The Radiation Bound for a Klein-Gordon Field on a Static Spherical Spacetime

Thesis submitted to

The University of Dublin

for the degree of

Doctor of Philosophy

March 2013

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Declaration

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Summary

We establish a *well-posed Cauchy problem* in Minkowski $(\mathbb{R}^4, \eta)$, associated with a radiating Klein-Gordon field $\psi(x) = e^{zt}\tilde{\psi}(x^i)$, in *curvature coordinates* $\{x^\mu = t, \rho, \theta, \phi\}$ on a static spherically symmetric spacetime $(\mathcal{M}, g)$. This is crucial to our primary concern with proving an optimal L^2-bound on the radiating field $\tilde{\psi}$, decaying at asymptotic spatial infinity, and manifests as the content of our main **Theorem 1**: *A proof of the Sommerfeld Finiteness Condition for K-G radiation on* $(\mathcal{M}, g)$.

In Chapter 1 we provide a context for this problem: we introduce the notion of well-posedness, we outline the relevant *Hilbert* and associated *Sobolev* function spaces and aspects of coupled *Einstein-Matter* systems. We define the *tortoise coordinate* $r(\rho)$ for mass parameter $\alpha^2(r) \equiv \frac{d\rho}{dr}$ and the *Synge isothermal metric* $ds_g^2 = \alpha^2(-dt^2 + dr^2) + \rho^2 d\Omega^2$ and show how Euler-Lagrange K-G : $\Box_g \psi(x) = 0$, distills to an *elliptic-Helmholtz* PDE in all $\mathbb{R}^3$ outside a ball of radius $2R_0$, for a radiating field $u(\mathbf{x}) = \frac{\rho}{r}\tilde{\psi}(x^i)$ with radial potential V: $-\Delta u + (\frac{1}{r^2} - \frac{\alpha^2}{\rho^2})\not{\Delta} u + Vu + z^2 u = 0$.

In Chapter 2 we detail the structure of the complete spacetime $(\mathcal{M}, g)$: as a perfect fluid 'star' inducing the Schwarzschild vacuum $(\mathcal{M}, g_0)$ and form appropriate coordinate charts and metrics for the physical system. In Chapter 3 we develop the functional analytic methods to establish the Sommerfeld bound on $u(\mathbf{x})$: using antecedent results and theorems of the respective authors cited, we construct suitable Hilbert and Sobolev *energy norms* and bounds. With this set-up we formulate the dual well-posed Cauchy problem as a hyperbolic wave equation in Minkowski spacetime $(\mathbb{R}^4, \eta)$ and with the structures we have developed prove some ancillary lemmas using Spectral Theory, with particular focus on a novel *light-cone argument* in Minkowski $(\mathbb{R}^4, \eta)$, that supports the eventual proof of **Theorem 1**.

Acknowlegements

I would first like to thank my supervisor Professor John Stalker for taking me on as his student in the first instance, I have a long held (stargazers $^*_*$), curiosity in the sombre subject of gravity, and Ph.D positions on the ground, in Ireland now at any rate, are as sparse as the matter content (baryonic) in our universe – I didn't always appreciate the empirical facts. John never once failed to provide, with a reassuring certainty, the answers to all of the 'rational' questions I asked him, concerning both Classical General Relativity and some of the trickier mathematical techniques demanded by the subject now, of which he is an acknowledged expert. Formulating the *proper* question on time, that is to say, Killing time, was always my enduring difficulty. His particular qualities of quiet unobtrusiveness and remarkable patience (as one, perhaps, awaiting elsewhere ($\mathcal{I}^+$?), a `txt msg` ... from the horizon), I greatly and sincerely appreciated, his 'pedagogy' besides, however, left me all but trapped on *the far side* of it, with the loss and recovery of information and reason becoming the leitmotif of this apprenticeship.

I must also acknowledge here my profound gratitude to my steadfast and devoted parents, both as constant as the Northern star, who have always encouraged/tolerated and supported my elliptic and indeed hyperbolic odyssey(s), who make a vast indifferent universe very much *HOME* for me and who never lose faith, where there never appears to be (because there isn't), any clear destination in sight, also my brothers and their respective families for the same.

From the School of Mathematics I thank Professor Richard Timoney, F.T.C.D., internal examiner, Professors Donal O' Donovan, Head of School, Stefan Sint, Director of Postgraduate Studies, Paschalis Karageorgis and David Simms, for their advice, encouragement and scholastic guidance throughout, also Professor Peter Hogan, University College Dublin, for his expert and gentle introduction to the subject and the School of Theoretical Physics of the Dublin Institute for Advanced Studies for making their facilities available to me and of course, last but not least, the reliable, diligent and now over-burdened taxpayers of the Republic, who have supported my studies over the past several years, through the generous grants provided by Science Foundation Ireland and Trinity College Dublin, in this regard. I hope that I can repay (in some way) my considerable debt to all, in kind.

Abstract

We formulate the *covariant* Euler-Lagrange equation for a real, massless, spin-zero, Klein-Gordon (K-G) field $\psi(x)$ in *curvature coordinates* $\{t, \rho, \theta, \phi\}$ expressed by

$$\Box_g \psi(x) = g^{\mu\nu} \nabla_\nu \nabla_\mu \psi(x) = 0, \tag{1}$$

radiating with exponential time dependence of complex frequency z, so that

$$\psi(t, \rho, \theta, \phi) = e^{zt} \tilde{\psi}(\rho, \theta, \phi) \equiv e^{zt} \tilde{\psi}(x^i), \text{ with the restriction: } \quad Re\ z = \zeta > 0,$$

on a four dimensional connected spherically symmetric static spacetime, denoted by $(\mathcal{M}, g)$ with canonical metric form

$$ds^2 = -f(\rho)dt^2 + h(\rho)d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2}$$

The coordinate ρ is a radial parameter with the property that the 2-sphere for constant t, ρ has standard line element $ds^2_\Omega = \rho^2(d\theta^2 + \sin^2\theta d\phi^2)$ and with θ, ϕ the usual spherical polar coordinates on 2-spheres; thus curvature coordinates are canonical coordinates defined invariantly by the natural symmetries present. We are essentially concerned with spatial decay at asymptotic infinity, where we seek to determine an optimal L^2-bound on the radiating K-G field $\tilde{\psi}(x^i)$ – this is the so-called *Sommerfeld finiteness condition* of the radiating problem, the proof of which is contained in **Theorem 1**, the principal result of the thesis.

To develop the analysis we effect a transformation of the weakly interacting field $\tilde{\psi}(x^i)$, effectively defined on the background spacetime $(\mathcal{M}, g)$, to a field $u(\mathbf{x})$ which we can then interpret on a background Minkowski spacetime $(\mathbb{R}^4, \eta)$. This is to be achieved by using the so-called *tortoise coordinate* r in the differential relation

$$\frac{d\rho}{dr} = \alpha^2(r), \qquad \text{for} \qquad 0 \le r < \infty, \tag{3}$$

where $\alpha^2(r)$ is a mass parameter of the curved spacetime $\mathcal{M}$, so that metric (2) on $(\mathcal{M}, g)$ admits the alternative *Synge isothermal* form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \alpha^2(r)(-dt^2 + dr^2) + \rho^2(r)(d\theta^2 + \sin^2\theta d\phi^2). \tag{4}$$

The transformation of the radiating K-G field $\tilde{\psi}$ in (1), i.e.

$$\Box_g \left(\frac{r}{\rho}u(\mathbf{x})\right) = 0, \quad \text{for} \quad u(\mathbf{x}) = \frac{\rho}{r}\tilde{\psi}(x^i), \quad \text{and where} \quad r = |\mathbf{x}|, \tag{5}$$

thus generates the central equation of our investigation: an elliptic-Helmholtz PDE for the radiating field $u(\mathbf{x})$, outside a closed ball B_r of radius $2R_0$ in $\mathbb{R}^3$–Euclidian space and given by

$$-\Delta u(\mathbf{x}) + \left(\frac{1}{r^2} - \frac{\alpha^2(r)}{\rho^2}\right) \not\Delta u(\mathbf{x}) + V(r)u(\mathbf{x}) + z^2 u(\mathbf{x}) = 0. \tag{6}$$

The transformation from curved to flat spacetime induces the angular Laplacian term $\frac{\alpha^2}{\rho^2}\not\Delta u$, in addition an associated asymptotically bounded repulsive potential $V(r)$, such that

$$V(r) = \frac{1}{\rho}\frac{d^2\rho}{dr^2}, \tag{7}$$

emerges naturally as a consequence of this transformation. The outgoing K-G radiation decays on the *exterior* vacuum Schwarzschild spacetime $(\mathcal{M}, g_0)$ where $\rho \geq \rho_0 > 2M_s$ and for tortoise $r = r_e(\rho)$, in which case we have

$$(i) \quad \frac{d\rho}{dr} = \alpha^2(r_e) = 1 - \frac{2M_s}{\rho} \qquad \text{and} \qquad (ii) \quad 0 \leq V(r) \leq \frac{U}{r^2}, \quad \text{for} \quad 0 < r < \infty.$$

Solving (i) yields the exterior tortoise or *Wheeler coordinate*

$$r_e(\rho) = \rho + 2M_s \log(\rho - 2M_s) - 2M_s + const.$$

To further assist the analysis we *modify* the $\mathbb{R}^3$- homogeneous Helmholtz equation (6), to the inhomogeneous form for $\tilde{u}(\mathbf{x})$, as presented in (8) below

$$-\Delta\tilde{u}(\mathbf{x}) + \left(\frac{1}{r^2} - \frac{\alpha^2(r)}{\rho^2}\right) \not{\Delta}\tilde{u}(\mathbf{x}) + \tilde{V}(r)\tilde{u}(\mathbf{x}) + z^2\tilde{u}(\mathbf{x}) = f(\mathbf{x}). \tag{8}$$

This is associated with the construction of a smooth artificial "source" function $f(\mathbf{x})$, necessarily compactly supported in the ball defined by B_r. For our purposes $0 < r < 2R_0$, where R_0 is just a finite fixed radius. The potential $\tilde{V}(r)$ agrees with $V(r)$ for $r > R_0$ and satisfies its bounds everywhere. Since $u(\mathbf{x})$ and $\tilde{u}(\mathbf{x})$ agree for $r > 2R_0$ any estimates we prove for $|\tilde{u}(\mathbf{x})|$ also apply to $|u(\mathbf{x})|$ for any $r > 2R_0$. Crucially, this construction enables a *well-posed* hyperbolic wave equation representation on Minkowski $(\mathbb{R}^4, \eta)$, associated with the dynamics of the transformed radiating field $u(\mathbf{x})$ of Eq. (8), and described by

$$\frac{\partial^2}{\partial t^2}w(\mathbf{x},t) - \left[\Delta + \left(\frac{1}{r^2} - \frac{\alpha^2}{\rho^2}\right)\not{\Delta} + V(r)\right] w(\mathbf{x},t) = 0, \tag{9}$$

with smooth initial Cauchy data: $w(0,\mathbf{x}) = 0$, $\partial_t w(0,\mathbf{x}) = f(\mathbf{x})$. The inhomogeneous elliptic-Helmholtz equation (8), derives from the standard Laplace transform of the hyperbolic wave equation (9), that is, where

$$\mathcal{L}\{w(\mathbf{x},t)\} = \int_0^\infty w(\mathbf{x},t)e^{-zt}dt = \tilde{w}(\mathbf{x}) \equiv \tilde{u}(\mathbf{x}). \tag{10}$$

By finding the optimal L^2-bound on $\tilde{u}(\mathbf{x})$, in tortoise coordinate $r = |\mathbf{x}|$, on Minkowski spacetime $(\mathbb{R}^4, \eta)$, we easily infer the bound on $\tilde{\psi}(x^i)$ in curvature coordinates $\{\rho, \theta, \phi\}$ at asymptotic spatial infinity, i.e., on the exterior Schwarzschild spacetime $(\mathcal{M}, g_0)$. To summarise this procedure: we transform the weakly interacting radiating field $\tilde{\psi}(x^i)$ on $(\mathcal{M}, g)$ to a field $u(\mathbf{x})$ on $(\mathbb{R}^4, \eta)$, we then modify $u(\mathbf{x})$ in the ball B_r yielding the compactly supported inhomogeneous 'source' term $f(\mathbf{x})$, we solve the associated well-posed hyperbolic i.v.p. (9) for $w(\mathbf{x}, t)$ and Laplace transform it to get the modified $\tilde{u}(\mathbf{x})$. Our analysis works if r is big enough, in which case the modified $\tilde{u}(\mathbf{x})$ of the inhomogeneous equation (8), is the same as $u(\mathbf{x})$ of the homogeneous equation (6).

Having established various results for well-posedness of $\Box_g \psi(x) = 0$, we then constructing suitable Hilbert and associated Sobolev space metric norms in $(\mathcal{M}, g)$ and using related theorems developed by John Stalker and A. Shadi Tahvildar-Zadeh, we prove the $\mathbb{R}^3$-Euclidian *energy norm* equalities. Following these authors by using conservation of energy in $\mathbb{R}^3$ on $w(\mathbf{x}, t)$, in conjunction with the domain of dependence and finite speed of propagation property of the hyperbolic wave equation (9), in a *light-cone* argument on $(\mathbb{R}^4, \eta)$, we arrive at the result of **Theorem 1**: *A proof the Sommerfeld finiteness condition for scalar field* $\tilde{\psi}$ *on* $(\mathcal{M}, g)$, *interpreted as a field* $u(\mathbf{x})$ *on* $\mathbb{R}^3$, such that the finiteness condition is the spherically restricted $L^2(S^2)$-space decay estimate for the transformed outgoing radiating field $u(\mathbf{x}) \sim \frac{1}{r} e^{zr}$ and expressed in the bound

$$\|u(\mathbf{x})\|_{L^2(S^2)} \leq C_1(z) \|f(\mathbf{x})\|_{L^2(\mathbb{R}^3)} \frac{1}{r} e^{-\mathrm{Re} z r} \leq C \|u(\mathbf{x})\|_{L^2(\mathbb{R}^3)} \frac{1}{r} e^{-\zeta r}.$$

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Chapter 1

Introduction

Unfathomablemind, now beacon, now sea.

Molloy, Samuel B. Beckett.

1.1 Outline of the Problem

1.1.1 The $(\mathcal{M}, g)$ Spacetime

A massless spin-zero Klein-Gordon field $\psi(x)$ radiates with exponential time dependence of complex frequency z, on a four dimensional connected spherically symmetric static spacetime $(\mathcal{M}, g)$ of canonical metric form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -f(\rho)dt^2 + h(\rho)d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (1.1)$$

in *curvature coordinates* $\{t, \rho, \theta, \phi\}$; with covariant Euler-Lagrange equation for ψ on $(\mathcal{M}, g)$ given by

$$\Box_g \psi(x) = g^{\mu\nu}\nabla_\nu\nabla_\mu\psi(x) = 0. \qquad (1.2)$$

Informally a manifold $\mathcal{M}$ possesses a symmetry if its geometry is invariant under a certain transformation that maps $\mathcal{M}$ into itself; i.e., if the metric is the same, in some sense, from one point to another. Symmetries of the metric are called *isometries.* Independence of the metric components on one or more coordinates implies the existence of isometries. For example if $\partial_{\sigma_*} g_{\mu\nu} = 0$ for some fixed σ_* (but for all of μ and ν), there will be a symmetry under translation along coordinate x^{σ_*}, i.e.

$$\partial_{\sigma_*} g_{\mu\nu} = 0 \quad \Rightarrow \quad x^{\sigma_*} \rightarrow x^{\sigma_*} + a^{\sigma_*} \quad \text{is a symmetry.}$$

All of the metric components of (1.1) are independent of coordinate $t = x^0$, writing vector

$$\xi = \partial_t \quad \Rightarrow \xi^\mu \;=\; (\partial_0)^\mu \;=\; \delta_0^\mu,$$

(in component notation as ξ^μ), we say that ξ^μ generates the isometry; the transformation under which the geometry is invariant is expressed infinitesimally as a motion in the direction of ξ^μ, it is called a *Killing vector field.* If ξ^μ satisfies *Killing's equation*

$$\nabla_\mu \xi^\nu - \nabla_\nu \xi^\mu = 0,$$

it is then always possible to find a coordinate system in which $\xi = \partial_t$. Killing fields on a manifold are in one-to-one correspondence with continuous symmetries of the metric on the manifold and *every Killing vector implies the existence of conserved quantities associated with geodesic motion.* Appendix B and [1] provides more detail. When there is a timelike Killing vector we can write the metric in a form where it is independent of the timelike coordinate as in the metric (1.1), and *Noether's theorem* implies a conserved energy quantity. We state here an important theorem due to Birkhoff [2]: *any static spherically symmetric spacetime possesses a timelike*

Killing field. The ideas introduced here will surface in greater detail throughout the exposition.

More precisely on our $\mathcal{M}$ we assume a time-like action of $\mathbb{R}$ and a space-like action of $SO(3,\mathbb{R})$ commuting with it. These actions are without fixed points, except that at most one $\mathbb{R}$-orbit is allowed to be $SO(3,\mathbb{R})$-fixed, this is the time axis. A spacetime is said to be *spherically symmetric* if its isometry group contains a subgroup isomorphic to the group $SO(3,\mathbb{R})$, and the orbits of this subgroup are two-dimensional spheres so that the isometries may be interpreted physically as rotations, and thus a spherically symmetric spacetime is one whose metric remains invariant under rotations. The spacetime metric induces a metric on each orbit 2-sphere which, because of the rotational symmetry must be a multiple of the metric of a unit 2-sphere and is completely characterised by the total area A of the 2-sphere. Following closely on Synge's analysis of spherically symmetric spacetimes [3], we choose curvature coordinate ρ defined by

$$\rho = \sqrt{\frac{A}{4\pi}},$$

so that in spherical coordinates (θ,ϕ) the metric on each orbit 2-sphere takes the form

$$ds_{\Omega}^2 = \rho^2(d\theta^2 + \sin^2\theta d\phi^2).$$

This choice of coordinates achieves maximum simplicity for the Einstein-matter field equations of the problem. By forming the differential

$$\frac{d\rho}{dr} = \alpha^2(r), \qquad \text{for} \qquad 0 \leq r < \infty, \tag{1.3}$$

such that $\alpha^2(r)$ is an $\mathbb{R} \times SO(3,\mathbb{R})$ invariant on $\mathcal{M}$ (interpreted as a mass parameter of the curved spacetime), it is possible to express the canonical metric (1.1) alternatively, in terms of Synge's *isothermal coordinates* by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \alpha^2(r)(-dt^2 + dr^2) + \rho^2(r)(d\theta^2 + \sin^2\theta d\phi^2). \tag{1.4}$$

In Chapter 2, Section 2.2.1, we will show that $\alpha^2(r) = -\xi_\mu\xi^\mu$ for the timelike Killing field ξ^μ which generates the $\mathbb{R}$-action. Importantly for this problem, we note that the quotient space $\mathcal{Q}$, which we define by

$$(\mathcal{Q}, \overline{g}) \equiv (\mathcal{M}, g)/SO(3, \mathbb{R}), \tag{1.5}$$

yields a *conformally flat* two-dimensional Lorentzian manifold in the (t, r)-plane, i.e.

$$d\overline{s}^2 = \overline{g}_{ab}dy^a dy^b = \alpha^2(r)(-dt^2 + dr^2),$$

isothermal coordinates can always be introduced on a compact domain of a regular 2-dimensional manifold. The conformally flat representation of $(\mathcal{Q}, \overline{g})$ is crucial to our problem and it is on this basis we develop all our subsequent analysis (exploiting known theorems in $\mathbb{R}^n$-spaces) culminating in the *light-cone* argument of the hyperbolic initial value problem (equation (9) of **Abstract**) in Minkowski spacetime $(\mathbb{R}^4, \eta)$.

Solving the differential equation (1.3) yields an explicit form for the so-called *tortoise coordinate* $r(\rho)$, of which two distinct forms emerge in the spacetime structure we choose. These are labelled the *interior tortoise* $r_i(\rho)$ and the *exterior tortoise* $r_e(\rho)$; we configure $(\mathcal{M}, g)$ so that the former holds in a non-vacuum region $0 < \rho \leq \rho_0$ which contains a spherically symmetric distribution of homogeneous fluid matter, contained in a volume

$$\overline{V}(\rho_0) = \frac{4}{3}\pi\rho_0^3,$$

the latter obtains in the vacuum region $\rho_0 < \rho < \infty$ where the tortoise junction condition at the fluid/vacuum interface is just $r_i(\rho_0) = r_e(\rho_0)$. As we will see in Section 1.1.2, r is an increasing function of ρ so that $0 < r_i(\rho) \leq r_i(\rho_0)$ and $r_e(\rho_0) \leq r_e(\rho) < \infty$. .

1.1.2 The Radiating K-G Equation

We seek an estimate or bound on the spatial decay of an outgoing, weakly interacting, radiating field $\tilde{\psi}(x^i)$, i.e.

$$\psi(x) = e^{zt}\tilde{\psi}(x^i), \qquad x^i = \{\rho, \theta, \phi\}, \qquad \text{and} \qquad Re\ z = \zeta > 0,$$

satisfying the Euler-Lagrange equation (1.2) and which effectively decays on the *exterior vacuum Schwarzschild* spacetime $(\mathcal{M}, g_0)$. The stratagems we conduct to find the bound on $\tilde{\psi}$ form the body of the thesis, culminating in the proof of our main theorem: **Theorem 1**.

This bound or estimate is known in the literature as the *Sommerfeld finiteness condition* of the well-posed radiating problem, and understood at asymptotic spatial infinity. The finiteness condition is usually encountered in conjunction with a second condition known as the *Sommerfeld radiation condition* proper, which is concerned with bounds on the radial derivative of a radiating field $u(\mathbf{x})$ say, and typically presented for $u(\mathbf{x})$ in $\mathbb{R}^3$ as the bound on $|\left(\frac{\partial}{\partial r} + \frac{1}{r} + z\right) u(\mathbf{x})|$; the proof of this second condition relies on the result established in **Theorem 1**, it is not however treated in this work, requiring analysis in L^p-spaces, but could feasibly form part of a future extension of it. Together these two conditions prescribe the asymptotic behaviour of the solutions of exterior boundary value problems for certain classes of partial differential equations, usually describing the oscillatory behaviour of physical systems and moreover, guarantee the uniqueness of their solutions. More often than not in the literature, these radiation conditions appear as hypotheses of the investigations concerning them. It is the goal of this thesis, however, following a long history of previous endeavors [4], to pursue a proof of the finiteness condition in L^2-Hilbert space, specifically for the massless Klein-Gordon field radiating on the static, spherically symmetric spacetime $(\mathcal{M}, g)$.

Assuming appropriate extremums for $\delta\psi(x)$, from an action principle on $(\mathcal{M}, g)$ we find the massless K-G equation

$$\Box_g \psi(x) = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \psi(x) \right) = 0, \tag{1.6}$$

which we express in the isothermal coordinates of (1.4).

We note here that the K-G equation can be alternatively expressed in *conformally invariant* form

$$\Box_{\tilde{g}} \Phi(x) - \frac{1}{6} \tilde{R} \Phi(x) = 0, \tag{1.7}$$

where $\Phi(x)$ is the massless scalar field and $\tilde{R}$ the *Ricci scalar* on $(\mathcal{M}, \tilde{g})$. The conformally invariant equation is discussed in the Appendix B. Very briefly, for the reader familiar with the field equations of General Relativity in the form

$$\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} = \tilde{T}^{(\Phi)}_{\mu\nu}, \tag{1.8}$$

a subsequent perturbation calculation decouples the *weakly interacting* Klein-Gordon field $\Phi(x)$ with energy-momentum tensor $\tilde{T}^{(\Phi)}_{\mu\nu}$ from $(\mathcal{M}, \tilde{g})$. This yields the weak field approximation, i.e.

$$\tilde{g}^{\mu\nu} \tilde{T}^{(\Phi)}_{\mu\nu} = \tilde{T} \approx 0, \tag{1.9}$$

and consequently, by contracting equations (1.8) with $\tilde{g}^{\mu\nu}$, i.e.

$$\tilde{g}^{\mu\nu}R_{\mu\nu} - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{g}_{\mu\nu}\tilde{R} = -\tilde{R} = \tilde{g}^{\mu\nu}\tilde{T}^{(\Phi)}_{\mu\nu} = \tilde{T} = 0, \tag{1.10}$$

we see the Ricci scalar $\tilde{R} \approx 0$ so that we are solving a conformally invariant radiating K-G equation

$$\Box_{\tilde{g}}\Phi(x) = 0.$$

The weak field approximation immediately implies here

$$\tilde{R}_{\mu\nu} = 0, \tag{1.11}$$

solving this in curvature coordinates $\{t, \rho, \theta, \phi\}$ for a static spherically symmetric spacetime yields the well known *exterior* or vacuum Schwarzschild spacetime $(\mathcal{M}, g_0)$ of mass parameter M_s, given in canonical form by

$$ds^2_{\text{ext}} = -\left(1 - \frac{2M_s}{\rho}\right)dt^2 + \left(1 - \frac{2M_s}{\rho}\right)^{-1}d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.12}$$

so that our wave equation (1.6) in the weak field case, $g \approx g_0$, is given by

$$\Box_{g_0}\psi(x) = 0. \tag{1.13}$$

In isothermal coordinates the exterior metric takes the form

$$ds^2_{\text{ext}} = (g_0)_{\mu\nu}dx^\mu dx^\nu = \alpha^2(r_e)(-dt^2 + dr^2) + \rho^2(r)(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.14}$$

with positive mass parameter $\alpha^2(r_e)$ got from the canonical form (1.12)

$$\alpha^2(r_e) = 1 - \frac{2M_s}{\rho} > 0, \tag{1.15}$$

in the domain of application $\rho \geq \rho_0 > 2M_s$. For this problem ds^2_{ext}represents the vacuum metric outside a compact spherically symmetric matter source, parametrised by M_s, confined to the sphere radius of $\rho_0 > 2M_s$, i.e.

$$M_s = m(\rho_0) = 4\pi \int_0^{\rho_0} d_F(\overline{\rho})\rho^2 d\rho,$$

where $d_F(\overline{\rho})$ is the fluid energy density, this model is discussed in more detail in Chapter 2. The explicit form of the exterior tortoise coordinate $r_e(\rho)$ is given by

$$r_e(\rho) = \int_{\rho_0}^{\infty} \left(1 - \frac{2M_s}{\rho}\right)^{-1} d\rho = \rho - 2M_s + 2M_s \log(\rho - 2M_s) + k_1, \tag{1.16}$$

an increasing function of ρ. An expedient use of metric coefficients $(g_0)_{\mu\nu}$ in isothermal coordinates finds

$$\Box_{g_0}\psi(x) = \frac{\partial^2\psi}{\partial t^2} - \frac{1}{\rho^2}\frac{\partial}{\partial r}\left(\rho^2 \frac{\partial\psi}{\partial r}\right) - \frac{\alpha^2(r_e)}{\rho^2} \not{\Delta}\psi = 0. \tag{1.17}$$

We note here, in the interests of clarity, what we will later prove in Section 2.2.3, that the metric ds^2_{int}, describing the *interior* spherically symmetric spacetime $(\mathcal{M}, g_i)$ i.e., in the non-vacuum region $0 < \rho \leq \rho_0$ of a spherically symmetric distribution of homogeneous fluid matter can be similarly expressed in isothermal coordinate form

$$ds^2_{\text{int}} = (g_i)_{\mu\nu} dx^\mu dx^\nu = \alpha^2(r_i)(-dt^2 + dr^2) + \rho^2(r)(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.18}$$

again with positive mass parameter $\alpha^2(r_i)$ obtained from Synge's incompressible fluid sphere solution [3]. What we have then is the *complete* Schwarzschild field for the interior $0 \leq \rho < \rho_0$ and exterior $\rho_0 < \rho < \infty$, describing the entire spacetime $(\mathcal{M}, g) \equiv (\mathcal{M}, g_i) \cup (\mathcal{M}, g_0)$ for $0 < \rho < \infty$, i.e.

$$ds^2 = \alpha^2(r)(-dt^2 + dr^2) + \rho^2(r)(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.19}$$

and for $0 < r < \infty$ the K-G equation on all $(\mathcal{M}, g)$ is then

$$\Box_g \psi(x) = \frac{\partial^2 \psi}{\partial t^2} - \frac{1}{\rho^2}\frac{\partial}{\partial r}\left(\rho^2 \frac{\partial \psi}{\partial r}\right) - \frac{\alpha^2(r)}{\rho^2} \not{\Delta} \psi = 0. \tag{1.20}$$

We note the conformally invariant form (1.7) of K-G on $(\mathcal{M}, g_i)$ would look like

$$\Box_{g_i} \Upsilon(x) - \frac{1}{6}\{3p - d_F(\overline{\rho})\}\Upsilon(x) = 0, \tag{1.21}$$

for fluid pressure $p(\rho)$ and density $d_F(\overline{\rho})$, this has the form of a massive K-G field.

1.1.3 The Initial Value Problem

For an outgoing radiating field on $(\mathcal{M}, g)$ with complex exponential time dependence of frequency z, for Re $z = \zeta > 0$, expressed in curvature coordinates $\{t, \rho, \theta, \phi\}$ with $x^0 = t$ we have

$$\psi(x^\mu) = e^{zt}\tilde{\psi}(x^i) \quad \Rightarrow \quad \psi_{tt}(x) = z^2 e^{zt} \tilde{\psi}(\rho, \theta, \phi). \tag{1.22}$$

Transforming ψ of equation (1.20) to $u(\mathbf{x})$, via the *tortoise coordinate* $r = |\mathbf{x}|$ where

$$u(\mathbf{x}) = \frac{\rho}{r}\tilde{\psi}(x^i), \tag{1.23}$$

with $\psi = e^{zt}\tilde{\psi}$, we eventually arrive at a dual flat space inhomogeneous elliptic-Helmholtz equation representation of equation (1.6) above, expressed by

$$-\Delta \tilde{u}(\mathbf{x}) + F(r)\not{\Delta}\tilde{u}(\mathbf{x}) + \tilde{V}(r)\tilde{u}(\mathbf{x}) + z^2 \tilde{u}(\mathbf{x}) = f(\mathbf{x}). \tag{1.24}$$

It is the Minkowski spacetime representation of the K-G equation (1.20) on $(\mathcal{M}, g)$ and the central equation of our investigation, describing the radiating field $\tilde{u}(\mathbf{x})$ in all $\mathbb{R}^3$. The radial coefficient of the $\not{\Delta}\tilde{u}(\mathbf{x})$ term and the radial 'potential' functions

$$F(r) = \frac{1}{r^2} - \frac{\alpha^2}{\rho^2}, \qquad \tilde{V}(r) = \frac{\rho''(r)}{\rho(r)}, \tag{1.25}$$

will be seen to emerge as a consequence of the transformation (1.23), which maps the scalar field $\tilde{\psi}$ on $(\mathcal{M}, g)$ to the field u, now interpreted on the flat spacetime $(\mathbb{R}^4, \eta)$. As we will demonstrate in Section 3.3.2, the construction of the smooth compactly supported 'source' function $f(\mathbf{x})$ enables the radiating problem on the complete Schwarzschild spacetime $\mathcal{M}$, as outlined, to be interpreted by the $\mathbb{R}^3$-inhomogeneous Helmholtz equation (1.24). In very stark terms we map the conformally flattened spherical interior matter source of $\mathcal{M}$ into the compact domain of f in $\mathbb{R}^3$, and the conformally flattened exterior vacuum into everywhere else outside the compact set, in this way we solve the radiating problem as a well-posed problem in $(\mathbb{R}^4, \eta)$. In Chapter 2, Section 3.3.1, we show the effective potential $V(r)$ is well behaved, repulsive and asymptotically bounded, i.e.

$$V(r)\,|_{r=r_e} = \frac{\rho''(r_e)}{\rho(r_e)} = \frac{2M_s}{\rho^3}\left(1 - \frac{2M_s}{\rho}\right) > 0, \qquad \rho \geq \rho_0 > 2M_s, \tag{1.26}$$

with an upper bound given by

$$0 \leq V(r) \leq \frac{U}{r^2}, \qquad \text{for} \quad 0 < r < \infty. \tag{1.27}$$

Our transformed radiating problem for the modified field $\tilde{u}(\mathbf{x})$ in all $\mathbb{R}^3$, governed by the inhomogeneous elliptic-Helmholtz equation of (1.24), is interpreted on the $\mathbb{R}^{3+1}$ Minkowski spacetime in coordinates $\{t, r, \theta, \phi\}$ for $0 < r < \infty$, as a hyperbolic initial value problem for the spacetime function $w(\mathbf{x}, t)$ given by

$$\frac{\partial^2}{\partial t^2} w(\mathbf{x}, t) - \left[\Delta + \left(\frac{1}{r^2} - \frac{\alpha^2(r)}{\rho^2}\right) \not{\Delta} + V(r)\right] w(\mathbf{x}, t) = 0, \tag{1.28}$$

with initial Cauchy data, $w(0, \mathbf{x}) = 0$ and $\partial_t w(0, \mathbf{x}) = f(\mathbf{x})$.

In Chapter 3 we will show that this is because (1.28) can be associated with the inhomogeneous elliptic-Helmholtz equation of (1.24) in the following way: writing the differential operator

$$-B \equiv \Delta + \left(\frac{1}{r^2} - \frac{\alpha^2}{\rho^2}\right) \not{\Delta} + V(r), \tag{1.29}$$

so that equation (1.24) is the Laplace transform $\mathcal{L}$ of equation (1.28), i.e.

$$\mathcal{L}\{w_{tt}(\mathbf{x}, t)\} + B \cdot \mathcal{L}\{w(\mathbf{x}, t)\} = B \cdot \tilde{u}(\mathbf{x}) + z^2\tilde{u}(\mathbf{x}), \tag{1.30}$$

where

$$\mathcal{L}\{w(\mathbf{x}, t)\} = \int_0^\infty w(\mathbf{x}, t)e^{-zt}dt = \tilde{u}(\mathbf{x}). \tag{1.31}$$

Proving this by Hilbert space spectral methods we then establish a well-posed Cauchy problem for the hyperbolic wave equation of (1.28). Exploiting the domain of dependence and finite speed of propagation properties of the initial value problem in $(\mathbb{R}^4, \eta)$, following Stalker *et al.* [5], we employ the *light-cone argument* of Section 3.5 to find, finally, a spherically restricted optimal L^2-bound on the field $u(\mathbf{x})$, in other words the Sommerfeld radiation bound for the outgoing radiating K-G field.

