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# Integrable lattice models for strongly correlated electrons



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**A thesis submitted for the degree of Ph.D.  
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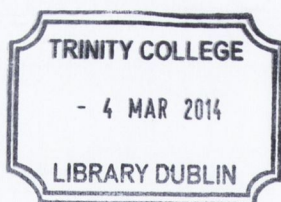


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Thesis 10279

## Summary

This thesis is dedicated to the study of a new family of integrable lattice models for strongly correlated electrons, namely the Hubbard-Shastry models. The techniques of exactly solvable models, and in particular those of integrable systems, are used. The models are constructed through the quantum inverse scattering method and are studied in one dimension through their Bethe ansatz solution.

The main results of this thesis are as follows:

1. The construction of the Hubbard-Shastry lattice models. It is found that there are two subfamilies of hermitian models, each of which have two special points where the models are also parity invariant. These are the Hubbard model, the A- and B-models, and the  $\mathfrak{su}(2|2)$  spin chain.
2. The thermodynamic Bethe ansatz study of the Hubbard-Shastry models. The thermodynamic Bethe ansatz equations are obtained and simplified, and their study in various limits is used to reveal properties of the models that are otherwise hidden.
3. A detailed study of the zero temperature properties of the A- and B-models. The ground state phase diagrams are obtained and excitations above each phase are examined. It is found that the A-model behaves as an itinerant ferromagnet while the B-model is a Mott insulator of spin-singlets.
4. The development of a general formalism for the study of the thermodynamics of Bethe ansatz solvable models. New results include general temperature expressions for the dressing of energy, spin and charge of excitations, and general temperature expressions for the thermodynamic variables.

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In addition I would like to thank my family for their love and support since the day I was born. I wish to dedicate this thesis to my parents John and Mary Quinn.

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# Chapter 1

## Introduction

The quest to understand strongly correlated electrons in low-dimensional systems represents an important frontier in the field of condensed matter physics [1]. There exist a range of materials whose behaviour is strongly correlated and for which a proper understanding has yet to be established. Prominent in this list are the high- $T_c$  superconductors [2], itinerant ferromagnets [3] and heavy fermion systems [4].

The complete description of a solid is a complex many body problem. Simplifications and exact methods are crucial for progress to be made. In particular, exactly solvable models provide a bedrock on which more realistic models can be constructed and examined. The most important such model is that of free particles. It can be solved in any dimension for both bosons and fermions. The physics of the majority of metals can be understood by considering the effect of interactions on an exact free solution. An important example is Landau's much celebrated theory of the Fermi liquid [5].

Strongly correlated materials however are those which cannot be understood by reducing the complexity to a non-interacting picture. The collective behaviour of the constituents can drive the material into unexpected phases, and interacting exactly solvable models are invaluable for getting insight into such effects. The Hubbard model [6] is a prominent example. Indeed it has become a cornerstone of investigations into strongly correlated electronic behaviour. The model is defined on a lattice and its Hamiltonian has two contributions: a kinetic term which describes the hopping of electrons and an interaction term which captures



## 1. INTRODUCTION

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Coulomb repulsion by penalising doubly occupied sites. The model is integrable in one dimension, and describes a Mott insulator [7], which can be seen through its Bethe ansatz exact solution [8, 9].

The Hubbard model is an effective tight binding model in which all the physics is projected onto a single band of electrons. That systems of strongly correlated electrons can be accurately described by such an effective lattice model is an assumption that is central to this thesis. This hypothesis has been the basis for great amount of work over recent decades, in particular since the discovery of unconventional superconductivity by Müller and Bednorz in 1986 [10], and subsequent arguments put forward that this physics can be captured by such single band tight binding models [11, 12]. The subject of strongly correlated electron behaviour is nevertheless a field with many open problems. This is due in no small part to the great difficulty involved in extracting information from even such relatively simple models.

In this thesis we investigate a new family of models for strongly correlated electrons which are exactly solvable in one dimension. These are the Hubbard-Shastry lattice models which were introduced by the author in [13]. In particular we focus on two members of this family which we refer to as the A-model and the B-model. Like the Hubbard model they describe a single band of electrons interacting on a lattice, but the models are quite distinct and the physics they exhibit is different. We will see that in one-dimension the A-model exhibits features of itinerant ferromagnetism, while the B-model is a special Mott insulator in which the electrons are paired into spin-singlets.

The A- and B-models have Hamiltonians which are more involved than that of the Hubbard model. Both have correlated hopping terms, where the magnitudes of the hopping amplitudes depend on the occupation of the lattice sites by other electrons. In addition they have extra effective interactions that take into account processes such as nearest neighbour Coulomb interaction, spin exchange of two neighbouring electrons, and the pair hopping of two electrons from one site to a neighbouring site. The relative contributions of these terms are strongly constrained by the integrability of the models. Like the Hubbard model they both have one free coupling constant, after we discount the overall normalisation of the Hamiltonian.



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We do not attempt to directly physically motivate the models. Rather we are interested in them as they are exactly solvable and so have the potential to provide insight into some class of non-perturbative physics.

In one dimension the models can be obtained from Shastry's R-matrix, the same object that governs the integrability of the Hubbard model [14]. Normally an R-matrix depends on two spectral parameters through their difference, and is thus invariant under shifts of the spectral parameters. Shastry's R-matrix is special however as it is of a non-difference form [15], and so the actual value of the spectral parameters are important. This was noted in particular by [16, 17], who used this extra variable to construct a one-parameter extension of the Hubbard model. Their extension is hermitian for purely imaginary values of this parameter and is in fact also hermitian along another line for complex values of the parameter. The model is parity invariant only for special distinct values of the parameter however and these correspond, in addition to the Hubbard model, to the A- and B-models, and the  $\mathfrak{su}(2|2)$  spin chain.

The models are thus constructed to be exactly solvable in one-dimension by Bethe ansatz [18]-[22]. In Bethe ansatz solvable models the complexity of calculating the spectrum is reduced to the solution of the Bethe equations, a set of polynomials whose degree scales linearly with the length of the system. Each solution corresponds to an eigenstate of the model and gives directly the associated energy eigenvalue. In the thermodynamic limit the roots of these equations, in general complex, align into patterns on the complex plane known as Bethe strings. This gives rise to a physical picture: the different types of Bethe strings can be understood as the particle content of the model. The free energy can thus be written and its minimisation gives access to the equilibrium state. In this way one goes from the full information about the spectrum that the Bethe equations provide to the physically interesting region.

In this thesis we focus on determining the ground state properties, which appear in the zero temperature limit, as these shed most insight into the behaviour of the models. By working in the grand canonical ensemble we can control the density and magnetisation of the ground state, by coupling to them through a chemical potential  $\mu$  and magnetic field  $B$  respectively. The ground state phase

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diagrams are determined and the behaviour of excitations above each phase are investigated.

We pay particular attention to the dressing of the spin and charge of the excitations, which is quite non-trivial in general. A comment is in order here. The zero temperature long-range physics of many one-dimensional models of electrons may be captured by the Luttinger liquid, see e.g. [23]. In the Luttinger liquid spin-charge separation is exhibited, the excitations carry either spin or charge and propagate at different velocities. Let us stress however that individual quasi-particle excitations need not exhibit spin-charge separation, even at zero temperature. In general these carry both spin and charge, the magnitude of which varies with the momentum of the excitation. This is compatible with the Luttinger liquid description as the excitations therein are wave-like. They are composed of infinitely many excited quasi-particles carrying infinitesimal energies, and in this limit spin-charge separation may be seen. The non-trivial dressing of spin and charge beyond the Luttinger regime does not appear to have received much attention. This leads us to speculate that these features may account for some of the anomalous behaviour observed in experimental studies of strongly correlated materials. It would be interesting to consider issues such as the unusual temperature dependence of resistivity, the Hall coefficient, and spin excitations in this context, see e.g. [24, 25].

Let us now summarise the results of our investigations of the A- and B-models.

### 1.1 Summary of the A-model

First we comment on the model's Hamiltonian. The kinetic term is of a complicated correlated hopping type, and does not reduce to free fermions for any value of the coupling constant. Moreover, the parity invariance of the kinetic term is not straightforward, despite the fact that the spectrum of the model is explicitly parity invariant. It is necessary to supplement the naive parity transformation with a change of basis generated by a non-trivial unitary transformation. In the limit of weak coupling constant the model reduces to the  $\mathfrak{su}(2|2)$ -spin chain [26], the model of graded permutations. In the strong coupling limit the Hubbard on-site interaction is the dominant term in the Hamiltonian. Here the model behaves



as a ferromagnetic  $t$ - $J$  model however, in contrast with the antiferromagnetic behaviour of the Hubbard model in this limit.

The ground state of the model is spin-polarised for the full range of electron density and as such the model describes an itinerant ferromagnet. It contains only spin-up electrons for a magnetic field  $B > 0$ , only spin-down electrons for  $B < 0$  and exactly at  $B = 0$  there is degeneracy between all levels of magnetisation. This behaviour is reminiscent of the ferromagnetic Heisenberg spin chain. That the behaviour extends to all fillings however is quite non-trivial, especially considering that the Hamiltonian of the model contains only nearest neighbour terms. The phase diagram is presented in Figure 5.3 for  $\mu < 0$  and  $B > 0$ . The density increases for increasing  $\mu$  either side of half-filling. There is a range of  $\mu$  for which the ground state is half-filled indicating that this is an insulating phase.

