold M is defined near $p = 0$ by

$$w = z\bar{z} + \lambda\left(z^2 + \bar{z}^2\right) + Q(z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_N, \bar{z}_N) + \mathrm{O}(3), \tag{3.3}$$

or equivalently by

$$\begin{aligned} w =& H_{0,0}(X) + \bar{z}H_{0,1}(X) + zH_{1,0}(X) + z\bar{z}(1 + H_{1,1}(X)) \\ &+ (\lambda + H_{2,0}(X))z^2 + (\lambda + H_{0,2}(X))\bar{z}^2 + \mathrm{O}\left(|z|^3\right), \end{aligned} \tag{3.4}$$

where $H_{0,0}(X)$, $H_{1,0}(X)$, $H_{0,1}(X)$, $H_{1,1}(X)$, $H_{2,0}(X)$, $H_{0,2}(X)$ are smooth functions vanishing at $X = 0$.

We prove the following lemma:

Lemma 3.1.1. *Let $M \subset \mathbb{C}^2$ be a real smooth submanifold defined near $p = 0$ by $w = az + b\bar{z} + O\left(|z|^2\right)$. Then*

$$T_0^c M \neq \emptyset \Longleftrightarrow b = 0. \tag{3.5}$$

Proof. We need to solve the equations $\partial f = \overline{\partial} f = 0$ at the point $z = w = 0$. We compute:

$$\partial f|_0 = \frac{\partial f}{\partial z}(0)dz + \frac{\partial f}{\partial w}(0)\,dw = -dw + adz, \quad \overline{\partial} f|_0 = \frac{\partial f}{\partial \bar{z}}(0)d\bar{z} + \frac{\partial f}{\partial \overline{w}}(0)d\overline{w} = bd\bar{z}. \tag{3.6}$$

We obtain $adz = dw$ and $bd\bar{z} = 0$. It follows that $p = 0$ is a CR singularity if and only if $b = 0$. □

We make a change of coordinates depending smoothly on $X \approx 0 \in \mathbb{C}^{N-2}$ preserving the CR singularity $p = 0$:

Proposition 3.1.2. *There exists a biholomorphic change of coordinates in (z,w) depending smoothly on $X \approx 0 \in \mathbb{C}^{N-2}$ that sends (3.4) to a submanifold defined by*

$$w = z\bar{z} + \lambda(X)\left(z^2 + \bar{z}^2\right) + O\left(|z|^3\right), \tag{3.7}$$

preserving the CR singularity $p = 0$. Here $0 \leq \lambda(X) < \frac{1}{2}$ for $X \approx 0 \in \mathbb{C}^{N-2}$ and $\lambda(0) = \lambda$.

Proof. We consider a local defining function for M near $p = 0$

$$\begin{aligned} f(z,X,w) = &-w + H_{0,0}(X) + \bar{z}H_{0,1}(X) + zH_{1,0}(X) + z\bar{z}(1 + H_{1,1}(X)) \\ &+ (\lambda + H_{2,0}(X))z^2 + (\lambda + H_{0,2}(X))\bar{z}^2 + \mathrm{O}\left(|z|^3\right). \end{aligned} \tag{3.8}$$

Each fixed $X \approx 0 \in \mathbb{C}^{N-2}$ defines us a real submanifold in $\mathbb{C}^2$ which may not have a CR singularity at the point $z = w = 0$ because $H_{0,1}(X)$ may be different than 0 (see Lemma 2.1). Therefore we need to make a change of coordinates in (z,w) depending smoothly on $X \approx 0 \in \mathbb{C}^{N-2}$ that perturbs the CR singularity $p = 0$. We consider the following equation

$$0 = \frac{\partial f}{\partial \bar{z}} = H_{0,1}(X) + (1 + H_{1,1}(X))z + B(z,\bar{z},X), \tag{3.9}$$

where $B(z,\bar{z},X)$ is a smooth function. Since $H_{1,1}(0) = 0$, by applying the implicit function theorem we obtain a smooth solution $z_0 = z_0(X)$ for (3.9). By making the translation $(w',z') = (w, z + z_0(X))$, the equation (3.4) becomes