In seeking to prove these bounded estimates, we are led to a carefully detailed consideration of the Cauchy problem in Chapter 3, for the dynamics of the scalar field $w(t, \mathbf{x})$ associated with appropriately defined generic initial data $w(0, \mathbf{x})$ and $w_t(0, \mathbf{x})$ prescribed on a domain of dependence, described by a spherically restricted spacelike hypersurface $D(\Sigma_0)$ in $(\mathbb{R}^4, \eta)$.

1.1.4 The Complete Schwarzschild Spacetime

The Klein-Gordon field ψ is weakly interacting and as alluded to in Section 1.1 can be interpreted as a test field on the entire static spherically symmetric spacetime

$(\mathcal{M}, g)$ with metric form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -f(\rho)dt^2 + h(\rho)d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2),$$

in *curvature coordinates* $\{t, \rho, \theta, \phi\}$ or by the equivalent isothermal form

$$ds^2 = \alpha^2(r)(-dt^2 + dr^2) + \rho^2(r)(d\theta^2 + \sin^2\theta d\phi^2),$$

in tortoise $r(\rho)$. We are ultimately concerned with the spatial decay of outgoing radiation, that is to say, in the exterior *vacuum* spacetime $(\mathcal{M}, g_0)$ where $R_{\mu\nu} = 0$ and

$$ds^2_{\text{ext}} = -\left(1 - \frac{2M_s}{\rho}\right)dt^2 + \left(1 - \frac{2M_s}{\rho}\right)^{-1} d\rho^2 + \rho^2 d\Omega^2. \tag{1.32}$$

The original interpretation of (1.32) was that it modelled the gravitational field in a vacuum region outside a spherically symmetric matter source M_s, a star for example, and (1.32) was considered only in some coordinate range $\rho > \rho_0$, for a $\rho_0 > 2M_s$. The metric was matched at $\rho = \rho_0$ to a static interior metric satisfying the coupled Einstein-Euler system of (1.8) in the interior $\rho \leq \rho_0$. This latter metric is of the form (1.32), but with $M_s = M_s(\rho)$ such that $M_s \to 0$ as $\rho \to 0$. A natural problem poses itself if we do away with the star M_s altogether and consider (1.32) for *all* values of ρ. At $\rho = 2M_s$, the metric appears to be singular in (t, ρ)-coordinates, however, with a change of coordinates, this metric can be extended regularly as a solution of $R_{\mu\nu} = 0$ beyond $\rho = 2M_s$. That is, there exists a manifold $\mathcal{M}$ that contains both a region $\rho > 2M_s$ and a region $0 < \rho < 2M_s$, separated by a regular (null) hypersurface, the *event horizon* $\mathcal{H}^+$. The metric element (1.32) is valid everywhere except on $\mathcal{H}^+$, where it must be rewritten in regular coordinates.

The hypersurface $\mathcal{H}^+$ is characterised by an exceptional global property: it defines the boundary of the region of spacetime that can send signals to null infinity $\mathcal{I}^+$, or in physical interpretation, to distant observers. In general, the set of points that *cannot* send signals to null infinity $\mathcal{I}^+$ is known as the *black hole* region of spacetime, where $0 < \rho < 2M_s$ and $\mathcal{H}^+$ is the *event horizon.* The global geometry of the extended spacetime $\mathcal{M}$ was clarified by J.L. Synge and later by Kruskal and Szekeres. These issues took a while to sort out in my thesis:

> A person will be imprisoned in a room with a door that is unlocked and opens outwards; as long as it does not occur to him to push rather than pull it. [6]

I started with the idea of radiation on a black hole spacetime, to which I became morbidly attached, notwithstanding that the radiation decays in the asymptotic vacuum Schwarzschild region $(\mathcal{M}, g_0)$ and was easier solved for a nice matter source, which could be conformally flattened by a positive mass parameter $\alpha^2(r)$, that admits a timelike Killing field, as enabled by the perfect fluid interior solution $(\mathcal{M}, g_i)$. *Birkhoff's theorem* [2] guarantees that the Schwarzschild metric is the unique vacuum solution with spherical symmetry.

So we follow the interpretation of the field outside a static spherically symmetric distribution of matter, parametrised by positive constant M_s. Of course its energy-momentum tensor must represent a physically realistic matter field, in the sense that it must necessarily describe a positive energy density that dominates any interior stresses present, or more precisely, it must satisfy a *dominant energy condition* [7]. The positive character of energy density dominates gravitation theory and this important idea is treated in more detail in Appendix B. The choice of gravitating source does not affect the statement of Theorem 1, and there is no loss in generality if we elect to generate the exterior vacuum spacetime $(\mathcal{M}, g_0)$ with a spherically symmetric distribution of *perfect fluid*, centred on curvature coordinate $\rho = 0$, as describe below. In this way the complete description for $(\mathcal{M}, g)$ is made possible. Put simply, the perfect fluid source is chosen to avoid tedious, unnecessary (and possibly hazardous!) mathematical arguments arising from the coordinate peculiarities and other subtleties of the extended vacuum Schwarzschild topology. As mentioned above the Schwarzschild black hole exterior or naked singularity spacetimes are also described by metric (1.32), but present difficulties when transformed, in the manner described by the tortoise coordinate $r(\rho)$, to the complete Minkowski setting $(\mathbb{R}^4, \eta)$. In particular, as we will see in Section 3.1, for $\alpha^2 < 0$ negative energy interpretations arise, which happens when the Killing vector field $\xi^\mu = \partial_t$ goes from being *timelike* to *spacelike* in the black hole region $0 < \rho \leq 2M_s$.

A perfect fluid is defined as one for which there are no forces between its constituent particles and no heat conduction or viscosity in the fluid's instantaneous rest frame. The interior Schwarzschild solution describing this model, as discovered by Synge, is treated in greater detail in Section 2.2.2 . As Synge has proven and comprehensively discussed in Chapter VII of his classic text [3], concerning spherically symmetric fields, a static spherically symmetric perfect fluid solution can *always* be matched to the Schwarzschild vacuum across a spherical surface and can thus be used as an interior solution in a stellar model. This is because the free boundary condition for fluids is zero pressure and the spherically symmetric Schwarzschild vacuum has zero pressure, and so can be matched for continuous radial pressure across the stellar boundary, we will return to this idea in 2.2.2. As we will see there, the fluid solution is an exact solution of the Einstein field equations in which the gravitational field is produced entirely by the mass, momentum and stress energy density of the fluid and as such, confers a degree of simplicity on the analysis, another reason why we put it to use here [3].

In Section 2.2.2 we will also consider in detail the appropriate formulation for the *interior tortoise coordinate* $r_i(\rho)$, corresponding to interior mass parameter $\alpha^2(r_i)$ where

$$\frac{d\rho}{dr} = \alpha^2(r_i), \qquad \text{for} \qquad 0 < \rho \leq \rho_0,$$

this is then solved to achieve a conformally flat representation for the quotient manifold $\mathcal{M}/SO(3,\mathbb{R})$ of the interior spherically symmetric spacetime $(\mathcal{M}, g_i)$, and should match the vacuum solution at the stellar surface ρ_0, i.e., where $r_i(\rho_0) = r_e(\rho_0)$.

We follow this procedure to represent the original problem of gravitating perfect fluid star and radiating scalar field $\tilde{\psi}(x^i)$ on the complete Schwarzschild spacetime $(\mathcal{M}, g)$. Ultimately this manifests as the well-posed hyperbolic i.v.p. of equation (1.28), when mapped via the isothermal metric into the entire (t, r)-plane of the spherically symmetric Minkowski spacetime $(\mathbb{R}^4, \eta)$

$$ds_\eta^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) = -dt^2 + d\mathbf{r}.d\mathbf{r}.$$

In Section 3.4, using conservation of energy and prior analysis [5] we solve our hyperbolic i.v.p.,

$$w_{tt}(\mathbf{x}, t) + B \cdot w(\mathbf{x}, t) = 0, \qquad w(0, \mathbf{x}) = 0, \quad \partial_t w(0, \mathbf{x}) = f(\mathbf{x}),$$

to establish L^2-bounds on $u(\mathbf{x})$, in the region $r \geq 2R_0$, exterior to the ball B_r where compactly supported generic initial datum $f(\mathbf{x})$ lives.

1.1.5 Theorem 1: The Sommerfeld Radiation Bound in $\mathbb{R}^3$

Here we summarise the terminology and content of **Theorem 1**: our proof of the Sommerfeld radiation bound. A weakly interacting K-G field $\psi(x)$ governed by Euler-Lagrange $\Box_g\psi(x) = 0$, radiates with complex frequency z, i.e., $\psi(x) = e^{zt}\tilde{\psi}(x^i)$ on a static spherically symmetric spacetime $(\mathcal{M}, g)$ with metric form

$$ds^2 = -f(\rho)dt^2 + h(\rho)d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)$$

in curvature coordinates $\{t, \rho, \theta, \phi\}$. The outgoing radiation effectively decays on the exterior vacuum Schwarzschild spacetime $(\mathcal{M}, g_0)$. Without loss of generality we choose to generate the exterior Schwarzschild vacuum with a spherically symmetric compactly supported perfect fluid source of mass M_s centred at $\rho = 0$. Using a tortoise coordinate representation $r(\rho)$ where

$$\frac{d\rho}{dr} = \alpha^2(r), \qquad 0 \le r < \infty,$$

we form the equivalent isothermal metric form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \alpha^2(r)(-dt^2 + dr^2) + \rho^2(r)(d\theta^2 + \sin^2\theta d\phi^2),$$

so that K-G is

$$\Box_g\psi(x) = \frac{\partial^2\psi}{\partial t^2} - \frac{1}{\rho^2}\frac{\partial}{\partial r}\left(\rho^2\frac{\partial\psi}{\partial r}\right) - \frac{\alpha^2(r)}{\rho^2}\not{\Delta}\psi = 0.$$

The mass parameter/conformal factor $\alpha^2(r)$ assumes two distinct forms in $\mathcal{M}$:

$\alpha^2(r_i)$ in the Schwarzschild *interior* $(\mathcal{M}, g_i)$, $\quad 0 \le \rho < \rho_0, \quad r(\rho) = r_i,$

$\alpha^2(r_e)$ in the Schwarzschild *exterior* $(\mathcal{M}, g_0)$, $\quad \rho_0 \le \rho < \infty, \quad r(\rho) = r_e.$

Under the transformation $u(\mathbf{x}) = \frac{\rho}{r}\tilde{\psi}(\rho, \theta, \phi)$ the K-G equation $\Box_g \psi$, becomes

$$-\Delta u(\mathbf{x}) + \left(\frac{1}{r^2} - \frac{\alpha^2}{\rho^2}\right) \not{\Delta} u(\mathbf{x}) + V(r)u(\mathbf{x}) + z^2 u(\mathbf{x}) = 0,$$

which is, in particular, non-trivially satisfied for $r = |\mathbf{x}| > 2R_0 > 0$. We want the Sommerfeld radiation bound on $\psi(x) = e^{zt}\tilde{\psi}(x^i)$ which decays on the vacuum region of $(\mathcal{M}, g)$, i.e., on the Schwarzschild exterior $(\mathcal{M}, g_0)$, there we have

$$u(\mathbf{x}) \equiv u_e(\mathbf{x}) = \frac{\rho}{r_e}\tilde{\psi}_e(\rho, \theta, \phi),$$

and $|u_e(\mathbf{x})|$ is the bound sought in the region $2R_0 < |\mathbf{x}| < \infty$. We find

$$\alpha^2(r)|_{r=r_e} \equiv 1 - \frac{2M_s}{\rho}, \qquad 2M_s < \rho_0 < \rho < \infty,$$

and also the repulsive potential V, given there by

$$V(r)\,|_{r=r_e} \equiv \frac{1}{\rho}\frac{d^2\rho}{dr^2} = \frac{2M_s}{\rho^3}\left(1 - \frac{2M_s}{\rho}\right), \qquad 2M_s < \rho_0 < \rho < \infty,$$

with upper bound

$$0 \leq V(r) \leq \frac{U}{r^2}, \qquad 0 \leq r < \infty,$$

similarly we have

$$\left.\frac{\alpha^2(r)}{\rho^2(r)}\right|_{r_e} = \frac{1}{\rho^2}\left(1 - \frac{2M_s}{\rho}\right) \leq \frac{C}{r^2}, \qquad 2M_s < \rho_0 < \rho < \infty,$$

so that

$$F(r) = \frac{1}{r^2} - \frac{\alpha^2}{\rho^2} \leq \frac{C}{r^2},$$

for generic constants C. We are mainly interested in the asymptotic behaviour of $u(\mathbf{x})$ near infinity, and so we assume without loss of generality that $R' > 2R_0 > |z|^{-1}$. **Theorem 1** then establishes the optimal $L^2(\mathbb{R}^3)$-bound on $u(\mathbf{x})$, in other words we prove the *Sommerfeld finiteness condition.*

Theorem 1 *Let* $u(\mathbf{x}) \in L^2(\mathbb{R}^3)$ *be a solution of the elliptic-Helmholtz equation given by*

$$-\Delta u(\mathbf{x}) + \left(\frac{1}{r^2} - \frac{\alpha^2(r)}{\rho^2}\right) \not{\Delta} u(\mathbf{x}) + V(r)u(\mathbf{x}) + z^2 u(\mathbf{x}) = 0, \qquad Re\ z = \zeta > 0,$$

outside a ball B_r*, for* $2R_0 < r < \infty$*. Suppose that* $V(r)$ *is a positive potential satisfying the bounds*

$$0 \leq V(r) < \frac{U}{r^2},$$

for all $r = |\mathbf{x}| > 2R_0$ *and suppose also that* $\alpha^2(r)$ *is a mass parameter of the static spherically symmetric spacetime* $(\mathcal{M}, g)$ *with isothermal metric*

$$ds^2 = \alpha^2(r)(-dt^2 + dr^2) + \rho^2 d\Omega^2.$$

There are then $R' > 0$ *and* $C_1(|z|)$ *constants such that the following spherically restricted bound is satisfied for* $r \geq R'$

$$\|u(\mathbf{x})\|_{L^2(S^2)} \leq C_1(z)\|u\|_{L^2(\mathbb{R}^3)}\frac{1}{r}e^{-\zeta r},$$

that is, we prove the Sommerfeld radiation bound on the field $u(\mathbf{x})$ *outside a compact source in* $\mathbb{R}^3$.

It is clear enough that the decay rate is optimal in the case of a spherically symmetric potential. We note that the dependence of the constant $C_1(z)$ on the argument of z prevents the use of this theorem to derive estimates on analytic functions of the Schrödinger operator $-\Delta + V(r)$ by contour integration. Before dealing with the details of this proof it is worthwhile to first locate our problem within the wider historical context of the Sommerfeld radiation conditions, as presented in the mathematical literature heretofore.

1.2 Review of the Sommerfeld Radiation Problem

In order for a partial differential equation to represent a plausible model of the physical problem it must necessarily be *well-posed.* Well-posed problems consist of a PDE in a suitable domain together with a set of initial and/or boundary conditions (or other auxiliary conditions) that possess the following three fundamental properties:

(i) *Existence*: At least one solution exists that satisfies all three conditions.

(ii) *Uniqueness*: At most one solution exists.

(iii) *Stability*: The unique solution depends in a stable manner on the data of the problem. If the data are changed a little, then the corresponding solution changes only a little.

In accordance with these properties and in particular with the requirements of (i) and (ii), the mathematical physicist Arnold Sommerfeld, in a comprehensive treatise on the partial differential equations of mathematical physics [8], first introduced the eponymous *condition of radiation* for a scalar field $u(\mathbf{x})$ in $\mathbb{R}^3$, satisfying the Helmholtz equation $-\Delta u + \kappa^2 u = 0$. This key extract is reproduced here:

> ... With increasing domain the eigenvalues become closer and closer; for an infinite domain they are dense everywhere; we then deal with a *continuous spectrum of eigenvalues.* Let us consider, e.g., the interior of a sphere of radius a for vanishing boundary values. For the case of purely radial oscillations its eigenvalues are given by the equation
>
> $$\psi_0(\kappa_\nu a) = 0, \qquad \psi_0(\rho) = \frac{\sin\rho}{\rho}.$$

Hence $\kappa_\nu a = \nu\pi$ and the difference of successive eigenvalues is

$$\Delta\kappa_\nu = \frac{\pi}{a} \to 0 \qquad \text{for} \qquad a \to \infty.$$

We may therefore consider the function $\psi_0(\kappa r)$ which is everywhere regular and vanishes at infinity as an *eigenfunction of infinite space.* Thus, if we have an acoustic or an optical problem in which the prescribed sources are in the finite domain (with a discrete or a continuous distribution), and which is to be solved for a given wave number κ, then we can always add the function ψ_0 to the solution. Hence oscillation problems (in contrast to potential problems) are *not* determined *uniquely* by their prescribed sources in the finite domain. This paradoxical result shows that the condition of *vanishing* at infinity is not sufficient, and that we have to replace it by a stronger condition at infinity. We call it the *condition of radiation*: the sources must be *sources*, not *sinks*, of energy. The energy which is radiated from the sources must scatter to infinity; *no energy may be radiated from infinity into the prescribed singularities of the field* (plane waves are excluded since for them even the condition $u = 0$ fails to hold at infinity). For our special eigenfunctions

$$\psi_0(\kappa r) = \frac{1}{2\iota}\left(\frac{e^{\iota k|\mathbf{r}|}}{r} - \frac{e^{-\iota k|\mathbf{r}|}}{r}\right)$$

the state of affairs is simple: for the time dependence $\exp(-\iota\omega t)$, $e^{\iota\kappa r}/r$ is a *radiated*, $e^{-\iota\kappa r}/r$ an *absorbed*, $\psi_0(\kappa r)$ a *standing* wave (nodal surfaces $\kappa r = \nu\pi$). By excluding absorption from infinity we exclude the addition of the eigenfunction $\psi_0(\kappa r)$. Hence the permissible singularities are restricted to the "outgoing" form

$$u = C\frac{e^{\iota k|\mathbf{r}|}}{|\mathbf{r}|}$$

For these singularities we have the condition

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - \iota k u\right) = 0,$$

It is called the general *condition of radiation* and we shall apply it to all acoustic and electrodynamic oscillation problems that are generated by sources in the finite domain.

By imposing this *condition of radiation*, the *existence* and *uniqueness* of the solutions of certain exterior boundary value problems, i.e., requirements (i) and (ii) of the well-posed problem are met. Such problems generally describe wave propagation phenomena, usually where incident and scattered radiation is being considered. Expressed mathematically, these problems usually take the form of an *exterior Dirichlet* or *Neumann* problem for the Helmholtz equations describing them. For scattered radiation the condition is applied at asymptotic spatial infinity, and when added to the statement of the boundary values, singles out the unique solution, in physical applications, which represents the "outgoing" radiation field only.

In the quoted extract Sommerfeld instances the spherical wave solution $u(\mathbf{r})$ which is *outgoing* with a plus sign and *incoming* with a minus sign. The outgoing wave satisfies the radiation condition, as he defines it, but the incoming one does not. For the outgoing solution in $\mathbb{R}^3$ we then have two conditions:

(i) $\lim_{r\to\infty} |ru(\mathbf{r})|$ is bounded, this is the *finiteness condition.*

(ii) $\lim_{r\to\infty} |r(\partial/\partial r - \imath k)u(\mathbf{r})| \to 0$, this is the *radiation condition proper.*

The *Sommerfeld radiation conditions*, applied as boundary conditions at spatial infinity, guarantee the uniqueness of solutions, and are so devised to concur with the phenomenological experience (this is Physics after all!) that no energy is radiated into the field sources from infinity. Sommerfeld's proof of this is however restricted to *Green's function* solutions in $\mathbb{R}^3$ only [9].

1.2.1 Some Helmholtz Equations

When treated in the literature, the Sommerfeld radiation conditions usually assume the status of hypotheses in the investigations concerning them, that is to say, they are imposed as *a priori conditions* of the problem, and thus the *uniqueness* criterion of the well-posed problem is met. This is particularly true when the frequency parameter z is allowed to be complex. This case was first treated by F. V. Atkinson [10] in 1949. In that paper the author addresses the fact that the two conditions advanced by Sommerfeld do not purport to be rigorous mathematical arguments and goes on to present a proof of the proposition that the Sommerfeld conditions do indeed confer uniqueness on the solutions of the radiating Helmholtz problem.

In a challenging 1959 paper [11] the functional analyst Tosio Kato, also dealt with the problem of the asymptotic behaviour, for $|\mathbf{x}| \to \infty$, of solutions of the n-dimensional reduced wave equation, expressed by

$$\Delta u(\mathbf{x}) + q(\mathbf{x})u(\mathbf{x}) = 0, \tag{1.33}$$

for our purposes we write this in the form

$$-\Delta u(\mathbf{x}) + p(\mathbf{x})u(\mathbf{x}) - \lambda u(\mathbf{x}) = 0, \tag{1.34}$$

with $\lambda \in \mathbb{C}$ and $p(\mathbf{x}) > 0$, considered in a domain $|\mathbf{x}| \geq 2R_0 > 0$.

As Kato's work is the starting point of our dissertation in some respects, and motivates the pursuit of its central problem, that of finding an L^2-bound on the solution $u(\mathbf{x})$ of a Helmholtz equation with a positive potential $V(\mathbf{x})$, we will expend some effort on its explication here. We have that $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ is the variable real vector of length $|\mathbf{x}| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}} = r$, and R_0 an arbitrary but fixed radial constant. By a *solution* is meant a complex-valued function $u(\mathbf{x})$, of class C^2 for $|\mathbf{x}| \geq R_0 > 0$, which satisfies (1.34). Amongst other things, Kato estimates the rate of growth of the quantity

$$\|S_r u\|_{L^2(S^{n-1})} = \left[\int_{|\mathbf{x}|=r} |u(\mathbf{x})|^2 \, dS\right]^{\frac{1}{2}}, \tag{1.35}$$

constructed from an arbitrary *spherically restricted* solution of (1.34), written as $S_r u(\mathbf{x})$, the integrals being surface integrals over the spherical surface defined by $|\mathbf{x}| = r$. As stated by Kato the estimation of the quantities (1.35) requires different methods according to the properties of the complex valued function as defined by

$$q(\mathbf{x}) = \lambda - p(\mathbf{x}),$$

Kato distinguishes between various cases of $q(\mathbf{x})$. The case of interest for us, that is for $\lambda \in \mathbb{C}$, with *positive* real part, and for a positive real potential $p(\mathbf{x}) > 0$, is covered in the *Example* 7, *Section* 6 of the 1959 paper, where $\mathcal{R}e\ e^{i\theta}q(\mathbf{x}) \leq 0$ and $|\theta| < \frac{1}{2}\pi$, we reproduce the extract here:

> Let
>
> $$q(\mathbf{x}) = \lambda - p(\mathbf{x})$$
>
> where $p(\mathbf{x})$ is real and non-negative whereas λ is a non-real constant. By symmetry it suffices to consider the case in which $\lambda = |\lambda|e^{i\phi}$ with $0 < \phi < \pi$. Then
>
> $$\mathcal{R}e\ e^{i\theta}q(\mathbf{x}) \leq 0,$$
>
> is satisfied by any θ such that $\frac{1}{2}\pi - \phi < \theta < \frac{1}{2}\pi$. We have
>
> $$s(r) = -\sup \mathcal{R}e\ e^{i\theta}q(\mathbf{x}) \geq \delta = -|\lambda|\cos(\theta + \phi)$$

so that $\delta \cos\theta$ takes its largest value $|\lambda| \sin^2 \frac{1}{2}\phi$ for $\theta = \frac{1}{2}(\pi - \phi)$ [ed. with Kato's radiation bound $M(r)$ given by]

$$M(r) \leq c_0 \exp\{-|\lambda|^{1/2} r \sin\frac{\phi}{2}\} = c_0 \exp\{-\mathcal{R}e(-\lambda)^{\frac{1}{2}} r\} \ > 0,$$

where $(-\lambda)^{1/2}$ is to be chosen in such a way that its real part is positive. As is easily seen, the equality holds for some spherically symmetric solutions of $\Delta u(\mathbf{x}) + q(\mathbf{x})u(\mathbf{x}) = 0$, when $n = 1$ or $n = 3$ and $p(\mathbf{x}) = 0$.

Importantly in this example the L^2 estimate on the spherically restricted $u(\mathbf{x})$, i.e.

$$||S_r u||_{L^2(S^{n-1})} \leq c_0 \exp\{-\mathcal{R}e(-\lambda)^{1/2} r\}, \tag{1.36}$$

is obtained without assuming anything *in advance* about the asymptotic behaviour of the solution $u(\mathbf{x})$. In conformity with this principle, the paper makes extensive use of *differential* rather than *integral* inequalities. The results obtained in Kato's paper were subsequently shown to be limited on several fronts – namely only L^2-control of the angular variables is provided there and there is no proof of the derivative estimate, this being the *strong* form of the radiation condition.

John G. Stalker and Shadi Tahvildar-Zadeh have recently established sharper decay estimates than those provided by Kato. The Stalker and Tahvildar-Zadeh paper [5] is similarly concerned with a Helmholtz equation in $\mathbb{R}^n$, also modified with a well behaved potential $V(\mathbf{x})$ (this corresponds to Kato's choice of $p(\mathbf{x})$), and given by

$$-\Delta u(\mathbf{x}) + V(\mathbf{x})u(\mathbf{x}) + z^2 u(\mathbf{x}) = 0, \qquad \text{Re } z = \zeta > 0, \tag{1.37}$$

with $V(\mathbf{x})$ satisfying more general conditions than the potential treated by Kato, namely, for potentials satisfying the upper and lower bounds U and L, respectively

$$\frac{-L}{r^2} \leq V(r) \leq \frac{U}{r^\eta}, \qquad \text{with} \qquad L < \frac{(n-2)^2}{4} \qquad \text{and} \qquad \eta > 1.$$

It succeeds in amending the aforementioned limitations of Kato's paper; to wit the authors prove that the L^2 functions $u(\mathbf{x})$, which solve their Helmholtz equation (1.37) outside a compactly supported source in $\mathbb{R}^n$, do indeed satisfy the strong form of the Sommerfeld radiation condition – an improved L^2 decay estimate for $u(\mathbf{x})$ is presented and in addition L^∞ control is also provided there.

The modified Euler-Lagrange equation in $\mathbb{R}^3$

$$-\Delta u(\mathbf{x}) + (\frac{1}{r^2} - \frac{\alpha^2}{\rho^2})\not{\Delta} u(\mathbf{x}) + V(r)u(\mathbf{x}) + z^2 u(\mathbf{x}) = f(\mathbf{x}), \qquad \text{Re } z > 0, \qquad (1.38)$$

associated with the compactly supported inhomogeneous term $f(\mathbf{x})$, and which is the central equation of this investigation, models the outgoing radiating Klein-Gordon field on the static spherically symmetric spacetime $(\mathcal{M}, g)$, in particular, outside a static spherically symmetric incompressible fluid star, when this system is transformed to the Minkowski spacetime $(\mathbb{R}^4, \eta)$. Equation (1.38) differs from the Helmholtz equation treated by both Kato and Stalker (for $\mathbb{R}^n$), essentially in its angular Laplacian term, the analysis of the influence of this term on the L^p and L^∞ solutions would form part of a broader investigation.

1.3 Review of Functional Analysis

For most classes of partial differential equations it is not possible to write down a tidy formula that solves them. However, there are many techniques for ascertaining *existence*, *uniqueness* and *stability* or other quantitative features of such solutions. These techniques come mainly from the analysis of function spaces. This section is a brief summary of aspects of the theory and terminology of function spaces used in this problem.

Formally, a function space is a *normed* space X, the elements of which are functions (with some fixed domain and range). Most of the standard function spaces of analysis are also *complete normed spaces* known as *Banach spaces.* The norm $\|f\|_X$ of a function f in X is the function spaces way of measuring how 'large' f is. It is common for the norm to be defined by a simple formula and for the space X to consist precisely of those functions f for which the resulting definition is sensible and finite. Thus, the mere fact that a function f belongs to a space X can already convey some qualitative information about that function. For example, it may imply some regularity, decay or boundedness on the function f. The *actual* value of the norm $\|f\|_X$ makes such information quantitative. That is to say it may tell us *how* regular f is, *how* much decay it has or *by which constant* it is bounded. We will be concerned in particular with the square integrable functions f, for which we have the L^2-norm $\|f\|_{L^2}$, and with the quantitative bounds of such functions, that is, we want to know what is the *smallest* $C \geq 0$ such that $\|f\|_{L^2} \leq C$ for all (or almost all), $x \in \mathbb{R}^n$.

Continuous functions on a compact domain $\mathcal{D}$ are bounded, so the most natural norm to place on this space is the supremum norm. The supremum norm is the norm associated with uniform convergence – it gives simultaneous control on the size of $|f(x)|$ for all $x \in \mathcal{D}$. However, this means that if there is a tiny set of x for which $|f(x)|$ is very large, then $\|f\|_\infty$ is very large, even if a typical value of $|f(x)|$ is much smaller. In a physical field theory that produces, for example, an L^∞-bound for its field magnitude, this L^∞-bound then possesses an advantage in its capacity to *falsify* hypotheses in the following way: imagine a physical field of magnitude greater than the theoretical L^∞-bound was detected experimentally, assuming a rigorous mathematical analysis we would then be forced to review the parameters and assumptions of the field theory. The L^2-bound which we establish for the scalar field $\tilde{\psi}(x^i)$ as a root mean square average or average value over a spherical surface could not do this; it is however, the easiest useful bound to find in our problem and it is possible to use *bootstrap* and/or other techniques to ascertain L^p and L^∞-bounds from the L^2-bound. These are desirable for the reason exampled above and because it is sometimes advantageous to work with norms that are less influenced by the values of a function on small sets. The *L^p-norm* of a function f given by

$$\|f\|_p \equiv (\int |f(x)|^p dx)^{\frac{1}{p}},$$

is defined for $1 \leq p < \infty$ and for any measurable f. The function space L^p is the class of measurable functions for which the above norm is finite. You might say informally that while the L^∞ norm is concerned solely with the "height" of a function, the L^p norms are concerned with a combination of the "height" and "width" of a function. Particularly important among these norms is the aforementioned L^2-norm space. This is a *Hilbert space* endowed with exceptional symmetries, these are briefly outlined in the Appendix A, on aspects of Functional Analysis. It is often very useful, particularly in the analysis of boundary value problems, to make use of the Hilbert space structure, or at least a Banach space structure of the function spaces from which the solutions are taken. Doing so makes it possible to apply the results of Functional Analysis to the theory of linear and non-linear partial differential equations. In many cases these methods are the only ones available in other cases they lead to more definitive results. Our problem ultimately distills to a well-posed *Cauchy initial value problem* and we rely heavily on the Hilbert space structure in our analysis.

1.3.1 Sobolev Spaces: Norms and Energy Conservation

The Lebesque norms control, to some extent, the height and width of a function, but say nothing about regularity; a function in L^p, for example, need not be differentiable or even continuous. To incorporate such information we often use the notion of *Sobolev norms* $\|f\|_{W^{k,p}}$, defined for $1 \leq p \leq \infty$ and $k \geq 0$ and denoted by

$$\|f\|_{W^{k,p}} = \sum_{j=0}^{k} \left\| \frac{\partial^i f}{\partial x^j} \right\|_p < \infty. \tag{1.39}$$

The *Sobolev Space* $W^{k,p}(\Omega)$ is the space of functions on a domain Ω for which this norm is finite. Thus a function lies in $W^{k,p}$ if it and its first k derivative all belong to L^p. Importantly *we do not require f to be k times differentiable in the usual sense, but rather in the weaker sense of distributions.* We need to consider these generalized differentiable functions because without them the space $W^{k,p}$ would not be complete. These function spaces adapted to the study of partial differential equations were first introduced by S.L. Sobolev of Moscow State University (Lomonosov - now a dodgy part of town). As defined here in (3.1.2), a Sobolev space is simply a vector space of functions equipped with a norm that is a combination of L^p-norms of the function itself as well as its derivatives up to a given order, the derivatives understood in a suitably weak sense in order to make the space a complete *Banach space.* Intuitively then, a Sobolev space is a Banach or Hilbert space of functions with sufficiently many derivatives for some application domain such as partial differential equations, and equipped with a norm that measures both the size and smoothness of a member function. Solutions of many classes of partial differential equations reside naturally in Sobolev spaces rather than in classical spaces of continuous functions. Sobolev spaces can be considered as one of the main tools that made possible the wide development of PDEs in the last several decades and are essential to our analysis here [12]. We use mainly the $W^{1,2}(\Omega) \equiv \mathbf{H}^1(\Omega)$ norm, interpreting it as the conserved "energy" associated with a function

$$\|\psi(x)\|_{W^{1,2}(\Omega)} = \left[\int_\Omega \left(|\psi(x)|^2 + \sum_{i=1}^{n}\left|\frac{\partial\psi(x)}{\partial x^i}\right|^2\right) dx\right]^{\frac{1}{2}} = \|\psi(x)\|_{\mathbf{H}^1(\Omega)},$$

where $\psi(x)$ is the solution of the Klein-Gordon equation in spacetime $(\mathcal{M}, g)$

$$g^{\nu\mu}\nabla_\mu\nabla_\nu\psi(x) = \psi_{tt}(x) + A\cdot\psi(x) = 0.$$

The domain of the spatial operator A, defined on some initial domain Σ_0, is contained within the closure of $D(\Sigma_0)$, the domain of dependence, under this norm. This norm is also known as the *first Sobolev norm* of ψ. We provide some formal notation for definitions of various norms on the Sobolev spaces in the Appendix A where we also explain the formal Sobolev space notation used throughout this work.