Now we come to the excitations above the ground state. At half-filling all excitations carrying charge are gapped. Gapless excitations only exist for  $B = 0$  and these are magnons and their bound states, and at low energies they have quadratic dispersion. The simplest excitation carrying charge is an electron-like excitation. This is gapped at half-filling but becomes gapless with linear dispersion away from half-filling. It carries the charge and spin of an electron for all fillings and all values of  $B$  and its spin is aligned with that of the spin-polarised ground state. There is no elementary excitation of the opposite spin however. Such an excitation should in fact be regarded as a compound excitation of the aligned electronic excitation and a magnon. There are further excitations which are bound states of electronic and magnonic excitations but these are gapped throughout the parameter space. The excitations that behave as magnons at half-filling retain their quadratic dispersion away from half-filling but get dressed with some charge in addition to their spin. As consequence of the quadratic dispersion, the spectrum of low-lying excitations cannot be linearised at any filling in zero magnetic field and the system cannot be approximated by a Luttinger liquid. This is evidenced by the existence of the gapless electronic excitation away from half-filling and the lack of spin-charge separation. In the presence of a magnetic field the dressed magnons become gapped, and there are then no low-lying excitations that decrease the magnetisation of the ground state without decreasing the filling.



### 1.2 Summary of the B-model

In the B-model each of the interaction terms appearing in the Hamiltonian appear at the same order of magnitude. Unlike the Hubbard and A-models, there is no value of the coupling constant for which the Hubbard interaction is dominant. This is a consequence of a hidden dynamical  $\mathfrak{su}(2|2)$  symmetry that the model exhibits for all values of the coupling. This includes, in addition to two sets of bosonic  $\mathfrak{su}(2)$  generators that act independently on the charge and spin degrees of freedom, a set of fermionic generators that mixes these degrees of freedom. In the weak coupling limit this dynamical symmetry becomes singular and the model reduces to free fermions. In the strong coupling limit on the other hand the model is equivalent to the Essler-Korepin-Schoutens (EKS) model [27] with its coupling  $U = 4$ .

The B-model describes a Mott insulator, like the Hubbard model, but the form of the ground state is quite different. The ground state phase diagram is presented in Figure 6.2. For  $B = 0$  the magnetisation is zero and all of the electrons are paired into spin-singlets. For  $\mu < -2$  the energy cost of having any particles in the ground state is too high but as  $\mu$  increases beyond this value the ground state starts to fill with spin-singlets. The density increases with increasing  $\mu$  up to some value  $\mu_0 < 0$  at which point the ground state becomes half-filled. Beyond  $\mu_0$  the density does not change for a range of  $\mu$ , which indicates that in this regime the model is insulating. Although the spin-singlets are paired electrons they have no binding energy. The pairs can be broken by an introducing an arbitrarily small field  $B$ . The ground state then contains a mix of spin-singlet pairs and spin-polarised electrons, and is magnetised. When  $B$  reaches  $2 + \mu$  all of the spin-singlets pairs get broken and the ground state becomes maximally spin-polarised.

The B-model has gapless excitations spin carrying so long as the ground state is not spin polarised. There are also gapless charge carrying excitations away from half-filling. In general all excitations carry both spin and charge to some extent. A special regime however is at half-filling with  $B = 0$ , where the ground state has zero magnetisation and is in an insulating phase. Here a strong form of spin-charge separation emerges: all excitations are scattering states of gapless

spinons and gapped holons. The picture is similar to that of the Hubbard model [9], the difference is that here the spinons are dressed electrons and the holons are dressed spin-singlets whereas in the Hubbard model the spinons are dressed magnons and it is the holons that are dressed electrons. If one goes away from half-filling or introduces a magnetic field then the spin and charge become energy dependent. Keeping the magnetic field zero and going away from half-filling the dressed spin-singlet becomes gapless. It remains pure spin but the charge it carries increases, and increases further with increasing energy. In particular, away from half-filling the magnitude of the charge carried by a particle excitation is greater than that carried by a hole excitation. The electrons are also gapless here but get dressed with a charge in addition to their spin. The dressed charge of an excited electron does go to zero however as its dressed energy goes to zero, and so these excitations are compatible with the spin-charge separated excitations of a Luttinger liquid. If on the other hand a magnetic field is applied at half-filling, then the spin-singlets get dressed with some spin in addition to their charge, but with a polarisation opposite to that of the applied field. Here the dressed electrons remain pure spin, and so the magnetic field has the effect of depleting the charge carriers. Let us conclude by remarking that the dressed electrons display an “hourglass” dispersion and that away from half-filling and in the absence of a magnetic field it can be clearly seen that the spin-singlets are held together by spin-spin interactions.

### 1.3 Extended models

The interactions of the A- and B-models just discussed are strongly constrained by their integrability. Physically it is interesting to consider the effect of decoupling the relative strengths of the interactions. Perturbing some interactions may allow new phases to develop, and understanding these effects may reveal much about the microscopic physics that gives rise to such phases.

For example, recall that in the B-model the coupling constant enters in such a way that all interactions are of the same order of magnitude. As the Hubbard interaction is generally thought to be the dominant interaction it would be interesting to examine the effect of increasing it. Indeed the resulting two parameter



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model interpolates between the B-, Hubbard, EKS, and  $t$ - $J$  models and so it might be a useful toy model for studying transitions between the phases they exhibit. Another direction would be to investigate if there is a perturbation to the B-model that gives a binding energy to the paired electrons that make up the ground state. As we know that the electrons are held together by spin-spin interactions, it would be natural in this regard to consider the effect of increasing the magnitude of the anti-ferromagnetic spin exchange interaction. As to the A-model, let us recall that the phase diagram was very simple. For now one can only speculate on what new and interesting behaviour variations of the interactions would open up.

Naturally it is to be expected that in general such extensions destroy the integrability of the models. There do however exist certain special cases where they do not. In particular it is possible to decouple the Hubbard interaction for both the weak coupling limit of the A-model and the strong coupling limit of the B-model while preserving integrability. This is fortunate as the Hubbard interaction is perhaps the most physically important and it is desirable to be able to control it.

Remarkably, these integrable extensions of the A- and B-models are in fact the same model up to the overall sign of the Hamiltonian. Furthermore they are equivalent to the EKS model. The formalism that we use to study the thermodynamics of the Hubbard-Shastry models can be naturally adopted to the EKS model, and allows us to obtain simplified thermodynamic Bethe ansatz equations. We obtain the ground state phase diagram and find that the role played by the decoupled Hubbard interaction is quite straightforward, it amounts to a shifting of the chemical potential. For the B-model in particular we see that this has no dramatic effect on the physics.

### 1.4 Outline of the thesis

Let us conclude the introduction with an outline of the thesis. The work is based primarily on the author's two papers [13, 28].

Chapter 2 contains an investigation of the subject of integrable lattice models for strongly correlated electrons. First the Hubbard model is introduced. Then



considerations are made on what extensions of the model may give rise to interesting physics, and it is here that the origin of the Hubbard-Shastry models is identified. Symmetry plays an important role in the discussion and we proceed to discuss dynamical symmetry. The quantum inverse scattering method is outlined and is used to construct the Hubbard-Shastry models. The Bethe equations are examined and the hermitian and parity invariant models are identified.

In chapter 3 a general formalism for the exact solution of Bethe ansatz solvable models is presented. Focus is placed on obtaining thermodynamic quantities and to investigating excitations about equilibrium. The zero temperature limit is also considered in some detail.

The foundations for the study of the thermodynamics of the Hubbard-Shastry models are then made in chapter 4. In particular the thermodynamic Bethe ansatz equations are obtained, simplified, and examined in various limits.

Chapters 5 and 6 contain detailed studies of the A- and B-models respectively. General features are first examined at the level of the both the Hamiltonian and the Bethe equations. The model's exact solutions are used to construct the ground state phase diagram and to investigate excitations at zero temperature.

The integrable extensions of the A- and B-models are studied in chapter 7. Their equivalence is shown, and the TBA equations and ground state phase diagram are obtained.

In appendix A our conventions, definitions and notations are summarised. Appendix B presents the formalism that we use for graded vector spaces. The symmetry algebra  $\mathfrak{su}(2|2)_{\mathcal{C}}$  is discussed in appendix C and Shastry's R-matrix is explicitly presented. The solution to the algebraic limit of the TBA equations is presented in appendix D. Finally in appendix E it is shown how the  $t$ - $J$  model appears from a general model in the limit of strong Coulomb repulsion.

## 1. INTRODUCTION

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# Chapter 2

## Integrable lattice models

There is great difficulty involved in determining the properties of an arbitrary lattice model. Complexity grows rapidly with system size and exact diagonalisation is only possible for very small systems. Beyond models which are essentially non-interacting, it is difficult to get a clear picture of a model's physics. In one dimension however a special class of models enjoy the property of integrability. This which allows direct access to their physics by means of an exact solution.

In this thesis we focus our attention on lattice models of strongly correlated electron systems. This leads us to consider lattice models with four states per site: the site can be either empty, occupied by an electron with spin-up or spin-down, or the site can be doubly occupied containing both a spin-up and a spin-down electron. Due to the fermionic nature of electrons these states generate a four dimensional graded vector space. Let us denote this space by  $\mathcal{V}$  and write its basis as

$$|e_1\rangle \leftrightarrow |0\rangle, \quad |e_2\rangle \leftrightarrow \mathbf{c}_\uparrow^\dagger \mathbf{c}_\downarrow^\dagger |0\rangle, \quad |e_3\rangle \leftrightarrow \mathbf{c}_\uparrow^\dagger |0\rangle, \quad |e_4\rangle \leftrightarrow \mathbf{c}_\downarrow^\dagger |0\rangle, \quad (2.1)$$

where  $\mathbf{c}_\sigma^\dagger$ ,  $\mathbf{c}_\sigma$  are the canonically anticommuting fermionic operators which respectively create and annihilate electrons of spin  $\sigma = \uparrow, \downarrow$ , and  $|0\rangle$  is the vacuum state. The states  $\{|e_1\rangle, |e_2\rangle\}$  are graded with respect to  $\{|e_3\rangle, |e_4\rangle\}$ , and by convention we call the first pair even and the second pair odd. Our conventions for graded vector spaces follow closely those of [29, 9] and are presented in Appendix B. In this thesis we focus on models on a periodic one-dimensional lattice of length  $L$ ,



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with Hilbert space given by the  $L$ -fold tensor product of  $\mathcal{V}$

$$\mathcal{H} = \mathcal{V} \otimes \dots \otimes \mathcal{V}. \quad (2.2)$$

Such models are well suited to capture the physics arising from the strong correlations of electrons.