$$w = zC_{1,0}(X) + z\bar{z}(1 + C_{1,1}(X)) + (\lambda + C_{2,0}(X))z^2 + (\lambda + C_{0,2}(X))\bar{z}^2 + \mathrm{O}\left(|z|^3\right), \tag{3.10}$$

where $C_{1,0}(X)$, $C_{1,1}(X)$, $C_{2,0}(X)$, $C_{0,2}(X)$ are smooth functions vanishing at $X = 0$. Let $\gamma(X) = 1 + C_{1,1}(X)$, $\Lambda_1(X) = \lambda + C_{2,0}(X)$, $\Lambda_2(X) = \lambda + C_{0,2}(X)$. In the new coordinates $(w,z) := ((w - C_{1,0}(X)z)/\gamma(X), z)$, the equation (3.10) becomes

$$w = z\bar{z} + \Lambda_1(X)z^2 + \Lambda_2(X)\bar{z}^2 + \mathrm{O}\left(|z|^3\right). \tag{3.11}$$

Next, we consider a map $\Theta(X)$ such that $\Lambda_2(X)e^{-2i\Theta(X)} \geq 0$. Changing the coordinates $(w,z) := \left(w, ze^{i\Theta(X)}\right)$, we can assume $\Lambda_2(X) \geq 0$. Changing again the coordinates $(w,z) := \left(w + (\Lambda_1(X) - \Lambda_2(X))z^2, z\right)$ we obtain (3.7). □

We write

$$M: w = z\bar{z} + \lambda(X)\left(z^2 + \bar{z}^2\right) + P(z,X) + iK(z,X), \tag{3.12}$$

where $P(z,X)$ and $K(z,X)$ are real smooth functions. We prove an extension of Lemma 1.1 from [24]:

Proposition 3.1.3. *There exists a holomorphic change of coordinates in (z,w) depending smoothly on $X \approx 0 \in \mathbb{C}^{N-2}$ in which K and its partial derivatives in z and $\bar{z}$ of order less or equal to l vanish at $z = 0$.*

Proof. By making the substitution $(z'(X), w'(X)) = (z, w + B(z,X,w))$ and by (3.12) it follows that

$$M: w' = q(z,X) + P(z,X) + iK(z,X) + \Re B(z,X,w) + i\Im B(z,X,w), \tag{3.13}$$

where $q(z,X) = z\bar{z} + \lambda(X)\left(z^2 + \bar{z}^2\right)$. We want to make the derivatives in z of order less than l of $i(K(z,X) + \Im B(z,X,w))$ vanish at $z = 0$. By multiplying (3.13) by $i = \sqrt{-1}$, our problem is reduced to the following general equation

$$\Re B(z,X,q(z,X) + P(z,X) + iK(z,X)) = f(z,\bar{z},X), \tag{3.14}$$

where $f(z,\bar{z},X)$ is a real formal power series in $(z,\bar{z},X)$ with cubic terms in z and $\bar{z}$ with coefficients depending smoothly on $X \approx 0 \in \mathbb{C}^{N-2}$. We write

$$f(z,\bar{z},X) = \sum_{m=3}^{l} f_m(z,\bar{z},X), \; f_m(z,\bar{z},X) = \sum_{j_1+j_2=m} c^m_{j_1,j_2}(X) z^{j_1}\bar{z}^{j_2}, \; c^m_{j_1,j_2}(X) = \overline{c^m_{j_2,j_1}(X)},$$
$$B(z,X,w) = \sum_{m=3}^{l} B_m(z,X,w), \quad B_m(z,X,w) = \sum_{j_1+2j_2=m} b^m_{j_1,j_2}(X) z^{j_1} w^{j_2}. \tag{3.15}$$

We solve inductively (3.14) by using the following lemma:

Lemma 3.1.4. *The equation (3.14) has a unique solution with the normalization condition* $\Im B_m(0,X,u) = 0$.