Chapter 2

The Klein-Gordon Field on the Static Spacetime

2.1 The Einstein-Matter System

Einstein's theory of gravitation, as finally formulated by him in 1915, represented a revolutionary development in the foundations of natural philosophy [13]. Geometry, inertia and gravity were unified as aspects of a single theoretical structure: the *Lorentzian* metric, and expressed through the symmetric tensor $g_{\mu\nu}$, assumed to exist on a 4-dimensional continuum or differentiable manifold $\mathcal{M}$ known as *spacetime*. The n-dimensional generalization of the object where Riemannian or Lorentzian metrics naturally live is the manifold. Loosely speaking, a Lorentzian metric is one that "looks locally like the Minkowski metric", just as a Riemannian metric looks locally Euclidian. Spacetime containing matter is thus described as a *pseudo-Riemannian manifold*, with the gravitational force manifesting as a curvature of its geometry.

Thus General Relativity allows the spacetime continuum not to be $\mathbb{R}^4$ but instead to be a general manifold $\mathcal{M}$, which may very well be topologically inequivalent to $\mathbb{R}^4$. We call the pair $(\mathcal{M}, g)$ a Lorentzian manifold. Properly put, the unknown in the Einstein equations is not just g but the pair $(\mathcal{M}, g)$. Manifolds are the structures obtained by consistently smoothly pasting together local coordinate systems. General Relativity postulates that this four-dimensional Lorentzian manifold $(\mathcal{M}, g)$, endowed with metric g, is to satisfy the acclaimed *Einstein field equations*

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}, \tag{2.1}$$

giving a geometrodynamic theory of gravitation. We take the Einstein equations as a basic axiom; $G_{\mu\nu}$ is the *Einstein* tensor expressed in terms of $R_{\mu\nu}$ and R, the *Ricci* curvature tensor and scalar curvature of metric g, respectively. $T_{\mu\nu}$ denotes a symmetric 2-tensor on $\mathcal{M}$ known as the *stress-energy-momentum tensor* of matter in the manifold and we set the gravitational coupling constant $\kappa = 8\pi G$, where G is Newton's gravitational constant.

The tensor-field equations (2.1) must be coupled to "matter equations" satisfied by a set of matter fields $\{\Psi_i\}$ defined on $\mathcal{M}$, together with a constitutive relation determining $T_{\mu\nu}$ from $\{g, \Psi_i\}$. These equations and relations are stipulated by the relevant continuum field theory describing the matter. For our set-up in the Schwarzschild exterior, the energy-momentum tensor $T^{(\psi)}_{\mu\nu}$ is that of the massless, spin-zero scalar Klein-Gordon field $\psi(x)$.

To construct an action for General Relativity, we must define a Lagrangian L which is a scalar under general coordinate transformations and which depends on $g_{\mu\nu}$ (these are now the dynamical fields) and its derivatives. The simplest non-trivial scalar that can be constructed from the metric and its derivatives is the Ricci scalar R, which depends on $g_{\mu\nu}$ and its first- and second-order derivatives. In fact, R is the *only* scalar derivable from the metric tensor that depends on derivatives no higher than second order. From our knowledge of gravitation as a manifestation of spacetime curvature, we might expect L to be derived from the curvature tensor. Thus, in searching for the simplest plausible variational principle for gravitation, one is led to the *Einstein-Hilbert* action

$$S_{EH} = \int_{\mathcal{R}} R\sqrt{-g}d^4x \ ,$$

with Lagrangian density $\mathcal{L}_{EH} = R\sqrt{-g}$. Considering a variation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, where $\delta g_{\mu\nu}$ and its first derivative vanish on the boundary $\partial\mathcal{R}$ of the region $\mathcal{R}$. For arbitrary variation of *Einstein-Hilbert* action, such that $\delta S_{EH} = 0$, a standard, if tedious, textbook calculation, i.e.

$$\delta S_{EH} = \delta \int_{\mathcal{R}} R\sqrt{-g}d^4x = \int_{\mathcal{R}} (R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu}\sqrt{-g}d^4x = 0, \qquad (2.2)$$

yields the *vacuum* Einstein field equations of (2.1), i.e., with energy-momentum $T_{\mu\nu} = 0$. To develop the variational notion for other (non-gravitational) fields, we simply add an extra term to the action to give

$$S = \frac{1}{2\kappa}S_{EH} + S_M = \int_{\mathcal{R}} \left(\frac{1}{2\kappa}\mathcal{L}_{EH} + \mathcal{L}_M\right) d^4x,$$

where S_M is the matter action and the factor $\frac{1}{2\kappa}$ is chosen for convenience. Again the corresponding field equations are got by a variation in the metric tensor, such that $g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$, and with vanishing on boundary $\partial\mathcal{R}$, so that we have

$$\delta S = \frac{1}{2\kappa}\frac{\delta}{\delta g^{\mu\nu}}\mathcal{L}_{EH} + \frac{\delta}{\delta g^{\mu\nu}}\mathcal{L}_M = 0.$$

From (2.2) we have

$$\frac{\delta}{\delta g^{\mu\nu}}\mathcal{L}_{EH} = \sqrt{-g}G_{\mu\nu},$$

and by asserting the definition for non-gravitational energy-momentum

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \mathcal{L}_M, \tag{2.3}$$

we will then recover the field equations of (2.1) above.

2.1.1 Conservation of Energy-Momentum I

The quantities $T_{\mu\nu}$ defined in (2.3) are clearly tensorial. From the definition we see $T_{\mu\nu}$ is a symmetric tensor, as required by the full Einstein equations. More important, however, we show how it obeys the covariant conservation equation

$$\nabla_\mu T^{\mu\nu} = 0. \tag{2.4}$$

From the definition in (2.3), the variation in the matter action resulting from a variation in the metric is given by

$$\delta S_M \equiv \int_{\mathcal{R}} \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} \delta g^{\mu\nu} d^4x = \frac{1}{2} \int_{\mathcal{R}} T_{\mu\nu} \delta g^{\mu\nu} \sqrt{-g} d^4x = -\frac{1}{2} \int_{\mathcal{R}} T^{\mu\nu} \delta g_{\mu\nu} \sqrt{-g} d^4x. \tag{2.5}$$

Consider making an infinitesimal general coordinate transformation

$$x'^\mu = x^\mu + \xi^\mu(x),$$

where $\xi^\mu(x)$ is an infinitesimal smooth vector field. Since the action S_M is, by construction, a covariant scalar, then we must have $\delta S_M = 0$ under the coordinate transformation. Following a standard text book calculation [34], requiring some familiarity with variational and tensor calculus, we find

$$\delta S_M = \int_{\mathcal{R}} T^{\mu\nu} \left(\nabla_\mu \xi_\nu\right) \sqrt{-g} d^4x = 0, \tag{2.6}$$

and then using Leibnitz' theorem for covariant differentiation of a product, we write

$$\delta S_M = \int_{\mathcal{R}} \nabla_\mu (T^{\mu\nu}\xi_\nu)\sqrt{-g}d^4x - \int_{\mathcal{R}} (\nabla_\mu T^{\mu\nu})\xi_\nu \sqrt{-g}d^4x = 0. \tag{2.7}$$

We use the divergence theorem (Appendix B) to write the first integral as a surface integral over the boundary $\partial\mathcal{R}$ in the usual manner. Assuming that the functions $\xi^\nu(x)$ vanish on the boundary $\partial\mathcal{R}$ this surface integral vanishes, leaving only the second integral in (2.7) . Since the $\xi^\mu(x)$ are arbitrary, however, one immediately finds that

$$\nabla_\mu T^{\mu\nu} = 0, \tag{2.8}$$

as required.

We note here that the existence of a timelike Killing vector ξ^μ allows us to define a conserved energy for the entire spacetime. Given ξ^μ and a conserved $T^{\mu\nu}$ we can construct a current J^μ_T that is automatically conserved

$$(\nabla_\mu \xi_\nu)T^{\mu\nu} + \xi_\nu(\nabla_\mu T^{\mu\nu}) = 0, \tag{2.9}$$

the first term vanishes by Killing equation, the second by energy-momentum conservation. For timelike ξ_ν we can integrate over a spacelike hypersurface Σ_t to define total energy [14]

$$\mathcal{E}_T = \int_{\Sigma_t} J^\mu_T n_\mu \sqrt{\gamma} d^3x, \tag{2.10}$$

where γ_{ij} is induced metric on Σ, n_μ a normal vector to Σ and in adapted coordinate

$$\xi_\nu = g_{\nu\mu}\xi^\mu = g_{\nu\mu}\delta^\mu_0 = g_{\nu o} = g_{00}\delta^0_\nu = \alpha^2.$$

This result fits with our analysis of energy conservation in Section 3.1.2 where the induced metric on the spacelike hypersurface $t = const.$ is given by

$$d\sigma = \alpha\rho^2 d\Omega dr \quad \text{and} \quad d\sigma'' = \alpha^{-1}d\sigma = \rho^2 d\Omega dr,$$

and the conserved current J_T^0 in Killing direction ξ_0 is

$$J_T^0 = \xi_\nu T^{0\nu} = \xi_0 T^{00} = \frac{1}{2}\psi_t^2 + \frac{1}{2}\psi_r^2 + \frac{\alpha^2}{2\rho^2}|\nabla\psi|^2,$$

so that

$$\mathcal{E}_T = \frac{1}{2}\int_{\Sigma_t=\sigma''}\left(\psi_t^2 + \psi_r^2 + \frac{\alpha^2}{\rho^2}|\nabla\psi|^2\right)\rho^2 d\Omega dr.$$

2.1.2 The Klein-Gordon Field

This classical spacetime field $\psi(x)$ does not correspond to any familiar type of matter and indeed no such field has yet been observed directly in nature. On the other hand there are indirect indications that such scalar fields play an important role in current cosmological and quantum field theories. We work with the Euler-Lagrange equation for the massless spin-zero wave field $\psi(x)$ on the *static spherically symmetric spacetime* $(\mathcal{M}, g)$. This particular system is one of the simplest to deal with from a mathematical point of view, and perhaps offers valuable insights into the dynamics of more complicated radiations on other families of related spacetimes. The premise here being that knowledge of a one-component field acts as a reliable guide to the behaviour of multi-component fields (such as electromagnetic A_μ or gravitational $g_{\mu\nu}$), with the main advantage gained in studying a non-gravitational test field, such as the weakly interacting $\psi(x)$, is that we can ignore the contributions of the field energy to the manifold geometry [15]; this fact will be illustrated in a subsequent simple perturbation calculation in Section 2.1.5, where the manifold $(\mathcal{M}, g)$, representing the static, spherically symmetric spacetime will thus be seen to be effectively the unperturbed *vacuum Schwarzschild exterior* background $(\mathcal{M}, g_0)$, on which the radiating field $\tilde{\psi}(x^i)$ decays asymptotically at infinity.

A considerable body of research exists on investigations of scalar fields on curved spacetimes, concerned with ideas of well-posedness, metric stability, decay and scattering [16] [17], and as a means of probing various *cosmic censorship conjectures* [18]. According to the extensive literature, such investigations in Schwarzschild $(\mathcal{M}, g_0)$ can provide, with a degree of simplicity, useful qualitative information in respect of the stability and asymptotic properties of various model black hole spacetimes or their related stellar collapse scenarios [19] [15]. In particular, the massless Klein-Gordon equation has been used heuristically to demonstrate the essentials of the Schwarzschild metric perturbation problem and the related stability of static *black hole spacetimes*, as it shares a similar structure to the mathematically more complicated *Regge-Wheeler* equation of that problem [17] . These types of wave equations persist as open research topics in classical General Relativity and there are a number of excellent introductions and reviews of the subject available [20] [21].

Although our primary goal in this work is to obtain an optimal bound for the radiating scalar field $\tilde{\psi}(x^i)$, in other words, to prove the Sommerfeld finiteness condition, we will also need to address the critical issue of well-posedness in this regard also. This demands some rigour in the analysis of the Hilbert and the associated Sobolev spaces arising in this problem.

2.1.3 The Coupled K-G Field

Consider the action S_ψ for a real scalar field $\psi(x)$ varying the action with respect to the inverse metric $g^{\mu\nu}$, rather than the field ψ, we obtain

$$\begin{aligned}\delta S_\psi &= \int_{\mathcal{R}} \left[\frac{1}{2}\delta g^{\mu\nu}(\nabla_\mu\psi)(\nabla_\nu\psi)\sqrt{-g} + \frac{1}{2}g^{\mu\nu}(\nabla_\mu\psi)(\nabla_\nu\psi)\delta(\sqrt{-g})\right] d^4x \\ &= \int_{\mathcal{R}} \left[\frac{1}{2}(\nabla_\mu\psi)(\nabla_\nu\psi) - \frac{1}{2}g_{\mu\nu}\{\frac{1}{2}g^{\rho\sigma}(\nabla_\rho\psi)(\nabla_\sigma\psi)\}\right] \delta g^{\mu\nu}\sqrt{-g}d^4x, \quad (2.11)\end{aligned}$$

comparing this expression with (2.5) we see the energy-momentum tensor for real scalar field $\psi(x)$ is

$$T^{(\psi)}_{\mu\nu} = \nabla_\mu\psi\nabla_\nu\psi - \frac{1}{2}g_{\mu\nu}\nabla_\sigma\psi\nabla^\sigma\psi. \quad (2.12)$$

On contraction of the Einstein equations (2.1) with $g^{\mu\nu}$, in the absence of matter, i.e. $T_{\mu\nu} = 0$, we get

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0, \quad \Rightarrow \quad g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R = 0, \quad (2.13)$$

this is the vacuum solution, with Ricci tensor

$$R_{\mu\nu} = 0. \quad (2.14)$$

The system of Equations (2.14) are known as the *Einstein vacuum field equations*, solving for a spherically symmetric vacuum spacetime yields the Schwarzschild field solution

$$ds^2_{\text{ext}} = -\left(1 - \frac{2M_s}{\rho}\right)dt^2 + \left(1 - \frac{2M_s}{\rho}\right)^{-1}d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

The perturbation argument for the weakly interacting scalar field ψ shows that our problem effectively presents as a radiating field on the exterior vacuum Schwarzschild spacetime $(\mathcal{M}, g_0)$.

A physical solution of the field equations, in other words a spacetime, consists of the manifold $\mathcal{M}$ together with its metric g. Two spacetimes are physically equivalent, in other words, give rise to the same gravitational field if their respective metrics can be transformed into each other. Mathematically, we should regard the physical solutions of the Einstein field equations (2.1), as equivalence classes of spacetimes possessing metrics which are related by legitimate coordinate transformations.

2.1.4 Conservation of Energy-Momentum II

We note that in classical field theory it is common to define also a *canonical* energy-momentum tensor, based on *Noether's theorem* which states that every continuous symmetry of a Lagrangian implies a corresponding conservation current and consequently its invariance under canonical spacetime translations $x^\mu \to x^\mu + a^\mu$ leads to the *conservation of energy-momentum*. The free massless scalar-field Lagrangian $\mathcal{L}$ in *flat* spacetime $(\mathbb{R}^4, \eta)$ with coordinates $\{x^\mu = t, r, \theta, \phi\}$ and metric

$$ds_\eta^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) = -dt^2 + d\mathbf{r}.d\mathbf{r},$$

is given by

$$\mathcal{L} = \eta^{\mu\nu} \partial_\nu \psi(x) \partial_\mu \psi(x), \tag{2.15}$$

this is used to obtain the flat energy-momentum tensor in the the standard way, i.e.

$$\begin{aligned} T_{\mu\nu}^{(\psi)} &= \frac{\partial \mathcal{L}}{\partial(\partial^\mu \psi)} \partial_\nu \psi - \mathcal{L} \eta^{\mu\nu}, \\ &= \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} \eta^{\mu\nu} \partial_\sigma \psi \partial^\sigma \psi. \end{aligned} \tag{2.16}$$

This quantity is conserved under canonical spacetime translations and as a consequence of Noether's theorem, we arrive at the *energy-momentum conservation law*

$$\partial^{\mu} T^{(\psi)}_{\mu\nu} = 0. \tag{2.17}$$

The above relations embody the conservation of energy-momentum at a differential level. Integrating (2.17) between homologous hypersurfaces and applying the Minkowski-space version of the divergence theorem we obtain global balance laws. If $T_{\alpha\beta}$ is assumed to be compactly supported, then, integrating between $t = t_1$ and $t = t_2$ we obtain

$$\int_{t=t_1} T_{0\alpha} dx^1 dx^2 dx^3 = \int_{t=t_2} T_{0\alpha} dx^1 dx^2 dx^3.$$

With respect to the chosen Lorentz frame, the zeroth component of the above equation represents the *conservation of total energy*, while the remaining components represent *conservation of total momentum.*

We need a method to obtain field equations of physical systems in General Relativity when the corresponding equations are known in Special Relativity. In this instance (mimicking established practice), we invoke the so-called *Principle of minimal gravitational coupling* – this is a simplicity principle of limited application which is adequate to the task here as the Klein-Gordon $\psi(x)$ is a weakly interacting spin-zero scalar field. Essentially minimal coupling says we should not add unnecessary terms in making the transition from the Special to the General theory. Since in local geodesic coordinates, the covariant derivative reduces to the ordinary derivative, we can also express this principle as follows: in local geodesic coordinates the equations of motion are those of Special Relativity. In practice this means that no terms explicitly containing the curvature tensor should be added in making this transition from a flat to a curved spacetime. Guided by this principle we substitute the curved metric $g^{\mu\nu}$ for the flat metric $\eta^{\mu\nu}$ and in addition we ought to replace partial derivatives by their covariant counterparts – for the scalar field $\psi(x)$ these are in fact equivalent. Consequently, for our system the field Lagrangian L, energy-momentum tensor $T_{\mu\nu}$ and conservation law respectively are given by

$$L = g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi, \tag{2.18}$$

$$T^{(\psi)}_{\mu\nu} = \nabla_\mu\psi\nabla_\nu\psi - \frac{1}{2}g^{\mu\nu}L, \tag{2.19}$$

$$\nabla^\mu T^{(\psi)}_{\mu\nu} = 0. \tag{2.20}$$

Importantly, equations (2.20), which we also derived in Section 2.1.1, as equations (2.8), is the analogue of Minkowski energy-momentum conservation equations (2.17) above. We note $\nabla^\mu = g^{\mu\nu}\nabla_\nu$ refers to the covariant connection

$$\nabla_\lambda T_{\mu\nu} = \partial_\lambda T_{\mu\nu} - \Gamma^\sigma_{\lambda\mu}T_{\sigma\nu} - \Gamma^\sigma_{\lambda\nu}T_{\mu\sigma},$$

associated with metric g. Energy-momentum conservation is in agreement with the Einstein field equations through the contracted *Bianchi identities*:

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} \equiv \nabla_{[e}R_{ab]cd},$$

these identities imply

$$\nabla_b(R^{bc} - \frac{1}{2}g^{bc}R) = 0, \quad \text{or} \quad \nabla_b G^{bc} = 0 \quad \Rightarrow \quad \nabla_b T^{bc} = 0. \tag{2.21}$$

It should be noted that the minimal coupling principle is not a symmetry principle and its *raison d'être* is really just its ease of application in aspects of General Relativity, it is *not* a general law and cannot be applied to all the equations of Physics. It does not, for example, apply to the equation of motion of the spin $\mathbf{S}$ of a rigid body – in local geodesic coordinates, with proper time τ, it would tell us $d\mathbf{S}/d\tau = 0$ since this *is* the spin equation in the absence of gravitation and it is wrong! Tidal torques are present in the true equation of spin, and motion depends on the Riemann tensor R, minimal coupling is thus generally unreliable and only to be used as a last resort. We remind ourselves that we are also solving here the equation for the conformally invariant massless Klein-Gordon field $\Psi(x)$ on a non-vacuum Lorentzian manifold $(\mathcal{M}, \tilde{g})$, given by

$$\Box_{\tilde{g}}\Phi(x) - \frac{1}{6}\kappa\tilde{R}\Phi(x) = 0, \tag{2.22}$$

this equation is discussed in greater detail in Appendix B.

From the *Principle of least action*, a standard variational calculation of the action functional $\mathcal{A}_\psi$ on the manifold $(\mathcal{M}, g)$, corresponding to the real massless scalar field $\psi(x) : \mathcal{M} \to \mathbb{R}$, and associated with a generic 4-volume form of measure $d\mu_g$, i.e.

$$\delta\mathcal{A}_\psi \equiv \delta\left[\frac{1}{2}\int_{\mathcal{M}} \mathcal{L}\left(\psi(x), \partial_\mu\psi(x)\right) d\mu_g\right] = 0, \tag{2.23}$$

engenders the Euler-Lagrange wave equation on $(\mathcal{M}, g)$, introduced by equation (1.6) in Sec. 1.1, as

$$\Box_g\psi(x) = \frac{1}{\sqrt{|g|}}\partial_\mu\left[\sqrt{|g|}g^{\mu\nu}\partial_\nu\psi(x)\right] = 0, \tag{2.24}$$

which is explicitly derived in Section 3.1.1. Thus, following the minimal coupling principle, the *coupled* Einstein-matter system on our manifold $(\mathcal{M}, g)$, for the massless, zero-spin Klein-Gordon field $\psi(x)$ can be written

$$\begin{cases} G_{\mu\nu} &= \kappa T^{(\psi)}_{\mu\nu} \equiv \kappa(\nabla_\mu\psi\nabla_\nu\psi - \frac{1}{2}g_{\mu\nu}\nabla_\sigma\psi\nabla^\sigma\psi), \\ \Box_g\psi &= g^{\mu\nu}\nabla_\nu\nabla_\mu\psi = 0. \end{cases} \tag{2.25}$$

2.1.5 Decoupling the Einstein-Matter System

Exact solutions of coupled Einstein-matter systems are rare indeed. However, if the deviation from a known exact solution is 'small' in some sense, an approximate solution may be useful. In their original work [17], Regge and Wheeler studied perturbations of the metric directly by introducing the metric form

$$g_{\mu\nu} = g^b_{\mu\nu} + h_{\mu\nu},$$

where $g^b_{\mu\nu}$ is the known exact or *background* solution and $|h_{\mu\nu}|$ is considered 'small' in some sense, so that only the terms *linear* in $h_{\mu\nu}$ are retained in all subsequent calculations. This provides a considerable simplification for the equations involved. For this problem we assume the scalar field $\psi(x)$ is weakly coupled to the gravitational field $g_{\mu\nu}$, that is, the effect of its energy-momentum on the spacetime $\mathcal{M}$ may be neglected, this simplifying approximation has proven to be surprisingly robust. Consider the equation

$$\mathcal{E}(g) = 0, \tag{2.26}$$

for an unknown function g (which, more generally, may be a collection of functions or tensor fields, etc.) In the case of interest, g is the spacetime metric possibly together with variables describing the matter distribution and $\mathcal{E}(g)$ represents Einstein's field equations as expressed by Eq. (2.1). Suppose an exact solution g^b of (2.26) is known and suppose also that we are interested in studying situations where the deviation from g^b is considered 'small'. What we would like to have then is a one-parameter family $g(s)$ of exact solutions, which we express by

$$\mathcal{E}[g(s)] = 0, \tag{2.27}$$

and where s measures the size of perturbations in the sense that, (i) $g(s)$ depends differentiably on s and (ii) $g(0) = g^b$. Thus small s corresponds to small deviations from a background metric $g^b(0)$ and a knowledge of $g(s)$ for small s would give us an exact perturbed solution, equation (2.27) however, may still prove too difficult to solve. Nevertheless, we can derive a much simpler equation from (2.27) by differentiating it with respect to s and then setting s equal to zero, so that

$$\frac{d}{ds}\mathcal{E}[g(s)]\bigg|_{s=0} = 0. \tag{2.28}$$

Equation (2.28) is a linear equation for the function

$$h = \frac{dg}{ds}\bigg|_{s=0}, \tag{2.29}$$

in other words it can be expressed in the form

$$\mathcal{F}(h) = 0, \tag{2.30}$$

where $\mathcal{F}$ just represents a linear operator. If we can solve Eq. (2.30) then $g^b + sh$ should yield a good approximation to $g(s)$ for sufficiently small s, and thus issues of physical interest can be more easily investigated.

Here we will denote a one-parameter family of spacetimes by $(\mathcal{M}, g(s))$ associated with a parametrised scalar field which we label by $\psi(s)$, depending differentiably on this parameter s. For each s the field quantities $g(s)$ and $\psi(s)$ satisfy the coupled Einstein-scalar system of Eq. (2.25) above. For small perturbations of the background spacetime we expand the parametrized field functions $g(s)$ and $\psi(s)$ in a perturbation expansion about $s = 1$, which we then write as

$$\begin{cases} g(s) &= g_b + s g_1 + s^2 g_2 + \cdots, \\ \psi(s) &= \psi_b + s\psi_1 + s^2\psi_2 + \cdots. \end{cases} \tag{2.31}$$

Importantly for small perturbations we will ignore all terms of order s^2 and higher in the above perturbation expansions. It is clear from Eq. (2.25) that $T_{\mu\nu}(s)$ is quadratic in $\psi(s)$. For $s = 0$ in the expansion we write

$$g(s) = g(0) \equiv g_b,$$

this is just the unperturbed or *background* spacetime. The scalar field $\psi(s)$ is then written as $\psi(0) = \psi_b$, which is, of course, just zero here as this represents the unperturbed spacetime. Differentiating the Einstein-scalar system of Eq. (2.25) above, with respect to s, and setting $s = 0$ we find

$$\frac{\partial}{\partial s} T^{(\psi)}(s)\bigg|_{s=0} = 0. \tag{2.32}$$

We see also that

$$\frac{\partial}{\partial s} G_{\mu\nu}(s)\bigg|_{s=0} = \frac{\partial}{\partial g} G_{\mu\nu}(s) \cdot \frac{\partial}{\partial s} g(s)\bigg|_{s=0} = \frac{\partial}{\partial g} G_{\mu\nu}(s) \cdot g_1\bigg|_{s=0} = 0, \tag{2.33}$$

contained in this equation is an expression for Regge-Wheeler gravitational waves, this detailed calculation is not, however, part of the investigation here [35].

Taking the coupled system of Eq. (2.25) and again differentiating with respect to s and then setting $s = 0$ in its perturbation expansion, we find the expansion in Eq. (2.31) yields

$$\frac{\partial}{\partial s} \left(\Box_{g(s)} \psi(s)\right)\bigg|_{s=0} = \Box_{g(0)} \psi'(0) = \Box_{g_b} \psi_1 = 0. \tag{2.34}$$

This provides our decoupled equation of motion, describing a weakly interacting, massless Klein-Gordon field $\psi_1(x)$ on the background spacetime $(\mathcal{M}, g_b)$. Relabelling to let $\psi_1 = \psi(x)$ now represent the scalar field, we arrive at an equivalent expression for the wave equation of the weakly interacting field $\psi(x)$

$$\Box_{g_b}\psi(x) \equiv \frac{1}{\sqrt{|g_b|}}\partial_\mu[\sqrt{|g_b|}g_b^{\mu\nu}\partial_\nu\psi(x)] = 0. \tag{2.35}$$

This is consistent with the derivation of the Euler-Lagrange equation for the scalar field $\psi(x)$ that is a stationary point of the action functional $\mathcal{A}_\psi$, propagating on the complete Schwarzschild background $(\mathcal{M}, g_b)$, which was discussed in Section 2.1.3. The Einstein-matter equations when contracted with metric tensor $g^{\mu\nu}$ where $T = g^{\mu\nu}T_{\mu\nu}$ give

$$g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R = \kappa g^{\mu\nu}T^{(\psi)}_{\mu\nu} \quad \Rightarrow \quad R = -\kappa T, \tag{2.36}$$

so that the conformal Klein-Gordon equation becomes

$$\Box_g\psi(x) + \frac{1}{6}\kappa T\psi(x) = 0. \tag{2.37}$$

Using the perturbation expansion of Eq. (2.31) above, for energy-momentum scalar $T(s)$ quadratic in s, and ignoring s^2 terms and higher for the small perturbation approximation, we find $T \approx 0$ and thus $g \approx g_s$ and again we return the wave equation on background spacetime $(\mathcal{M}, g_s)$. For time dependence $\psi(x) = e^{zt}\tilde{\psi}(\rho, \theta, \phi)$, in curvature coordinates, we have the originating Helmholtz-type equation of our investigation.

2.2 The $(\mathcal{M}, g)$ Spacetime Structure

. . . I must have fallen asleep, for all of a sudden there was the moon, a huge moon framed in the window. Two bars divided it in three segments, of which the middle remained constant, while little by little the right gained what the left lost. For the moon was moving from left to right, or the room was moving from right to left, or both together perhaps, or both were moving from left to right, but the room not so fast as the moon, or from right to left, but the moon not so fast as the room. But can one speak of right and left in such circumstances? That movements of an extreme complexity were taking place seemed certain, and yet what a simple thing it seemed. . .

Molloy, S.B.

The manifold $(\mathcal{M}, g)$ represents a four-dimensional, connected, spherically symmetric static spacetime. Indulging the oxymoron, this symmetry brings profound simplifications to certain features of classical General Relativity and these spacetimes continue to provide a rich source of investigation and speculation, albeit for the mathematician rather than the physicist. We seek all solutions of Einstein's equation which describe the completer gravitational field of a static, spherically symmetric spacetime. First we define more precisely the meaning of the terms "static" and spherically "symmetric" and then choose a convenient coordinate system for analysing this class of spacetimes

A spacetime is said to be *stationary* if there exists a one-parameter group of *isometries* ϕ_t whose orbits are *timelike* curves. This group of isometries expresses the "time translation symmetry" of the spacetime. Equivalently, a stationary spacetime is one which possesses a *timelike Killing vector field* ξ^{μ}. The spacetime is said to be *static* if it is stationary and if in addition, there exists a (spacelike) hypersurface Σ, which is orthogonal to the orbits of the isometry ϕ_t. By *Frobenius's theorem* [14] this is equivalent to the requirement that the hypersurface-orthogonal timelike Killing vector field ξ^μ satisfy

$$\xi_{[\mu}\nabla_\nu\, \xi_{\omega]} = \xi_\mu \nabla_\nu \xi_\omega + \xi_\omega \nabla_\mu \xi_\nu + \xi_\nu \nabla_\omega \xi_\mu - \xi_\mu \nabla_\omega \xi_\nu - \xi_\nu \nabla_\mu \xi_\omega - \xi_\omega \nabla_\nu \xi_\mu = 0. \quad (2.38)$$

The condition of hypersurface orthogonality for the metric can be best seen by introducing convenient coordinates for static spacetimes as follows. If $\xi^\mu \neq 0$ everywhere on the hypersurface Σ, then in a neighbourhood of Σ, every point will lie on a unique orbit of ξ^μ which passes through Σ. Assuming $\xi^\mu \neq 0$, we choose arbitrary local spacelike coordinates $\{x^i\}$ on Σ, and label each point p in the neighbourhood of Σ by the parameter, t, of the orbit which starts from Σ and ends at p, and the coordinates $\{x^1,\, x^2,\, x^3\}$ of the orbit *at* Σ. Precisely because the theory *is* invariant under coordinate transformations, allows us to choose whatever coordinate system is most convenient to perform the desired analysis. Since this coordinate system employs the Killing parameter t as one of the coordinates, the metric components in this coordinate basis will be *independent* of t. Furthermore, since the surface Σ_t (defined as the set of points whose "time coordinate" has the value t) is the image of Σ under the isometry ϕ_t, it follows that each Σ_t is also orthogonal to ξ^μ. Thus in these coordinates the metric components take the canonical form

$$ds^2 = -\alpha^2(x^1, x^2, x^3)dt^2 + h_{ij}(x^1, x^2, x^3)dx^i dx^j, \quad (2.39)$$

with time independent coefficient

$$\alpha^2(x^i) = -\xi_\mu \xi^\mu > 0, \quad (2.40)$$

and the absence of $dtdx^i$ cross terms expresses the orthogonality of the Killing vector ξ^μ, to the hypersurface Σ and $g_{\mu\nu}\xi^\mu\xi^\nu < 0$ for timelike ξ^μ.