### 2.1 The Hubbard model

A central role in the study of strongly correlated electronic behaviour is played by the Hubbard model [6]. The model has a kinetic term that captures the hopping of electrons around the lattice and an interaction term that penalises having a spin-up electron and a spin-down electron on the same site. In this way the model describes mobile electrons subject to Coulomb repulsion. The model exhibits a metal insulator transition, becoming an insulator at half-filling. It is known as a Mott insulator [7] as the insulating phase is driven purely by electronic correlations. Moreover, the model is integrable in one dimension and its exact solution [8] provides a clear understanding of the nature of the Mott transition. The book [9] provides an extensive review of the one-dimensional Hubbard model.

The Hamiltonian of the one-dimensional Hubbard model is

$$\mathbf{H} = \sum_{j=1}^L \mathbf{H}_{j,j+1} = \sum_{j=1}^L (\mathbf{T}_{j,j+1} + 4\mathbf{u} \mathbf{V}_{j,j+1}^H), \quad (2.3)$$

where  $L$  is the length of the chain and the kinetic and potential terms are given by

$$\mathbf{T}_{j,k} = - \sum_{\sigma=\uparrow,\downarrow} (\mathbf{c}_{j,\sigma}^\dagger \mathbf{c}_{k,\sigma} + \mathbf{c}_{k,\sigma}^\dagger \mathbf{c}_{j,\sigma}), \quad (2.4)$$

$$\mathbf{V}_{j,k}^H = \frac{1}{2} (\mathbf{n}_{j,\uparrow} - \frac{1}{2})(\mathbf{n}_{j,\downarrow} - \frac{1}{2}) + \frac{1}{2} (\mathbf{n}_{k,\uparrow} - \frac{1}{2})(\mathbf{n}_{k,\downarrow} - \frac{1}{2}) - \frac{1}{4}. \quad (2.5)$$

Here the fermionic operators carry an index to label the site on which they act. The operator  $\mathbf{n}_{j,\sigma} = \mathbf{c}_{j,\sigma}^\dagger \mathbf{c}_{j,\sigma}$  is the local particle number operator for electrons of spin  $\sigma$  at site  $j$ , and in what follows we also use  $\mathbf{n}_j = \mathbf{n}_{j,\uparrow} + \mathbf{n}_{j,\downarrow}$ . The coupling constant  $\mathbf{u}$  is  $\mathbf{u} = U/4t$ , and the hopping parameter  $t$  is set to 1.

## 2.2 Extended Hubbard models

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The Hamiltonian (2.3) has a lot of symmetry. Beyond possessing parity and translational invariance it is invariant under a spin  $\mathfrak{su}(2)$  and a charge  $\mathfrak{su}(2)$ , the generators of which are given respectively by

$$\begin{aligned} \mathbf{S}^+ &= \sum_{j=1}^L \mathbf{S}_j^+, & \mathbf{S}^- &= \sum_{j=1}^L \mathbf{S}_j^-, & \mathbf{S}^z &= \sum_{j=1}^L \mathbf{S}_j^z, \\ \boldsymbol{\eta}^+ &= \sum_{j=1}^L (-1)^j \boldsymbol{\eta}_j^+, & \boldsymbol{\eta}^- &= \sum_{j=1}^L (-1)^j \boldsymbol{\eta}_j^-, & \boldsymbol{\eta}^z &= \sum_{j=1}^L \boldsymbol{\eta}_j^z. \end{aligned} \quad (2.6)$$

where

$$\begin{aligned} \mathbf{S}_j^+ &= \mathbf{c}_{j,\uparrow}^\dagger \mathbf{c}_{j,\downarrow}, & \mathbf{S}_j^- &= \mathbf{c}_{j,\downarrow}^\dagger \mathbf{c}_{j,\uparrow}, & \mathbf{S}_j^z &= \frac{1}{2}(\mathbf{n}_{j,\uparrow} - \mathbf{n}_{j,\downarrow}), \\ \boldsymbol{\eta}_j^+ &= \mathbf{c}_{j,\downarrow}^\dagger \mathbf{c}_{j,\uparrow}^\dagger, & \boldsymbol{\eta}_j^- &= \mathbf{c}_{j,\uparrow} \mathbf{c}_{j,\downarrow}, & \boldsymbol{\eta}_j^z &= \frac{1}{2}(\mathbf{n}_j - 1). \end{aligned} \quad (2.7)$$

To gain control over the electron density and the magnetisation it is useful to consider the more general Hamiltonian

$$\mathbf{H} - \mu \mathbf{N} - 2B \mathbf{S}^z, \quad (2.8)$$

with  $\mathbf{N} = \sum_{j=1}^L \mathbf{n}_j = 2\boldsymbol{\eta}^z + L$ . Here the chemical potential  $\mu$  and magnetic field  $B$  break the charge and spin  $\mathfrak{su}(2)$  respectively. Each leave a  $\mathfrak{u}(1)$  subalgebra unbroken as  $[\mathbf{H}, \mathbf{N}] = 0$  and  $[\mathbf{H}, \mathbf{S}^z] = 0$ . We will see in section 2.4 that the  $\mathfrak{su}(2)$  symmetries have been dynamically broken. Let us observe that

$$\mathbf{V}^H \equiv \sum_{j=1}^L \mathbf{V}_{j,j+1}^H = \sum_{j=1}^L \mathbf{n}_{j,\uparrow} \mathbf{n}_{j,\downarrow} - \frac{1}{2} \mathbf{N}. \quad (2.9)$$

The Hubbard Hamiltonian is most commonly presented in a form that hides the charge  $\mathfrak{su}(2)$  symmetry with  $\mu = -2u$  and  $B = 0$  in (2.8).

## 2.2 Extended Hubbard models

The Hubbard model is an effective model for the behaviour of electrons in a solid. For a given material the effective model on  $\mathcal{H}$ , given by (2.2), can be obtained by integrating out all degrees of freedom except for those corresponding to a single band of electrons, and in the process projecting the physics onto  $\mathcal{H}$ . In practice this is a very difficult task, especially if the electrons exhibit strong correlations and so mean field techniques cannot be used. It is of course not to be expected that the resulting Hamiltonian is that of the Hubbard model (2.3), in general



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all possible processes will enter the effective model with some coefficient. While the physics of the Hubbard model may be typical of a class of behaviour on  $\mathcal{H}$ , it is likely that there is a range of behaviour that it does not capture. In this thesis we identify and study exactly solvable models which may be viewed as representatives for other possible classes of behaviour.

Let us consider a general model on  $\mathcal{H}$ . If the model is completely arbitrary then there is little that can initially be said. Physics is described by the languages we have at our disposal, for example symmetry, geometry and topology. Here we wish to focus on symmetry<sup>1</sup>, in particular as we are considering models in one dimension. Firstly, as electrons have spin-1/2 it is natural to have an  $\mathfrak{su}(2)$  symmetry on the states  $|e_3\rangle$  and  $|e_4\rangle$  of (2.1). For the other two states  $|e_1\rangle$  and  $|e_2\rangle$  to be related by symmetry it is also natural to have an  $\mathfrak{su}(2)$  acting on them, as indeed was the case for the Hubbard model (2.6). An interesting question now is whether all four states can be related by symmetry? Here we recall that the states  $\{|e_3\rangle, |e_4\rangle\}$  are graded with respect to  $\{|e_1\rangle, |e_2\rangle\}$ , i.e. one set is fermionic while the other is bosonic. Thus a superalgebra is needed and the natural candidate is  $\mathfrak{su}(2|2)$ , which has, along with the two  $\mathfrak{su}(2)$  algebras, a set of fermionic generators that act between them. The symmetry  $\mathfrak{su}(2|2)$  is however very restrictive, it constrains the Hamiltonian to act as graded permutations on states at neighbouring sites [26]. For a model which has two states per site, such as the Heisenberg model, there is not much extra freedom beyond permutations, but in the present case these fall far short of taking full advantage of the richness of the Hilbert space.

It is a remarkable fact that  $\mathfrak{su}(2|2)$  admits an exceptional central extension  $\mathfrak{su}(2|2)_c$ . The extension explicitly breaks the supersymmetry but the algebra nevertheless mixes the four states (2.1). Interestingly the Hubbard model can be directly related to  $\mathfrak{su}(2|2)_c$  [31]. Moreover, there is a family of models that originate from this exceptional symmetry algebra. These are the Hubbard-Shastry models which are the main subject of this thesis. A discussion on  $\mathfrak{su}(2|2)_c$  is presented in appendix C and the relationship to the Hubbard-Shastry models is developed in section 2.6.