Proof. We define the weight of z to be 1 and the weight of w to be 2. We say that the polynomial $B_m(z,X,w)$ has weight m if $B_m(tz,X,t^2w) = t^m B_m(z,X,w)$. Let $\mathbb{B}_m$ be the space of all such homogeneous holomorphic polynomials in (z,w) of weight m satisfying the normalization condition with coefficients depending smoothly on $X \approx 0 \in \mathbb{C}^{N-2}$ and let $\mathbb{F}_m$ be the space of all homogeneous polynomials $f_m(z,\bar{z},X)$ of bidegree (k,l) in $(z,\bar{z})$ with $k+2l = m$ with coefficients depending smoothly on $X \approx 0 \in \mathbb{C}^{N-2}$. We can rewrite (3.14) as follows

$$B_m(z,X,q(z,X)+P(z,X)+iK(z,X)) = B_m(z,X,q(z,X)) + \mathrm{O}\left(|z|^{m+1}\right). \tag{3.16}$$

In order to solve (3.16) it is enough to prove that we have a linear invertible transformation

$$\varphi(X) : \mathbb{B}_m \ni B_m(z,X,w) \mapsto \Re B_m(z,X,q(z,X)) \in \mathbb{F}_m, \tag{3.17}$$

depending smoothly on $X \approx 0 \in \mathbb{C}^{N-2}$. By Lemma 1.1 from the paper [24] of Kenig-Webster, it follows that $\varphi(X)$ is invertible for $X = 0 \in \mathbb{C}^{N-2}$. By the continuity it follows that $\varphi(X)$ is invertible. If it is necessary we shrink the range of $X \approx 0 \in \mathbb{C}^{N-2}$. □

The proof is completed now by induction and by using Lemma 2.4. □

3.1.2 Preliminary Preparations

Let $w = u+iv$ and $I_\varepsilon := (-\varepsilon,\varepsilon) \subset \mathbb{R}$, for $0 < \varepsilon << 1$. We assume that M is defined by (3.12) and satisfies the properties of Proposition 2.3.

In order to define a family of attached discs to the manifold M, we define the following domain

$$D_{X,r} = \{z \in \mathbb{C};\; v = 0,\; q(z,X)+P(z,X) \leq u < \varepsilon\}, \tag{3.18}$$

where $u = r^2$. By similar arguments as in the paper [17] of Huang, it follows that $D_{X,r}$ is a simply connected bounded set of $\mathbb{C}$. Therefore there exists a unique mapping $r\sigma_{X,r} : \Delta \to D_{X,r}$ such that $\sigma_{X,r}(0) = 0$ and $\sigma'_{X,r}(0) > 0$. Then, for $0 < r << 1$ we can define the following family of curves depending smoothly on $X \approx 0 \in \mathbb{C}^{N-2}$

$$\gamma_{X,r} = \left\{z \in \mathbb{C};\; q(z,X)+P(z,X) = r^2\right\}. \tag{3.19}$$

Next, we define the following family of analytic discs

$$\left\{\left(r\sigma_{X,r}, X, r^2\right)\right\}_{X\approx 0 \in \mathbb{C}^{N-2},\; 0<r<<1}. \tag{3.20}$$

The family of analytic discs shrinks to $\{0\} \times \mathscr{O} \times \{0\}$ as $r \mapsto 0$, where $0 \in \mathscr{O} \subset \mathbb{C}^{N-2}$ and fills up the following domain

$$\widetilde{M}_0 = \left\{(z,X,u) \in \mathbb{C} \times \mathbb{C}^{N-2} \times \mathbb{R};\; \|X\| << 1,\; q(z,X)+P(z,X) \leq u\right\}. \tag{3.21}$$

3.1.3 The Hilbert Transform on a Variable Curve

Let $\gamma_{X,r}$ given by (3.19), where r is taken very small. For a function $\varphi_{X,r}(\theta)$ defined on $\gamma_{X,r}$ we define its Hilbert transform $\mathrm{H}_{X,r}[\varphi_{X,r}]$ to be the boundary value of a function holomorphic inside $\gamma_{X,r}$, with its imaginary part vanishing at the origin. For more informations about Hilbert's transform we mention here the book [15] of Helmes.