A spacetime is *spherically symmetric* if its isometry group contains a subgroup isomorphic to the group $SO(3,\mathbb{R})$, and the orbits of this subgroup, i.e., the collection of points resulting from the action of the subgroup on a given point, are two-dimensional spheres. In this sense the $SO(3,\mathbb{R})$ isometries may then be interpreted physically as rotations, and so a spherically symmetric spacetime is one whose metric remains invariant under rotations. The spacetime metric induces a metric on each orbit 2-sphere which, because of rotational symmetry, must be a multiple of the metric of a unit 2-sphere, and is thus completely characterised by the total area A of the 2-sphere. It proves convenient to introduce the $\mathbb{R} \times SO(3,\mathbb{R})$-invariant function

$$\rho \equiv \sqrt{A/4\pi}, \tag{2.41}$$

where $SO(3,\mathbb{R})$ represents the two-sphere $\mathbb{S}^2$ of radius ρ, which is a sub-manifold embedded in $\mathcal{M}$. These spheres are said to *foliate* the $\mathbb{R}^3$-space and in spherical coordinates (θ,ϕ), the metric induced on the orbit 2-sphere takes the familiar spherical form

$$ds_\Omega^2 = \rho^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.42}$$

In flat three-dimensional Euclidian space, ρ is the distance from the surface to the centre of the sphere. In curved space a sphere need not have a centre, the manifold structure could be, for example, described by $\mathbb{R} \times \mathbb{S}^2$; and even if it does have a centre ρ need not bear any relation to the distance to the centre. Nevertheless, we refer to ρ as the "radial coordinate" of the sphere and this is how it is used in Synge's *curvature coordinates* $\{x^\mu = t, \rho, \theta, \phi\}$, defined invariantly by the symmetries present. If a spacetime is both static and spherically symmetric, and if the static Killing field ξ^μ is unique, then ξ^μ must be orthogonal to the orbit 2-spheres and invariant under all rotational isometries. However, this requires its projection onto any orbit sphere to vanish, since a non-vanishing vector field on a sphere cannot be invariant under all rotations. Thus the orbit spheres lie wholly within the hypersurface Σ_t, orthogonal to ξ^μ. Spherical symmetry can be defined rigorously in terms of Killing vector fields as follows: A spacetime is said to be spherically symmetric if and only if it admits three linearly independent *spacelike* Killing vector fields X_a whose orbits are closed (i.e. topological circles) obeying a Lie algebra

$$[X_i, X_j] = \epsilon_{ikm} X_m,$$

with spherical representation in (θ, ϕ) as follows

$$\begin{aligned} X_1 &= \sin\phi\frac{\partial}{\partial\phi} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}, \\ X_2 &= \cos\phi\frac{\partial}{\partial\theta} - \cot\phi\sin\phi\frac{\partial}{\partial\phi}, \\ X_3 &= \frac{\partial}{\partial\phi}, \end{aligned}$$

generating the group of motions of the sphere S^2 and a coordinate system exists in which the Killing vectors take on a standard form. Convenient local coordinates on the spacetime $\mathcal{M}$ may be chosen as follows. We select a sphere on $\Sigma = \Sigma_0$, and choose standard spherical coordinates (θ, ϕ) on it. We "carry" these coordinates to the other spheres of Σ by means of geodesics orthogonal to the 2-sphere, we choose (ρ, θ, ϕ) as local coordinates in Σ_t, and finally we choose $\{t, \rho, \theta, \phi\}$ as local *curvature coordinates* for the spacetime, according to the prescription described in Eq. (2.39). the metric on $\mathcal{M}$ in these coordinates is then

$$ds^2 = -f(\rho)dt^2 + h(\rho)d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.43}$$

2.2.1 The Schwarzschild Solution

A now standard textbook calculation, first presented in 1916 by Karl Schwarzschild (at the time he was serving on the Russian front, where he was to die the same year), yields the static, spherically symmetric, asymptotically flat *vacuum solution* of the Einstein field equations $R_{\mu\nu} = 0$, exterior to a spherical source of matter. In curvature coordinates $\{t, \rho, \theta, \phi\}$, with mass parameter M_s describing the spacetime geometry $(\mathcal{M}, g_0)$, outside a spherical source of matter, for $\rho > 2M_s$, we have the famous *Schwarzschild exterior solution*

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2M_s}{\rho}\right)dt^2 + \left(1 - \frac{2M_s}{\rho}\right)^{-1} d\rho^2 + \rho^2 d\Omega^2, \tag{2.44}$$

where $d\Omega^2 \equiv (d\theta^2 + \sin^2\theta d\phi^2)$ is the usual metric on the 2-sphere. From standard arguments in the weak field regime (as $\rho \to \infty$) the invariant scalar M_s, which functions as a parameter, can be interpreted as the conventional Newtonian mass of the gravitating object producing the vacuum Schwarzschild field. Because the metric of (2.44) contains only a single constant of integration M_s, this implies that the metric exterior to the spherical body, parametrised by M_s, is completely independent of the composition of that body and *Birkhoff's theorem: A spherically symmetric vacuum solution in the Schwarzschild exterior spacetime is necessarily static* [2] guarantees the unique spherically symmetric vacuum solution. From (2.44) we see

$$g^{00} = -\left(1 - \frac{2M_s}{\rho}\right)^{-1}, \quad g^{11} = \left(1 - \frac{2M_s}{\rho}\right), \quad g^{22} = \frac{1}{\rho^2}, \quad g^{44} = \frac{1}{\rho^2\sin^2\theta},$$

it follows that $x^0 = t$ is timelike and $x^1 = \rho$ is spacelike as long as $\rho > 2M_s$ and both $x^2 = \theta$ and $x^3 = \phi$ are spacelike. Since the metric is independent of t with no cross term in dt, it follows the solution is static and t is the invariantly defined world time of $(\mathcal{M}, g)$. The coordinate ρ is a radial parameter which has the property that the 2-sphere $t =$ constant, $\rho =$ constant has the standard line element

$$ds_\Omega^2 = \rho^2(d\theta^2 + \sin^2\theta d\phi^2),$$

from which it follows that the $\mathbb{R} \times SO(3, \mathbb{R})$-invariant 2-sphere surface area is

$$A(\rho) = 4\pi\rho^2,$$

and θ and ϕ are usual spherical polar coordinates on the spheres. Thus curvature coordinates $\{x^\mu\}$ are canonical coordinates defined invariantly by the symmetries present. The timelike Killing vector field ξ^μ is hypersurface-orthogonal to the family of hypersurfaces Σ_t where t is constant, it is clear from (2.44) that the $\mathbb{R} \times SO(3, \mathbb{R})$-invariant conformal factor is

$$\alpha^2 = -\xi_\mu\xi^\mu = \left(1 - \frac{2M_s}{\rho}\right) > 0, \qquad \rho > 2M_s. \tag{2.45}$$

The *Schwarzschild vacuum solution* (2.44) abstracted away from any source, for all values of ρ, is degenerate at $\rho = 2M_s$ and at $\rho = 0$. The value $\rho = 2M_s$ is called the *Schwarzschild radius*, it is the null hypersurface $\mathcal{H}^+$ known as the *event horizon*. The hypersurface $\rho = 2M_s$ turns out to be a removable coordinate singularity, indicated by the Riemann invariant

$$R_{\mu\nu\sigma\omega}R^{\mu\nu\sigma\omega} = \frac{48M_s^2}{\rho^6},$$

which is finite at $\rho = 2M_s$. It blows up as $\rho \to 0$, which is an *intrinsic* or *real* singularity. The vacuum solution bifurcates at the horizon 2-sphere, separating the manifold $\mathcal{M}$ into two disconnected components:

$$\text{I.} \quad 2M_s < \rho < \infty \quad \text{and} \quad \text{II.} \quad 0 < \rho < 2M_s,$$

Inside the region II the coordinates t and ρ reverse their character, with t now being spacelike and ρ timelike. It follows that the topology of the Schwarzschild solution is not simply Euclidian. This fact causes fundamental problems, particularly with energy interpretations, in other words, with everything!, when we transfer the curved spacetime radiating K-G system to the Minkowski spacetime in $\{t, r, \theta, \phi\}$. Issues arise in the fully extended $0 < \rho < \infty$ vacuum spacetime $(\mathcal{M}, g)$, for example, where the Killing vector $\xi^\mu = \partial/\partial t$ goes from being timelike to spacelike at the event horizon, effecting *inter alia* our Sobolev-norm energy argument when $\alpha^2 < 0$. We avoid the disaster (Gk. pun!), with impunity by inducing the exterior vacuum with the incompressible perfect fluid stellar source.

2.2.2 The Interior Schwarzschild Solution

> You boil it in sawdust: You salt it in glue:
> You condense it with locusts and tape:
> Still keeping one principal object in view –
> to preserve its symmetrical shape.
> Lewis Carroll [22]

Without loss of generality we choose our gravitating source to be a *white dwarf*. It is a matter of indifference (I just like white dwarfs) insofar as the statement and proof of **Theorem 1** is concerned, i.e., the proof of an L^2-bound on the radiating Klein-Gordon field $\tilde{\psi}$ decaying at asymptotic spatial infinity on the manifold $(\mathcal{M}, g)$ which *is* the vacuum Schwarzschild spacetime $(\mathcal{M}, g_0)$. Following Birkhoff's theorem, a neutral static black hole or naked singularity spacetime or an alternative static spherically symmetric compact matter source would equally induce the vacuum

$(\mathcal{M}, g_0)$. However, with this particular choice of gravitating source as incompressible perfect fluid sphere, we simplify the mathematics in our bid to represent the curved spacetime problem for $\tilde{\psi}(x^i)$ in terms of $u(\mathbf{x})$ on the Minkowski manifold $(\mathbb{R}^4, \eta)$. A manifold endowed with an affine or metric geometry is said to be *maximal* if every geodesic emanating from an arbitrary point of the manifold, either can be extended to infinite values of the affine parameter along the geodesic in both directions or terminates on an intrinsic singularity. If in particular, all geodesics emanating from any point can be extended to infinite values of the affine parameters in both directions, the manifold is said to be *geodesically complete.* Clearly a geodesically complete manifold is maximal. Minkowski spacetime provides a trivial example of a geodesically complete manifold. Neither the Schwarzschild $(\mathcal{M}, g_0)$ nor the Eddington-Finkelstein advanced or retarded extensions therein, is in fact maximal. The Kruskal solution is maximal but contains intrinsic singularities. Our manifold $(\mathcal{M}, g)$, fluid interior and vacuum exterior, is geodesically complete and can be fully mapped into the Minkowski manifold. Real stars evolve, and it may happen that a star eventually collapses, shrinking down to below $\rho = 2M_s$ and further into a singularity, resulting in a static black hole – this dynamical scenario is by no means a necessary endpoint of stellar evolution, and as we will see in the explicit form of the interior tortoise coordinate $r_i(\rho)$, the collapse scenario is proscribed in the geometric setup of this problem as it stands. The appropriate general form of the metric for the spherical fluid star is again in a canonical symmetric form in curvature coordinates $\{t, \rho, \theta, \phi\}$ given by

$$ds^2_{\text{int}} = -e^{2a(\rho)}dt^2 + e^{2b(\rho)}d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{2.46}$$

with the unknown functions $a(\rho)$ and $b(\rho)$, are now required to satisfy the non-vacuum solutions of the full Einstein field equations in the star's *interior*. As shown in Synge [3], the Einstein tensor in curvature coordinates is of the form

$$\begin{aligned}
G_{tt} &= \frac{1}{\rho^2}e^{2(a-b)}\left(2\rho\partial_\rho b(\rho) - 1 + e^{2b}\right)\\
G_{\rho\rho} &= \frac{1}{\rho^2}e^{2(a-b)}\left(2\rho\partial_\rho a(\rho) - 1 - e^{2b}\right)\\
G_{\theta\theta} &= \rho^2 e^{-2b}\left(\partial_\rho^2 a(\rho) + (\partial_\rho a(\rho))^2 - \partial_\rho a(\rho)\partial_\rho b(\rho) + \frac{1}{\rho}[\partial_\rho a(\rho) - \partial_\rho b(\rho)]\right)\\
G_{\phi\phi} &= \sin^2\theta G_{\theta\theta} \qquad (2.47)
\end{aligned}$$

For a perfect fluid model there are no forces between the particles, no heat conduction and no viscosity in the instantaneous rest frame where the components of the energy-momentum $\mathbf{T}$ for a perfect relativistic fluid are given by

$$[T^{(s)}_{\mu\nu}] = \begin{bmatrix} d(\rho) & 0 & 0 & 0 \\ 0 & p(\rho) & 0 & 0 \\ 0 & 0 & p(\rho) & 0 \\ 0 & 0 & 0 & p(\rho) \end{bmatrix},$$

where $d(\rho)$ and $p(\rho)$ are the proper mass density and isotropic pressure respectively. We have the Einstein-matter field equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T^{(s)}_{\mu\nu} \quad \text{or} \quad R_{\mu\nu} = -\kappa(T^{(s)}_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}). \qquad (2.48)$$

Pursuing a similar perturbation argument to that used on the exterior spacetime, we do not have an energy-momentum contribution from weakly interacting $\tilde{\psi}$ in the stellar interior, thus for this model the energy-momentum tensor is given by

$$T^{(s)}_{\mu\nu} = [d_F(\overline{\rho}) + p(\rho)]\, u_\mu u_\nu - p(\rho)g_{\mu\nu}, \quad \text{and} \quad g_{\mu\nu}u^\mu u^\nu = -1. \qquad (2.49)$$

The energy density $d_F(\overline{\rho})$ and pressure $p(\rho)$ will be functions of ρ alone and with the four-velocity pointing in a time-like direction (for a static solution), normalised to $u^\mu u_\nu = -1$. This is the simplest analytic interior solution for a relativistic star, there is no physical justification for the constant density assumption, it is believed that the interiors of dense neutron stars are of nearly uniform density and so it is borderline realistic. Our star is in a state of *hydrostatic equilibrium*, the fluid is at rest and thus we have the following four-velocities u_μ

$$u_t = u_0 = (-g_{00})^{\frac{1}{2}}, \qquad u_\rho = u_1 = 0, \qquad u_\theta = u_2 = 0, \qquad u_\phi = u_3 = 0,$$

and as we will see for $g_{00} < 0$, with the requisite positive energy density component

$$T^{(s)}_{00} = -d_F(\overline{\rho})g_{00} > 0. \tag{2.50}$$

(see Appendix B on *energy conditions*). Following through the analysis we have

$$ds^2_{\text{int}} = -e^{2a(\rho)}dt^2 + \left(1 - \frac{2Gm(\rho)}{\rho}\right)^{-1} d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.51}$$

The component of $g_{\rho\rho}$ is an obvious generalisation of the Schwarzschild case, but the g_{tt} equation, i.e.

$$\frac{1}{\rho^2}e^{-2b}\left(2\rho\partial_\rho a(\rho) + 1 - e^{2b}\right) = 8\pi G\rho \ , \tag{2.52}$$

yields

$$\frac{d}{d\rho}m(\rho) = 4\pi\rho^2 d_F(\rho),$$

which on integration gives

$$m(\rho) = 4\pi \int_0^\rho d_F(\rho)\rho^2 d\rho.$$

Our star extends to a radius ρ_0 at the star's surface, after which we are in Schwarzschild vacuum, or more precisely, we encounter the weakly coupled radiating scalar field

$\tilde{\psi}$. In order that the metrics, exterior and interior, match at the radius ρ_0, the mass parameter M_s is such that

$$M_s = m(\rho_0) = 4\pi \int_0^{\rho_0} d_F(\rho)\rho^2 d\rho, \tag{2.53}$$

which can be (more or less) interpreted as the mass contained within a sphere of radius ρ_0, the star's self-gravitating mass. For the simple and semi-realistic model of an incompressible perfect fluid star: the density $d_F(\overline{\rho})$ is a constant out to the surface of the star, after which it vanishes, i.e.

$$d_F(\rho) = \begin{cases} d_F(\overline{\rho}) & \text{if } 0 < \rho < \rho_0, \\ 0 & \text{if } \rho > \rho_0 > 2M_s. \end{cases}$$

For $\rho > \rho_0$, $d_F(\rho)$ and $p(\rho)$ are both zero of course and the mass parameter, $m(\rho) = m(\rho_0) \equiv M_s$, that is

$$m(\rho) = \begin{cases} \frac{4}{3}\pi d_F(\overline{\rho})\rho^3 & \text{for } \rho \leq \rho_0, \\ \frac{4}{3}\pi d_F(\overline{\rho})\rho_0^3 \equiv M_s & \text{for } \rho > \rho_0. \end{cases}$$

The *Tolman-Oppenheimer-Volkoff* equation of hydrostatic equilibrium (the details of which are not strictly relevant to this work [23] [24]) is given by

$$\frac{d}{d\rho}p(\rho) = -\frac{[d_F(\overline{\rho}) + p(\rho)][Gm(\rho) + 4\pi G\rho^3 p(\rho)]}{\rho[\rho - 2Gm(\rho)]}, \tag{2.54}$$

and relates $p(\rho)$ to $d_F(\rho)$, since $m(\rho)$ is related to $d_F(\rho)$ via (2.53). Integrating (2.54) for a constant density $d_F(\overline{\rho})$ yields

$$p(\rho) = d_F(\overline{\rho}) \left[\frac{\rho_0\sqrt{\rho_0 - 2GM_s} - \sqrt{\rho_0^3 - 2GM_s\rho^2}}{\sqrt{\rho_0^3 - 2GM_s\rho^2} - 3\rho_0\sqrt{\rho_0 - 2GM_s}} \right]. \tag{2.55}$$

Finally we get the metric component $g_{tt} = -e^{2a(\rho)}$

$$e^{a(\rho)} = \frac{3}{2}\left(1 - \frac{2GM_s}{\rho_0}\right)^{1/2} - \frac{1}{2}\left(1 - \frac{2GM_s\rho^2}{\rho_0^3}\right)^{1/2}, \qquad \rho < \rho_0.$$

We see that the pressure increases near the core of the star as expected and also there is zero pressure at the free boundary ρ_0, i.e., $p(\rho_0) = 0$. We can clearly see how, for a star of fixed radius ρ_0, the central pressure $p(0)$ will need to be greater than infinity if its mass exceeds

$$M_{\max} = \frac{4}{9G}\rho_0, \tag{2.56}$$

if we try to squeeze a greater mass than this inside a radius ρ_0, classical General Relativity admits no static solutions; a star that shrinks to such a size must inevitably keep shrinking, eventually forming a black hole. We have shown this result for the rather strong assumption that the density is constant but it continues to hold with this assumption considerably weakened. This is summarised in *Buchdahl's theorem*: *Any reasonable static, spherically symmetric interior solution has* $M_s < \frac{4\rho_0}{9G}$ [25]. This result makes sense; if we imagine that there is some maximum sustainable density in nature, the most massive object would have that density everywhere and we assume as much for the fluid interior solution of our stellar model which could be a *white dwarf*.

2.2.3 The Interior Tortoise Coordinate $r_i(\rho)$

Using the perfect fluid model we are now in a position to consider in more detail the geometry of this interior. Our tortoise coordinate

$$r_e(\rho) = \rho - 2M_s + 2M_s \log(\rho - 2M_s) + k_1 \;, \tag{2.57}$$

is restricted to the exterior region where $\rho > \rho_0 > 2M_s$ and writing $r_e(\rho_0) \equiv r_0$ the exterior tortoise coordinate is sensibly defined in the range $r_0 < r_e(\rho) < \infty$. Following Synge's analysis [3] the complete Schwarzschild field for an incompressible perfect fluid sphere is given by the two metrics:

Interior ($\rho < \rho_0$):

$$ds_i^2 = \left(1 - q\rho^2\right)^{-1} d\rho^2 + \rho^2 d\Omega^2 - \left(\frac{3}{2}\sqrt{1 - q\rho_0^2} - \frac{1}{2}\sqrt{1 - q\rho^2}\right)^2 dt'^2, \tag{2.58}$$

Exterior ($\rho > \rho_0^2$):

$$ds_e^2 = \left(1 - \frac{2M_s}{\rho}\right)^{-1} d\rho^2 + \rho^2 d\Omega^2 - \left(1 - \frac{2M_s}{\rho}\right) dt'^2, \tag{2.59}$$

where we define

$$q \equiv \frac{1}{3}\kappa d_F(\overline{\rho}) \equiv \frac{2GM_s}{\rho_0^3}, \tag{2.60}$$

and we note

$$g_{00}(\rho) = g_{tt}(\rho) = -\left(\frac{3}{2}\sqrt{1 - q\rho_0^2} - \frac{1}{2}\sqrt{1 - q\rho^2}\right)^2 < 0,$$

the metric ds_i^2 is manifestly static (independent of coordinate t), thus admitting the timelike Killing vector $\xi^\mu = \partial/\partial t$. We also note that at the star's surface, where $\rho = \rho_0$, we find

$$e^{2b(\rho)} \mid_{\rho=\rho_0} = \left(1 - \frac{2M_s}{\rho}\right)^{-1} = \left(1 - q\rho^2\right)^{-1}. \tag{2.61}$$

When we effect the transformation

$$\frac{\rho}{r}\tilde{\psi}(\rho, \theta, \phi) = u(\rho, \theta, \phi),$$

to the (t, r)-plane in Minkowski spacetime we need to express this interior metric ds_i^2 in conformally flat form where $r = r_i(\rho)$ is its associated interior tortoise coordinate, so that it can be suitably matched to the conformally flat exterior metric in the (t, r)-plane, expressed in tortoise coordinate $r_e(\rho)$. In this way the (t, r)-light-cone plane in $(\mathbb{R}^4, \eta)$ is fully covered by coordinates $(t, r_i \cup r_e)$. This is to be achieved in the following way: for the interior tortoise $r_i(\rho)$ we write

$$\beta^2 = 1 - q\rho^2 \quad \text{and} \quad c_1 = \frac{3}{2}\sqrt{1 - q\rho_0^2}, \tag{2.62}$$

so that the interior metric may be written

$$\begin{aligned} ds_i^2 &= \frac{d\rho^2}{\beta^2} + \rho^2 d\Omega^2 + (c_1 - \frac{1}{2}\beta)^2 dt^2 \\ &= (c_1 - \frac{1}{2}\beta)^2 \left(\beta^{-2}(c_1 - \frac{1}{2}\beta)^{-2} d\rho^2 - dt^2\right) + \rho^2 d\Omega^2, \end{aligned} \tag{2.63}$$

by choosing

$$dr_i^2 = \beta^{-2}(c_1 - \frac{1}{2}\beta)^{-2} d\rho^2, \tag{2.64}$$

we get

$$ds_i^2 = (c_1 - \frac{1}{2}\beta)^2 \left(-dt^2 + dr_i^2\right) + \rho^2 d\Omega^2. \tag{2.65}$$

The quotient spacetime $(\mathcal{Q}, \tilde{g}_i) \equiv (\mathcal{M}, g_i)/(SO(3, \mathbb{R}))$ is now in conformally flat form with coordinates t and r_i now assigned on quotient manifold $\mathcal{Q}$ such that

$$\tilde{g}_{ab} dy^a dy^b = \alpha^2(r_i)(-dt^2 + dr_i^2), \tag{2.66}$$

in the stellar interior where $\rho < \rho_0$ and $\rho_0 > 2GM_s$, this implies the $\mathbb{R} \times (SO(3, \mathbb{R}))$-invariant

$$\alpha^2(r_i) = -\xi^\mu \xi_\mu = -g_{tt} = \left(\frac{3}{2}\sqrt{1 - q\rho_0^2} - \frac{1}{2}\sqrt{1 - q\rho^2}\right)^2 > 0, \tag{2.67}$$

so $\xi^\mu \xi_\mu < 0$ and the time-like Killing field $\xi^\mu = \partial_t$ in these coordinates. From the condition imposed by equation (2.64) we have

$$\frac{dr_i}{d\rho} = \frac{1}{\beta \cdot (c_1 - \frac{1}{2}\beta)}, \tag{2.68}$$

and with $\beta(\rho) = \sqrt{1 - q\rho^2}$ we can form the integral

$$r_i(\rho) = -\frac{1}{\sqrt{q}} \int \frac{1}{\sqrt{(1-\beta^2)} \cdot (c_1 - \frac{1}{2}\beta)} d\beta, \tag{2.69}$$

using standard substitutions, completing squares, etc., we find

$$\begin{aligned} r_i(\rho) \;&=\; \frac{2}{\sqrt{q}} \cdot \frac{1}{\sqrt{1-4c_1^2}} \left[\log\left(\sqrt{(1-4c_1^2)} \cdot (\beta - 2c_1) \right) \right. \\ &\left. \quad - \log\left(2 \left(\sqrt{(1-4c_1^2)} \cdot \sqrt{1-\beta^2} - 2c_1\beta + 1 \right) \right) \right] + k_2, \end{aligned}$$

with k_2 the arbitrary constant of integration. We solve for k_2 by noting the condition at our coordinate origin

$$r_i(\rho)\,|_{\rho=0} = 0,$$

and using the constants stated in Eq. (2.62) above, we have

$$\begin{aligned} r_i(0) \;&=\; \frac{2}{\sqrt{q(9q\rho_0^2 - 8)}} \cdot \log \left| \frac{\sqrt{(9q\rho_0^2 - 8)} \cdot \left(1 - 3\sqrt{1 - q\rho_0^2}\right)}{2(1 - 3\sqrt{1 - q\rho_0^2})} \right| + k_2 \\ &=\; \frac{1}{\sqrt{q(9q\rho_0^2 - 8)}} \cdot \log \left| \frac{9q\rho_0^2 - 8}{4} \right| + k_2 = 0, \end{aligned} \tag{2.70}$$

and thus we find for the arbitrary constant

$$k_2 = -\frac{1}{\sqrt{qC_s}} \cdot \log \left| \frac{C_s}{4} \right|, \qquad \text{with} \quad C_s = 9q\rho_0^2 - 8. \tag{2.71}$$

The exterior tortoise coordinate at $\rho = \rho_0$ is given by

$$r_e(\rho_0) = \rho_0 + 2M_s \log(\rho_0 - 2M_s) - 2M_s + k_1, \tag{2.72}$$

and matches to the interior tortoise coordinate $r_i(\rho_0)$ there. We have

$$\begin{aligned} r_i(\rho) &= \frac{2}{\sqrt{q(9q\rho_0^2 - 8)}} \cdot \log\left|\frac{\sqrt{(9q\rho_0^2 - 8)} \cdot \left(\sqrt{1 - q\rho^2} - 3\sqrt{1 - q\rho_0^2}\right)}{2(\sqrt{(9q^2\rho_0^2 - 8q)} \cdot \rho - 3\sqrt{1 - q\rho_0^2} \cdot \sqrt{1 - q\rho^2} + 1)}\right| + k_2 \\ &= \frac{2}{\sqrt{q(9q\rho_0^2 - 8)}} \cdot \log\left|\frac{\sqrt{1 - q\rho^2} - 3\sqrt{1 - q\rho_0^2}}{\sqrt{q(9q\rho_0^2 - 8)} \cdot \rho - 3\sqrt{1 - q\rho_0^2} \cdot \sqrt{1 - q\rho^2} + 1}\right|, \end{aligned} \tag{2.73}$$

and also

$$\begin{aligned} r_i(\rho_0) &= \frac{2}{\sqrt{q(9q\rho_0^2 - 8)}} \cdot \log\left|\frac{2\sqrt{1 - q\rho_0^2}}{\sqrt{q(9q\rho_0^2 - 8)} \cdot \rho_0 + 3q\rho_0^2 - 2}\right|, \\ &= \rho_0 - 2M_s + 2M_s \log(\rho_0 - 2M_s) + k_1. \end{aligned} \tag{2.74}$$

We observe here that $r_i(\rho)$ is a *sensible* coordinate, it is real and finite in respect of the condition of Buchdal's theorem as stated in (2.56), i.e., for total fluid mass

$$M_s = M_{\text{max}} < \frac{4}{9G}\rho_0,$$

using (2.60) this implies

$$9\left(\frac{2GM_s}{\rho_0^3}\right)\rho_0^2 - 8 > 0,$$

or in the q-notation of the interior metric

$$9q\rho_0^2 - 8 > 0,$$

thus hydrostatic equilibrium is necessarily maintained –a collapse scenario is proscribed by this model. Realistically the gravitating source could be a maximum density *white dwarf* – supported by electron degeneracy pressure. We easily find the value of the arbitrary constant of (2.74) to be given by

$$k_1 = r_i(\rho_0) + 2M_s - \rho_0 - 2M_s \log(\rho_0 - 2M_s),$$

and thus we now have a full expression for the exterior tortoise coordinate

$$\begin{aligned} r_e(\rho) &= \rho + r_i(\rho_0) - \rho_0 + 2M_s \log\left|\frac{\rho - 2M_s}{\rho_0 - 2M_s}\right| \\ &= r_0 + \rho + 2M_s \log|\rho - 2M_s| \,. \end{aligned} \tag{2.75}$$

We have shown here how it *is* possible to account for the spatial extent of this arbitrary interior model when we map the conformally flat quotient manifold $\mathcal{Q} \equiv \mathcal{M}/SO(3,\mathbb{R})$, where $r = r_i \cup r_e$ to the (t, r)-plane of spacetime $(\mathbb{R}^4, \eta)$. The interior solution is mapped into the compact set B_r for $0 < r \leq 2R_0$. We have presented a sensible tortoise coordinate $r_i(\rho)$ in the interior domain of $\mathcal{M}$, the tortoise exterior coordinate $r_e(\rho)$ covers the exterior domain and with $r_e(\rho_0) = r_i(\rho_0)$ we cover the entire $(\mathbb{R}^4, \eta)$ manifold so that our radiating problem on $\mathcal{M}$ can be represented in Minkowski spacetime. Crucially, as we will see, the spacelike domain where initial datum $f(\mathbf{x})$ lives, can be explicitly defined.

We see also that the Killing field ξ^μ is clearly just ∂_t and the quotient space $\mathcal{Q} \equiv \mathcal{M}/SO(3,\mathbb{R})$ is a two-dimensional Lorentzian sub-manifold whose metric is now in *conformally flat* form, i.e.

$$ds_i^2 = \alpha^2(r_i)(-dt^2 + dr^2).$$

These coordinates specify a conformal mapping from the 2-dimensional pseudo-Riemannian manifold $\mathcal{Q}$, into the flat Minkowski (t, r)-plane; isothermal coordinates can always be introduced on a compact domain of a regular 2-dimensional manifold. Importantly, the causal structure of the spacetime, defined by its light cones is preserved here – the new coordinates must have both a "timelike" and "radial" part. Following the previous discussion on symmetric spacetimes, we know that these coordinates are unique up to translations in the t and r coordinates and rotations in the spherical θ and ϕ coordinates.

The volume form element for the spacetime $\mathcal{M}$, in isothermal coordinates is given by

$$d\mu_g \equiv \sqrt{|g|}d^4x = \alpha^2(r)\rho^2(r)d\Omega drdt. \tag{2.76}$$

We also have the *induced volume* form on the Cauchy space-like hypersurfaces Σ_t, of constant $x^0 \equiv t$ given by

$$d\sigma = \alpha(r)\rho^2(r)d\Omega dr. \tag{2.77}$$

We note two other volume forms on these hypersurfaces that will be useful in this analysis, namely

$$d\sigma^{'} = \alpha(r)d\sigma = \alpha^2(r)\rho^2 d\Omega dr \qquad \text{and} \qquad d\sigma^{''} = \alpha^{-1}(r)d\sigma = \rho^2 d\Omega dr \,. \tag{2.78}$$

The significance of the choice of volume form $d\sigma^{''}$ will become clear in the next section.

Chapter 3

The Elliptic-Helmholtz Problem

3.1 The Well-posed Cauchy Problem for $\psi(x)$

Although we do not deal directly with a Cauchy initial value problem on the static spacetime $(\mathcal{M}, g)$ – our problem presents as a hyperbolic initial value problem for a function $w(t, \mathbf{x})$ on $(\mathbb{R}^4, \eta)$, from which we then ascertain a bound on the transformed field $u(\mathbf{x})$ in $\mathbb{R}^3$. Nonetheless, a careful consideration of the form of a related well-posed Cauchy problem on $(\mathcal{M}, g)$ yields some useful results. Specifically we examine aspects of well-posedness criteria for the originating wave equation $\Box_g \psi(x) = 0$ in particular, with regard to the properties of a self-adjoint Hilbert space operator A, which enters the initial value problem set-up in the form

$$\partial_t^2 \psi(x) + A \cdot \psi(x) = 0, \qquad \psi(0, x) = h(x), \quad \partial_t \psi(0, x) = g(x). \tag{3.1}$$

We will explain the provenance of this form, and the relevance of the properties of operator A will become clear as we proceed through the exposition. Ordinary Cauchy evolution determines a solution of a partial differential equation only within the *domain of dependence* $D(\Sigma)$ of the initial data surface. We see this in an explicit formula for the solution u of the $\mathbb{R}^3$-wave equation

$$u_{tt}(t,\mathbf{x}) = c^2\Delta u(t,\mathbf{x}), \qquad \text{with data on } \Sigma, \quad u(0,\mathbf{x}) = \phi(\mathbf{x}), \quad u_t(0,\mathbf{x}) = \psi(\mathbf{x}),$$

which is given by *Kirchoff's formula*

$$u(t_0,\mathbf{x}_0) = \frac{1}{4\pi c^2 t_0}\int\!\!\int_S \psi(\mathbf{x})dS + \frac{\partial}{\partial t_0}\left[\frac{1}{4\pi c^2 t_0}\int\!\!\int_S \phi(\mathbf{x})dS\right],$$

where S is a sphere of centre $\mathbf{x}_0$ and radius ct_0. From this we see the value of $u(t_0,\mathbf{x}_0)$ depends only on the values of $\psi(\mathbf{x})$ and $\phi(\mathbf{x})$ for $\mathbf{x}$ *on* the spherical surface $S = \{|\mathbf{x}-\mathbf{x}_0|\} = ct_0$ but not on the values of $\psi(\mathbf{x})$ and $\phi(\mathbf{x})$ *inside* the sphere. This statement can be inverted to say that the values of ψ and ϕ at a spatial point $\mathbf{x}_1$ influence the solution only on the surface $S = \{|\mathbf{x}-\mathbf{x}_1|\} = ct$ of the light cone that emanates from $(\mathbf{x}_1, 0)$.

In original work by Wald [18], a physically sensible, fully deterministic dynamical evolution prescription is given for the case of a massless Klein-Gordon field propagating in an arbitrary static spacetime, i.e., one with arbitrary singularities consistent with staticity, in other words, a non-globally hyperbolic spacetime. Wald shows that the problem of defining the dynamics can then be translated into the mathematical problem of finding self-adjoint extensions of the spatial part of the wave operator A, as presented in Eq. (3.1); effectively this amounts to establishing a well-posed *Cauchy problem* for the scalar field. This is a well studied classical problem, and it is known that for positive A, self-adjoint extensions necessarily exist, Wald chooses the natural *Friedrichs extension.* The dynamical evolution prescription is then defined and shown to satisfy the following properties: (1) solutions are uniquely determined in spacetime by their initial Cauchy data, (2) where ordinary dynamical evolution is defined, i.e., in the usual domain of dependence of the initial data surface, the results coincide with the evolution prescription, (3) smooth initial data of compact support yields smooth spacetime solutions. Wald's motivation for this prescription is that if *cosmic censorship* is abandoned, deterministic dynamics is still possible in such non-globally hyperbolic spacetimes.