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<sup>1</sup>The following discussion is motivated by a similar consideration in [30]



For now let us make some general considerations on the form of possible nearest neighbour Hamiltonians acting on  $\mathcal{H}$ . First we introduce some additional interactions complementing the Hubbard interaction  $\mathbf{V}^H$ . These are

$$\begin{aligned}\mathbf{V}_{j,k}^{CC} &= \boldsymbol{\eta}_j^z \boldsymbol{\eta}_k^z - \frac{1}{4} = \frac{1}{4} (\mathbf{n}_j - 1)(\mathbf{n}_k - 1) - \frac{1}{4}, \\ \mathbf{V}_{j,k}^{SS} &= \frac{1}{2} (\mathbf{S}_j^+ \mathbf{S}_k^- + \mathbf{S}_j^- \mathbf{S}_k^+) + \mathbf{S}_j^z \mathbf{S}_k^z \\ &= \frac{1}{2} (\mathbf{c}_{j,\uparrow}^\dagger \mathbf{c}_{j,\downarrow} \mathbf{c}_{k,\downarrow}^\dagger \mathbf{c}_{k,\uparrow} + \mathbf{c}_{j,\downarrow}^\dagger \mathbf{c}_{j,\uparrow} \mathbf{c}_{k,\uparrow}^\dagger \mathbf{c}_{k,\downarrow}) + \frac{1}{4} (\mathbf{n}_{j,\uparrow} - \mathbf{n}_{j,\downarrow}) (\mathbf{n}_{k,\uparrow} - \mathbf{n}_{k,\downarrow}), \\ \mathbf{V}_{j,k}^{PH} &= \frac{1}{2} (\boldsymbol{\eta}_j^+ \boldsymbol{\eta}_k^- + \boldsymbol{\eta}_j^- \boldsymbol{\eta}_k^+) = \frac{1}{2} (\mathbf{c}_{j,\uparrow}^\dagger \mathbf{c}_{j,\downarrow} \mathbf{c}_{k,\downarrow} \mathbf{c}_{k,\uparrow} + \mathbf{c}_{k,\uparrow}^\dagger \mathbf{c}_{k,\downarrow} \mathbf{c}_{j,\downarrow} \mathbf{c}_{j,\uparrow}).\end{aligned}$$

The charge-charge interaction  $\mathbf{V}^{CC}$  has the interpretation of nearest neighbour Coulomb repulsion because physically one should think of a site with one electron on it as being charge neutral, and so a site with electron density less than one is positively charged, and a site with electron density greater than one is negatively charged. The spin-spin interaction  $\mathbf{V}^{SS}$  is the familiar spin-exchange term of the Heisenberg XXX spin chain. The pair hopping interaction  $\mathbf{V}^{PH}$  relates to the simultaneous hopping of two electrons from one site to a neighbouring site. Let us remark these interactions are expressed naturally in terms of the generators of the two  $\mathfrak{su}(2)$  algebras (2.6).

Now we write a general nearest neighbour Hamiltonian for a one-dimensional lattice of length  $L$

$$\begin{aligned}\mathbf{H} &= \sum_{j=1}^L \left( \mathbf{T}_{j,j+1} + \kappa_H \mathbf{V}_{j,j+1}^H + \kappa_{CC} \mathbf{V}_{j,j+1}^{CC} + \kappa_{SS} \mathbf{V}_{j,j+1}^{SS} + \kappa_{PH} \mathbf{V}_{j,j+1}^{PH} \right), \\ \mathbf{T}_{j,k} &= - \sum_{\sigma} \left[ \mathbf{c}_{j,\sigma}^\dagger \mathbf{c}_{k,\sigma} (\tau_0 + \tau_1 \mathbf{n}_{j,-\sigma} + \tau_2 \mathbf{n}_{k,-\sigma} + \tau_3 \mathbf{n}_{j,-\sigma} \mathbf{n}_{k,-\sigma}) \right. \\ &\quad \left. + \mathbf{c}_{k,\sigma}^\dagger \mathbf{c}_{j,\sigma} (\bar{\tau}_0 + \bar{\tau}_1 \mathbf{n}_{j,-\sigma} + \bar{\tau}_2 \mathbf{n}_{k,-\sigma} + \bar{\tau}_3 \mathbf{n}_{j,-\sigma} \mathbf{n}_{k,-\sigma}) \right].\end{aligned}\tag{2.10}$$

We see that there is much freedom in this general expression. It could be made even more general, for example the spin-spin interaction  $\mathbf{V}_{j,j+1}^{SS}$  could be made anisotropic by decoupling the  $\mathbf{S}_j^z \mathbf{S}_{j+1}^z$  term within it, or one could even consider breaking the spin symmetry of the kinetic term. A complete study, say by numeric methods, of (2.10) is an overly daunting task. The Hubbard-Shastry models

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provide slices in the parameter space and, through their integrability, clear insight into the corresponding physics.

Let us mention here some other integrable models defined on  $\mathcal{H}$ . The EKS model [27] is an integrable extension of the  $\mathfrak{su}(2|2)$  spin chain in which the Hubbard interaction is decoupled. We will discuss this model in some detail in chapter 7. Another is the supersymmetric-U model [32, 33]. This model is based on the four dimensional representation of  $\mathfrak{su}(2|1)$  and so does not have a symmetry acting between the charge degrees of the freedom. Its quantum generalisation has also been investigated [34, 35]. Then there are the Alcaraz-Bariev models [36] discussed in the Outlook in chapter 8. These are based on the quantum deformation of  $\mathfrak{su}(2|2)_C$  [37], and can be regarded as quantum generalisations of the Hubbard-Shastry models. Discussions similar to this can be found in the introduction to [37, 38].

### 2.3 Opposite models

Given a Hamiltonian one can study either the model determined by  $\mathbf{H}$  or  $-\mathbf{H}$ . For example instead of the Hubbard model one could consider

$$\mathbf{H}^{\text{attr.H}} = \sum_{j=1}^L \left( -\mathbf{T}_{j,j+1} - 4\mathbf{u} \mathbf{V}_{j,j+1}^H \right). \quad (2.11)$$

This is the attractive Hubbard model and its physical properties are very different from the repulsive case, see e.g. [9]. For the Hubbard-Shastry models, which are not constructed purely from physical reasoning but rather to have rich symmetry on  $\mathcal{H}$ , one has a choice for the sign of the Hamiltonian. With a view to capturing some interesting physics the sign is generally chosen so that there is a cost for a doubly occupied site relative to a singly occupied site, mirroring Coulomb repulsion. Nevertheless it may be of interest to consider both possibilities.

Although the properties depend strongly on the choice of sign the two cases are closely related. Let us demonstrate this first for the case of the Hubbard model. Under a unitary transformation generated by

$$\mathbf{U}_1(\alpha) = \exp \left( i\alpha \sum_{j=1}^L j (\mathbf{n}_{j,\uparrow} + \mathbf{n}_{j,\downarrow}) \right) \quad (2.12)$$



with  $\alpha$  an arbitrary real parameter, the kinetic term in (2.11),  $k = j + 1$ , transforms as

$$\mathbf{T}_{j,k} \rightarrow \mathbf{U}_1^\dagger(\alpha) \mathbf{T}_{j,k} \mathbf{U}_1(\alpha) = - \sum_{\sigma=\uparrow\downarrow} (e^{i\alpha} \mathbf{c}_{j,\sigma}^\dagger \mathbf{c}_{k,\sigma} + e^{-i\alpha} \mathbf{c}_{k,\sigma}^\dagger \mathbf{c}_{j,\sigma}), \quad (2.13)$$

while the interaction  $\mathbf{V}^H$  is invariant. Thus the Hamiltonian (2.11) is unitary equivalent to

$$\sum_{j=1}^L (\mathbf{T}_{j,j+1} - 4\mathbf{u} \mathbf{V}_{j,j+1}^H) \quad (2.14)$$

under  $\mathbf{U}_1(\pi)$ . In momentum space the transformation  $\mathbf{U}_1(\alpha)$  corresponds to a shift  $p \rightarrow p' = p + \alpha$  as (2.13) is diagonalised as  $-2 \sum_p \cos(p + \alpha) (\mathbf{n}_{p,\uparrow} + \mathbf{n}_{p,\downarrow})$ . The attractive Hubbard model is conventionally written as (2.14) so that states with  $p' = 0$  then minimize the kinetic part of the Hamiltonian.

For the general Hamiltonian (2.10) the unitary transformation  $\mathbf{U}_1(\pi)$  maps  $\mathbf{T}_{j,j+1} \rightarrow -\mathbf{T}_{j,j+1}$  while leaving the interaction terms invariant. Thus we choose to write the Hamiltonian that has the opposite spectrum as

$$\tilde{\mathbf{H}} = -\mathbf{U}_1^\dagger(\pi) \mathbf{H} \mathbf{U}_1(\pi). \quad (2.15)$$

In what follows we refer to  $\tilde{\mathbf{H}}$  as the *opposite* Hamiltonian.

Let us remark that for the Hubbard-Shastry models a particularly direct relationship between the models and their opposite models appears at the level of the thermodynamic Bethe ansatz equation and this is discussed in section 4.5.

## 2.4 Dynamical symmetry

Symmetry has played an important role in our discussion. Before we progress we would like to discuss *dynamical* symmetries. We call a symmetry dynamical if the generators have explicit time dependence but are nevertheless conserved.

For example, suppose a set of symmetry generators  $\mathbf{A}_j$  do not commute with the Hamiltonian of a system but rather satisfy

$$[\mathbf{H}, \mathbf{A}_j] = \lambda_j \mathbf{A}_j, \quad (2.16)$$

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for some scalars  $\lambda_j \in \mathbb{R}$ . Then recalling Heisenberg's equation of motion for an operator  $\mathbf{O}(t)$

$$\frac{d\mathbf{O}(t)}{dt} = \frac{\partial\mathbf{O}(t)}{\partial t} + i[\mathbf{H}, \mathbf{O}(t)], \quad (2.17)$$

we note that the set of operators

$$\tilde{\mathbf{A}}_j(t) = e^{-i\lambda_j t} \mathbf{A}_j \quad (2.18)$$

are conserved, i.e. that they have zero total time derivative. If the operators  $\tilde{\mathbf{A}}_j(t)$  satisfy the same symmetry algebra as the  $\mathbf{A}_j$ , then we say that the system has a dynamical symmetry, or that the symmetry is dynamically broken. The dynamical symmetry lifts the spectral degeneracy of states related by the symmetry generators in an exact way. If say  $|\psi\rangle$  is an eigenstate of  $\mathbf{H}$  with energy  $E_\psi$ , then the states  $\tilde{\mathbf{A}}_j(t)|\psi\rangle$  are eigenstates with energy  $E_\psi + \lambda_j$ . As an equation, this statement takes the form

$$\mathbf{H}|\psi\rangle = E_\psi|\psi\rangle \quad \Rightarrow \quad \mathbf{H}(\tilde{\mathbf{A}}_j(t)|\psi\rangle) = (E_\psi + \lambda_j)(\tilde{\mathbf{A}}_j(t)|\psi\rangle). \quad (2.19)$$

In addition this shows that hermiticity of the Hamiltonian requires  $\lambda_j \in \mathbb{R}$ .