For $\alpha \in (0,1)$ we define the following Banach spaces:

$$\begin{aligned}\mathscr{C}^{\alpha} &:= \left\{ u : \gamma_{X,r} \longrightarrow \mathbb{R};\ \|u\|_{\alpha} := \sup_{x\in\gamma_{X,r}} + \sup_{\substack{x,y\in\gamma_{X,r}\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} < \infty \right\},\\ \mathscr{C}^{k,\alpha} &:= \left\{ u : \gamma_{X,r} \longrightarrow \mathbb{R};\ \|u\|_{k,\alpha} := \sum_{|\beta|\leq k} \left\| \mathrm{D}^{\beta} u \right\|_{\alpha} < \infty \right\}.\end{aligned} \tag{3.22}$$

Let $X := \{x_2, y_2, \ldots, x_N, y_N\}$. The following result can be proved by using the same lines as in Kenig-Webster's paper [24] (Theorem 2.5) or from Kenig-Webster's paper [25]:

Proposition 3.1.5. *As $r \to 0$ and $X \approx 0 \in \mathbb{C}^{N-2}$ we have*

$$\|\mathscr{H}_{X,r}\|_{j,\alpha} = O(r),\ \textit{for all}\ j \leq l-2; \quad \left\| \left(\partial_X^{|I|} \partial_r^{s} \right) \mathscr{H}_{X,r} \right\|_{j,\alpha} = O(1),\ \textit{for all}\ j+2s \leq l-4,\ I \in \mathbb{N}^{N-2}. \tag{3.23}$$

3.1.4 An Implicit Functional Equation

During this section we work in the Holder space $\left(\mathscr{C}^{j,\alpha}, \|\cdot\|_{j,\alpha}\right)$. We employ ideas developed by Huang-Krantz in [17], Huang in [17], Kenig-Webster in [24], [25] and we define the following auxiliary hypersurface

$$M_0 = \left\{ (z,X,u) \in \mathbb{C} \times \mathbb{C}^{N-2} \times \mathbb{R};\ \|X\| << 1,\ q(z,X) + P(z,X) = u < \varepsilon \right\}, \tag{3.24}$$

where $\varepsilon > 0$ is small enough and $w = u + iv$. We would like to find a map of the following type

$$T = T[X] := \left(z\left(1 + \mathscr{F}(z,X,r)\right), \mathscr{B}(z,X,r) \right) \tag{3.25}$$

such that $T(M_0) \subseteq M$. Here $\mathscr{F}$, $\mathscr{B}$ are holomorphic functions in z and smooth in (X,r). It follows that

$$\mathscr{B}(z,X,r)\,|_{\gamma_{X,r}} = (q+P+iK)\left(z + z\mathscr{F}(z,X,r), X\right)|_{\gamma_{X,r}}, \tag{3.26}$$

where $\gamma_{X,r}$ is the curve defined by (3.19). By using the Hilbert transform on the curve $\gamma_{X,r}$ and by dividing by r^2 the equation (3.26), it follows that there exists a smooth function $V(X,r)$ such that

$$\begin{aligned} q\left(z(1+\mathscr{F}(z,X,r)), X\right)|_{\gamma_{X,r}} &= -P\left(z(1+\mathscr{F}(z,X,r)), X\right)|_{\gamma_{X,r}} \\ &\quad - \mathscr{H}_{X,r}\left[K\left(z(1+\mathscr{F}(z,X,r)), X\right)\right]|_{\gamma_{X,r}} + V(X,r). \end{aligned} \tag{3.27}$$