3.1.1 The Euler-Lagrange Equation for $\psi(x)$

By considering a variational of the action functional on the static spherically symmetric spacetime $(\mathcal{M}, g)$

$$\mathcal{A}_{\psi} = \frac{1}{2}\int_{\mathcal{M}} \mathcal{L} d\mu_g,$$

with Lagrangian density corresponding to the real massless scalar field $\psi(x) : \mathcal{M} \to \mathbb{R}$

$$\mathcal{L} = g^{\mu\nu}\nabla_{\nu}\psi(x)\nabla_{\mu}\psi(x), \tag{3.2}$$

we deduce the Euler-Lagrange equation in the standard way

$$\Box_g \psi(x) = 0.$$

In our chosen isothermal coordinate system $\{t, \rho, \theta, \phi\}$ i.e., with tortoise coordinate $r(\rho)$, we found

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = -\alpha^2(r)dt^2 + \alpha^2(r)dr^2 + \rho^2(r)d\Omega^2,$$

on $\mathcal{M}$, from which we construct our scalar Lagrangian density, i.e.

$$\mathcal{L} = -\alpha^{-2}(r)\psi_t^2 + \alpha^{-2}(r)\psi_r^2 + \rho^{-2}|\nabla\psi|^2, \tag{3.3}$$

where $\nabla\psi$ is the unit-spherical gradient. Thus the action integral introduced in Eq. (2.23), associated with the natural volume measure, given in Eq. (2.76) of Section 2.1.2 by

$$d\mu_g \equiv \sqrt{|g|}d^4x = \alpha^2(r)\rho^2(r)d\Omega drdt, \qquad d\Omega \equiv \sin\phi d\theta d\phi,$$

can now be expressed as

$$\mathcal{A}_\psi = \frac{1}{2}\int \left(-\alpha^{-2}\psi_t^2 + \alpha^{-2}\psi_r^2 + \rho^{-2}|\nabla\psi|^2\right) d\mu_g. \tag{3.4}$$

Performing the δ-variation on the action functional $\mathcal{A}_\psi$, via the spacetime scalar fields $\psi(x)$ in the usual manner, we get

$$\begin{aligned}\delta\mathcal{A}_\psi &= \frac{1}{2}\int_{\mu_g} \left(-\alpha^{-2}\delta\psi_t^2 + \alpha^{-2}\delta\psi_r^2 + \rho^{-2}\delta|\nabla\psi|^2\right) d\mu_g \\ &= \int_{\mu_g} \left(-\alpha^{-2}\psi_t\delta\psi_t + \alpha^{-2}\psi_r\delta\psi_r + \rho^{-2}|\nabla\psi|\delta|\nabla\psi|\right) \alpha^2\rho^2 d\Omega drdt \\ &= \underbrace{\int}_{V_1} -\psi_t\delta\psi_t\rho^2 d\Omega drdt + \underbrace{\int}_{V_2} \psi_r\delta\psi_r\rho^2 d\Omega drdt + \underbrace{\int}_{V_3} |\nabla\psi|\delta|\nabla\psi|\alpha^2 d\Omega drdt.\end{aligned}$$

We remind ourselves of this procedure by calculating these three integrals, V_1, V_2 and V_3 in full. For the V_1- integral we have

$$\psi_t\delta\psi_t = \psi_t\frac{\partial}{\partial t}\delta\psi, \tag{3.5}$$

and so

$$\psi_t\frac{\partial}{\partial t}\delta\psi = \frac{\partial}{\partial t}\left(\psi_t\delta\psi\right) - \delta\psi\psi_{tt}. \tag{3.6}$$

Integrating this expression with respect to the measure $\alpha^{-2}(r)d\mu_g$ and noting that $\delta\psi(x) = 0$, on the hypersurface extremum, but is otherwise arbitrary, we find

$$\int_\Omega\int_r \left(\int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left(\psi_t\delta\psi\right) dt\right) \rho^2 d\Omega dr = \int_\Omega\int_r \left(\psi_t\delta\psi\right)|_{-\infty}^{\infty} \cdot \rho^2 d\Omega dr = 0, \tag{3.7}$$

and thus the V_1- integral is

$$-\int_r\int_\Omega\int_t\left(\frac{\partial}{\partial t}(\psi_t\delta\psi)-\psi_{tt}\cdot\delta\psi\right)\rho^2 d\Omega drdt=\int_{V_1}\psi_{tt}\cdot\delta\psi\rho^2 d\Omega drdt. \tag{3.8}$$

For the V_2- integral we have

$$\rho^2\psi_r\delta\psi_r=\rho^2\psi_r\frac{\partial}{\partial r}\delta\psi, \tag{3.9}$$

so that

$$\rho^2\psi_r\frac{\partial}{\partial r}\delta\psi=\frac{\partial}{\partial r}\left(\rho^2\psi_r\delta\psi\right)-\rho^2\psi_{rr}\delta\psi-2\rho\rho_r\psi_r\delta\psi, \tag{3.10}$$

and we integrate with respect to the measure $drd\Omega dt$

$$\int_t\int_\Omega\int_r\left(\frac{\partial}{\partial r}\left(\rho^2\psi_r\delta\psi\right)-\rho^2\psi_{rr}\delta\psi-2\rho\rho_r\psi_r\delta\psi\right)drd\Omega dt. \tag{3.11}$$

Bearing in mind that we can integrate into the coordinate $r=r_i=0$, in the stellar interior, i.e., from Eq. (2.73), where

$$r(\rho)\mid_{\rho=0}=\frac{2}{\sqrt{q(9q\rho_0^2-8)}}\cdot\log\left|\frac{\sqrt{1-q\rho^2}-3\sqrt{1-q\rho_0^2}}{\sqrt{q(9q\rho_0^2-8)}\cdot\rho-3\sqrt{1-q\rho_0^2}\cdot\sqrt{1-q\rho^2}+1}\right|=0,$$

and as with the previous calculation at the extremum we find

$$\int_r\frac{\partial}{\partial r}\left(\rho^2\psi_r\delta\psi\right)dr=\rho^2\psi_r\delta\psi\mid_{r=0}^{r=\infty}=0, \tag{3.12}$$

giving the V_2 -integral as

$$\int_t\int_\Omega\int_r\left(-\psi_{rr}-\frac{2}{\rho}\rho_r\psi_r\right)\cdot\delta\psi\rho^2 drd\Omega dt. \tag{3.13}$$

For the final V_3 -integral

$$\begin{aligned}\delta|\nabla\psi|^2 &= \delta(\psi_\theta^2)+(\sin^2\theta)^{-1}\delta(\psi_\phi)\\ &= \psi_\theta\delta\psi_\theta+(\sin^2\theta)^{-1}\psi_\phi\delta\psi_\phi,\end{aligned} \tag{3.14}$$

giving integrals with measure $d\Omega = \sin\theta d\theta d\phi$

$$\int_t \int_r \left(\int_\phi \int_\theta \psi_\theta \delta\psi_\theta \sin\theta d\theta d\phi + \int_\theta (\sin^2\theta)^{-1} \int_\phi \psi_\phi \delta\psi_\phi d\phi \sin\theta d\theta \right) \alpha^2 dr dt, \tag{3.15}$$

noting

$$\psi_\theta \delta\psi_\theta = \psi_\theta \frac{\partial}{\partial\theta}\delta\psi = \frac{\partial}{\partial\theta}(\psi_\theta \delta\psi) - \delta\psi\psi_{\theta\theta},$$

and integrating by parts at extremum we find for the first angular integral in Eq. (3.15) above

$$\int_\phi \int_\theta \psi_\theta \delta\psi_\theta \sin\theta d\theta d\phi = [\sin\theta \psi_\theta \delta\psi]_{\theta,\phi} - \int_\phi \int_\theta (\psi_\theta \frac{\cos\theta}{\sin\theta} + \psi_{\theta\theta}) \delta\psi \sin\theta d\theta d\phi,$$

with the boundary term $[\sin\theta\psi_\theta\delta\psi]_{\theta,\phi} = 0$, similarly for the second integral in Eq. (3.15) we find

$$\int_\theta (\sin^2\theta)^{-1} \int_\phi \psi_\phi \delta\psi_\phi d\phi \sin\theta d\theta = [\psi_\theta \delta\psi]_{\theta,\phi} - \int_\theta (\sin^2\theta)^{-1} \int_\phi \psi_{\phi\phi} \delta\psi d\phi \sin\theta d\theta,$$

adding these expressions together under the full integral to get the V_3-integral

$$\int_t \int_r \int_\Omega \left(\frac{1}{\sin\theta} \partial_\theta (\sin\theta \psi_\theta) + \frac{1}{\sin^2\theta} \psi_{\theta\theta} \right) \cdot \delta\psi d\Omega \alpha^2 dr dt. \tag{3.16}$$

Adding all of the above integral contributions, i.e., $\int_{V_1} + \int_{V_2} + \int_{V_3}$, to find

$$\delta\mathcal{A}_\psi = \int_t \int_r \int_\Omega \left(\psi_{tt} - \psi_{rr} - \frac{2}{\rho}\rho_r \psi_r - \frac{\alpha^2}{\rho^2} \not{\Delta}\psi \right) \cdot \delta\psi \rho^2 d\Omega dr dt = 0, \tag{3.17}$$

where

$$\not{\Delta}\psi(x) \equiv \psi_{\theta\theta} + \frac{\cos\theta}{\sin\theta}\psi_\theta + \frac{1}{\sin^2\theta}\psi_{\phi\phi},$$

and because $\delta\psi \neq 0$ in general, we deduce the Euler-Lagrange equation on $(\mathcal{M}, g)$ as

$$\Box_g \psi(x) = \frac{\partial^2 \psi(x)}{\partial t^2} - \frac{1}{\rho^2} \frac{\partial}{\partial r} \left(\rho^2 \frac{\partial \psi(x)}{\partial r} \right) - \frac{\alpha^2}{\rho^2} \not{\Delta}\psi(x) = 0. \tag{3.18}$$

Equation (3.18) emerges, of course, from a direct substitution of the metric coefficients of ds^2 in the decoupled wave equation (2.35) on the manifold $(\mathcal{M}, g)$, derived in Section 2.1.5, with $\sqrt{|g|} = \alpha^2 \rho^2 \sin\theta$, etc., i.e.

$$\begin{aligned}
\Box_g \psi(x) &= (\sqrt{|g|} g^{\mu\nu} \psi_{,\nu})_{,\mu} \\
&= \partial_0(\sqrt{|g|} g^{00} \psi_0) + \partial_1(\sqrt{|g|} g^{11} \psi_1) + \partial_2(\sqrt{|g|} g^{22} \psi_2) + \partial_3(\sqrt{|g|} g^{33} \psi_3) \\
&= -\partial_t(\alpha^2 \rho^2 \sin\theta (\frac{1}{\alpha^2}) \psi_t) + \partial_r(\alpha^2 \rho^2 \sin\theta (\frac{1}{\alpha^2}) \psi_r) \\
&+ \partial_\theta(\alpha^2 \rho^2 \sin\theta (\frac{1}{\rho^2}) \psi_\theta) + \partial_\phi(\alpha^2 \rho^2 \sin\theta (\frac{1}{\rho^2 \sin^2\theta}) \psi_\phi) \\
&= -\psi_{tt} + \frac{1}{\rho^2}(\rho^2 \psi_r)_{,r} + \frac{\alpha^2}{\rho^2 \sin\theta}(\sin\theta \psi_\theta)_{,\theta} + \frac{\alpha^2}{\rho^2 \sin^2\theta} \psi_{\phi\phi}. \qquad (3.19)
\end{aligned}$$

Switching to conventional partial differential notation we expressing the spatial operator above by

$$A \equiv -\frac{1}{\rho^2} \frac{\partial}{\partial r}(\rho^2 \frac{\partial}{\partial r}) - \frac{\alpha^2}{\rho^2} \Delta, \qquad (3.20)$$

so that we may express our massless Klein-Gordon equation on $(\mathcal{M}, g)$ in the concise form

$$\Box_g \psi(x) = \left(\frac{\partial^2}{\partial t^2} + A \right) \cdot \psi(x) = 0. \qquad (3.21)$$

As we discussed in Subsection 3.1, on the related Cauchy problem, the properties of this spatial operator A bear significantly on the analysis of our Helmholtz problem, this will become clearer in the next section.

The symmetric property:

Using routine integration by parts over the hypersurface Σ_t

$$\begin{aligned}
\int_{\Sigma_t} \psi(x) A \phi(x) d\sigma'' &= -\int_\Omega \int_r \psi \left(\frac{1}{\rho^2} \frac{\partial}{\partial r}(\rho^2 \phi_r) + \frac{\alpha^2}{\rho^2} \Delta \phi \right) \rho^2 d\Omega dr \\
&= -\int_\Omega \int_r \psi \frac{\partial}{\partial r}(\rho^2 \phi_r) dr d\Omega - \int_r \alpha^2 \int_\Omega \psi \Delta \phi d\Omega dr. \qquad (3.22)
\end{aligned}$$

For the first part of the integral in Eq. (3.22) above we have

$$\int_r \psi\frac{\partial}{\partial r}(\rho^2\phi_r)dr = (\psi\rho^2\phi_r)\,|_0^\infty - \int_r \phi_r\psi_r\rho^2 dr = -\int_r \phi_r\psi_r\rho^2 dr, \tag{3.23}$$

with boundary term zero this is clearly symmetric under $\phi(x) \leftrightarrow \psi(x)$ interchange. For the second part of the integral in Eq. (3.22) we have for the function $\tilde{\phi}(x) \in C_0^\infty(\Sigma_t)$

$$\int_\Omega \psi(x)\not{\Delta}\tilde{\phi}(x)d\Omega = \int_\phi\int_\theta\left(\frac{\psi}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\tilde{\phi}}{\partial\theta} + \frac{\psi}{\sin^2\theta}\frac{\partial^2\tilde{\phi}}{\partial\phi^2}\right)\sin\theta d\theta d\phi. \tag{3.24}$$

Integrating Eq. (3.24) by parts, again we find

$$= (\psi\sin\theta\frac{\partial\tilde{\phi}}{\partial\theta})\Big|_0^\pi + (\psi\frac{\partial\tilde{\phi}}{\partial\phi})\Big|_0^{2\pi} - \int_\phi\int_\theta\left(\frac{\partial\tilde{\phi}}{\partial\theta}\frac{\partial\psi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial\tilde{\phi}}{\partial\phi}\frac{\partial\psi}{\partial\phi}\right)\sin\theta d\theta d\phi, \tag{3.25}$$

the first two terms here are zero and so the integral of Eq. (3.25) is again clearly symmetric under the $\tilde{\phi}(x) \leftrightarrow \psi(x)$ interchange, and so for the entire integral.

The positive definite property:

We simply replace $\phi(x)$ (or $\tilde{\phi}(x)$) with $\psi(x)$ in the previous integrals of Eq. (3.23) and Eq. (3.25) above, to find

$$\begin{aligned}
\int_{\Sigma_t}\psi(x)A\psi(x)d\sigma'' &= \int_r\int_\phi\int_\theta\left(\rho^2\psi_r^2 + \alpha^2(\psi_\theta^2 + \frac{1}{\sin^2\theta}\psi_\phi^2)\right)\sin\theta d\theta d\phi dr \\
&= \int_r\int_\phi\int_\theta\left(\psi_r^2 + \frac{\alpha^2}{\rho^2}(\psi_\theta^2 + \frac{1}{\sin^2\theta}\psi_\phi^2)\right)\rho^2\sin\theta d\theta d\phi dr \\
&= \int_{\Sigma_t}\left(\psi_r^2 + \frac{\alpha^2}{\rho^2}|\not{\nabla}\psi|^2\right)d\sigma'' \geq 0,
\end{aligned} \tag{3.26}$$

with equality only for $\psi(x) \equiv 0$.

In summary we have demonstrated the symmetric and positive definite property of the extended operator A, sometimes written as A_E, with respect to the Hilbert space inner product of L^2-functions, associated with the modified induced volume form $d\sigma'' = \alpha^{-1}(r)d\sigma$, i.e.

$$\langle \psi(x), A\phi(x) \rangle_{L^2(\Sigma, \alpha^{-1} d\sigma)} \equiv \langle \phi(x), A\psi(x) \rangle_{L^2(\Sigma, \alpha^{-1} d\sigma)},$$

and

$$\langle \psi(x), A\psi(x) \rangle_{L^2(\Sigma, \alpha^{-1} d\sigma)} \geq 0. \tag{3.27}$$

The simplest way to see why we make this choice of measure is to note, as in Wald's paper [16], that equation (3.21) can be written as the $\mathcal{H}$-space inner product

$$\langle \psi, A\chi \rangle_{L^2(\Sigma, \alpha^{-1} d\sigma)} = \langle \chi, A\psi \rangle,$$

and also

$$\langle \psi, A\psi \rangle_{L^2(\Sigma, \alpha^{-1} d\sigma)} = \int_{\Sigma_t} \left(\psi_r^2 + \frac{\alpha^2}{\rho^2} |\nabla \psi|^2 \right) d\sigma'' = Q_A(\psi) \geq 0. \tag{3.28}$$

In the notation of Wald [16] the operator

$$A \equiv -\frac{1}{\rho^2} \frac{\partial}{\partial r}(\rho^2 \frac{\partial}{\partial r}) - \frac{\alpha^2}{\rho^2} \Delta \equiv -\alpha D^a(\alpha D_a),$$

i.e.

$$\partial_t^2 \psi = \alpha D^a(\alpha D_a \psi),$$

and acts on the Hilbert space $\mathcal{H}$ of square integrable functions on hypersurface Σ with $\alpha^2(r) = -\xi^\mu \xi_\mu$, where ξ^μ is the static Killing field with Killing parameter t and D_a denotes the derivative operator on the hypersurface Σ. In this notation we show that operator A is positive and self-adjoint on $L^2(\Sigma, \alpha^{-1}d\sigma)$:

$$\begin{aligned}
\langle \phi, A\psi \rangle_{L^2(\Sigma,\alpha^{-1}d\sigma)} &= -\int_\Sigma \phi D^a(\alpha D_a \psi) d\sigma \\
&= \int_\Sigma \alpha D^a \phi D_a \psi d\sigma \\
&= \int_\Sigma \alpha D_a \phi D^a \psi d\sigma \\
&= -\int_\Sigma D^a(\alpha D_a \phi) \psi d\sigma \\
&= \langle A\phi, \psi \rangle_{L^2(\Sigma,\alpha^{-1}d\sigma)},
\end{aligned} \tag{3.29}$$

and this will only work with measure $d\sigma'' = \alpha^{-1}d\sigma$. It is then a classical result [27] that equation (3.21) is well-posed if the initial domain of A consists of sufficiently smooth functions. The quantity

$$Q_A(\psi) \equiv \langle \psi, A\psi \rangle_{L^2(\Sigma,\alpha^{-1}d\sigma)} \equiv \int_{\Sigma_t} \left(\psi_r^2 + \frac{\alpha^2}{\rho^2} |\nabla \psi|^2 \right) d\sigma'' > 0, \tag{3.30}$$

is called the quadratic form and assists the construction of the appropriate metric norms for the problem, which we will see in the next section.

3.1.2 Sobolev Energy Norms

We previously defined the canonical energy-momentum tensor of the massless Klein-Gordon field $\psi(x)$ by

$$T_{\mu\nu}^{(\psi)} \equiv \nabla_\mu \psi \nabla_\nu \psi - \frac{1}{2} g_{\mu\nu} \mathcal{L}(\psi, \partial_\mu \psi), \tag{3.31}$$

associated with the Lagrangian density

$$\mathcal{L} = \alpha^{-2}(r)\psi_t^2 + \alpha^{-2}(r)\psi_r^2 + \rho^{-2}(r)|\nabla \psi|^2, \tag{3.32}$$

using these quantities we form the following tensor component

$$T_{00} = \frac{1}{2}\psi_t^2 + \frac{1}{2}\psi_r^2 + \frac{\alpha^2(r)}{2\rho^2}|\nabla\psi|^2 > 0, \tag{3.33}$$

T_{00} is clearly positive in $(\mathcal{M}, g)$ and in classical field theory is usually interpreted as an *energy density*, whose integration over an entire volume form is interpreted as the total *energy* of the propagating spacetime scalar field $\psi(x)$. If we choose to integrate over the volume form, in this case given by $d\sigma'' \equiv \rho^2 d\Omega dr$, we may then form the scalar integral

$$\mathcal{E}(\psi) = \int_{\sigma''} T_{00}\rho^2 d\Omega dr = \frac{1}{2}\int_{\sigma''}\left(\psi_r^2 + \frac{\alpha^2}{\rho^2}|\nabla\psi|^2\right)\rho^2 d\Omega dr + \frac{1}{2}\int_{\sigma''}\psi_t^2\rho^2 d\Omega dr. \tag{3.34}$$

The form of these integrals motivates the construction of the following normed *Sobolev function spaces* on the space-like hypersurfaces Σ_t, associated with the induced volume form $d\sigma''$ which are defined as follows:

$\mathbf{H^1}(\Sigma_t)$: Denotes the completion of smooth compactly supported functions f on the hypersurface Σ_t with respect to the norm:

$$\|f\|^2_{\mathbf{H^1}(\Sigma_t)} \equiv \int_{\Sigma_t} |f_r|^2 + \frac{\alpha^2}{\rho^2}|\nabla f|^2 d\sigma'', \tag{3.35}$$

$\mathbf{H^0}(\Sigma_t)$: Denotes the completion of smooth compactly supported functions on Σ_t with respect to the norm:

$$\|f\|^2_{\mathbf{H^0}(\Sigma_t)} \equiv \int_{\Sigma_t} |f|^2 d\sigma''. \tag{3.36}$$

We note that the quadratic form introduced in Eq. (3.30), can be expressed in terms of the Sobolev norm of Eq. (3.35) as follows

$$Q_A(\psi) \equiv \int_{\Sigma_t}\left(\psi_r^2 + \frac{\alpha^2}{\rho^2}|\nabla\psi|^2\right) d\sigma'' \equiv \langle\psi, A\psi\rangle_{L^2(\Sigma,\alpha^{-1}d\sigma)} \equiv \|\psi\|^2_{\mathbf{H^1}(\Sigma_t)}. \tag{3.37}$$

The Sobolev function spaces are discussed in more detail in the Appendix A. Moreover, using these Sobolev norms with Eq. (3.34) above we may form, what I will tentatively call an *energy norm* and define here by

$$\mathcal{E}(\psi) \equiv \frac{1}{2}\left(\|\psi\|^2_{\mathbf{H}^1(\Sigma_t)} + \|\psi_t\|^2_{\mathbf{H}^0(\Sigma_t)}\right) = \int_{\Sigma_t} T_{00} d\sigma'' > 0. \tag{3.38}$$

Differentiating the quantity $\mathcal{E}(\psi)$, formed by Eq. (3.38), with respect to the local time parameter t, i.e.

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{E}(\psi) &= \frac{1}{2}\frac{\partial}{\partial t}\left(\|\psi\|^2_{\mathbf{H}^1(\Sigma_t)} + \|\psi_t\|^2_{\mathbf{H}^0(\Sigma_t)}\right) \\
&= \frac{1}{2}\frac{\partial}{\partial t}\left(\langle\psi, A\psi\rangle_{L^2(\Sigma,\alpha^{-1}d\sigma)} + \langle\psi_t,\psi_t\rangle_{L^2(\Sigma,\alpha^{-1}d\sigma)}\right) \\
&= \langle\psi_t, A\psi\rangle + \langle\psi_{tt},\psi_t\rangle \\
&= \langle\psi_{tt} + A\psi, \psi_t\rangle \\
&= \langle 0, \psi_t\rangle = 0.
\end{aligned} \tag{3.39}$$

This scalar quantity defined by $\mathcal{E}(\psi)$ in Eq.(3.34) above, is thus shown to be conserved under the flow of the timelike Killing field $\frac{\partial}{\partial t}$ in $\mathcal{M}$, and following *Noether's theorem* we identify this positive conserved quantity $\mathcal{E}(\psi)$ with the total energy of the Klein-Gordon field $\psi(x)$. We see our choice of measure $d\sigma''$ also assists in our definition of appropriate Sobolev norms on the spacelike hypersurfaces Σ_t above.

3.2 The Radiating Problem on $(\mathbb{R}^4, \eta)$

For time dependence $\psi(x) = e^{zt}\tilde{\psi}(x^i)$ in curvature coordinates $\tilde{\psi}(x^i) \equiv \tilde{\psi}(\rho,\theta,\phi)$ our hyperbolic K-G equation

$$\frac{\partial^2\psi}{\partial t^2} + A\cdot\psi = 0 \qquad \text{where} \qquad A \equiv -\frac{1}{\rho^2}\frac{\partial}{\partial r}(\rho^2\frac{\partial}{\partial r}) - \frac{\alpha^2}{\rho^2}\not{\Delta}, \tag{3.40}$$

becomes an elliptic-Helmholtz equation for $\tilde{\psi}(x^i)$ on $(\mathcal{M}, g)$, expressed as

$$z^2\tilde{\psi}(x^i) + A\tilde{\psi}(x^i) = 0. \tag{3.41}$$

Our strategy here is first to transform the curved spacetime equation (3.41) on $(\mathcal{M}, g)$ to an equation we can then interpret on Minkowski $(\mathbb{R}^4, \eta)$. In this way we will be able to apply some analysis developed by Stalker and Tahvildar-Zadeh [5] for $(\mathbb{R}^4, \eta)$, in order to obtain the desired bound on the radiating field $u(\mathbf{x})$ in $\mathbb{R}^3$. This strategy is of course possible only if the manifold $\mathcal{M}$ has at least the same topology as $\mathbb{R}^4$. We assume this to be the case and denote the local coordinates on $\mathbb{R}^4$ by the same letters as those on $\mathcal{M}$, namely $(t, r, \Omega) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^2$ and a time slice hypersurface in $\mathbb{R}^4$ is again denoted by Σ_t. We now define a spacetime function, $u(t, \mathbf{x}) : \mathbb{R}^4 \to \mathbb{R}$, by the following radial transformation:

$$u(t, r, \Omega) \equiv \frac{\rho(r)}{r}\psi(t, r, \Omega), \tag{3.42}$$

with ρ and $r = |\mathbf{x}|$ having their usual interpretation as areal and tortoise coordinate respectively, $\Omega(\theta, \phi)$ are the usual $\mathbb{S}^2$ angular coordinates. Using this transformation in Eq. (3.41) we derive an alternative flat spacetime Euler-Lagrange equation for the field $u(\mathbf{x})$. After a careful consideration of the interior Schwarzschild geometry $(\mathcal{M}, g_i)$, we can modify the radiating Helmholtz problem for $u(\mathbf{x})$ in $\mathbb{R}^3$ such that $0 \leq |\mathbf{x}| < \infty$ and thus transform the radiating problem to a well-posed hyperbolic initial value problem on the entire flat Minkowski spacetime $(\mathbb{R}^4, \eta)$.

3.2.1 The Euler-Lagrange Equation for $u(t, \mathbf{x})$

We have the Euler-Lagrange equation for $\psi(x)$ from Eq. (3.19), which we express in terms of the transformed field $u(t, \mathbf{x})$ on the spacetime $(\mathcal{M}, g)$ with volume form

$\sqrt{|g|} = \alpha^2(r)\rho^2(r)\sin\theta$, etcetera, i.e.

$$\begin{aligned}
\Box_g\left(\frac{r}{\rho}u(t,\mathbf{x})\right) &= \frac{\partial}{\partial x^\mu}\left[\sqrt{|g|}g^{\mu\nu}\left(\frac{r}{\rho}u\right)_{,\nu}\right] \\
&= -\partial_t\left[\sqrt{|g|}\left(\frac{1}{\alpha^2}\right)\left(\frac{r}{\rho}u\right)_{,t}\right] + \partial_r\left[\sqrt{|g|}\left(\frac{1}{\alpha^2}\right)\left(\frac{r}{\rho}u\right)_{,r}\right] \\
&+ \partial_\theta\left[\sqrt{|g|}\left(\frac{1}{\rho^2}\right)\left(\frac{r}{\rho}u\right)_{,\theta}\right] + \partial_\phi\left[\sqrt{|g|}\left(\frac{1}{\rho^2\sin^2\theta}\right)\left(\frac{r}{\rho}u\right)_{,\phi}\right],
\end{aligned}$$

and an easy calculation then gives

$$\frac{\partial^2}{\partial t^2}u(t,\mathbf{x}) + \frac{1}{r\rho}\frac{\partial}{\partial r}\left[\rho^2\partial_r\left(\frac{r}{\rho}u(t,\mathbf{x})\right)\right] + \frac{\alpha^2}{\rho^2}\Delta u(t,\mathbf{x}) = 0. \tag{3.43}$$

We need to massage this clumsy radial second term into something resembling the flat spherically symmetric radial Laplacian. To this end we write

$$\begin{aligned}
\frac{1}{r\rho}\partial_r\left[\rho^2\partial_r\left(\frac{ru}{\rho}\right)\right] &= \frac{1}{r\rho}\partial_r\left[\rho r\partial_r u + \rho^2 u\partial_r(\frac{r}{\rho})\right] \\
&= \frac{r}{\rho}\left[\frac{1}{r^2}\partial_r\left(r^2\partial_r u\right)\frac{\rho}{r} + \partial_r u\partial_r\left(\frac{r}{\rho}\right)\right] + 2^{nd}\text{ term.}
\end{aligned} \tag{3.44}$$

It is clear enough that our radial Laplacian is contained within the derivatives of the first term and so we write

$$\frac{r}{\rho}\left[\frac{1}{r^2}\partial_r\left(r^2\partial_r u\right)\frac{\rho}{r} + \partial_r u\partial_r\left(\frac{r}{\rho}\right)\right] + 2^{nd}\text{ term} = \frac{1}{r^2}\partial_r\left(r^2\partial_r u\right) + \frac{r}{\rho}\partial_r u\partial_r\left(\frac{\rho}{r}\right) + 2^{nd}\text{ term.} \tag{3.45}$$

We now have the flat radial Laplacian plus something else, which an easy calculation reveals to be simply

$$\frac{r}{\rho}\partial_r u\partial_r(\frac{\rho}{r}) + \frac{1}{r\rho}\partial_r\left[\rho^2 u\partial_r(\frac{r}{\rho})\right] = \frac{\rho''(r)}{\rho(r)}u \equiv V(r)u, \tag{3.46}$$

and so we can write our Euler-Lagrange equation for u as the flat radial wave equation with an angular term and a radial "potential" V(r), that is

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial r^2} - \frac{2}{r}\frac{\partial u}{\partial r} - \frac{\alpha^2(r)}{\rho^2}\not\Delta u + V(r)u = 0 \; . \tag{3.47}$$

Writing

$$B \equiv -\frac{1}{r^2}\partial_r\left(r^2\partial_r\right) - \frac{\alpha^2(r)}{\rho^2(r)}\not\Delta + V(r), \tag{3.48}$$

we can express the wave equation (3.47) more compactly by

$$\frac{\partial^2}{\partial t^2}u(t,\mathbf{x}) + B \cdot u(t,\mathbf{x}) = 0. \tag{3.49}$$

With the assumed harmonic time dependence of our scalar field, and abusing notation a little, by having the same letter u describe both the time dependent and time harmonic field, so that $u(t,\mathbf{x}) \equiv e^{zt}u(\mathbf{x})$ and thus $u_{tt} = z^2e^{zt}u(\mathbf{x})$, we can express the hyperbolic equation (3.47) as an elliptic equation, in terms of the *flat* Laplacian, as follows

$$-\Delta u(\mathbf{x}) + \left(\frac{1}{r^2} - \frac{\alpha^2}{\rho^2}\right)\not\Delta u(\mathbf{x}) + V(r)u(\mathbf{x}) + z^2u(\mathbf{x}) = 0, \qquad \zeta = \text{Re } z > 0. \tag{3.50}$$

We note the emergence of the "potential" $V(r)$ as a relic of the spherically symmetric spacetime curvature, this is the *intrinsic Gaussian curvature* of the 2-sphere of radius $\rho(r)$ with metric $d\rho^2 + \rho^2 d\Omega^2$, and determinant $|g_2|$, and can be expressed in terms of the Schwarzschild Riemann curvature tensor $R_{\theta\phi\theta\phi}$ by

$$\frac{\rho''(r)}{\rho(r)} = \frac{1}{|g_2|}R_{\theta\phi\theta\phi}.$$

A simple calculation for the spherically symmetric potential (required later) gives

$$V(r) = \frac{\rho''(r)}{\rho(r)} = \frac{2M_s}{\rho^3}\left(1 - \frac{2M_s}{\rho}\right) > 0, \qquad \text{where} \qquad \rho \geq \rho_0 > 2M_s. \tag{3.51}$$

We have now explicit representations for the A and B operators of the wave equations in their respective $\psi(x)$ and $u(t, \mathbf{x})$ representations, i.e.

$$\begin{cases} \psi_{tt}(x) + A \cdot \psi(x) & = 0, \\ u_{tt}(t, \mathbf{x}) + B \cdot u(t, \mathbf{x}) & = 0, \end{cases} \tag{3.52}$$

because

$$\int_{\Sigma_t} \psi \psi_{tt} d\sigma'' = \int_{\Sigma_t} \frac{r}{\rho} u \frac{\partial^2}{\partial t^2} \left(\frac{r}{\rho} u \right) \rho^2 dr d\Omega = \int_{\Sigma_t} u \frac{\partial^2 u}{\partial t^2} r^2 dr d\Omega,$$

this implies the following equivalence on the respective spacelike hypersurfaces Σ_t

$$\int_{\Sigma_t} (\psi \psi_{tt} + \psi A \psi) d\sigma'' = \int_{\Sigma_t} (u u_{tt} + u B u) d\mathbf{x} = 0, \tag{3.53}$$

and we easily deduce the equivalence of their quadratic forms, that is

$$Q_A(\psi) \equiv \int_{\Sigma_t} \psi A \psi d\sigma'' = \int_{\Sigma_t} u B u d\mathbf{x} \equiv Q_B(u), \tag{3.54}$$

alternatively we may write

$$Q_A(\psi) \equiv \langle \psi, A\psi \rangle_{L^2(\Sigma, \alpha^{-1} d\sigma)} \equiv \langle\, u, Bu \rangle_{L^2(\mathbb{R}^3)} \equiv Q_B(u), \tag{3.55}$$

and integrating by parts for

$$B = -\Delta + (\frac{1}{r^2} - \frac{\alpha^2}{\rho^2}) \not{\Delta} + V(r), \tag{3.56}$$

with $\mathbb{R}^3$-Euclidian measure $d\mathbf{x} = r^2 dr d\Omega$, the quadratic form $Q_B(u)$ is deduced as follows

$$\begin{aligned} Q_B(u) &= -\int_{\Sigma_t} u \frac{1}{r^2} \partial_r \left(r^2 \partial_r u \right) r^2 dr d\Omega - \int_{\Sigma_t} \frac{\alpha^2}{\rho^2} u \not{\Delta} u r^2 dr d\Omega + \int_{\Sigma_t} V(r) u^2 r^2 dr d\Omega \\ &= \int_{\Sigma_t} \left(u_r^2 + \frac{\alpha^2}{\rho^2} |\not{\nabla} u|^2 + V(r) u^2 \right) d\mathbf{x}. \end{aligned} \tag{3.57}$$

Since the quadratic forms corresponding to A and B are equivalent, we can easily identify self-adjointness and the Sobolev space metrics based on them, which we do in the next section.