More explicitly, let us consider a Hamiltonian  $\mathbf{H}^0$  with a spin  $\mathfrak{su}(2)$  symmetry. The generators (2.6) satisfy

$$[\mathbf{S}^z, \mathbf{S}^\pm] = \pm\mathbf{S}^\pm, \quad [\mathbf{S}^+, \mathbf{S}^-] = 2\mathbf{S}^z, \quad (2.20)$$

and commute with  $\mathbf{H}^0$ . In we introduce a magnetic field to the model coupling to the spin as

$$\mathbf{H} = \mathbf{H}^0 - 2B\mathbf{S}^z, \quad (2.21)$$

then

$$[\mathbf{H}, \mathbf{S}^\pm] = \mp 2B\mathbf{S}^\pm, \quad [\mathbf{H}, \mathbf{S}^z] = 0, \quad (2.22)$$

and the  $\mathfrak{su}(2)$  symmetry is explicitly broken. The symmetry is not completely destroyed however, it is rather dynamically broken. The conserved dynamical symmetry generators are

$$\tilde{\mathbf{S}}^+(t) = e^{2iBt}\mathbf{S}^+, \quad \tilde{\mathbf{S}}^-(t) = e^{-2iBt}\mathbf{S}^-, \quad \tilde{\mathbf{S}}^z(t) = \mathbf{S}^z, \quad (2.23)$$

and can be easily seen to satisfy the  $\mathfrak{su}(2)$  algebra (2.20).



In an analogous way a model with a charge  $\mathfrak{su}(2)$  symmetry, such as the models we consider in this thesis, has this symmetry dynamically broken if a chemical potential is introduced to the model. Let us conclude this discussion by mentioning that in chapters 6 and 7 we will meet two models which have a dynamical  $\mathfrak{su}(2|2)$  symmetry.

## 2.5 Quantum inverse scattering method

Having made some general remarks we come to the core of this chapter which is the construction of integrable lattice models on  $\mathcal{H}$ . In this section we review the setup of the quantum inverse scattering method [20, 21, 9], the technique we use to achieve this. The resulting models come equipped with a set of commuting operators that allow for their exact solution. Our discussion is close in spirit to that of [29, 39, 9]. We will deal generally with graded vector spaces  $\mathcal{B}$  and will use the convention of the fermionic R-operator [39]. The R-matrix lies at the heart of the method, and in the next section we will use the  $\mathfrak{su}(2|2)_C$  invariant Shastry's R-matrix [14] to construct the Hubbard-Shastry models.

For a given graded vector space  $\mathcal{V}$ , an R-matrix is an operator

$$\mathbf{R} = R_{\beta\delta}^{\alpha\gamma} \mathbf{e}_\alpha^\beta \otimes \mathbf{e}_\gamma^\delta : \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V} \otimes \mathcal{V} \quad (2.24)$$

for which

$$\mathbf{R}_{jk} = (-1)^{\epsilon_\beta(\epsilon_\gamma + \epsilon_\delta)} R_{\beta\delta}^{\alpha\gamma} \mathbf{E}_{j_\alpha}^\beta \mathbf{E}_{k_\gamma}^\delta \quad (2.25)$$

satisfies the Yang-Baxter equation

$$\mathbf{R}_{12}(z_1, z_2) \mathbf{R}_{13}(z_1, z_3) \mathbf{R}_{23}(z_2, z_3) = \mathbf{R}_{23}(z_2, z_3) \mathbf{R}_{13}(z_1, z_3) \mathbf{R}_{12}(z_1, z_2), \quad (2.26)$$

an operator equation on  $\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3$ . Here  $\mathbf{R}$  and  $\mathcal{V}$  are given indices to indicate which spaces the R-matrices act on in the triple tensor product. The R-matrix is endowed with two spectral parameters, which we label by  $z_j$  for each  $\mathcal{V}_j$  on which they act. Let us make some restrictions for our discussion here. We will consider only R-matrices which reduce to the graded permutation operator when

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the spectral parameters are set equal. We further require that the R-matrix satisfies the compatibility condition of Kulish and Sklyanin [40]

$$\epsilon(\mathbf{R}(z_1, z_2)) = 0, \quad (2.27)$$

where  $\epsilon$  gives the grading of the operator (B.5).

From the R-matrix we construct an operator, called the transfer matrix, acting on  $\mathcal{H}$  that will be the generating function for a set of commuting operators which will provide the integrability of the model. Indeed from this set we will obtain the model's Hamiltonian. Let us first introduce an auxiliary space, denoted as  $\mathcal{V}_a$ , in addition to the  $\mathcal{V}_j$  composing  $\mathcal{H}$  with  $j$  labelling the site. The transfer matrix is defined as

$$\mathbf{t}(\vec{z}; z_a) = \text{str}_a \mathbf{R}_{1a}(z_1, z_a) \mathbf{R}_{2a}(z_2, z_a) \dots \mathbf{R}_{La}(z_L, z_a), \quad (2.28)$$

where  $\vec{z} = \{z_1, z_2, \dots, z_L\}$ , and it obeys the important property that

$$[\mathbf{t}(\vec{z}; z_a), \mathbf{t}(\vec{z}; z'_a)] = 0. \quad (2.29)$$

Expanding  $\mathbf{t}(\vec{z}; z'_a)$  about  $z_a$  gives

$$\mathbf{t}(\vec{z}; z'_a) = I_0 + (z'_a - z_a)I_1 + (z'_a - z_a)^2 I_2 + \dots \quad (2.30)$$

and thus the commuting set  $[I_j, I_k] = 0$  for all  $j, k \geq 0$ . Let us prove eq. (2.29). Considering the object

$$\mathbf{T}(\vec{z}; z_a) = \mathbf{R}_{1a}(z_1, z_a) \mathbf{R}_{2a}(z_2, z_a) \dots \mathbf{R}_{La}(z_L, z_a), \quad (2.31)$$

we see that it satisfies

$$\mathbf{R}_{a_1 a_2}(z_{a_1}, z_{a_2}) \mathbf{T}_{a_2}(\vec{z}; z_{a_2}) \mathbf{T}_{a_1}(\vec{z}; z_{a_1}) = \mathbf{T}_{a_1}(\vec{z}; z_{a_1}) \mathbf{T}_{a_2}(\vec{z}; z_{a_2}) \mathbf{R}_{a_1 a_2}(z_{a_1}, z_{a_2}) \quad (2.32)$$

as a consequence of the Yang-Baxter equation (2.26) and the compatibility condition (2.27). Multiplying this equation by  $\mathbf{R}_{a_1 a_2}^{-1}(z_{a_1}, z_{a_2})$  and taking the super trace one obtains equation (2.29) upon identifying  $z_{a_1}$  as  $z_a$  and  $z_{a_2}$  as  $z'_a$ .

To define an integrable lattice model acting on the Hilbert space  $\mathcal{H}$  we select an operator from the commuting set and choose it to be a Hamiltonian. Let us



## 2.5 Quantum inverse scattering method

restrict ourselves to a homogeneous chain, for which the spectral parameters at each site are set equal to some value  $z_0$ :  $\vec{z}_0 = \{z_0, \dots, z_0\}$ . A useful object is  $\log \mathbf{t}(\vec{z}_0; z_a)$ , and expanding it about  $z_0$  gives

$$\log \mathbf{t}(\vec{z}_0; z_a) = \log \mathbf{t}(\vec{z}_0; z_0) + (z_a - z_0) \mathbf{t}(\vec{z}_0; z_0)^{-1} \left. \frac{d\mathbf{t}(\vec{z}_0; z)}{dz} \right|_{z=z_0} + \mathcal{O}((z_a - z_0)^2).$$

First note that  $\mathbf{t}(\vec{z}_0; z_0) = \text{str}_a(\mathbf{P}_{1a} \dots \mathbf{P}_{La})$  and is thus proportional to the shift operator. We will identify the second term as a Hamiltonian

$$\mathbf{H} = i\mathcal{N} \mathbf{t}(\vec{z}_0; z_0)^{-1} \left. \frac{d\mathbf{t}(\vec{z}_0; z)}{dz} \right|_{z=z_0}, \quad (2.33)$$

where  $\mathcal{N}$  represents the freedom to choose a normalisation. We have that

$$\begin{aligned} \left. \frac{d\mathbf{t}(\vec{z}_0; z)}{dz} \right|_{z=z_0} &= \sum_{j=1}^L \text{str}_a \mathbf{P}_{1a} \dots \mathbf{P}_{j-1a} \left. \frac{d}{dz} \mathbf{R}_{ja}(z_0, z) \right|_{z=z_0} \mathbf{P}_{j+1a} \dots \mathbf{P}_{La} \\ &= \text{str}_a \mathbf{P}_{j+1a} \dots \mathbf{P}_{La} \mathbf{P}_{1a} \dots \mathbf{P}_{j-1a} \left. \frac{d}{dz} \mathbf{R}_{ja}(z_0, z) \right|_{z=z_0} \\ &= \sum_{j=1}^L \mathbf{P}_{j+2j+1} \dots \mathbf{P}_{Lj+1} \mathbf{P}_{1j+1} \dots \mathbf{P}_{j-1j+1} \text{str}_a \mathbf{P}_{j+1a} \left. \frac{d}{dz} \mathbf{R}_{ja}(z_0, z) \right|_{z=z_0} \end{aligned}$$

using the cyclicity of the super trace and the identity (B.11), and similarly that

$$\begin{aligned} \mathbf{t}(\vec{z}_0; z_0)^{-1} &= \frac{1}{\dim(\mathcal{V})^2} \text{str}_a(\mathbf{P}_{La} \dots \mathbf{P}_{1a}) \\ &= \frac{1}{\dim(\mathcal{V})^2} \text{str}_a(\mathbf{P}_{ja} \mathbf{P}_{j+1a}) \mathbf{P}_{j-1j+1} \dots \mathbf{P}_{1j+1} \mathbf{P}_{Lj+1} \dots \mathbf{P}_{j+2j+1}, \end{aligned}$$