We follow Huang-Krantz's strategy from [17] and we define the following functional

$$\Omega(\mathscr{F},X,r) = \frac{q\left(z(1+\mathscr{F}),X\right) + P\left(z(1+\mathscr{F}),X\right)}{r^2}|_{\gamma_{X,r}}, \tag{3.28}$$

where $\mathscr{F} = \mathscr{F}(z,X,r)$. By linearizing in $\mathscr{F} = 0$ the functional defined in (3.28), the equation (3.27) becomes

$$1 + \Omega'(\mathscr{F},X,r) + \Omega_1(\mathscr{F},X,r) + \frac{1}{r^2}\mathscr{H}_{X,r}\left[K\left(z(1+\mathscr{F}),X\right)\right]|_{\gamma_{X,r}} - \frac{V(X,r)}{r^2} = 0, \tag{3.29}$$

where $\mathscr{F} = \mathscr{F}(z,X,r)$ and $\Omega_1(\mathscr{F}(z,X,r),X,r)$, are terms that are coming from the Taylor expansion of

$P(z,X)$ and

$$\Omega'(\mathscr{F},X,r)=\frac{2}{r^2}\Re\left\{(q+P)_z(z,X)z\mathscr{F}\right\}|_{\gamma_{X,r}}. \tag{3.30}$$

We put the normalization condition $V(X,r)=r^2$. In order to find a solution $\mathscr{F}$ in the Holder space $\left(\mathscr{C}^{j,\alpha},\|\cdot\|_{j,\alpha}\right)$ for (3.29), we need to study the regularity properties of the functional Ω. We consider the following notation

$$\mathscr{C}_{X,r}(z)=\frac{2}{r^2}\Re\left\{(q+P)_z(z,X)z\right\}|_{\gamma_{X,r}}. \tag{3.31}$$

Since $\mathscr{C}_{X,r}(z)\neq 0$ for $|r|<<1$, $X\approx 0\in\mathbb{C}^{N-2}$, we can write $\mathscr{C}_{X,r}(z)=\mathscr{A}(z,X,r)\mathscr{B}(z,X,r)$ with

$$\mathscr{A}(z,X,r)=|\mathscr{C}_{X,r}(z)|,\quad \mathscr{B}(z,X,r)=\frac{\mathscr{C}_{X,r}(z)}{|\mathscr{C}_{X,r}(z)|}. \tag{3.32}$$

Then $\ln\mathscr{B}(z,X,r)$ is a well-defined smooth function in (z,X,r). Among the lines developed by Huang-Krantz in [17], we define the following function

$$\mathscr{C}^{\star}(z,X,r)=\frac{e^{i\mathscr{H}_{X,r}(\ln\mathscr{B}(z,X,r))}}{\mathscr{A}(z,X,r)}. \tag{3.33}$$

Then $\mathscr{C}^{\star}$ is a smooth positive function and $D(z,X,r):=\mathscr{C}^{\star}(z,X,r)\mathscr{C}(z,X,r)$ is holomorphic in z, smooth in (X,r). We write $D(z,X,r)\mathscr{F}(z,X,r)\equiv U(z,X,r)+\sqrt{-1}\mathscr{H}_{X,r}[U(z,X,r)]$. Since $D(z,X,r)\neq 0$ we can rewrite (3.29) as follows

$$\begin{aligned} U(z,X,r)=&-C^{\star}(z,X,r)\left(\Omega_1\left(\frac{U(z,X,r)+i\mathscr{H}_{X,r}[U(z,X,r)]}{D(z,X,r)},X,r\right)\right)\\ &-C^{\star}(z,X,r)\frac{1}{r^2}\mathscr{H}_{X,r}\left[K\left(z\left(1+\frac{U(z,X,r)+i\mathscr{H}_{X,r}[U(z,X,r)]}{D(z,X,r)}\right),X\right)\right].\end{aligned} \tag{3.34}$$

We summarize all the precedent computations and we obtain the following regularity result