3.3 L^2-Decay for the Scalar Field

We have established the equivalence of the quadratic forms $Q_A(\psi)$ and $Q_B(u)$, corresponding to the operators A and B in the wave equations (3.52) and we can also identify their self-adjoint extensions A_E and B_E, and consequently the Sobolev space norms based on them. Such self -adjoint extensions are guaranteed to exist for positive, real, symmetric operators if their initial domains consist of sufficiently smooth functions, and in particular for smooth functions of compact support on Σ, these are then *well-posed* problems [26]. For

$$\begin{aligned}\int_{\Sigma_t} u(\mathbf{x}) B_E u(\mathbf{x}) d\mathbf{x} &= \int_{\Sigma_t} u(\mathbf{x}) \left(-\Delta + (\frac{1}{r^2} - \frac{\alpha^2}{\rho^2}) \not{\Delta} + V(r) \right) u(\mathbf{x}) d\mathbf{x} \\ &= \left\| \left(-\Delta + (\frac{1}{r^2} - \frac{\alpha^2}{\rho^2}) \not{\Delta} + V \right)^{1/2} u(\mathbf{x}) \right\|_{L^2(\mathbb{R}^3)}, \end{aligned} \tag{3.58}$$

we define the $\mathcal{H}^1$-norm by

$$\|u(\mathbf{x})\|_{\mathcal{H}^1} \equiv \|(B_E)^{1/2} u(\mathbf{x})\|_{L^2(\mathbb{R}^3)}. \tag{3.59}$$

Let $\mathcal{H}^s$ denote, what I will call here, the B-based Sobolev space, defined as the completion of smooth compactly supported functions on $\mathbb{R}^3/\{0\}$ with respect to the $\mathcal{H}^s$-norm given by

$$\|u(\mathbf{x})\|_{\mathcal{H}^s} \equiv \|(B_E)^{s/2} u(\mathbf{x})\|_{L^2(\mathbb{R}^3)}. \tag{3.60}$$

The $\mathcal{H}^s$-norm is defined by interpolation for $0 < s < 1$ between $\mathcal{H}^1$-norm and the $\mathcal{H}^0 \equiv L^2$-norm and we define the $\mathcal{H}^s$-norm for $-1 \leq s < 0$ by duality. We provide a detailed proof of this in Appendix A. For $s = 1$ we have from the quadratic form $Q_B(u)$ that

$$\|u(\mathbf{x})\|^2_{\mathcal{H}^1} \equiv \int_{\mathbf{x}\in\mathbb{R}^3} \left(|u_r(\mathbf{x})|^2 + \frac{\alpha^2}{\rho^2} |\nabla\!\!\!/ u(\mathbf{x})|^2 + V(r)|u(\mathbf{x})|^2 \right) d\mathbf{x}. \tag{3.61}$$

3.3.1 Asymptotics and Bounds

Asymptotics of radial coordinates r and ρ:

By considering the potential $V(r)$ in the range $0 \leq r(\rho) < \infty$ and letting $V(0) = 0$ and recalling the exterior tortoise coordinate as an increasing function of ρ, i.e.

$$r(\rho) = r_0 + \rho + 2M_s \log|\rho - 2M_s|, \qquad \rho > \rho_0 > 2M_s,$$

so that in the exterior Schwarzschild $(\mathcal{M}, g_0)$ we have

$$V(r) = \frac{\rho''(r)}{\rho(r)} = \frac{2M_s}{\rho^3}\left(1 - \frac{2M_s}{\rho}\right), \qquad \rho > \rho_0 > 2M_s, \tag{3.62}$$

in the asymptotic limit, as $\rho \to \infty$ then $r(\rho) \to \infty$ and we may express the following bounds

$$0 < V(r) = \frac{2M_s}{\rho^3}(1 - \frac{2M_s}{\rho}) \leq \frac{2M_s}{\rho^3} \leq \frac{C_1}{r^2}, \tag{3.63}$$

and likewise

$$0 < \frac{\alpha^2(r)}{\rho^2(r)} = \frac{1}{\rho^2}(1 - \frac{2M_s}{\rho}) \leq \frac{C_2}{r^2}. \tag{3.64}$$

For convenience we write the coefficient of $\Delta\!\!\!\!/$ as

$$F(r) \equiv \frac{1}{r^2} - \frac{\alpha^2(r)}{\rho^2(r)}, \tag{3.65}$$

we now find asymptotics on this radial function $F(r)$ which may illuminate our final result. We are only interested in the case where both r and ρ are considered *large*. We note the exterior tortoise coordinate

$$\begin{aligned} r(\rho) &= r_0 + \rho + 2M_s \ln(\rho/2M_s - 1), \\ &\approx \rho + 2M_s \ln(\rho/2M_s), \end{aligned}$$

as $\rho \to \infty$,

$$\frac{2M_s}{\rho} \to 0,$$

and

$$2M_s\rho \log(\rho/2M_s - 1)/\rho \to 0.$$

Hence we find

$$\log(1 - 2M_s/\rho) \to 0,$$

$$\frac{r}{\rho} = 1 + 2M_s \log(\rho/2M_s - 1)/\rho \to 1,$$

$$\log(r/\rho) \to 0,$$

and

$$\frac{r}{\rho} \to 0.$$

But we have

$$\rho - r + 2M_s \log(r/2M_s) = 2M_s \log(\rho/r) + 2M_s \log(1 - 2M_s/\rho),$$

so that

$$\rho - r + 2M_s \log(r/2M_s) \to 0.$$

We also have $\alpha = \sqrt{1 - 2M_s/\rho}$, so that

$$\begin{aligned} F(r) &= \frac{1}{r^2} - \frac{\alpha^2}{\rho^2} \\ &= \frac{1}{r^2} - \frac{1}{\rho^2}\left(1 - \frac{2M_s}{\rho}\right) \\ &= \frac{\rho^2 - r^2 + 2M_s\rho}{r^2\rho^2}. \end{aligned}$$

As shown above

$$\rho = r - 2M_s \log(r/2M_s) + \delta(r),$$

where $\delta(r) \to 1$. Now

$$\frac{1}{r^2} - \frac{\alpha^2}{\rho^2} = -\left(\frac{r}{\rho}\right)^2 \frac{4M_s \log(r/4M_s) + u(r)}{r^3},$$

where

$$\begin{aligned} u(r) &= 4M_s^2 \frac{\log^2(r/2M_s)}{r} + \frac{2M_s\rho}{r} \\ &\quad + \left[2 - 4M_s \frac{\log(r/2M_s)}{r} + \frac{\delta(r)}{r}\right] \delta(r). \end{aligned}$$

As $\rho \to \infty$

$$u(r) \to 2M_s,$$

and

$$\frac{u(r)}{\log(r/4M_s)} \to 0,$$

so

$$\frac{1/r^2 - \alpha^2/\rho^2}{r^{-3}\log(r/4M_s)} = -4M_s,$$

in other words

$$F(r) = \frac{1}{r^2} - \frac{\alpha^2}{\rho^2} = -\frac{4M_s}{r^3} \log\left(\frac{r}{4M_s}\right) + \text{error}, \tag{3.66}$$

and our $F(r)$ asymptotic follows

$$\lim_{r\to\infty} F(r) \;=\; \frac{1}{r^2} - \frac{\alpha^2}{\rho^2} = -\frac{4M_s}{r^3}\log\left(\frac{r}{4M_s}\right). \tag{3.67}$$

Bounds on u($\mathbf{x}$):

We use these bounds in the $\mathcal{H}^1$-norm as follows

$$\|u(\mathbf{x})\|^2_{\mathcal{H}^1} \equiv \int_{\mathbf{x}\in\mathbb{R}^3}\left(|u_r(\mathbf{x})|^2 + \frac{\alpha^2}{\rho^2}|\nabla\!\!\!/ u(\mathbf{x})|^2 + V(r)|u(\mathbf{x})|^2\right) d\mathbf{x}$$

$$\begin{aligned}
\|u(\mathbf{x})\|^2_{\mathcal{H}^1} &\leq \int_{\mathbb{R}^3}\left(|u_r|^2 + \frac{\alpha^2(r)}{\rho^2(r)}|\nabla\!\!\!/ u|^2 + V_{max}(r)u^2\right) d\mathbf{x} \\
&\leq \int_{\mathbb{R}^3}\left(|u_r|^2 + \frac{C_2}{r^2}|\nabla\!\!\!/ u|^2 + \frac{C_1}{r^2}u^2\right) d\mathbf{x} \\
&\leq C\int_{\mathbb{R}^3}\left(|u_r|^2 + \frac{1}{r^2}|\nabla\!\!\!/ u|^2 + \frac{1}{r^2}u^2\right) d\mathbf{x} \\
&\leq C\int_{\mathbb{R}^3}\left(|\nabla u|^2 + \left(\frac{u}{r}\right)^2\right) d\mathbf{x},
\end{aligned} \tag{3.68}$$

with C being used here to label the generic bound on this integral.

We now define the ancillary Sobolev norms $\dot{H}^s$ in the following way

$$\begin{aligned}
\|u\|_{\dot{H}^s} &\equiv \left\|-\Delta^{\frac{s}{2}}u\right\|_{L^2(\mathbb{R}^3)} \\
&= \left\langle -\Delta^{\frac{s}{2}}u, -\Delta^{\frac{s}{2}}u\right\rangle_{L^2(\mathbb{R}^3)} \\
&= \left\langle -\Delta^{s}u, u\right\rangle_{L^2(\mathbb{R}^3)}
\end{aligned} \tag{3.69}$$

In the $\dot{H}^1$ case, for example, we have the differential identity:

$$u\Delta v = \nabla \cdot (u\nabla v) - (\nabla u) \cdot (\nabla v), \tag{3.70}$$

integrating over a domain Ω with volume measure $d\tau$ and using the *Divergence Theorem* on the dot product to the right, where $\nabla u \cdot \mathbf{n}$ is the directional derivative $\frac{\partial u}{\partial n}$, we obtain *Green's First Identity*:

$$\langle \Delta v, u \rangle \equiv \int_{\Omega} u\Delta v d\tau = \int_{\partial\Omega} u\frac{\partial v}{\partial n} ds - \int_{\Omega} (\nabla u) \cdot (\nabla v) d\tau, \tag{3.71}$$

and we can easily deduce *Green's Second Identity*:

$$\int_{\Omega} (u\Delta v - v\Delta u)\, d\tau = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) ds. \tag{3.72}$$

Choosing suitable boundary conditions on u and v, such that the integral over the surface $\partial\Omega$ vanishes, it is clear that $-\Delta$ is *positive definite* and *formally self adjoint*, so that

$$\begin{aligned} \|u(\mathbf{x})\|_{\dot{H}^1} &\equiv \left\| -\Delta^{\frac{1}{2}} u(\mathbf{x}) \right\|_{L^2(\Omega)} \\ &= \left\langle -\Delta^{\frac{1}{2}} u(\mathbf{x}), -\Delta^{\frac{1}{2}} u(\mathbf{x}) \right\rangle_{L^2(\Omega)} \\ &= \langle -\Delta u, u \rangle_{L^2(\Omega)} \\ &= -\int_{\Omega} u\Delta u d\tau \\ &= \int_{\Omega} |\nabla u|^2 d\mathbf{x}, \end{aligned} \tag{3.73}$$

that is

$$\|u\|^2_{\dot{H}^1} \equiv \int_{\Omega} |\nabla u|^2 d\mathbf{x} \equiv \|\nabla u\|_{L^2(\Omega)}.$$

Hardy Inequality:

If f is an integrable function with non-negative values, then the celebrated *Hardy's inequality* states:

$$\int_0^\infty \left(\frac{1}{x}f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p f(x)^p dx, \tag{3.74}$$

with equality holding if and only if $f(x) = 0$ *almost everywhere.* This can be alternatively expressed for L^2-norms as

$$\left\|\frac{1}{r}u(r)\right\|_{L^2(\mathbb{R}^n)} \leq \frac{C}{n-2}\left\|\frac{\partial}{\partial r}u(r)\right\|_{L^2(\mathbb{R}^n)}, \tag{3.75}$$

for $n \geq 3$. The inequality was first published (without proof) in 1920 in a note by Hardy [37], with a proof provided in a later text [38]. Now writing the first part of the final integral in Eq. (3.68) in terms of the $\dot{H}^1$-norm, i.e.,

$$\int_r\int_\Omega |\nabla u|^2 r^2 dr d\Omega \equiv \|\nabla u\|^2_{L^2(\mathbb{R}^3)} \equiv \|u\|^2_{\dot{H}^1}, \tag{3.76}$$

and using the inequality (3.75) for the $n = 3$ case here, we can then express the second part of the integral in Eq. (3.68) as an L^2-norm in the following way, where $d\Omega$ denotes the solid angle in $\mathbb{R}^3$

$$\int_r\int_\Omega \left(\frac{u}{r}\right)^2 r^2 dr d\Omega \equiv \|\frac{u}{r}\|_{L^2(\mathbb{R}^3)}, \tag{3.77}$$

and we note the resulting inequalities

$$\left\|\frac{1}{r}u(r)\right\|_{L^2(\mathbb{R}^3)} < \left\|\frac{\partial}{\partial r}u(r)\right\|_{L^2(\mathbb{R}^3)} < \|\nabla u\|_{L^2(\mathbb{R}^3)} = \|u\|_{\dot{H}^1}. \tag{3.78}$$

Combining these inequalities we can form bounds for the $s = 1$ case, and as we will later prove in the spectral argument of **Lemma 2.2**, for the $0 \leq s \leq 1$ case. From Equation (3.68) we have that:

$$\|u\|^2_{\mathcal{H}^1} \leq \quad C\int_{\mathbb{R}^3}\left(|\nabla u|^2 + \left(\frac{u}{r}\right)^2\right) r^2 dr d\Omega \leq \quad C\|u\|^2_{\dot{H}^1}.$$

We state here again the bounds on $\mathcal{H}^s$, in the case $0 < s \leq 1$, in the form

$$\|u\|_{\mathcal{H}^s} < C^{s/2}\|u\|_{\dot{H}^s}, \tag{3.79}$$

which are got by interpolation and by duality for the case $-1 \leq s < 0$

$$\|u\|_{\dot{H}^s} < C^{-s/2}\|u\|_{\mathcal{H}^s}. \tag{3.80}$$

The full details of the proof of these interpolation norms and their duals is provided in Appendix A.5. These Sobolev bounds are required in a subsequent calculation to find for the L^2-decay of the field $u(\mathbf{x})$ on $(\mathbb{R}^4, \eta)$.

3.3.2 The Inhomogeneous Problem

We recall the transformation $u : \mathbb{R}^4 \to \mathbb{R}$, of the Euler-Lagrange equation for the scalar field $\psi(x)$ on the spacetime $(\mathcal{M}, g)$ to a flat wave equation in $\mathbb{R}^4$ for the field $u(t, \mathbf{x})$, associated with a radial potential $V(r)$, i.e.

$$u(t, r, \Omega) = \frac{\rho(r)}{r}\psi(t, r, \Omega), \qquad (t, r, \Omega) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^2.$$

In effecting this transformation we assumed the spherically symmetric manifold $(\mathcal{M}, g)$ to have the same topology as $\mathbb{R}^4$, denoting the coordinates on $\mathbb{R}^4$ by the same letters as those on $\mathcal{M}$. We subsequently provided a justification of this assumption in setting up our gravitating source as a static perfect fluid star, and demonstrated a conformally flat Minkowski representation in the (t, r)-plane for its local radial coordinate $0 < \rho < \rho_0$, expressed by the interior tortoise coordinate $r_i(\rho)$ and with the complex exponential time dependence we arrived at the Helmholtz problem.

It is of no consequence to the analysis of the Sommerfeld radiation conditions, which are after all, concerned with bounds on $u(\mathbf{x})$ at asymptotic infinity, whether we work with the homogeneous problem

$$-\Delta u(\mathbf{x}) + \left(\frac{1}{r^2} - \frac{\alpha^2}{\rho^2}\right) \measuredangle u(\mathbf{x}) + V(r)u(\mathbf{x}) + z^2 u(\mathbf{x}) = 0,$$

in an *exterior region of* $\mathbb{R}^3$ or with an inhomogeneous problem of the form

$$-\Delta \tilde{u}(\mathbf{x}) + \left(\frac{1}{r^2} - \frac{\alpha^2}{\rho^2}\right) \measuredangle \tilde{u}(\mathbf{x}) + \tilde{V}(r)\tilde{u}(\mathbf{x}) + z^2 \tilde{u}(\mathbf{x}) = f(\mathbf{x}),$$

in all of $\mathbb{R}^3$, for an appropriate $\tilde{V}(r)$ and provided the inhomogeneous term $f(\mathbf{x})$, which we construct presently, is compactly supported in $\mathbb{R}^3$. We note here, for a later spherical restriction application, that a subset of $\mathbb{R}^3$ is compact, if and only if it is closed and bounded, since smooth functions are de facto continuous, the support is always closed, that is, vanishing outside a bounded set.

Suppose that $u(\mathbf{x})$ satisfies the homogeneous problem above, for $r = |\mathbf{x}| > R_0$ say, and suppose that $V(r)$ also satisfies the bounds there. We choose an arbitrary smooth $\mathcal{C}^\infty$ function $\gamma(|\mathbf{x}|)$, such that

$$\gamma(r) = \begin{cases} 0 & \text{for } |\mathbf{x}| < R_0, \\ 1 & \text{for } |\mathbf{x}| > 2R_0, \end{cases} \tag{3.81}$$

and we modify $u(\mathbf{x})$ by setting

$$\tilde{u}(\mathbf{x}) = \gamma(r)u(\mathbf{x}),$$

so that

$$\tilde{u}(\mathbf{x}) = \begin{cases} 0 & \text{for } |\mathbf{x}| < R_0, \\ u(\mathbf{x}) & \text{for } |\mathbf{x}| > 2R_0, \end{cases} \tag{3.82}$$

and multiplying our homogeneous equation by γ to get

$$-\gamma\Delta u(\mathbf{x}) + F(r)\gamma\not\Delta u(\mathbf{x}) + \gamma V(r)u(\mathbf{x}) + \gamma z^2 u(\mathbf{x}) = 0, \tag{3.83}$$

looking at the first term here we can write

$$\begin{aligned} \Delta(\gamma u) &\equiv \partial_i\partial_j(\gamma u)\delta_{ij} = \partial_i\left(u\partial_j\gamma + \gamma\partial_j u\right)\delta_{ij} \\ &= \left(u\partial_i\partial_j\gamma + \partial_i u\partial_j\gamma + \partial_i\gamma\partial_j u + \gamma\partial_i\partial_j u\right)\delta_{ij} \\ &= u\Delta\gamma + 2\nabla\gamma.\nabla u + \gamma\Delta u, \end{aligned}$$

and so we may write Eq. (3.83) as

$$-\Delta(\gamma u) + F(r)\not\Delta(\gamma u) + V(r)(\gamma u) + z^2(\gamma u) = -2\nabla\gamma.\nabla u - \gamma\Delta u\,. \tag{3.84}$$

Our equation for the modified field $\tilde{u}$, in the exterior region is now expressed by

$$-\Delta\tilde{u} + (\frac{1}{r^2} - \frac{\alpha^2}{\rho^2})\not\Delta\tilde{u} + \tilde{V}(r)\tilde{u} + z^2\tilde{u} = f(\mathbf{x}), \tag{3.85}$$

where $\tilde{V}(r) = \gamma V(r)$ is a function which agrees with $V(r)$ for $r \equiv |\mathbf{x}| < 2R_0$, and satisfies the upper and lower bounds

$$0 \le V(r) \le \frac{U}{r}, \tag{3.86}$$

everywhere. The smooth 'source' function $f(\mathbf{x})$ is thus compactly supported in the annulus $R_0 < |\mathbf{x}| < 2R_0$ and is given by

$$f(\mathbf{x}) = -u(\mathbf{x})\Delta\gamma - 2\nabla u(\mathbf{x})\cdot\nabla\gamma. \tag{3.87}$$

It is clear from our definition of $\gamma(r)$ that the inhomogeneous problem in *all* $\mathbb{R}^3$ is equivalent to the homogeneous problem exterior to the annulus, i.e., for $|\mathbf{x}| > 2R_0$, where we find $f(\mathbf{x}) = 0$, $\tilde{u}(\mathbf{x}) = u(\mathbf{x})$ and $\tilde{V}(|\mathbf{x}|) = V(|\mathbf{x}|)$. Having flattened the white dwarf with tortoise r_i we map into the region $0 \leq R_0$ where $\gamma = 0$. In this way we isolate the interior radiating field $u_i(\mathbf{x})$ and so solve the problem in $(\mathbb{R}^4, \eta)$.

3.3.3 L^2-Bound on the Compact 'Source' $f(\mathbf{x})$

In this section we find the L^2-bound on our 'source' function $f(\mathbf{x})$, as constructed for the inhomogeneous Helmholtz equation (dropping tildes so $u = \tilde{u}$ etc.)

$$-\Delta u(\mathbf{x}) + F(r)\not{\Delta} u(\mathbf{x}) + V(r)u(\mathbf{x}) + z^2 u(\mathbf{x}) = f(\mathbf{x}).$$

For a combination of heuristic and pedagogic motives I will outline the attempt, which first suggests itself to reason, at finding this bound. As we will see the exegesis goes full circle, but we gather some useful material from the endeavour, which we put to use in the succeeding (successful) analysis, working with an *alternative metric* which we label by $d\tilde{\sigma}^2$.

By the *triangle inequality* for $f(\mathbf{x})$ in Eq (3.87) we have

$$\|f(\mathbf{x})\|_{L^2(\mathbb{R}^3)} \leq \|u(\mathbf{x})\Delta\gamma(r)\|_{L^2(\mathbb{R}^3)} + 2\|\nabla u(\mathbf{x}) \cdot \nabla\gamma(r)\|_{L^2(\mathbb{R}^3)}. \tag{3.88}$$

The first term to the right is clearly bounded by max $|\Delta\gamma(r)|\|u(\mathbf{x})\|_{L^2(\mathbb{R}^3)}$. From *Schwarz's inequality* we have, for the second term to the right, in equation (3.88) above, that

$$|\langle \nabla u \cdot \nabla\gamma\rangle|^2 \leq \|\nabla u\|^2_{L^2} \cdot \|\nabla\gamma\|^2_{L^2}, \tag{3.89}$$

and observing the dot product inequality, with complex conjugate $\overline{u}$ we can write

$$|\nabla u \cdot \nabla \gamma|^2 \leq (\nabla u \cdot \nabla \overline{u})(\nabla \gamma \cdot \nabla \gamma), \tag{3.90}$$

which holds point-wise, we then have the integral

$$\|\nabla u \cdot \nabla \gamma\|^2_{L^2(\mathbb{R}^3)} \equiv \int_{\mathbb{R}^3} |\nabla u \cdot \nabla \gamma|^2 d\mathbf{x} \leq \int_{\mathbb{R}^3} (\nabla \gamma \cdot \nabla \gamma)(\nabla u \cdot \nabla \overline{u}) d\mathbf{x}. \tag{3.91}$$

Using Green's first identity and integrating over the domain $\mathbb{R}^3$ we have

$$\int_{\mathbb{R}^3} \nabla \cdot (u \nabla \overline{u}) d\mathbf{x} = \int_{\partial \mathbb{R}^3} u \frac{\partial \overline{u}}{\partial n} ds = \int_{\mathbb{R}^3} \nabla u \cdot \nabla \overline{u} d\mathbf{x} + \int_{\mathbb{R}^3} u \Delta \overline{u} d\mathbf{x}, \tag{3.92}$$

where $\frac{\partial u}{\partial n} = \mathbf{n} \cdot \nabla u$, is the directional derivative in the outward normal direction and for convenience we write

$$\nabla \gamma(r) \cdot \nabla \gamma(r) \equiv \epsilon(r),$$

which is by definition supported in the ball B_r, of radius $2R_0$, so that in these integrals

$$\int_{\mathbb{R}^3} \epsilon(r) H\left[u(\mathbf{x})\right] d\mathbf{x} = \int_{B_{r=|2R_0|}} \epsilon(r) H\left[u(\mathbf{x})\right] r^2 dr \sin\theta d\theta d\phi \, , \tag{3.93}$$

and the surface integral in $\mathbb{R}^2$ is then simply

$$\pm \int_{\partial \mathbb{R}^3} u \frac{\partial \overline{u}}{\partial n} ds = \int_{\lambda|R_0|} u \frac{\partial \overline{u}}{\partial n} ds = 0 \, , \tag{3.94}$$

where $H\left[u(\mathbf{x})\right]$ is a continuous function of its arguments in the volume integral and we integrate for

$$\lambda |R_0| > 2|R_0| \, , \tag{3.95}$$

in the surface integral over this radius. The integral on the far right of Eq.(3.91) can then be written as

$$\begin{aligned}
&\int_{\mathbb{R}^3} \epsilon(r) \left(\nabla \cdot (u\nabla\overline{u}) - u\Delta\overline{u}\right) d\mathbf{x} \\
=& \int_{\mathbb{R}^3} \epsilon(r) \left(\frac{1}{2}\nabla\cdot\nabla|u|^2 - F(r)u\not{\Delta}\overline{u} - V(r)|u|^2 - z^2|u|^2\right) d\mathbf{x} \\
=& \frac{1}{2}\underbrace{\int_{\mathbb{R}^3}}_{S_1} \epsilon(r)\nabla\cdot\nabla|u|^2 d\mathbf{x} - \underbrace{\int_{\mathbb{R}^3}}_{S_2} \epsilon(r)F(r)u\not{\Delta}\overline{u}d\mathbf{x} - \underbrace{\int_{\mathbb{R}^3}}_{S_3} \epsilon(r)|u|^2\left(V(r) + \text{Re } z^2\right) d\mathbf{x}.
\end{aligned}$$

Using Green's identity again in the first integral $\int_{S_1}$ we have

$$\epsilon(r)\nabla\cdot\nabla|u|^2 = \nabla\cdot\left(\epsilon(r)\nabla|u|^2\right) - \nabla|u|^2\cdot\nabla\epsilon(r),$$

and integrating this over the entire space volume, for $\mathbf{x}\in\mathbb{R}^3$, and using the Divergence theorem over the surface of radius $\lambda|R_0|$ we find

$$\frac{1}{2}\int_{\mathbb{R}^3}\epsilon(r)\nabla\cdot\nabla|u|^2 d\mathbf{x} = \frac{1}{2}\int_{\mathbb{R}^3}\left(\nabla\cdot(\epsilon(r)\nabla|u|^2) - \nabla|u|^2\cdot\nabla\epsilon(r)\right) d\mathbf{x} = -\frac{1}{2}\int_{\mathbb{R}^3}\nabla|u|^2\cdot\nabla\epsilon(r)d\mathbf{x}.$$

Applying the same procedure again to the above integral on the extreme right we find

$$-\frac{1}{2}\int_{\mathbb{R}^3}\nabla|u|^2\cdot\nabla\epsilon(r)d\mathbf{x} = \frac{1}{2}\int_{\mathbb{R}^3}|u|^2\Delta\epsilon(r)d\mathbf{x}.$$

For the second integral $\int_{S_2}$, we have from a previous calculation (where we found for $\int_{\mathbb{R}^3}\psi\not{\Delta}\tilde{\phi}d\Omega$) that

$$\int_{\mathbb{R}^3}\epsilon(r)F(r)u\not{\Delta}\overline{u}d\mathbf{x} = \int_{B_{r=2|R_0|}}\epsilon(r)F(r)\left(\int_\Omega u\not{\Delta}\overline{u}d\Omega\right)r^2dr = -\int_{\mathbb{R}^3}\epsilon(r)F(r)|\not{\nabla}u|^2d\mathbf{x}. \tag{3.96}$$

We have a bound on $F(r)$ given by

$$F(r) \ = \ \frac{1}{r^2} - \frac{\alpha^2}{\rho^2} \ < \ \frac{C}{r^2},$$

using this with the Euclidian gradient in $\{r, \theta, \phi\}$ spherical coordinates

$$\nabla u \ = \ \nabla_r u + \frac{1}{r} \not\nabla u, \quad \text{so that} \quad |\nabla u|^2 \ = \ |\nabla_r u|^2 + \frac{1}{r^2} |\not\nabla u|^2,$$

to get

$$\int_{\mathbb{R}^3} \epsilon(r) F(r) \ |\not\nabla u|^2 \ d\mathbf{x} \leq C \int_{\mathbb{R}^3} \epsilon(r) \ \frac{1}{r^2} \ |\not\nabla u|^2 \ d\mathbf{x} \leq C \int_{\mathbb{R}^3} \epsilon(r) \ |\nabla u|^2 \ d\mathbf{x}. \qquad (3.97)$$

For convenience we drop the radial argument in $\epsilon(r)$, and by definition ϵ is compactly supported in a ball of radius $2R_0$. We have that

$$\nabla(\epsilon u) \ = \ u \nabla \epsilon + \epsilon \nabla u,$$

and from the *triangle inequality* we find

$$\|\nabla(\epsilon u)\|_{L^2}^2 \ \leq \ \|u \nabla \epsilon\|_{L^2}^2 + \|\epsilon \nabla u\|_{L^2}^2. \qquad (3.98)$$

Integrating over the solid region $\mathcal{D} \subset \mathbb{R}^3$, we can write this as

$$\int_{\mathcal{D}} |\nabla(\epsilon u)|^2 d\mathbf{x} \ \leq \ \int_{\mathcal{D}} |u \nabla \epsilon|^2 d\mathbf{x} + \int_{\mathcal{D}} |\epsilon \nabla u|^2 d\mathbf{x}. \qquad (3.99)$$

Using Green's First Identity on the product $u \nabla u \cdot \epsilon^2$ we have

$$\int_{\mathcal{D}} \nabla \cdot (\epsilon^2 u \nabla u) d\mathbf{x} = \int_{\mathcal{D}} \nabla(\epsilon^2 u) \cdot \nabla u d\mathbf{x} + \int_{\mathcal{D}} \epsilon^2 u \Delta u d\mathbf{x}, \qquad (3.100)$$

and applying the Divergence Theorem on the left hand side of (3.100) to find

$$\begin{aligned}
\int_{\partial\mathcal{D}} (\epsilon^2 u \nabla u) \cdot \mathbf{n} ds \ &= \ \int_{\mathcal{D}} \nabla(\epsilon^2 u) \cdot \nabla u d\mathbf{x} + \int_{\mathcal{D}} \epsilon^2 u \Delta u d\mathbf{x} \\
&= \ \int_{\mathcal{D}} \left(u \nabla \epsilon^2 + \epsilon^2 \nabla u \right) \cdot \nabla u d\mathbf{x} + \int_{\mathcal{D}} \epsilon^2 u \Delta u d\mathbf{x} \\
&= \ \int_{\mathcal{D}} 2\epsilon u \nabla \epsilon \cdot \nabla u + \int_{\mathcal{D}} \epsilon^2 \left| \nabla u \right|^2 d\mathbf{x} + \int_{\mathcal{D}} \epsilon^2 u \Delta u d\mathbf{x}.
\end{aligned}$$

The left hand side integral over the spherical surface $\partial\mathcal{D} = 2\lambda R_0$ as defined by (3.95) is zero, and so we now have

$$-\int_{\mathcal{D}} \epsilon^2 u \Delta u d\mathbf{x} - \int_{\mathcal{D}} 2\epsilon u \nabla\epsilon \cdot \nabla u = \int_{\mathcal{D}} \epsilon^2 \left|\nabla u\right|^2 d\mathbf{x}.$$

Using Eq. (3.100) then implies

$$-\int_{\mathcal{D}} \nabla \cdot (\epsilon^2 u \nabla u) d\mathbf{x} = \int_{\mathcal{D}} \epsilon^2 \left|\nabla u\right|^2 d\mathbf{x}, \tag{3.101}$$

we can see this in a 'rule of thumb' way, i.e., using self-adjointness of $\epsilon(r)$ and 'switching' sign of ∇ under change of position in brackets to get

$$\|\epsilon\nabla u\|_{L^2}^2 \equiv \langle \epsilon\nabla u, \epsilon\nabla u\rangle_{L^2} = -\left\langle \nabla\cdot(\epsilon^2\nabla u), u\right\rangle_{L^2}. \tag{3.102}$$

From our elliptic equation in $\mathbb{R}^3$ outside a ball B_r, we have

$$\nabla \cdot \nabla u \equiv \Delta u = F(r)\not{\Delta} u + V(r)u + z^2 u = 0 \ ,$$

and so we find

$$\nabla \cdot (\epsilon^2 \ \nabla u) = \epsilon^2 \left(\ F\not{\Delta} u + Vu + z^2 u\right) + 2\epsilon \ \nabla\epsilon \cdot \nabla u.$$

Recalling equation (3.102) we see that

$$\begin{aligned}
\|\epsilon\nabla u\|_{L^2}^2 &= \left\langle \ \epsilon^2(F\not{\Delta} u + Vu + z^2 u) + \ 2\epsilon\nabla\epsilon\cdot\nabla u \ , u\right\rangle_{L^2} \\
&= \left\langle \ \epsilon^2 F\not{\Delta} u, u\right\rangle_{L^2} + \left\langle \epsilon^2 Vu, u\right\rangle_{L^2} + \left\langle \epsilon^2 z^2 u, u\right\rangle_{L^2} + 2\left\langle \epsilon\nabla\epsilon\cdot\nabla u \ , u\right\rangle_{L^2} \\
&\leq \ \|\epsilon^2 F\not{\Delta} u\|_{L^2}\|u\|_{L^2} + 2\|\epsilon\nabla\epsilon\cdot\nabla u\|_{L^2}\|u\|_{L^2} \\
&\quad + \left(\max\left|\epsilon^2 V\right| + \max\left|\epsilon^2 z^2\right|\right)\|u\|_{L^2}^2.
\end{aligned} \tag{3.103}$$

For the last term of the second line above, we use the *Cauchy-Schwarz* inequality to get

$$\langle \epsilon\nabla\epsilon\cdot\nabla u \ , u\rangle_{L^2} \leq \|\epsilon\nabla\epsilon\cdot\nabla u\|_{L^2}\|u\|_{L^2}. \tag{3.104}$$