for any  $j$ . The Hamiltonian (2.33) thus decomposes to a sum of nearest neighbour terms  $\mathbf{H} = \sum_{j=1}^L \mathbf{H}_{jj+1}$  with

$$\begin{aligned} \mathbf{H}_{jj+1} &= \frac{i\mathcal{N}}{\dim(\mathcal{V})^2} \text{str}_a(\mathbf{P}_{ja} \mathbf{P}_{j+1a}) \text{str}_a \left( \mathbf{P}_{j+1a} \left. \frac{d}{dz} \mathbf{R}_{ja}(z_0, z) \right|_{z=z_0} \right) \\ &= i\mathcal{N} \mathbf{P}_{jj+1} \left. \frac{d}{dz} \mathbf{R}_{jj+1}(z_0, z) \right|_{z=z_0}, \end{aligned} \quad (2.34)$$

and  $\mathbf{H}_{LL+1} = \mathbf{H}_{L1}$ . In terms of the coefficients of the R-matrix (2.24) we have

$$\mathbf{H}_{jj+1} = i(-1)^{\epsilon_\beta + \epsilon_\alpha \epsilon_\beta + \epsilon_\alpha \epsilon_\gamma} \mathcal{N} \left. \frac{d}{dz} R_{\beta\delta}^{\gamma\alpha}(z_0, z) \right|_{z=z_0} \mathbf{E}_{j\alpha}^\beta \mathbf{E}_{j+1\gamma}^\delta. \quad (2.35)$$

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In conclusion, we now have the machinery to extract an integrable homogeneous lattice model with a nearest neighbour Hamiltonian from an R-matrix satisfying the Yang-Baxter equation (2.26). Often it is possible to go much further and to use the integrability to simplify the spectral problem. This is the subject of the algebraic Bethe ansatz, in which equation (2.32) plays a central role in determining the eigenvalues of the transfer matrix. One considers a general state and determines under which conditions it is an eigenstate of the transfer matrix. The conditions that are obtained are the Bethe ansatz equations. We will not go into any more detail here, but rather refer the reader to the texts [20, 21, 9].

### 2.6 Hubbard-Shastry models

The Hubbard-Shastry models are obtained from a particular R-matrix found by Shastry [14]. Originally this R-matrix was constructed to bring the Hubbard model into the realm of the quantum inverse scattering method. Later it was found [31] to be equivalent to the  $\mathfrak{su}(2|2)_C$  invariant R-matrix [41] which appears as a building block of the  $\text{AdS}_5 \times S^5$  scattering S-matrix playing an important role in the AdS/CFT spectral problem, see [42, 43] for a review.

Representations of  $\mathfrak{su}(2|2)_C$  depend on the values of the central elements of the algebra, which can be conveniently parametrized by variables  $x^\pm$  satisfying

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = 4i\mathbf{u}. \quad (2.36)$$

This constraint defines a torus, with  $\mathbf{u}$  a free parameter that characterises its elliptic modulus. We will restrict our attention to  $\mathbf{u}$  being real. The spectral parameters of Shastry's R-matrix  $\mathbf{R}(x_1^\pm, x_2^\pm)$  can be understood as the variables parametrising the two 4-dimensional fundamental representations upon which the R-matrix acts. Further details on the algebra  $\mathfrak{su}(2|2)_C$  and an explicit expression for Shastry's R-matrix are presented in appendix C.

Shastry's R-matrix gives rise to a family of models as any point  $x^\pm$  on the torus (2.36) can be used to obtain a homogeneous integrable model with a coupling constant  $\mathbf{u}$ . In particular the Hubbard model corresponds to the point  $x^- = 1/x^+$  (here  $x^+ \rightarrow \infty$ , see (H) in eqs. (2.43)) and  $\mathbf{u}$  is precisely the coupling constant that appears in the Hubbard Hamiltonian (2.3). We refer to the more general



family of models as to the Hubbard-Shastry models. In general however the corresponding Hamiltonian may be neither hermitian or parity invariant.

The transfer matrix (2.30) constructed from Shastry's R-matrix has been diagonalised by the algebraic Bethe ansatz [44, 45, 43]. The eigenvalues of  $\mathbf{t}$  are given by [46]

$$\begin{aligned}
 t(\vec{x}^\pm; x_a^\pm) &= e^{i\phi} \prod_{k=1}^N \frac{y_k - x_a^-}{y_k - x_a^+} \sqrt{\frac{x_a^+}{x_a^-}} + \\
 &+ e^{i\phi} \prod_{k=1}^N \frac{y_k - x_a^-}{y_k - x_a^+} \sqrt{\frac{x_a^+}{x_a^-}} \left[ \frac{x_a^+ + \frac{1}{x_a^+} - y_k - \frac{1}{y_k}}{x_a^+ + \frac{1}{x_a^+} - y_k - \frac{1}{y_k} - \frac{2i}{g}} \right] \prod_{j=1}^L \left[ \frac{(x_a^- - x_j^-)(1 - x_a^- x_j^+)}{(x_a^+ - x_j^-)(1 - x_a^+ x_j^+)} \frac{x_a^+}{x_a^-} \right] \\
 &- e^{i\phi} \prod_{k=1}^N \frac{y_k - x_a^-}{y_k - x_a^+} \sqrt{\frac{x_a^+}{x_a^-}} \left[ \frac{x_a^+ + \frac{1}{x_a^+} - y_k - \frac{1}{y_k}}{x_a^+ + \frac{1}{x_a^+} - y_k - \frac{1}{y_k}} \right] \prod_{j=1}^L \frac{x_a^+ - x_j^+}{x_a^+ - x_j^-} \sqrt{\frac{x_j^-}{x_j^+}} \times \\
 &\times \left\{ \prod_{j=1}^M \frac{w_j - x_a^+ - \frac{1}{x_a^+} - 2iu}{w_j - x_a^+ - \frac{1}{x_a^+} + 2iu} + \prod_{k=1}^N \frac{y_k + \frac{1}{y_k} - x_a^+ - \frac{1}{x_a^+}}{y_k + \frac{1}{y_k} - x_a^+ - \frac{1}{x_a^+} + 4iu} \prod_{j=1}^M \frac{w_j - x_a^+ - \frac{1}{x_a^+} + 6iu}{w_j - x_a^+ - \frac{1}{x_a^+} + 2iu} \right\},
 \end{aligned} \tag{2.37}$$

where  $\phi \in \mathbb{R}$  is an arbitrary twist which should satisfy the condition  $\exp(i\phi L) = 1$  for a periodic spin chain,  $N$  is the number of electrons in the eigenstate, and  $y_k$  and  $w_j$  are the corresponding Bethe roots.<sup>1</sup> For a homogeneous chain, setting the spectral parameters at each site equal to the spectral parameter at the auxiliary site  $x_j^\pm = x_a^\pm \equiv x^\pm$  the eigenvalue (2.37) becomes that of the shift operator

$$t(x^\pm) = \prod_{k=1}^N e^{i\phi} \frac{y_k - x^-}{y_k - x^+} \sqrt{\frac{x^+}{x^-}}. \tag{2.38}$$

This is naturally identified with  $e^{i \sum_k p_k}$  and so the momentum is given by

$$e^{ip_k} = e^{i\phi} \frac{y_k - x^-}{y_k - x^+} \sqrt{\frac{x^+}{x^-}}, \quad -\pi \leq p_k \leq \pi. \tag{2.39}$$

The Bethe equations, which constrain the Bethe roots  $y_k$  so that (2.37) corresponds to an eigenstate of the transfer matrix, are given by

$$\begin{aligned}
 e^{ip_k L} &= \prod_{j=1}^M \frac{v_k - w_j + i\mathbf{u}}{v_k - w_j - i\mathbf{u}}, \quad k = 1, \dots, N \leq L, \\
 \prod_{j=1}^N \frac{w_k - v_j + i\mathbf{u}}{w_k - v_j - i\mathbf{u}} &= \prod_{j=1, j \neq k}^M \frac{w_k - w_j + 2i\mathbf{u}}{w_k - w_j - 2i\mathbf{u}}, \quad k = 1, \dots, M \leq N/2,
 \end{aligned} \tag{2.40}$$

<sup>1</sup>Our notations are related to the ones adopted in [9] as  $p_j \leftrightarrow k_j$ ,  $w_j \leftrightarrow \Lambda_j$ .

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where  $v_k = \frac{1}{2}(y_k + 1/y_k)$ .

We will return to examine the physical consequences of the Bethe equations in chapter 4. For now we will use them to investigate the space of Hubbard-Shastry models. In particular we consider what restrictions are placed on the parameters  $x^\pm$  by the requirements of hermiticity and parity invariance. First let us examine hermiticity. We observe from the expression for the Hamiltonian (2.33) that the model is hermitian if the eigenvalue of the shift operator (2.38) is a phase. Examining the Bethe equations (2.40) for  $N = 2$  and  $M = 1$  we see that there are two possibilities, either  $\bar{p}_1 = p_1$  and  $\bar{p}_2 = p_2$  or  $\bar{p}_1 = p_2$ , and correspondingly either  $\bar{v}_1 = v_1$  and  $\bar{v}_2 = v_2$  or  $\bar{v}_1 = v_2$ . Here bar denotes complex conjugation. Both possibilities lead to the condition

$$\frac{y - x^- \bar{y} - \bar{x}^-}{y - x^+ \bar{y} - \bar{x}^+} \sqrt{\frac{x^+ \bar{x}^+}{x^- \bar{x}^-}} = 1, \quad (2.41)$$

and the relations for  $v$  have two solutions:  $\bar{y} = y$  and  $\bar{y} = 1/y$ . Considering first  $\bar{y} = y$ , we see that eq. (2.41) implies that  $\frac{x^+}{x^-}$  is a phase. On the other hand, putting  $\bar{y} = 1/y$  into eq. (2.41) gives  $\frac{x^+}{x^-} = \frac{\bar{x}^+}{\bar{x}^-}$ , and compatibility with the constraint equation (2.36) implies that  $x^+ x^-$  is a phase. Thus we conclude that hermitian Hubbard-Shastry models correspond to two lines in the  $x^\pm$  parameter space, determined by either  $\frac{x^+}{x^-}$  or  $x^+ x^-$  being a phase.