Theorem 3.1.6. *The equation (3.34) has a unique solution in the Banach space $\left(\mathscr{C}^{j,\alpha},\ \|\cdot\|_{j,\alpha}\right)$ such that*

$$\|U\|_{j,\alpha}=O\left(r^{l-2}\right),\ \textit{for all } j\leq l-2;\quad \left\|\left(\partial_X^{|I|}\partial_r^s\right)U\right\|_{j,\alpha}=O\left(r^{l-s-2}\right),\ \textit{for all } j+2s\leq l-4,\ I\in\mathbb{N}^{N-2}. \tag{3.35}$$

Proof. The solution U and its uniqueness follows by applying the implicit function theorem. We denote by $\Lambda_1(U,X,r)$ and $\Lambda_2(U,X,r)$ the first and the second term from (3.34). It follows that

$$\|U\|_{j,\alpha}\leq\|\Lambda_1(U,X,r)\|_{j,\alpha}+\|\Lambda_2(U,X,r)\|_{j,\alpha}\leq\|\Lambda_1(U,X,r)\|_{j,\alpha}+\mathrm{O}\left(r^{l-2}\right)\leq C\|U\|_{j,\alpha}^2+\mathrm{O}\left(r^{l-2}\right),$$

for some $C>0$. It follows that $\|U\|_{j,\alpha}=\mathrm{O}\left(r^{l-2}\right)$.

The proof of the second regularity property goes after the previous line. Differentiating with r the equation (3.34) it follows that $\partial_r U=\partial_r\Lambda_1(U,X,r)+\partial_U\Lambda_1(U,X,r)[\partial_r U]+\partial_r\Lambda_2(U,X,r)+\partial_U\Lambda_2(U,X,r)[\partial_r U]$. By Proposition 2.3 and Proposition 2.5 we obtain that $\|\partial_r U\|_{j,\alpha}=\mathrm{O}\left(r^{l-2-1}\right)$. Since $P(z,X)=\mathrm{O}\left(z^3\right)$ and $K(z,X)=\mathrm{O}\left(z^l\right)$, by taking higher derivatives of x in (3.34) it follows that the differentiation of any order with $x\in X$ does not affect the estimates. Therefore the second estimates follow immediately. ☐

We write that

$$\mathscr{F}_{X,r}[\varphi_{X,r}]=\frac{U(z,X,r)+i\mathscr{H}_{X,r}[U(z,X,r)]}{D(z,X,r)}:=\varphi_{X,r}+i\mathscr{H}_{X,r}[\varphi_{X,r}], \tag{3.36}$$

where $\|\varphi_{X,r}\|_{j,\alpha}=\mathrm{O}\left(r^{l-2}\right)$, for all $j\leq l-2$ and $\left\|\left(\partial_X^{|I|}\partial_r^s\right)\varphi_{X,r}\right\|_{j,\alpha}=\mathrm{O}\left(r^{l-s-2}\right)$, for all $j+2s\leq l-4$, $I\in$

$\mathbb{N}^{N-2}$.

3.2 A Family of Analytic Discs and Proofs of Main Results

3.2.1 A Family of Analytic Discs

We construct a continuous mapping T defined on $\widetilde{M}_0$ into $\mathbb{C}^2$ that is holomorphic in z for each fixed $u = r^2$ and that maps slice by slice the hypersurface M_0 into M. Let $\varphi_{X,r}$ be the function defined by (3.36). Then

$$\mathscr{F}_{X,r}[\varphi_{X,r}] = \varphi_{X,r} + i\mathscr{H}_{X,r}[\varphi_{X,r}], \quad \mathscr{B}_{X,r}[\varphi_{X,r}] = (q+P+iK)(z+z\mathscr{F}_{X,r}[\varphi_{X,r}],X). \tag{3.37}$$