We now apply the following trick to the above norms, that is, for $p = \|u\|_{L^2}$ and $q = 2\|\epsilon\nabla\epsilon \cdot \nabla u\|_{L^2}$ and for $p, q \in \mathbb{R}$ we have the following inequality

$$\begin{aligned} 0 &\leq (p-q)^2 \\ \Rightarrow \frac{\beta^2}{\beta^2}(p \cdot q) = p \cdot q &\leq (\frac{1}{2}\beta^2)p^2 + (\frac{1}{2}\beta^{-2})q^2, \end{aligned}$$

where $\frac{1}{2}\beta^2 = C_1$ is a large constant and $\frac{1}{2}\beta^{-2} = c_2$ is a small constant. From inequality (3.104) we then have

$$\langle \epsilon\nabla\epsilon \cdot \nabla u \ , u \rangle_{L^2} \leq C_1\|u\|_{L^2}^2 + c_2\|\epsilon\nabla\epsilon \cdot \nabla u\|_{L^2}^2, \tag{3.105}$$

and this gives us a $\|u\|_{L^2}^2$ bound on (3.104) above. The above analysis brings us to a point where we still have to deal with the integral

$$\left\langle \ \epsilon^2 F(r) \not\Delta u, u \right\rangle_{L^2} = - \int_{\mathbb{R}^3} \epsilon(r) F(r) |\not\nabla u|^2 d\mathbf{x},$$

and thus in exploration we have arrived where we started! at the bound in equation (3.97)

$$\int_{\mathbb{R}^3} \epsilon(r) F(r) |\not\nabla u|^2 d\mathbf{x} \leq C \int_{\mathbb{R}^3} \epsilon(r) \ \frac{1}{r^2} \ |\not\nabla u|^2 \ d\mathbf{x} \leq C \int_{\mathbb{R}^3} \epsilon(r) \ |\nabla u|^2 \ d\mathbf{x}.$$

Metric Equivalence:

We find a way around this is as follows: we start again with the by now familiar equality

$$\|f(\mathbf{x})\|_{L^2(\mathbb{R}^3)} = \|u(\mathbf{x})\Delta\gamma(r)\|_{L^2(\mathbb{R}^3)} + 2\|\nabla u(\mathbf{x}) \cdot \nabla\gamma(r)\|_{L^2(\mathbb{R}^3)}, \tag{3.106}$$

it proves convenient to express the differential operators Δ and ∇ on the $\mathbb{R}^3$-Euclidian space with metric

$$d\sigma^2 = \sigma_{ij} dx^i dx^j = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{3.107}$$

in terms of an alternative metric given by

$$d\tilde{\sigma}^2 = \tilde{\sigma}_{ij} d\tilde{x}^i d\tilde{x}^j = dr^2 + \frac{\rho^2}{\alpha^2}\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{3.108}$$

In these coordinates we find the following differential forms in $d\tilde{\sigma}^2$

$$\nabla_{\tilde{\sigma}} u(\mathbf{x}) = u_r \mathbf{e}_r + \frac{\alpha}{\rho} u_\theta \mathbf{e}_\theta + \frac{\alpha}{\rho \sin\theta} u_\phi \mathbf{e}_\phi, \tag{3.109}$$

and

$$\begin{aligned} \Delta_{\tilde{\sigma}} u(\mathbf{x}) &= u_{rr} + \frac{\alpha^2}{\rho^2} u_{\theta\theta} + \frac{\alpha^2}{\rho^2 \sin^2\theta} u_{\phi\phi} \\ &= \Delta u - \frac{1}{r^2} \not{\Delta} u + \frac{\alpha^2}{\rho^2}\left(u_{\theta\theta} + \frac{1}{\sin^2\theta} u_{\phi\phi}\right) \\ &= \Delta u + \left(\frac{\alpha^2}{\rho^2} - \frac{1}{r^2}\right) \not{\Delta} u \\ &= \Delta u - F(r) \not{\Delta} u. \end{aligned} \tag{3.110}$$

Using these alternative operators we form this inequality

$$\|p(\mathbf{x})\|_{L^2(\mathbb{R}^3)} \leq \|u(\mathbf{x}) \Delta_{\tilde{\sigma}} \gamma(r)\|_{L^2(\mathbb{R}^3)} + 2\|\nabla_{\tilde{\sigma}} u(\mathbf{x}) \cdot \nabla_{\tilde{\sigma}} \gamma(r)\|_{L^2(\mathbb{R}^3)}, \tag{3.111}$$

the first term to the right of the inequality is clearly bounded by

$$\max|\Delta_{\tilde{\sigma}} \gamma(r)| \|u(\mathbf{x})\|_{L^2(\mathbb{R}^3)},$$

and as before we use the inequality independent of a coordinate basis given by

$$\|\epsilon_{\tilde{\sigma}} u\|^2_{\dot{H}^1} \equiv \|\nabla_{\tilde{\sigma}}(\epsilon_{\tilde{\sigma}} u)\|^2_{L^2} \leq \|\nabla_{\tilde{\sigma}}(\epsilon_{\tilde{\sigma}}) u\|^2_{L^2} + \|\epsilon_{\tilde{\sigma}} \nabla_{\tilde{\sigma}} u\|^2_{L^2}, \tag{3.112}$$

and of course where

$$\epsilon_{\tilde{\sigma}} \equiv \epsilon_{\tilde{\sigma}}(r) \equiv \nabla_{\tilde{\sigma}} \gamma(r). \nabla_{\tilde{\sigma}} \gamma(r) \equiv \nabla \gamma(r) \cdot \nabla \gamma(r) = \epsilon(r) \equiv \epsilon.$$

From equation (3.102) we also have

$$\|\epsilon\nabla_{\tilde{\sigma}}u\|_{L^2}^2 \equiv \langle \epsilon\nabla_{\tilde{\sigma}}u, \epsilon\nabla_{\tilde{\sigma}}u\rangle_{L^2} = -\left\langle \nabla_{\tilde{\sigma}}\cdot(\epsilon^2\nabla_{\tilde{\sigma}}u), u\right\rangle_{L^2}, \tag{3.113}$$

and similarly

$$\nabla_{\tilde{\sigma}}\cdot\left(\epsilon^2\nabla_{\tilde{\sigma}}u\right) = \epsilon^2\Delta_{\tilde{\sigma}}u + 2\epsilon\nabla_{\tilde{\sigma}}\epsilon\cdot\nabla_{\tilde{\sigma}}u, \tag{3.114}$$

so that using this in the expression of equation (3.113) above we find

$$\begin{aligned}\|\epsilon\nabla_{\tilde{\sigma}}u\|_{L^2}^2 &\equiv \langle \epsilon\nabla_{\tilde{\sigma}}u, \epsilon\nabla_{\tilde{\sigma}}u\rangle_{L^2} \\ &= -\left\langle \epsilon^2\Delta_{\tilde{\sigma}}u + 2\epsilon\nabla_{\tilde{\sigma}}\epsilon\cdot\nabla_{\tilde{\sigma}}u, u\right\rangle_{L^2}.\end{aligned} \tag{3.115}$$

Now we can use equation (3.110) to find the Laplacian

$$\Delta_{\tilde{\sigma}}u(\mathbf{x}) = \Delta u(\mathbf{x}) - F(r)\not{\Delta}u(\mathbf{x}) = V(r)u(\mathbf{x}) + z^2u(\mathbf{x}),$$

and so

$$\begin{aligned}\|\epsilon\nabla_{\tilde{\sigma}}u\|_{L^2}^2 &= \left\langle \epsilon^2Vu + \epsilon^2z^2u + 2\epsilon\nabla_{\tilde{\sigma}}\epsilon\cdot\nabla_{\tilde{\sigma}}u, u\right\rangle_{L^2} \\ &= \left\langle \epsilon^2Vu, u\right\rangle_{L^2} + \left\langle \epsilon^2z^2u, u\right\rangle_{L^2} + \langle 2\epsilon\nabla_{\tilde{\sigma}}\epsilon\cdot\nabla_{\tilde{\sigma}}u, u\rangle_{L^2} \\ &\leq \left(\max\left|\epsilon^2V\right| + \max\left|\epsilon^2z^2\right|\right)\|u\|_{L^2}^2 + \langle 2\epsilon\nabla_{\tilde{\sigma}}\epsilon\cdot\nabla_{\tilde{\sigma}}u, u\rangle_{L^2}.\end{aligned} \tag{3.116}$$

Using the Cauchy-Schwarz inequality as in equation (3.104) and applying the same method we use there to find

$$\langle \epsilon\nabla_{\tilde{\sigma}}\epsilon\cdot\nabla_{\tilde{\sigma}}u\ , u\rangle_{L^2} \leq C_1\|u\|_{L^2}^2 + c_2\|\epsilon\nabla_{\tilde{\sigma}}\epsilon\cdot\nabla_{\tilde{\sigma}}u\|_{L^2}^2, \tag{3.117}$$

and similarly for C_1 and c_2, the large and small constants respectively, and in exactly the same way we establish a bound on this norm, given here by

$$\|p(\mathbf{x})\|_{L^2(\mathbb{R}^3)}^2 \leq \|\epsilon\nabla_{\tilde{\sigma}}u\|_{L^2}^2 \leq C'\|u\|_{L^2}^2, \tag{3.118}$$

so that finally we have

$$\|\epsilon u\|^2_{\dot{H}^1(\tilde{\sigma})} \equiv \|\nabla_{\tilde{\sigma}}(\epsilon u)\|^2_{L^2} \leq \|\nabla_{\tilde{\sigma}}(\epsilon)u\|^2_{L^2} + \|\epsilon(\nabla_{\tilde{\sigma}}u)\|^2_{L^2} \leq C\|u\|^2_{L^2}. \tag{3.119}$$

We have the alternative metrics given by

$$\sigma(x,x) \equiv d\sigma^2 = \sigma_{ij}dx^i dx^j = dr^2 + r^2 d\Omega^2, \tag{3.120}$$

and

$$\tilde{\sigma}(x,x) \equiv d\tilde{\sigma}^2 = \tilde{\sigma}_{ij}d\tilde{x}^i d\tilde{x}^j = dr^2 + \frac{\rho^2}{\alpha^2}d\Omega^2. \tag{3.121}$$

If we can establish upper and lower bounds on these metrics, such that

$$d\sigma^2_{\text{lower}} \quad \leq \quad \left(d\tilde{\sigma}^2 = dr^2 + \frac{\rho^2}{\alpha^2}d\Omega^2\right) \quad \leq \quad d\sigma^2_{\text{upper}}, \tag{3.122}$$

in other words, for the generic constants k and K, if

$$k\sigma(x,x) \leq \tilde{\sigma}(x,x) \leq K\sigma(x,x) \tag{3.123}$$

holds, so that from the notion of *metric equivalence*

$$\|\epsilon u\|^2_{\dot{H}^1(\tilde{\sigma})} \leq K\|u\|^2_{L^2}, \tag{3.124}$$

we can form the following upper and lower bounds:

$$k\|w\|_{\dot{H}^s(\sigma)} \leq \|w\|_{\dot{H}^s(\tilde{\sigma})} \leq K\|w\|_{\dot{H}^s(\sigma)}, \tag{3.125}$$

and we can then say for the $s = 1$ case that

$$k\|w\|_{\dot{H}^1(\sigma)} \leq \|w\|_{\dot{H}^1(\tilde{\sigma})} \leq K\|w\|_{\dot{H}^1(\sigma)}, \tag{3.126}$$

and so it follows

$$\|p(\mathbf{x})\|^2_{L^2(\mathbb{R}^3)} \leq \|\epsilon u\|^2_{\dot{H}^1(\sigma)} \leq K\|\epsilon u\|^2_{\dot{H}^1(\tilde{\sigma})} \leq K'\|u\|^2_{L^2(\tilde{\sigma})} \leq K''\|u\|^2_{L^2(\sigma)}, \tag{3.127}$$

and in this way we establish the bound on $f(\mathbf{x})$.

We have previously found that

$$\frac{\alpha^2}{\rho^2} \leq \frac{C}{r^2},$$

i.e.

$$r^2 \leq C' \frac{\rho^2}{\alpha^2},$$

alternatively, in the domain of application where $\rho >> 2M_s$ and for $\rho < r$ we also have

$$\begin{aligned} \alpha^2 &= 1 - \frac{2M_s}{\rho} \\ \Rightarrow \frac{1}{\alpha^2} &= \frac{\rho}{\rho - 2M} \\ \Rightarrow \frac{\rho^2}{\alpha^2} &\approx \frac{\rho^3}{\rho} = \rho^2 \geq cr^2, \end{aligned} \tag{3.128}$$

and as $\rho \to \infty$ we have an upper bound given by

$$\frac{\rho^2}{\alpha^2} < Cr^2, \tag{3.129}$$

similarly

$$\begin{aligned} \frac{\alpha^2}{\rho^2} &= \frac{1}{\rho^2}\left(1 - \frac{2M_s}{\rho}\right), \\ &< \frac{1}{\rho^2}, \end{aligned} \tag{3.130}$$

and so for the lower bound we find

$$\frac{\rho^2}{\alpha^2} > \rho^2 \sim cr^2. \tag{3.131}$$

That is

$$cr^2 < d\sigma^2_{\text{lower}} \leq d\tilde{\sigma}^2(x,x) \leq d\sigma^2_{\text{upper}} < Cr^2, \tag{3.132}$$

and thus we establish metric equivalence, i.e.

$$c\sigma(x,x) \leq \tilde{\sigma}(x,x) \leq C\sigma(x,x),$$

with the foregoing reasoning we can conclude our bound, i.e.

$$\|f(\mathbf{x})\|_{L^2(\mathbb{R}^3)} \leq \|u(\mathbf{x})\Delta\gamma(r)\|_{L^2(\mathbb{R}^3)} + 2\|\nabla u(\mathbf{x}) \cdot \nabla\gamma(r)\|_{L^2(\mathbb{R}^3)} \leq C'' \|u\|^2_{L^2(\sigma)}. \quad (3.133)$$

Since $u(\mathbf{x})$ and $\tilde{u}(\mathbf{x})$ agree for $|\mathbf{x}| > 2R_0$, any estimates we prove for $|\tilde{u}(\mathbf{x})|$ apply equally to $|u(\mathbf{x})|$ provided $|\mathbf{x}| > 2R_0$.

3.4 The Hyperbolic Initial Value Problem

3.4.1 Classical results

In this section we establish two important lemmata, namely the results stated in (3.135) and (3.136) below:

Lemma 2.1 For $w(t,\mathbf{x})$ in the solution of the hyperbolic initial value problem

$$\frac{\partial^2}{\partial t^2}w(t,\mathbf{x}) + B \cdot w(t,\mathbf{x}) = 0; \qquad w(0,\mathbf{x}) = 0, \quad \partial_t w(0,\mathbf{x}) = f(\mathbf{x}), \quad (3.134)$$

the Laplace transform is given by

$$u(\mathbf{x}) = \int_0^\infty w(t,\mathbf{x})e^{-zt}dt. \quad (3.135)$$

Lemma 2.2 We prove the following conservation of energy result by using $\mathcal{H}^s$-norms and spectral analysis

$$\|w(t)\|^2_{\mathcal{H}^s} + \|\partial_t w(t)\|^2_{\mathcal{H}^{s-1}} = \|f\|^2_{\mathcal{H}^{s-1}}. \quad (3.136)$$

We recall that a *real*, *symmetric* and *positive* operator is self-adjoint if its initial domain consists of sufficiently smooth functions. The initial domain of B consists of the $C_c^\infty(\Sigma)$ functions, i.e., functions of compact support so that B is densely defined and it is then a classical result [27] that the Cauchy problem (3.134) is well-posed. We have then a problem for which a sensible, fully deterministic dynamical evolution prescription can be given [18]. The wave equation (3.134) is similar to the classical massless Klein-Gordon wave equation – the Hilbert space analysis leading to its solution was originally laid down by von Neumann [26], and as we have alluded to, used also by Wald [16] [18] to establish deterministic dynamics in non-globally hyperbolic spacetimes (those that admit naked singularities and are thus *geodesically incomplete*), as a challenge to a putative cosmic censor, who insists on global hyperbolicity for spacetimes and thus forbids the nakedness of singularities, clothing them with the event horizon hypersurface $\mathcal{H}^+$ etcetera.

These classical works of von Neumann, Wald et al. motivate the following approach to establishing the results for the well-posed problem here. The inhomogeneous equation of our radiating system in $\mathbb{R}^3$ can be expressed in the following way

$$B \cdot u(\mathbf{x}) + z^2 u(\mathbf{x}) = f(\mathbf{x}), \tag{3.137}$$

with the differential operator

$$B \equiv -\Delta + \left(\frac{1}{r^2} - \frac{\alpha^2}{\rho^2}\right) \not{\Delta} + V(r),$$

previously established as *real*, *symmetric* and *positive definite* and thus with the self-adjoint property guaranteed via the unique self-adjoint extension B_E [26]. There are many versions of the spectral theorem for a compact, self-adjoint operator on a Hilbert space $\mathcal{H}$. The following version, appropriate to the analysis here, states that:

Every bounded self-adjoint operator is unitarily equivalent to a multiplication operator on a suitable L^2-space.

This means that, given a bounded self-adjoint operator B, on a separable Hilbert space $\mathcal{H}$, we can always find a measure μ on a measure space M and a *unitary* operator $U : \mathcal{H} \to L^2(M, d\mu)$ so that

$$UBU^{-1} \cdot u(\mathbf{x}) = \beta(\mathbf{x})u(\mathbf{x}), \qquad \text{and} \qquad u(\mathbf{x}) \in L^2(M, d\mu), \tag{3.138}$$

for some bounded real-valued measurable function $\beta(\mathbf{x}) \in L^\infty(M, d\mu)$ on M. The formula given by Eq. (3.138) is called the "spectral representation" of the self-adjoint operator B.

Lemma 2.1 Consider the initial value problem with given initial data:

$$\frac{\partial^2}{\partial t^2} w(t, \mathbf{x}) + Bw(t, \mathbf{x}) = 0; \qquad w(0, \mathbf{x}) = 0, \quad \partial_t w(0, \mathbf{x}) = f(\mathbf{x}). \tag{3.139}$$

From the spectral theory of positive self-adjoint operators and with

$$w(t, \mathbf{x}) \in L^\infty(\mathbb{R}, \mathcal{H}^s) \quad \text{and} \quad \partial_t w(t, \mathbf{x}) \in L^\infty(\mathbb{R}, \mathcal{H}^{s-1}),$$

the elliptic-Helmholtz solution $u(\mathbf{x})$ admits an integral representation in z-space, for $z > 0$ given by

$$u(\mathbf{x}) = \int_0^\infty w(t, \mathbf{x}) e^{-zt} dt, \tag{3.140}$$

which is an $L^2(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$ solution of the homogeneous Helmholtz equation

$$-\Delta u(\mathbf{x}) + \left(\frac{1}{r^2} - \frac{\alpha^2}{\rho^2}\right) \not{\Delta} u(\mathbf{x}) + V(r)u(\mathbf{x}) + z^2 u(\mathbf{x}) = 0, \qquad \zeta = \text{Re } z > 0. \tag{3.141}$$

We prove this as follows:

Using the foregoing definition and notation for spectral representation we can define the following

$$UBU^{-1} \cdot u(\mathbf{y}) = \beta(\mathbf{y})u(\mathbf{y}), \tag{3.142}$$

and

$$Uf(B)U^{-1} \cdot u(\mathbf{y}) = f(\beta(\mathbf{y}))u(\mathbf{y}), \tag{3.143}$$

where

$$B = U^{-1}\beta(\mathbf{y})U, \qquad f(B) = U^{-1}[f \circ \beta(\mathbf{y})]U \qquad \text{and} \qquad UU^{-1} = \mathbb{I}, \tag{3.144}$$

and we note for later that

$$U \cdot u(\mathbf{x}) = \tilde{u}(\mathbf{y}) \qquad \text{and} \quad U \cdot f(\mathbf{x}) = \tilde{f}(\mathbf{y}). \tag{3.145}$$

We have the initial value problem of Eq. (3.139), i.e.

$$w_{tt}(t, \mathbf{x}) + B \cdot w(t, \mathbf{x}) = 0,$$

applying unitary operator U, this then implies

$$U \cdot w_{tt}(t, \mathbf{x}) + UB\mathbb{I} \cdot w(t, \mathbf{x}) = 0, \tag{3.146}$$

and also

$$U \cdot w_{tt}(t, \mathbf{x}) + \left(UBU^{-1}\right) U \cdot w(t, \mathbf{x}) = 0, \tag{3.147}$$

which we can express by

$$(Uw_{tt})(\mathbf{y}) + \beta(\mathbf{y})(Uw)(\mathbf{y}) = 0, \tag{3.148}$$

solving this linear second order ordinary differential equation we get

$$(Uw)(t, \mathbf{y}) = (Uw)(0, \mathbf{y}) \cos \sqrt{\beta(\mathbf{y})} + (Uw_t)(0, \mathbf{y}) \frac{\sin \sqrt{\beta(\mathbf{y})}t}{\sqrt{\beta(\mathbf{y})}}, \tag{3.149}$$

for initial values $w(0, \mathbf{x}) = 0$ and $\partial_t w(0, \mathbf{x}) = f(\mathbf{x})$, this is

$$\tilde{w}(t, \mathbf{y}) = \tilde{f}(\mathbf{y}) \frac{\sin \sqrt{\beta(\mathbf{y})}t}{\sqrt{\beta(\mathbf{y})}}, \tag{3.150}$$

or in terms of the operator B

$$
\begin{aligned}
w(t,\mathbf{x}) &= \cos\sqrt{B}t w(0,\mathbf{x}) + \frac{\sin\sqrt{B}t}{\sqrt{B}} w_t(0,\mathbf{x}) \\
&= \frac{\sin\sqrt{B}t}{\sqrt{B}} f(\mathbf{x}).
\end{aligned} \tag{3.151}
$$

We also have Eq. (3.141) which we write as

$$
B \cdot u(\mathbf{x}) + \mathbb{I} \cdot z^2 u(\mathbf{x}) = f(\mathbf{x}), \tag{3.152}
$$

following the spectral theorem we apply the unitary transformation U

$$
UB\mathbb{I} \cdot u(\mathbf{x}) + \mathbb{I} z^2 U \cdot u(\mathbf{x}) = U \cdot f(\mathbf{x}),
$$

and with unitarity $U^{-1}U = \mathbb{I}$ we form

$$
(UBU^{-1})\tilde{u}(\mathbf{y}) + \mathbb{I} \cdot z^2 \tilde{u}(\mathbf{y}) = \tilde{f}(\mathbf{y}),
$$

that is

$$
\beta(\mathbf{y})\tilde{u}(\mathbf{y}) + z^2 \tilde{u}(\mathbf{y}) = \tilde{f}(\mathbf{y}),
$$

so that

$$
\tilde{u}(\mathbf{y}) = \frac{\tilde{f}(\mathbf{y})}{\beta(\mathbf{y}) + z^2} = U \cdot u(\mathbf{x}). \tag{3.153}
$$

Applying the inverse unitary operation U^{-1} to this we get

$$
u(\mathbf{x}) = U^{-1} \cdot \frac{\tilde{f}(\mathbf{y})}{\beta(\mathbf{y}) + z^2}, \tag{3.154}
$$

now we integrate $\tilde{w}(t, \mathbf{y})$ with respect to t as follows

$$\begin{aligned}
\tilde{v}(\mathbf{y}) &= \int_0^\infty e^{-zt}\tilde{w}(t, \mathbf{y})dt \\
&= \int_0^\infty e^{-zt}\frac{\sin\sqrt{\beta(\mathbf{y})}\,t}{\sqrt{\beta(\mathbf{y})}}\tilde{f}(\mathbf{y})dt \\
&= \frac{\tilde{f}(\mathbf{y})}{2i\sqrt{\beta(\mathbf{y})}}\int_0^\infty \left(e^{-(z-i\sqrt{\beta(\mathbf{y})})t} - e^{-(z+i\sqrt{\beta(\mathbf{y})})t}\right) dt \\
&= \frac{\tilde{f}(\mathbf{y})}{\beta(\mathbf{y}) + z^2} \\
&= \tilde{u}(\mathbf{y}),
\end{aligned} \tag{3.155}$$

and thus we find

$$U \cdot u(\mathbf{x}) = \tilde{u}(\mathbf{y}) = \int_0^\infty e^{-zt}\tilde{w}(t, \mathbf{y})dt,$$

and the result of Lemma 2.1 follows

$$u(\mathbf{x}) = \int_0^\infty e^{-zt}\left[U^{-1} \cdot \tilde{w}(t, \mathbf{y})\right] dt = \int_0^\infty w(t, \mathbf{x})e^{-zt}dt. \tag{3.156}$$

Note that this representation of $u(\mathbf{x})$ is clearly consistent with the operation of Laplace transform $\mathcal{L}$ on the solution of Eq. (3.139), i.e.

$$\mathcal{L}\{\partial_t^2 w\} - \Delta\mathcal{L}\{w\} + \left(\frac{1}{r^2} - \frac{\alpha^2}{\rho^2}\right)\measuredangle\!\!\!\Delta\mathcal{L}\{w\} + V\mathcal{L}\{w\} = 0,$$

where

$$\mathcal{L}\{w(t, \mathbf{x})\} = \tilde{w}(z) = \int_0^\infty w(t, \mathbf{x})e^{-zt}dt,$$

and

$$\mathcal{L}\{\partial_t^2 w(t, \mathbf{x})\} = z^2\tilde{w}(z) - zw(0, \mathbf{x}) - w_t(0, \mathbf{x}) = z^2\tilde{w}(z) - f(\mathbf{x}),$$

which yields the elliptic equation in $\tilde{w}(z)$

$$-\Delta\tilde{w}(z) + \left(\frac{1}{r^2} - \frac{\alpha^2}{\rho^2}\right)\not{\Delta}\tilde{w}(z) + V(r)\tilde{w}(z) + z^2\tilde{w}(z) = f(\mathbf{x}),$$

that is where

$$u(\mathbf{x}) = \tilde{w}(z) \equiv \int_0^\infty w(t,\mathbf{x})e^{-zt}dt < \infty. \tag{3.157}$$

Lemma 2.2 Consider again the IVP of Eq. (3.139), with $0 \le s \le 1$ and for initial datum $f(\mathbf{x}) \in \mathcal{H}^{s-1}$; a solution $w(\mathbf{x},t)$ then exists which satisfies a conservation law given by

$$\|w(t)\|^2_{\mathcal{H}^s} + \|\partial_t w(t)\|^2_{\mathcal{H}^{s-1}} = \|f\|^2_{\mathcal{H}^{s-1}}. \tag{3.158}$$

We prove this as follows:

Using the spectral representation of Eq. (3.144), we have

$$F(B) = B^{s/2} \Rightarrow UF(B)U^{-1} = F(\beta(\mathbf{y})) = \beta^{s/2}(\mathbf{y}) \quad , \tag{3.159}$$

with the definition of the $\mathcal{H}^s$-norm, which we previously introduced, i.e.

$$\|\tilde{w}(t,\mathbf{y})\|^2_{\mathcal{H}^s} \equiv \|\beta^{s/2}\tilde{w}(t,\mathbf{y})\|^2_{L^2},$$

admitting a representation for the L^2-norm above, given by

$$B^{s/2}\tilde{w}(t,\mathbf{y}) = \beta^{\frac{s}{2}}(\mathbf{y})\frac{\tilde{f}(\mathbf{y})}{\sqrt{\beta}}\sin\sqrt{\beta}t = \beta^{(\frac{s-1}{2})}(\mathbf{y})\tilde{f}(\mathbf{y})\sin\sqrt{\beta}t. \tag{3.160}$$

With this we can write

$$\begin{aligned}
\|\tilde{w}(t,\mathbf{y})\|^2_{\mathcal{H}^s} &= \int_{\mathbb{R}^3}\left|\beta^{(\frac{s-1}{2})}\tilde{f}(\mathbf{y})\sin\sqrt{\beta}t\right|^2 d\mathbf{x} \\
&= \int_{\mathbb{R}^3}\left|\beta^{(\frac{s-1}{2})}\tilde{f}(\mathbf{y})\sin\sqrt{\beta}t\right|^2 d\mathbf{x} \\
&= \int_{\mathbb{R}^3}\left|\beta^{s-1}\tilde{f}(\mathbf{y})^2\sin^2\sqrt{\beta}t\right| d\mathbf{x},
\end{aligned} \tag{3.161}$$

and in similar fashion we find

$$\|\partial_t \tilde{w}(t,\mathbf{y})\|^2_{\mathcal{H}^{s-1}} = \int_{\mathbb{R}^3} \left|\beta^{(s-1)} \tilde{f}(\mathbf{y})^2 \cos^2 \sqrt{\beta} t\right| d\mathbf{x}, \tag{3.162}$$

adding these two integrals to get

$$\|\tilde{w}(t,\mathbf{y})\|^2_{\mathcal{H}^s} + \|\partial_t \tilde{w}(t,\mathbf{y})\|^2_{\mathcal{H}^{s-1}} = \int_{\mathbb{R}^3} \left|\beta^{(\frac{s-1}{2})} \tilde{f}(\mathbf{y})\right|^2 d\mathbf{x}, \tag{3.163}$$

and using the definition of these so called B-Sobolev norms, we find

$$\int_{\mathbb{R}^3} \left|\beta^{(\frac{s-1}{2})} \tilde{f}(\mathbf{y})\right|^2 d\mathbf{x} = \|B^{(\frac{s-1}{2})} f\|^2_{L^2} \equiv \|f\|^2_{\mathcal{H}^{s-1}}, \tag{3.164}$$

thus proving our second lemma,

$$\|w(t)\|^2_{\mathcal{H}^s} + \|\partial_t w(t)\|^2_{\mathcal{H}^{s-1}} = \|f\|^2_{\mathcal{H}^{s-1}} \,. \tag{3.165}$$

These lemmata will be used in the succeeding argument for L^2-spatial decay.

3.5 Light-Cone Argument for L^2-Decay

We now use a *light-cone argument* to find for optimal L^2-decay of the radiating field $u(\mathbf{x})$. From the $\mathcal{H}^s$ and $\dot{H}^s$ norm estimates for $u(\mathbf{x})$, deduced in Eq. (3.79) and Eq. (3.80) for $w(t) \in L^\infty(\mathbb{R}, \mathcal{H}^s)$, we have from the interpolation argument we used for $\|u\|_{\mathcal{H}^s}$ with $0 < s < 1$ (the proof of which is provided in Appendix A.5)

$$\|w(t)\|_{\mathcal{H}^s} < C^{s/2} \|w(t)\|_{\dot{H}^s}, \tag{3.166}$$

and clearly

$$\|w(t)\|_{\mathcal{H}^{s-1}} < C^{\frac{(s-1)}{2}} \|w(t)\|_{\dot{H}^{s-1}}. \tag{3.167}$$

Using the conservation law established in Eq. (3.158), in conjunction with the s-space estimates, we deduce the following estimate

$$\|w(t)\|_{\dot{H}^s(\mathbb{R}^3)} \leq C\|f\|_{\dot{H}^{s-1}(\mathbb{R}^3)}, \tag{3.168}$$

for $s = 1$ we then have

$$\|w(t)\|_{\dot{H}^1(\mathbb{R}^3)} \leq C\|f\|_{\dot{H}^0(\mathbb{R}^3)} \equiv C\|f\|_{L^2(\mathbb{R}^3)}, \tag{3.169}$$

and likewise for $s = 0$ we have

$$\|w(t)\|_{\dot{H}^0(\mathbb{R}^3)} \equiv \|w(t)\|_{L^2(\mathbb{R}^3)} \leq C\|f\|_{\dot{H}^{-1}(\mathbb{R}^3)}, \tag{3.170}$$

and thus

$$\|w(t)\|_{L^2(\mathbb{R}^3)} \cdot \|w(t)\|_{\dot{H}^1(\mathbb{R}^3)} \leq C^{'}\|f\|_{L^2(\mathbb{R}^3)}\|f\|_{\dot{H}^{-1}(\mathbb{R}^3)}. \tag{3.171}$$

Since $f \in L^2(\mathbb{R}^3)$ and is by definition compactly supported in the ball of radius $2R_0$, we have from using our interpolation argument (Appendix A.5) for the dual spaces $-1 \leq s < 0$ that

$$\|f\|_{\dot{H}^{-1}(\mathbb{R}^3)} \equiv \| - \Delta^{-\frac{1}{2}} f\|_{L^2(\mathbb{R}^3)} \leq C^{''}\|f\|_{L^2(\mathbb{R}^3)}, \tag{3.172}$$

using this in the inequality (3.171) we find

$$\|w(t)\|^{1/2}_{L^2(\mathbb{R}^3)} \cdot \|w(t)\|^{1/2}_{\dot{H}^1(\mathbb{R}^3)} \leq C^{'''}\|f\|_{L^2(\mathbb{R}^3)}. \tag{3.173}$$

We now need the result of an important lemma, which I will call the *ST-Z Restriction Lemma*, due to Stalker and Tahvildar-Zadeh and proven in their novel and challenging paper on the Helmholtz equation in $\mathbb{R}^n$ [5], where they provide a proof of the

Sommerfeld radiation conditions and not yet published at the time of going to press.

The ST-Z Restriction Lemma

For the spherical restriction operator S_r, from functions on $\mathbb{R}^n$ to functions on S^{n-1}, defined by

$$S_r u(\alpha) = u(r\alpha),$$

there is a constant $C_0(n)$ such that

$$\|S_r u(\mathbf{x})\|_{L^2(S^{n-1})} \leq C_0(n) \|u(\mathbf{x})\|^{1/2}_{L^2(\mathbb{R}^n)} \cdot \|u(\mathbf{x})\|^{1/2}_{\dot{H}^1(\mathbb{R}^n)} r^{-\frac{n-1}{2}}, \tag{3.174}$$

for all $u(\mathbf{x}) \in L^2(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$.