Next we consider the restriction imposed by parity invariance. At the level of the Bethe equations parity invariance amounts to a symmetry under  $p \rightarrow -p$ , and there are two ways this can be realised

$$\begin{aligned} p &\rightarrow -p, & y &\rightarrow -y, & w &\rightarrow -w, \\ p &\rightarrow -p, & y &\rightarrow -\frac{1}{y}, & w &\rightarrow -w. \end{aligned} \quad (2.42)$$

These transformations strongly constrain  $x^\pm$  and  $\phi$ , as  $p$  and  $y$  are related through (2.39). Requiring also compatibility with the constraint equation (2.36), one finds that there are four possible solutions leading to four parity invariant models, each of which satisfy the requirement to be hermitian

$$\begin{aligned} \text{(H)} & \quad x^+ = 1/x^- = \infty, & \phi &= \frac{2n-1}{2}\pi, \\ \text{(A)} & \quad x^+ = -1/x^- = i(\mathbf{u} + \sqrt{1 + \mathbf{u}^2}), & \phi &= n\pi, \\ \text{(B)} & \quad x^+ = -x^- = i(\mathbf{u} + \sqrt{1 + \mathbf{u}^2}), & \phi &= \frac{2n-1}{2}\pi, \quad n = 0, 1. \end{aligned} \quad (2.43)$$



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The fourth solution  $x^+ = x^- = \infty$  corresponds to the  $\mathfrak{su}(2|2)$  spin chain, but this point is highly degenerate and to get the correct Bethe equations for the model from (2.40) one should send the roots  $v_k$  and  $w_k$  to infinity in a proper way. Technically it is similar to the way we obtain the Bethe equations for the  $\mathfrak{su}(2|2)$  spin chain in section 7.2 from the weak coupling limit of the A-model. Indeed, as the  $\mathfrak{su}(2|2)$  spin chain appears in this limit we will not discuss it further for now. Later in chapter 7 the model will be discussed in some detail as the weak coupling limit of the A-model.

Now we obtain the dispersion relation for the models. These we get from the eigenvalues of the Hamiltonian (2.33). Let us be explicit and write the Hamiltonian in terms of the  $x^\pm$  parameters as

$$\mathbf{H} = i\mathcal{N}(x^\pm)\mathbf{t}(\bar{x}^\pm; x^\pm)^{-1} \frac{d\mathbf{t}(\bar{x}^\pm; x_a^\pm)}{dx_a^\pm} \Big|_{x_a^\pm = x^\pm}, \quad (2.44)$$

where  $x^-$  is considered as a function of  $x^+$  through the constraint (2.36) and the normalisation is chosen to be

$$\mathcal{N}(x^\pm) = \pm i \frac{x^+}{x^-} (x^+ - x^-) \sqrt{\frac{1 - x^{-2}}{1 - x^{+2}}}. \quad (2.45)$$

The sign of the Hamiltonian is fixed for each of the special models by requiring the Hubbard interaction term to have a positive coefficient for positive  $\mathbf{u}$ . Then from the eigenvalues of the transfer matrix (2.37) with  $N = 1$  we obtain the following dispersion relations

$$\begin{aligned} \text{(H)} \quad \mathcal{E}(p) &= 2 \sin(p + \phi) - 2\mathbf{u} &= -2 \cos(p) - 2\mathbf{u}, \\ \text{(A)} \quad \mathcal{E}(p) &= 2 \sin(p + \phi) - 2\sqrt{1 + \mathbf{u}^2} &= -2 \cos(p) - 2\sqrt{1 + \mathbf{u}^2}, \\ \text{(B)} \quad \mathcal{E}(p) &= 2 \cos(p + \phi) &= -2 \cos(p). \end{aligned} \quad (2.46)$$

Here we have chosen  $\phi_H = -\frac{\pi}{2}$ ,  $\phi_A = \pi$  and  $\phi_B = -\frac{\pi}{2}$  as these lead to the minus sign in front of the  $\cos p$  terms in the dispersion relations.

Consequently, the direct calculation from the R-matrix at  $x^- = 1/x^+$  does not give the Hubbard Hamiltonian (2.3) but rather one must perform the unitary transformation generated by  $\mathbf{U}_1(-\pi/2)$  to obtain it. Let us remark that it is this transformation that is responsible for the twisting that appears in the charge

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$\mathfrak{su}(2)$  generators (2.6). Similarly, to obtain the Hamiltonians for the A- and B-models one must apply  $\mathbf{U}_1(\pi)$  and  $\mathbf{U}_1(-\pi/2)$  respectively to the corresponding results from the R-matrix calculation.

These then are the Hubbard-Shastry models that will be of interest to us: the Hubbard model, the A-model, and the B-model. As the Hubbard model has already been extensively studied, we focus on the latter two. We construct their Hamiltonians through equation (2.35) from the explicit expression for Shastry's R-matrix (C.8), giving the Hamiltonian density as a  $16 \times 16$  matrix. To obtain the more physical form in terms of the fermion creation and annihilation operators  $\mathbf{c}_{k,\uparrow}^\dagger, \mathbf{c}_{k,\uparrow}, \mathbf{c}_{k,\downarrow}^\dagger, \mathbf{c}_{k,\downarrow}$  we rewrite it using the following identification

$$\begin{aligned} \mathbf{c}_\uparrow^\dagger &\equiv \mathbf{e}_1^3 - \mathbf{e}_4^2, & \mathbf{c}_\uparrow &\equiv \mathbf{e}_3^1 - \mathbf{e}_2^4, & \mathbf{c}_\downarrow^\dagger &\equiv \mathbf{e}_1^4 + \mathbf{e}_3^2, & \mathbf{c}_\downarrow &\equiv \mathbf{e}_4^1 + \mathbf{e}_2^3, \\ \mathbf{c}_\uparrow^\dagger \otimes I_4 &\leftrightarrow \mathbf{c}_{1,\uparrow}^\dagger, & \mathbf{c}_\uparrow \otimes I_4 &\leftrightarrow \mathbf{c}_{1,\uparrow}, & \mathbf{c}_\downarrow^\dagger \otimes I_4 &\leftrightarrow \mathbf{c}_{1,\downarrow}^\dagger, & \mathbf{c}_\downarrow \otimes I_4 &\leftrightarrow \mathbf{c}_{1,\downarrow}, \\ I^g \cdot I_4 \otimes \mathbf{c}_\uparrow^\dagger \cdot I^g &\leftrightarrow \mathbf{c}_{2,\uparrow}^\dagger, & I^g \cdot I_4 \otimes \mathbf{c}_\uparrow \cdot I^g &\leftrightarrow \mathbf{c}_{2,\uparrow}, \\ I^g \cdot I_4 \otimes \mathbf{c}_\downarrow^\dagger \cdot I^g &\leftrightarrow \mathbf{c}_{2,\downarrow}^\dagger, & I^g \cdot I_4 \otimes \mathbf{c}_\downarrow \cdot I^g &\leftrightarrow \mathbf{c}_{2,\downarrow}, \end{aligned}$$

where  $I_4$  is the  $4 \times 4$  identity matrix, and  $I^g = \sum_{i,j} (-1)^{\epsilon_i \epsilon_j} \mathbf{e}_i^i \otimes \mathbf{e}_j^j$  is the graded identity. Taking into account the above mentioned twists by  $\mathbf{U}_1(\phi)$ , the resulting expressions for the A-model and B-models are related to the general Hamiltonian (2.10) by setting

$$\begin{aligned} \kappa_{\text{H}} &= \frac{2 \cosh 2\nu}{\cosh \nu}, & \kappa_{\text{CC}} = -\kappa_{\text{SS}} = \kappa_{\text{PH}} &= \frac{2}{\cosh \nu}, \\ \tau_0 &= 1, & \tau_1 = \bar{\tau}_2 &= -1 - i \tanh \nu, & \tau_3 &= 2i \tanh \nu, \end{aligned} \quad (2.47)$$

for the A-model, and

$$\begin{aligned} \kappa_{\text{H}} &= -\kappa_{\text{CC}} = \kappa_{\text{SS}} = \kappa_{\text{PH}} = 2 \tanh \nu, \\ \tau_0 &= 1, & \tau_1 = \tau_2 &= -1 + \operatorname{sech} \nu, & \tau_3 &= -2\tau_1. \end{aligned} \quad (2.48)$$

for the B-model, where we have reparametrised the coupling constant as

$$\mathbf{u} = \sinh \nu. \quad (2.49)$$

Finally let us remark that a general expression for the Hubbard-Shastry Hamiltonian was derived in [16, 17], see eq. (12.229) of [9]. In the parametrization (12.109) used in [9] the A-model Hamiltonian (up to a unitary transformation)



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corresponds to  $l = i\pi/4$ , and the B-model corresponds to  $\mu = \pi/4$ . The line  $x^+x^-$  a phase corresponds to the line  $Re(\mu) = 0$ , and the line  $x^+/x^-$  a phase corresponds to the line  $Re(\mu) = \pi/4$ .

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# Chapter 3

## Exact solution

Integrable models have an exact solution that greatly reduces their complexity. The Bethe ansatz technique allows access to the non-perturbative behaviour of a model like no other method. By yielding analytic expressions it provides a clear picture of the physics exhibited.