We extend these functions to $\widetilde{M}_0$ by the Cauchy integral as follows

$$\begin{aligned}\mathscr{F}(\zeta,X,r) = \mathscr{C}(\mathscr{F}_{X,r}[\varphi_{X,r}])(\zeta) \equiv \frac{1}{2\pi i}\int_0^{2\pi}\frac{\mathscr{F}_{X,r}[\varphi_{X,r}](\theta)z_\theta(\theta,X,r)}{z(\theta,X,r)-\zeta}d\theta,\\ \mathscr{B}(\zeta,X,r) = \mathscr{C}(\mathscr{B}_{X,r}[\varphi_{X,r}])(\zeta) \equiv \frac{1}{2\pi i}\int_0^{2\pi}\frac{\mathscr{B}_{X,r}[\varphi_{X,r}](\theta)z_\theta(\theta,X,r)}{z(\theta,X,r)-\zeta}d\theta,\end{aligned} \tag{3.38}$$

where $z = z(\theta,X,r)$ is a parameterization of the curve $\gamma_{X,r}$ defined by (3.19).

We define T by (3.37). Then T is continuous by construction up to the boundary on each slice (X,r)=constant. In order to obtain the regularity of T, we have to bound the derivatives in (z,X,u) of $\mathscr{F}$ and $\mathscr{B}$. We state the following lemma:

Lemma 3.2.1. *For all $j+2s \le l-4$, $I \in \mathbb{N}^{N-2}$ as $r \mapsto 0$, we have*

$$\partial_\theta^j\partial_X^{|I|}\partial_r^s\mathscr{F}(z,X,r) = O\left(r^{l-s-2}\right), \quad \partial_z^j\partial_X^{|I|}\partial_r^s\mathscr{B}(z,X,r) = O\left(r^{l-s}\right). \tag{3.39}$$

The proof of the preceding lemma follows by the lines of Lemma 4.1 proof from Kenig-Webster's paper [24].

Theorem 3.2.2. *Let M defined by (3.12) with $P(z,X) = O(z^3)$, $K(z,X) = O(z^l)$, $l \ge 7$, T extended by (3.38). Then $\widetilde{M} = T\left(\widetilde{M}_0\right)$ is a complex manifold-with-boundary regularly foliated by discs embedded of class $\mathscr{C}^{\frac{l-7}{3}}$.*

Proof. Since $\partial_u = \frac{1}{2r}\partial_r$, it follows that

$$\partial_z^j\partial_X^{|I|}\partial_u^s\mathscr{F}_{X,r}(z,X,r) = \mathrm{O}\left(r^{l-2s-j-2}\right), \quad \partial_z^j\partial_X^{|I|}\partial_u^s\mathscr{F}_{X,r}(z,X,r) = \mathrm{O}\left(r^{l-2s}\right), \tag{3.40}$$

and these derivatives remain bounded for all $j+2s \le l-4$, $I \in \mathbb{N}^{N-2}$. It follows that the jacobian matrix DT of $T = T(X)$ is the identity matrix. □

3.2.2 Proof of Theorem 3.0.2

Let M, $\widetilde{M}$, T as in Theorem 3.3.2. Using the techniques from [26], [29] together with an extended reflection principle as in the paper [25] of Kenig-Webster, we construct smooth extension of T past every point of $M_0 - \{0\}$. By similar arguments as in the papers [24], [25] of Kenig-Webster, we obtain that $M \cup \widetilde{M}$ is a smooth manifold-with-boundary M in a neighborhood of the CR singular point $p = 0$.

3.2.3 Proof of Theorem 3.0.4

Since the hypersurface given by Theorem 2.1 is Levi-flat it follows each of our analytic discs is a reparameterization of an analytic disc contained inside. By dimension reasons it follows that the under the hypothesis of Theorem 3.0.3, the hypersurfaces given by Theorem 3.0.2 and Theorem 3.0.3 are the same.

3.2.4 Proof of Theorem 3.0.5

We can study now the hull of M near $p = 0$ when M is assumed to be real-analytic. The hypersurface M_0 defined by (3.24) is foliated by a the family of analytic discs defined by (3.20) and therefore $\widetilde{M}$ is foliated by the family of analytic discs defined by (3.38). By similar arguments as in Section 7 of the paper [23] of Huang-Yin we obtain our result. The author believes that the arguments from the paper [17] of Huang-Krantz or from the paper [17] of Huang can be adapted in order to prove the analyticity in our case.

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