The *spherical restriction* operator S_r confines functions $u(\mathbf{x})$ in $\mathbb{R}^n$ to lie on hyperspheres S^{n-1}; it is used to get from $\mathbb{R}^n$ to the embedded sphere of radius r, expressed in the notation used above by $S_r u(\alpha)$. The corresponding *Sobolev Restriction theorem* is false and so we must use the result of the *Restriction Lemma* instead [5]. The details of this Lemma are rather involved but the result is crucial to our analysis of the L^2-decay here. As the S_r operation will be used again later, it is worthwhile to give a simple example of its effect in $\mathbb{R}^3$ using spherical coordinates given by

$$\begin{cases} x_1 &= r\sin\theta\cos\phi, \\ x_2 &= r\sin\theta\sin\phi, \\ x_3 &= r\cos\theta, \end{cases}$$

to express the polar form integral

$$\int_{\mathbb{R}^3} f(\mathbf{x})d\mathbf{x} = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta) r^2 dr d\Omega.$$

If g is a function on the unit sphere $S^2 = \{x \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$, with angular coordinates $\gamma = (\sin\theta\cos\phi, \sin\theta\sin\phi)$ we define the surface element $d\sigma(\gamma)$ by

$$\int_{S^2} g(\gamma)d\sigma(\gamma) = \int_0^{2\pi}\int_0^{\pi} g(\gamma)\sin\theta d\theta d\phi,$$

resulting in the spherical restriction to a radius r, that is where

$$\int_{\mathbb{R}} f(\mathbf{x})d\mathbf{x} = \int_{S^2}\int_0^{\infty} f(r\gamma)r^2 dr d\sigma(\gamma).$$

We apply spherical restriction S_r to $u(\mathbf{x})$, as defined by the Laplace transform $\mathcal{L}[w(t,\mathbf{x})]$ of Eq. (3.157), and form the following restricted integral

$$S_r u(\mathbf{x}) = u(r\alpha) = \int_0^{\infty} S_r w(t,\mathbf{x})e^{-zt}dt. \tag{3.175}$$

Recall that $w(t,\mathbf{x})$ is the unique solution for the Cauchy problem

$$\partial_t^2 w(t,\mathbf{x}) + B\cdot w(t,\mathbf{x}) = 0,$$

with generic initial datum $f(\mathbf{x})$, so that

$$S_r w(t,\mathbf{x}) = \int_0^{\infty} S_r[f(\mathbf{x})]\frac{1}{\sqrt{\beta}}\sin\sqrt{\beta}t dt. \tag{3.176}$$

The spherically restricted function $S_r f(\mathbf{x})$ first defined in Eq. (3.5) is

$$S_r f(\mathbf{x}) = -S_r u(\mathbf{x})\Delta\gamma(r) - 2\nabla\left(S_r u(\mathbf{x})\right).\nabla\gamma(r),$$

and is compactly supported in the ball of radius $r = |\mathbf{x}| \leq 2R_0$ where $\gamma(r)$ is zero. The *finite speed of propagation* asserts that the value of the unique solution $w(t,\mathbf{x})$ of the hyperbolic wave equation (3.139) for $t > 0$, is determined only by Cauchy data *in* the ball $\{|\mathbf{x}_0 - \mathbf{x}| \leq ct\}$, which is the intersection of the solid light cone with the initial data hypersurface $\Sigma_{t=0}$ in Minkowski spacetime $(\mathbb{R}^4, \eta)$. The solid cone of this base is called the *domain of dependence* or the *past history of the vertex event* labelled $(\mathbf{x}, t)$; it is precisely the portion of the hypersurface cut off by the characteristics $\{|\mathbf{x}_0 - \mathbf{x}| = ct\}$. If $ct < r - 2R_0$ then

$$S_r w(t, \mathbf{x}) = 0,$$

and thus

$$S_r u(\mathbf{x}) = \int_{(r-2R_0)}^{\infty} S_r w(t, \mathbf{x}) e^{-zt} dt. \tag{3.177}$$

The homogeneous problem for $S_r u(\mathbf{x})$ is defined in all $\mathbb{R}^3$ outside the compact set B_r, the ball of radius $2R_0$; there is radiating field dynamics in $\mathbb{R}^3$, as it were, only when $|\mathbf{x}| > 2R_0$. That is to say, the spherically restricted function $S_r u(\mathbf{x})$ attains a non-trivial value only for $r - 2R_0 > 0$ and for the massless radiating Klein-Gordon field, as $u(\mathbf{x})$ in $\mathbb{R}^3$, this implies that $ct \geq r - 2R_0$. We see from Equation (3.177) above, the hyperbolic wave function $w(t, \mathbf{x})$ is integrated along the world-line parametrised by t, until it attains a non-trivial value after a *proper time* lapse of $t_1 = \frac{1}{c}|2R_0|$ at which point it enters its domain of dependence. It is this fact that brings about the improved L^2-decay estimate which we demonstrate presently.

Using the result of the *Restriction Lemma*, Eq. (3.174), for the case $n = 3$ in conjunction with the bound deduced from Eq. (3.173) above, on the now spherically restricted $w(t, \mathbf{x})$, we find for the $\|S_r u(\mathbf{x})\|$ estimate as follows

$$\|S_r w(t, \mathbf{x})\|_{L^2(S^2)} \leq \tilde{C}_0 \|w(t, \mathbf{x})\|_{L^2(\mathbb{R}^3)}^{1/2} \|w(t, \mathbf{x})\|_{\dot{H}^1(\mathbb{R}^3)}^{1/2} r^{-1} \leq C\|f\|_{L^2(\mathbb{R}^3)} r^{-1}, \tag{3.178}$$

with this we evaluate the norm for the spherically restricted field $u(\mathbf{x})$ as follows

$$\begin{aligned} \|S_r u(\mathbf{x})\|_{L^2(S^2)} &= \left\| \int_{r-2R_0}^{\infty} S_r w(t,\mathbf{x}) e^{-zt} dt \right\|_{L^2(S^2)} \\ \Rightarrow \|S_r u(\mathbf{x})\|_{L^2(S^2)} &\leq \int_{r-2R_0}^{\infty} \|S_r w(t,\mathbf{x})\|_{L^2(S^2)} e^{-zt} dt \\ \Rightarrow \|S_r u(\mathbf{x})\|_{L^2(S^2)} &\leq C \|f\|_{L^2(\mathbb{R}^3)} \frac{1}{r} \int_{r-2R_0}^{\infty} e^{-zt} dt, \end{aligned} \tag{3.179}$$

whence we arrive at the estimate for $\|u(\mathbf{x})\|_{L^2(S^2)}$ on evaluating the integral in t, which is the sought *Sommerfeld Radiation Bound*

$$\|S_r u(\mathbf{x})\|_{L^2(S^2)} \equiv \|u(\mathbf{x})\|_{L^2} \leq C|z|^{-1} e^{2R_0\zeta} \|f\|_{L^2(\mathbb{R}^3)} \frac{1}{r} e^{-\zeta r}. \tag{3.180}$$

This completes the proof of **Theorem 1** for the L^2-decay bound on $u(\mathbf{x})$, which we write concisely as

$$\|u(\mathbf{x})\|_{L^2(S^2)} \leq C' \|f\|_{L^2(\mathbb{R}^3)} \frac{1}{r} e^{-\zeta r} \leq C'' \|u\|_{L^2(\mathbb{R}^3)} \frac{1}{r} e^{-\zeta r}. \tag{3.181}$$

We easily get the K-G bound $|\tilde{\psi}(\rho,\theta,\phi)|$ in $(\mathcal{M},g)$ by using the exterior tortoise coordinate $r(\rho) = \rho + 2M_s \log(\rho/2M_s - 1)$ with the transformation $\tilde{\psi}(x^i) = \frac{r}{\rho} u(\mathbf{x})$.

In conformity with the standard structure of theses I conclude with a brief and temperate paragraph suggesting possible future extensions to this theorem. From previous work [5] on the flat Helmholtz equation in $\mathbb{R}^n$

$$-\Delta v(\mathbf{x}) + P(\mathbf{x}) v(\mathbf{x}) + z^2 v(\mathbf{x}) = 0, \tag{3.182}$$

for potentials $P(\mathbf{x})$ which satisfy the bounds

$$-\frac{L}{r^2} \leq P(\mathbf{x}) \leq \frac{U}{r^\eta}, \qquad L < \frac{(n-2)}{4}, \qquad \eta > 1, \tag{3.183}$$

outside a compact set in $\mathbb{R}^n$, and following a detailed analysis, an L^∞-bound is provided for a radiating solution $v(\mathbf{x})$ and given there by

$$|v(\mathbf{x})|_{L^\infty} \leq C' r^{-\frac{n-1}{2}} e^{-\zeta r}. \tag{3.184}$$

With the following rough heuristic argument we might anticipate a similar *radiation condition* for the radiating field $u(\mathbf{x})$ outside a compact set in $\mathbb{R}^3$ as the solution of

$$-\Delta u(\mathbf{x}) + F(r)\not{\Delta} u(\mathbf{x}) + V(r)u(\mathbf{x}) + z^2 u(\mathbf{x}) = 0, \qquad \mathrm{Re}\; z = \zeta > 0. \tag{3.185}$$

where the potential $V(r)$ satisfies the bound (3.183) for $n = 3$ and with $\eta = 2$, i.e.

$$0 \leq V(r) \leq \frac{U}{r^2}.$$

Very crudely, since $V(r)$ and $F(r)$ vanish at spatial infinity, from the asymptotic behaviour of the Hankel functions we would expect optimal radial decay in outgoing spherical form

$$g(|\mathbf{x}|) \sim y(\theta, \phi)\frac{e^{-\zeta r}}{r},$$

we have shown that as $\rho \to \infty$, $r(\rho) \to \infty$ we have an asymptotic

$$\lim_{r(\rho)\to\infty} F(r) \equiv \lim_{r(\rho)\to\infty} \left(\frac{1}{r^2} - \frac{\alpha^2}{\rho^2}\right) \sim -\frac{4M_s}{r^3}\log\left(\frac{r}{4M_s}\right) \sim \frac{1}{r^4},$$

in this regime $V(r)$ dominates $F(r)$ so that the elliptic-Helmholtz equation (3.185) is asymptotically equivalent, in some sense, to equation (3.182), so that the bound on $|u(\mathbf{x})|$ should be of the same character as the $|v(\mathbf{x})|$ bound. It is worth pointing out that the L^∞-bound has the capacity to falsify competing scalar field analyses in the following way: if a *gedanken experiment* recorded a measurement of the fields intensity at local curvature coordinates $\{R, \theta_R, \phi_R\}$ to be given by $|u(R)|^2$ say, and if the scalar field theory predicts an L^∞-bound, the supremum bound, to be less than $|u(R)|^2$, then the theory contradicts experiment and is thus falsified – some component of the argument is incorrect and requires reconsideration; an L^2-bound, which is essentially a root mean square bound, does not have this discriminating quality. It is feasible that an L^∞-bound can be ascertained by a so-called *bootstrap* argument, using the L^2-bound and *Young's inequality for convolutions* to get from L^2 to L^p and ultimately arrive at the L^∞-norm. This would be an extension of the L^2 result established here and require considerably more effort in establishing its proof.

As much of my time was spent or perhaps misspent on Killing time, in the same building where *he* was born on Westland Row, Dublin 2 (now in danger of collapsing under the heft of the spawning by-products of the 'knowledge' industry), it is as well to finish with this pertinent or perhaps impertinent quote from the timeless wit of Oscar Wilde, all depending dear reader on your relative frames of reference

> ... In the wild struggle for existence, we want to have something that endures, so we fill our minds with rubbish and facts. The mind of the thoroughly well-informed man is a dreadful thing. It is like a bric-à-brac shop, all monsters and dust, with everything priced above its proper value.
>
> Oscar Fingal O'Flahertie Wills Wilde.

Appendix A

Functional Analysis

A.1 Definitions of Metric Spaces

A distance function (or a metric) on a set M is a function $d : M \times M \to \mathbb{R}$ which to any pair of points $x, y \in M$ associates a real number $d(x, y)$, called the *distance* from x to y. To get a reasonable notion of distance, it has proven advantageous to require that the following three conditions are satisfied:

1. **Positive definite**

   $d(x, y) \geq 0$, for all $x, y \in M$, $\quad = 0 \Leftrightarrow x = y$.

2. **Symmetry**

   $d(x, y) = d(y, x)$ for all $x, y \in M$.

3. **Triangle**

   $d(x, z) \leq d(x, y) + d(y, x)$ for all $x, y, z \in M$.

Definition A.1

A *metric* on a set M is a function $d : M \times M \to \mathbb{R}$ satisfying the above three distance conditions.

A pair (M, d) consisting of a set M together with a specific metric d on M is called a *metric space.*

Definition A.2

Let $\mathcal{H}$ be a complex vector space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$; if $\mathcal{H}$ is a *Banach space*, in other words a complete normed vector space, with respect to the induced norm $\| \cdot \|$, it is then called a Hilbert space.

A *Hilbert space* is an inner product space $\mathcal{H}$ which is a Banach space with respect to the induced norm.

Definition A.3

If $A = A^*$, i.e., $\langle Ax, y \rangle = \langle x, Ay \rangle$, where A is an operator on $\mathcal{H}$ and for all $x, y \in \mathcal{H}$ then A is called *self-adjoint (or Hermitian).*

A.2 Function Inequalities

The L^2-Theory: The main idea is to regard orthogonality as if it were a geometric property. The inner product on (a, b) is defined

$$(f, g) \equiv \int f(x)\overline{g(x)}dx.$$

The L^2-norm of f

$$\|f\| \equiv (f, f)^{\frac{1}{2}} = \left[\int_b^a |f(x)|^2 dx\right]^{\frac{1}{2}},$$

and the quantity

$$\|f-g\| = \left[\int_b^a |f(x)-g(x)|^2 dx\right]^{\frac{1}{2}},$$

is a measure of the "distance" between two functions f and g, also called the L^2 metric.

Cauchy-Schwarz: *If $x, y \in X$, where X is an inner product space, then the following inequality holds*

$$|(x,y)| \leq \|x\| \|y\|.$$

Minkowski: *For any real number $p \geq 1$, and any pair of continuous functions $f, g \in C_0(\mathbb{R})$, the following inequality holds*

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

A.3 Sobolev Spaces

Definition 1: The Sobolev space $\mathbf{H}^1(\Omega)$ is defined by

$$\mathbf{H}^1(\Omega) = \left\{v \in L^2(\Omega) : \frac{\partial v}{\partial x_i} \in L^2(\Omega) \quad i = 1 \cdots n\right\}, \tag{A.1}$$

Ω is a general open subset of $\mathbf{R}^n$. The space $\mathbf{H}^1(\Omega)$ is equipped with the scalar product

$$(u,v) = \int_\Omega \left(uv + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\right) dx,$$

and the corresponding squared norm

$$\|v\|^2_{\mathbf{H}^1} = \int_\Omega \left(v^2 + \sum_{i=1}^{N} (\frac{\partial v}{\partial x_i})^2\right) dx. \tag{A.2}$$

Remark: By definition of the *distributional derivative* the following are equivalent:

(a) $v \in \mathbf{H}^1(\Omega)$

(b) $v \in L^2(\Omega)$ and there exists $g_1, g_2, \ldots, g_n \in L^2(\Omega)$ such that

$$\int_\Omega v \frac{\partial \phi}{\partial x_i} dx = -\int_\Omega g_i \phi dx, \qquad \forall\ \phi \in \mathcal{D},$$

Then, by definition, $\frac{\partial \phi}{\partial x_i} = g_i$ in a distributional sense. This definition can be naturally extended when replacing the $L^2(\Omega)$ space by a general $L^p(\Omega)$ space.

Definition 2: For any $1 \leq p \leq \infty$, the Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \left\{ v \in L^p(\Omega) : \frac{\partial v}{\partial x_i} \in L^p(\Omega) \quad i = 1 \cdots n \right\}. \tag{A.3}$$

The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|v\|_{W^{1,p}(\Omega)} = \left[\int_\Omega \left(|v|^p + \sum_{i=1}^{n} \left| \frac{\partial v}{\partial x_i} \right|^p \right) dx \right]^{\frac{1}{p}}. \tag{A.4}$$

When $p = 2$ the space $W^{1,2}(\Omega)$ is often denoted by $H^1(\Omega)$ and this is the notation we use throughout. We do not consider higher order derivatives here.

A.4 Summary of Sobolev Norms

In the simple notation encountered in the thesis, and with the interpolation spaces defined for $0 < s < 1$, we summarise the norms used as follows:

(i) $\|u(\mathbf{x})\|_{L^2}^2 \equiv \int |u(\mathbf{x})|^2 d\mathbf{x}$

(ii) $\|u(\mathbf{x})\|_{L^p}^p \equiv \int |u(\mathbf{x})|^p d\mathbf{x}$

(iii) $\|u(\mathbf{x})\|_{\dot{H}^s}^2 \equiv \|(-\Delta)^{s/2} u(\mathbf{x})\|_{L^2}^2$, for s = 0, $\dot{H}^0 \equiv \|u(\mathbf{x})\|_{L^2}^2$

(iv) $\|u(\mathbf{x})\|_{\dot{H}^1}^2 \equiv \|(-\Delta)^{1/2}u(\mathbf{x})\|_{L^2}^2 = \int \left|(-\Delta)^{1/2}u(\mathbf{x})\right|^2 d\mathbf{x} = \langle\sqrt{-\Delta}u(\mathbf{x}), \sqrt{-\Delta}u(\mathbf{x})\rangle$
$= \langle\Delta u(\mathbf{x}), u(\mathbf{x})\rangle = \int -u(\mathbf{x})\Delta u(\mathbf{x}) = \int |\nabla u(\mathbf{x})|^2 d\mathbf{x} \equiv \|\nabla u(\mathbf{x})\|_{L^2}$

(v) $\|u(\mathbf{x})\|_{H^s}^2 \equiv \|(1-\Delta)^{s/2}u(\mathbf{x})\|_{L^2}^2$, for s $= 0$, $H^0 \equiv \|u(\mathbf{x})\|_{L^2}^2$

(vi) $\|u(\mathbf{x})\|_{\mathcal{H}^s}^2 \equiv \|(-\Delta+V)^{s/2}u(\mathbf{x})\|_{L^2}^2$, for s $= 0$, $\mathcal{H}^0 \equiv \|u(\mathbf{x})\|_{L^2}^2$

(vii) We encounter a special case for the $\dot{H}^{1/2}$ norm $\|u(\mathbf{x})\|_{\dot{H}^{1/2}}^2$
$\equiv 2\pi \int (\zeta^2+\eta^2)^{1/2}|\tilde{u}(\zeta,\eta)|^2 d\zeta d\eta$

A.5 Lemma: Interpolation Bounds $0 \leq |s| \leq 1$:

Suppose that B is a non-negative self-adjoint operator on a Hilbert space $\mathcal{H}$, and that $u \in \mathcal{H}$. We assume without loss of generality that $\mathcal{H}$ is a real Hilbert space, since every complex Hilbert space is also a real Hilbert space, and every self-adjoint operator on the former is then a self-adjoint operator on the latter. Define

$$k_B(s,t) = \pi^{-1}\sin(\pi s) \min_{v+w=u} \left(e^{st-t}\|B^{1/2}v\|^2 + e^{st}\|w\|^2\right),$$

for all $s \in (0,1)$ and all real t. Then, for all

$$\|B^{1/2}u\| = \sqrt{\int_{-\infty}^{\infty} k_B(s,t)dt}, \tag{A.5}$$

for all $s \in (0,1)$.

Equation (A.5) is proved as follows. By spectral theory there is a set M with positive measure μ together with an isometry $U : \mathcal{H} \to L^2(M,\mu)$ and a non-negative measurable function β with the property that

$$B = U^{-1}M_\beta U,$$

where M_β is multiplication by β in $L^2(M,\mu)$. The fractional powers of B referred to above are *defined by*

$$B^{s/2} = U^{-1} M_{\beta^{s/2}} U.$$

Since U is an isometry

$$k_B(s,t) = \pi^{-1}\sin(\pi s) \min_{\tilde{v}+\tilde{w}=\tilde{u}} \left(e^{st-t}\|M_{\beta^{1/2}}\tilde{v}\|^2 + e^{st}\|\tilde{w}\|^2 \right),$$

where $\tilde{u} = Uu \in L^2(M,\mu)$. Now

$$e^{st-t}\|M_{\beta^{1/2}}\tilde{v}\|^2 + e^{st}\|\tilde{w}\|^2 = \int_{y\in M} \left(e^{st-t}\beta(y)\tilde{v}(y)^2 + e^{st}\tilde{w}(y)^2 \right) d\mu(y).$$

From the identity

$$a\xi^2 + b\eta^2 = \frac{ab(\xi+\eta)^2 + (a\xi - b\eta)^2}{a+b},$$

we see that

$$e^{st-t}\beta(y)\tilde{v}(y)^2 + e^{st}\tilde{w}(y)^2 = \frac{e^{2st-t}\beta(y)\tilde{u}(y)^2 + [e^{st-t}\beta(y)\tilde{v}(y) - e^{st}\tilde{w}(y)]^2}{e^{st-t}\beta(y) + e^{st}},$$

when $\tilde{v}(y) + \tilde{w}(y) = \tilde{u}(y)$. The quantity in brackets is zero for

$$\tilde{v}(y) = \frac{1}{1+e^{-t}\beta(y)}\tilde{u}(y),$$

and

$$\tilde{w}(y) = \frac{e^{-t}\beta(y)}{1+e^{-t}\beta(y)}\tilde{u}(y),$$

and positive for any other $\tilde{v}(y)$ and $\tilde{w}(y)$ such that $\tilde{v}(y)+\tilde{w}(y) = \tilde{u}(y)$. These choices therefore minimise the integrand above for each $y \in M$. It follows that

$$k_B(s,t) = \pi^{-1}\sin(\pi s)e^{st} \int_{y\in M} \frac{e^{-t}\beta(y)}{1+e^{-t}\beta(y)}\tilde{u}(y)^2 d\mu(y).$$

Integrating this

$$\int_{-\infty}^{\infty} k_B(s,t)dt = \pi^{-1}\sin(\pi s)\int_{\infty}^{\infty} e^{st}\int_{y\in M}\frac{e^{-t}\beta(y)}{1+e^{-t}\beta(y)}\tilde{u}(y)^2 d\mu(y)dt.$$

Reversing the order of integration,

$$\int_{-\infty}^{\infty} k_B(s,t)dt = \pi^{-1}\sin(\pi s)\int_{y\in M}\beta(y)^2\tilde{u}(y)^2\int_{-\infty}^{\infty}\frac{e^{st-t}\beta(y)^{1-s}}{1+e^{-t}\beta(y)}dtd\mu(y).$$

Making the change of variable $r = 1/(1+e^{-t}\beta(y))$,

$$\int_{-\infty}^{\infty}\frac{e^{st-t}\beta(y)^{1-s}}{1+e^{-t}\beta(y)}dt = \int_0^1 r^{s-1}(1-r)^{-s}dr = \Gamma(s)\Gamma(s-1) = \pi\csc\pi s.$$

Evaluating the inner integrals then,

$$\int_{\infty}^{\infty} k_B(s,t)dt = \int_{y\in M}\beta(y)^s\tilde{u}(y)^2 d\mu(y) = \|M_{\beta^{s/2}}\tilde{u}\| = \|B^{s/2}u\|^2.$$

Suppose now that we have a pair of non-negative self-adjoint operators A and B, such that

$$\|A^{1/2}v\| \leq \|B^{1/2}v\|,$$

for all $v \in \mathcal{H}$. Then

$$e^{st-t}\|A^{1/2}v\|^2 + e^{st}\|w\|^2 \leq e^{st-t}\|B^{1/2}v\|^2 + e^{st}\|w\|^2,$$

and hence

$$k_A(s,t) \leq k_B(s,t),$$

for all $s \in (0,1)$ and for all real t. It then follows from equation (A.5), and from its counterpart for A, that

$$\|A^{s/2}u\| \leq \|B^{s/2}u\|.$$

If, instead of $\|A^{1/2}v\| \leq \|B^{1/2}v\|$, we have

$$\|A^{1/2}v\| \leq C\|B^{1/2}v\|,$$

then we apply the same argument, not to A and B but rather to $C^{-1}A$ and B, obtaining

$$\|A^{s/2}u\| \leq C^s\|B^{s/2}u\|.$$

Appendix B

Aspects of General Relativity

B.1 Covariant Calculus

Scalar Gradient:

$$\nabla\phi = (\nabla_a\phi)\,\mathbf{e}^a = (\partial_a\phi)\,\mathbf{e}^a.$$

Divergence:

$$\nabla.\mathbf{v} \equiv \nabla_a v^a + \Gamma^a_{ab}v^b = \frac{1}{\sqrt{|g|}}\partial_a\left(\sqrt{|g|}v^a\right).$$

Laplacian:

$$\Delta\phi \equiv \nabla_a\nabla^a\phi = \frac{1}{\sqrt{|g|}}\partial_a\left(\sqrt{|g|}g^{ab}\partial_b\phi\right).$$

Stokes Theorem:

$$\int_{\mathcal{M}} d^n x\sqrt{|g|}\nabla_\mu V^\mu = \int_{\partial\mathcal{M}} d^{n-1}y\sqrt{|\gamma|}n_\mu V^\mu.$$

Metric Identities:

$$g \equiv \det[g_{\mu\nu}]; \quad \delta\sqrt{-g} = -1/2\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}; \quad \delta g = gg^{\mu\nu}\delta g_{\mu\nu} = -gg_{\mu\nu}\delta g^{\mu\nu}.$$

B.2 Killing Fields

Any vector K_μ that satisfies $\nabla_{(\mu}K_{\nu)} = 0$ implies that the scalar quantity $K_\nu p^\nu$ is conserved along a geodesic trajectory:

$$\nabla_{(\mu}K_{\nu)} = 0 \quad \Rightarrow p^\nu \nabla_\mu (K_\nu p^\nu) = 0. \tag{B.1}$$

The existence of a timelike Killing vector allows us to define a conserved energy for the entire spacetime. Given a Killing vector K_ν and a conserved energy-momentum tensor $T_{\mu\nu}$ we can construct a current $J_T^\mu = K_\nu T^{\mu\nu}$ that is automatically conserved

$$\nabla_\mu J_T^\mu = (\nabla_\mu K_\nu)T^{\mu\nu} + K_\nu(\nabla_\mu T^{\mu\nu}) = 0.$$

The first term vanishes by virtue of equation B.1 and the second by conservation of $T^{\mu\nu}$.

B.3 Energy Conditions

It is sometimes useful to think about Einstein's equation without specifying the theory of matter from which $T_{\mu\nu}$ is derived. This allows a great deal of arbitrariness; consider for example the question of what metrics obey Einstein's equation? In the absence of some constraint on $T_{\mu\nu}$ the answer is any metric at all; simply take the metric of choice, compute the Einstein tensor $G_{\mu\nu}$ for this metric and demand that $G_{\mu\nu} = T_{\mu\nu}$. It will automatically be conserved, by the *Bianchi identity.* Our real concern is with the existence of solutions to Einstein's equations in the presence of " realistic" sources of energy-momentum. One strategy is to consider specific kinds of sources, such as scalar fields, dust or electromagnetic fields. It is advantageous to understand properties of Einstein's equations that hold for a variety of different sources and so *energy conditions* that limit the arbitrariness of $T_{\mu\nu}$ are imposed. We give two examples here : the **Weak** and **Dominant** energy conditions.

These are coordinate invariant restrictions on $T_{\mu\nu}$. Therefore we must construct scalars from $T_{\mu\nu}$, typically accomplished by *contracting* with arbitrary timelike vectors t^μ or null vectors l^μ. For example the **Weak Energy Condition** is

$$T_{\mu\nu}t^\mu t^\nu \geq 0.$$

For a perfect fluid in coordinate ρ

$$T_{\mu\nu} = \{d(\rho) + p(\rho)\}u_\mu u_\nu \ + p(\rho)g^{\mu\nu},$$

because the pressure is isotropic $T_{\mu\nu}t^\mu t^\nu$ will be nonnegative for all timelike vectors t^μ if both $T_{\mu\nu}u_\mu u_\nu \geq 0$ and $T_{\mu\nu}l_\mu l_\nu \geq 0$ for some null vector l^μ. We therefore evaluate

$$T_{\mu\nu}u^\mu u^\nu = d(\rho), \qquad T_{\mu\nu}l^\mu l^\nu\{d(\rho) + p(\rho)\}(u_\mu l^\mu)^2.$$

The weak energy condition therefore implies $d(\rho) \geq 0$ and $d(\rho)+p(\rho) \geq 0$. These are simply the reasonable-sounding requirements that the energy density be nonnegative and the pressure not too large compared to the energy density.

The Dominant Energy Condition includes the weak energy condition ($T_{\mu\nu}t^\mu t^\nu \geq$

0, for all timelike vectors t^μ), as well as the additional requirement that $T^{\mu\nu}t_\mu$ is a non-spacelike vector (namely that $T_{\mu\nu}T^\nu_\lambda t^\mu t^\lambda \leq 0$). For a perfect fluid, these conditions together are equivalent to the simple requirement that $d(\rho) \geq |p(\rho)|$; the energy density must be nonnegative, and greater than or equal to the magnitude of the pressure.

Most ordinary classical forms of matter, including scalar fields, obey the dominant energy condition. The energy conditions are not, strictly speaking, related to energy conservation; the Bianchi identity guarantees that $\nabla_\mu T^{\mu\nu} = 0$ regardless of whether we impose any additional constraints on $T^{\mu\nu}$. Rather, they serve to prevent other properties that we think of as "unphysical" such as energy propagating faster than light, or empty space spontaneously decaying into compensating regions of positive and negative energy.

B.4 The Conformally Invariant K-G Equation

Here we briefly remind ourselves that an equation for a field $\phi(x)$ is said to be *conformally invariant* if there exists a number $s \in \mathbb{R}$ (called the *conformal weight* of the field) such that ϕ is a solution with a metric $g_{\mu\nu}$, if and only if $\tilde{\phi} = \Omega^s(x^\mu)\phi$ is a solution with metric

$$\tilde{g}_{\mu\nu} = \Omega^2(x^\mu)g_{\mu\nu},$$

and where the conformal factor $\Omega(x^\mu)$ is a smooth, strictly positive function, so a conformal transformation is essentially a local change in scale.

The Lagrangian density of a massive (massless case $m \neq 0$) scalar field in curved spacetime is given by

$$\mathcal{L} = \sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{1}{2}m^2\phi^2 - \lambda R\phi^2\right).$$

We include a direct coupling to the Riemann scalar R parametrized by dimensionless constant λ. In the literature there are two favourite choices for the value of λ: *minimal coupling* simply turns off the direct interaction with R and as we will see *conformal coupling* sets

$$\lambda = \frac{n-2}{4(n-1)},$$

making the scalar field theory invariant under conformal transformations. Many equations for physical fields are conformally invariant and the study of the behaviour of equations under conformal transformations is also useful for many mathematical purposes. Conformal transformations occur in many contexts in General Relativity, in particular in the definition of asymptotic flatness where the transformation brings infinitely remote points to a finite distance [7]. At these points the metric ds^2 is meaningless, but the conformal metric $d\tilde{s}^2$ is regular; it is the conformal structure that proves important for studying the general properties of a spacetime, this is because it determines the causal properties of the neighbourhood of a point, including the properties of null cones and it also emphasises the influence of curvature, through the Ricci scalar R, on the test field dynamics. In the interests of generality it is worth noting that the massless Klein-Gordon equation for a field $\phi(x)$, i.e.

$$\Box_g\phi(x) = g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi(x) = 0, \tag{B.2}$$

is not *conformally invariant* on a general n-dim Lorentzian manifold $(\mathcal{M}, g)$ unless dim $\mathcal{M} = 2$ and noting also for n-dim that $g^{\alpha\beta}g_{\alpha\beta} = n$. Using a transformation

$\tilde{\phi} = \Omega^s \phi$ in the conformal coordinates

$$\Box_{\tilde{g}} \tilde{\phi} = \tilde{g}^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \tilde{\phi}, \tag{B.3}$$

and a simple, if quite tedious, calculation eventually yields

$$\begin{aligned} \Box\tilde{\phi} &= \Omega^{s-4} s(s+n-3)\phi g^{\alpha\beta} \nabla_\alpha \Omega \nabla_\beta \Omega \\ &+ \Omega^{s-3} s\phi g^{\alpha\beta} \nabla_\alpha \nabla_\beta \Omega \\ &+ \Omega^{s-3} (2s+n-2) g^{\alpha\beta} \nabla_\alpha \Omega \nabla_\beta \phi \\ &+ \Omega^{s-2} g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi, \end{aligned} \tag{B.4}$$

now we choose $s = 1 - n/2$ in which case the $\nabla_\alpha \Omega \nabla_\beta \phi$ term is eliminated. Using the Ricci scalar conformal transformation $\tilde{R} = \tilde{g}^{\alpha\beta} \tilde{R}_{\alpha\beta}$ which is given by

$$\begin{aligned} \tilde{R} &= \Omega^{-2}\{R - 2(n-1) g^{\alpha\beta} \nabla_\alpha \nabla_\beta \log \Omega\} \\ &+ \Omega^{-2}(n-2)(n-1) g^{\alpha\beta} \nabla_\alpha (\log \Omega) \nabla_\beta \log \Omega \\ &= \Omega^{-2} R - 2\Omega^{-3}(n-1) g^{\alpha\beta} \nabla_\alpha \nabla_\beta \Omega \\ &+ \Omega^{-4}(n-2)(4-n) g^{\alpha\beta} \nabla_\alpha \Omega \nabla_\beta \Omega. \end{aligned} \tag{B.5}$$

Using the Ricci scalar conformal transformation (B.5) we add the term $\lambda R \phi$ to equation (B.4) with

$$\lambda = -\frac{n-2}{4(n-1)},$$

following Wald [14] for the conformally invariant wave equation for n-dimensional spacetime, with weight $s = 1 - n/2$ we have:

$$\left(\tilde{g}^{ab} \tilde{\nabla}_a \tilde{\nabla}_b - \frac{n-2}{4(n-1)} \tilde{R}\right) \left[\Omega^{1-n/2} \phi\right] = \Omega^{-1-n/2} \left[g^{ab} \nabla_a \nabla_b - \frac{n-2}{4(n-1)} R\right] \phi, \tag{B.6}$$

the right hand side of Equation (B.7) is in the physical metric and the left hand side is in the conformal metric, in our case, $n = 4$ and $s = -1$, and we find

$$\left(\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b - \frac{1}{6}\tilde{R}\right)[\Omega^{-1}\phi] = \Omega^{-3}\left[g^{ab}\nabla_a\nabla_b - \frac{1}{6}R\right]\phi, \tag{B.7}$$

and for the vacuum static spacetime $(\mathcal{M}, g_0)$ the Riemann scalar $R = 0$, so that the conformal K-G equation is

$$\left(\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b - \frac{1}{6}\tilde{R}\right)\Omega^{-1}\phi = \Omega^{-3}g^{ab}\nabla_a\nabla_b\phi, \tag{B.8}$$

a conformally invariant generalization to curved geometry of the K-G equations in flat spaces.

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