An efficient way to determine thermodynamic quantities and the excitation spectrum of integrable models in the thermodynamic limit was proposed by C. N. Yang and C. P. Yang for the case of the Bose gas with the delta-function interaction [19]. For the further development of this technique see [20, 22, 47]. In this approach one starts with the Bethe ansatz equations and describes their solutions in the thermodynamic limit. Complex roots of the Bethe equations for large length spin chains arrange themselves in regular patterns called Bethe strings. Assuming that all Bethe roots form strings of various kinds one can bring Bethe equations into a set of equations involving only the real parts of the roots. The procedure of classifying the strings and deriving the Bethe equations for the strings is called the string hypothesis. Equilibrium can then be accessed by minimising the free energy, which can be written explicitly in terms of the densities of the strings rapidities. This yields a set of coupled non-linear integral equations, called the Thermodynamic Bethe Ansatz (TBA) equations, from which the equilibrium and excitation properties of the model can be calculated. For the Hubbard model the string hypothesis and TBA equations were derived by Takahashi [48].

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In this chapter we review this technique, presenting it in a very general form applicable to a wide class of integrable models. Firstly in section 3.1 we outline the procedure by which one obtains the TBA equations. Here we rely on the ability to make a string hypothesis and so we restrict our attention to models for which this is possible.

Next in section 3.2 we present a detailed treatment of the study of excitations about equilibrium. There are some novel features of the formalism so let us briefly outline these. Firstly, individual particle and hole excitations are considered and explicit non-zero temperature expressions for dressed quantities such as energy, momentum, spin and charge are obtained. The resulting expressions include an important contribution that did not appear in previous studies which were restricted to particle-hole excitations [19, 20]. Secondly, the dressing of spin and charge of excitations [49, 50] is examined in some detail and a formula for the induced charge of the system that results from an excitation is provided. Thirdly, the focus is put on working with the Bethe strings as opposed to the roots of the original Bethe equations, and this allows a natural extension of the formalism for excitations presented in e.g. [19, 20] for the Bose gas, a model for which the Bethe roots do not form Bethe strings. Fourthly and perhaps the most convenient aspect of the formalism, when considering excitations at zero temperature the need to explicitly deal with mode numbers [51, 52] is overcome. Instead, possible restrictions on allowed excitations are related to properties of the kernels appearing in the Thermodynamic Bethe Ansatz equations.

The calculation of thermodynamic variables is next discussed in section 3.3. In particular we consider the charge densities, the susceptibilities, and the specific heat. We obtain expressions for them, valid at any temperature, in terms of quantities that satisfy sets of integral equations. These make it possible to evaluate derivatives of the free energy, without explicitly having to take the derivative numerically.

Finally in section 3.4 we consider the important limit of zero temperature. Here the equilibrium state becomes the ground state and the formalism simplifies dramatically.



### 3.1 Thermodynamic Bethe ansatz

#### Bethe equations

Let us consider a one-dimensional lattice model with Hamiltonian  $\mathbf{H}$ , on chain of length  $L$ , whose spectrum is determined by a set of nested Bethe ansatz equations

$$(-1)^{\varphi_a} = e^{iLp_a(v_{a,k})} \prod_b \prod_{j=1}^{N_b} S_{ab}(v_{a,k} - v_{b,j}). \quad (3.1)$$

We use indices  $a, b$  and so on to distinguish between the different types of Bethe roots corresponding to the different levels of the Bethe ansatz, and the indices  $j, k$  are used to distinguish between different roots of the same type. Here  $\varphi_a$  is a constant which appears in particular as we have not excluded self-scattering from the product on the right hand side. For periodic boundary conditions and in the absence of a twist,  $\varphi_a = 0$  if  $S_{aa}(0) = 1$  and  $\varphi_a = 1$  if  $S_{aa}(0) = -1$ . The roots are parametrised by a rapidity variable for which the scattering matrices are of a difference form:  $S_{ab}(v_{a,k}, v_{b,j}) = S_{ab}(v_{a,k} - v_{b,j})$ . For roots which do not carry momentum there is no term  $e^{iLp_a}$  in the Bethe equations, and by convention we set  $p_a = 0$  for such roots.

To each  $N_a$  appearing in the Bethe equations (3.1) there is a corresponding conserved charge  $\mathbf{N}_a$ . In general it may be convenient to consider a different, more physical, set of conserved charges<sup>1</sup>  $\mathbf{N}_i$  which are some linear combinations of the  $N_a$ . Let us write this as

$$\mathbf{N}_i = \sum_a w_a^i \mathbf{N}_a, \quad (3.2)$$

and we call  $w_a^i$  the charge of a Bethe root of type  $a$  under  $\mathbf{N}_i$ . We stress that  $a$  and  $i$  are indices that take values in two distinct sets.

Each solution of the Bethe equations corresponds to an eigenstate of the model. In fact as the charges  $\mathbf{N}_i$  are diagonal on these eigenstates, the Bethe equations give the spectrum of the more general model  $\mathbf{H} - \mu_i \mathbf{N}_i$ , where we have introduced chemical potentials  $\mu_i$  for each of the conserved charges, and the

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<sup>1</sup>See for example section 4 where charge and spin are the conserved quantities of interest and are related to the conserved quantities enumerating the Bethe equations through eq. (4.2).

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repeated index  $i$  is summed over. The energy of each eigenstate is determined by the corresponding solution of the Bethe equations through

$$E = \sum_a \sum_{j=1}^{N_a} \left( \mathcal{E}_a(v_{a,j}) - \mu_i w_a^i \right), \quad (3.3)$$

where  $\mathcal{E}_a$  is the dispersion relation for roots of type  $a$ .

Before we proceed let us make a general remark. Sometimes there are solutions to the Bethe equations which include roots at  $v = \infty$ . Such roots are insensitive to the finite roots within the solution and correspond to (dynamical) symmetries of the model: the addition of such a root to a solution corresponds to the action of some symmetry generator on the associated eigenstate. Sometimes it is useful to restrict attention to solutions which do not include roots at  $v = \infty$ . The eigenstates one obtains from the Bethe equations are then highest weight states of the model's symmetry algebra.

#### String hypothesis

In the limit of large  $L$  the roots of the Bethe equations align into regular patterns known as Bethe strings. We use greek indices  $\alpha, \beta, \gamma$  to label between different strings patterns. Each string can be parametrised by a single real parameter, which we write as  $v_{\alpha,k}$ , where again  $k$  distinguishes between different strings of the same type. The rapidity of each root composing the string is then some function of  $v_{\alpha,k}$ . Let us assume that in the  $L \rightarrow \infty$  limit every solution of the Bethe equations corresponds to a particular configuration of Bethe strings. For large spin chain length  $L$  the Bethe equations can be rewritten for string configurations by fusing together the Bethe equations

$$(-1)^{\varphi_\alpha} = e^{iLp_\alpha(v_{\alpha,k})} \prod_{\beta} \prod_{j=1}^{N_\beta} S_{\alpha\beta}(v_{\alpha,k} - v_{\beta,j}). \quad (3.4)$$

The Bethe equation for each string is obtained from the product of the equations of the roots composing the string, and the string S-matrices  $S_{\alpha\beta}$  are appropriate products of  $S_{ab}$  constructed by grouping together the Bethe roots on the right hand side into their string configurations. The number of equations in (3.4) is



much smaller than the number of equations in (3.1). That each solution of (3.4) nevertheless gives a solution of the full set (3.1) is the consistency condition for the string hypothesis. The set of real roots  $v_{\alpha,k}$  appearing in the string Bethe equations (3.4) are referred to as the string Bethe roots. As the Bethe strings provide a classification of the solutions of the Bethe equations we can regard them as the particle content of the model. Quantities for the Bethe strings such as  $\varphi_\alpha$ ,  $p_\alpha$ ,  $\mathcal{E}_\alpha$  and  $w_\alpha$  are found by summing the contributions of the roots composing the strings.

### Thermodynamic limit

To take the thermodynamic limit it is useful to introduce the counting function  $z_\alpha(v)$ . It is constructed by taking the logarithm of the string Bethe equations (3.4), and treating them as a function of the string Bethe root

$$L \sigma_\alpha z_\alpha(v) = \pi \varphi_\alpha + L p_\alpha(v) + \sum_\beta \sum_{j=1}^{N_\beta} \phi_{\alpha\beta}(v, v_{\beta,j}), \quad (3.5)$$

where

$$\phi_{\alpha\beta} \equiv \frac{1}{i} \log S_{\alpha\beta} \quad (3.6)$$

is the phase of the S-matrix. The counting functions allow one to enumerate the Bethe strings because  $\frac{L z_\alpha}{2\pi}$  evaluates to an integer on each of the  $\alpha$ -string Bethe roots. We refer to these integers as the mode number of the Bethe strings. For a given root  $v_{\alpha,k}$  we denote the corresponding mode number as  $I_{\alpha,k} \equiv \frac{L}{2\pi} z_\alpha(v_{\alpha,k})$ . Moreover the counting function  $L z_\alpha / 2\pi$  may evaluate to an integer for a rapidity which is not that of a string of type  $\alpha$ , and we call such mode numbers holes. Note the appearance of  $\sigma_\alpha$  in the definition of the counting function. For momentum carrying strings  $\sigma_\alpha = \text{sign}(\frac{dp_\alpha}{dv})$  and this guarantees that the counting function is an increasing function of  $v$ . For an  $\alpha$ -string which does not carry momentum,  $\sigma_\alpha$  is determined by requiring the counting function to be increasing. Let us remark that to fully define the counting function (3.5) it is necessary to specify the branch of the logarithm. This does not affect the study of equilibrium properties, and of course should not affect the physics, but it will be important for the study

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of excitations and we will return to this issue when we begin to discuss them in section 3.2.

Taking the thermodynamic limit  $L \rightarrow \infty$  with  $N_\alpha/L$  fixed, one gets equations for the densities of particles and holes

$$\rho_\alpha + \bar{\rho}_\alpha = \frac{1}{2\pi} \left| \frac{dp_\alpha}{dv} \right| + K_{\alpha\beta} \star \rho_\beta, \quad (3.7)$$

where  $L\rho_\alpha\Delta v$  (respectively  $L\bar{\rho}_\alpha\Delta v$ ) is the number of integers corresponding to particles (respectively holes) that the counting function  $Lz_\alpha/2\pi$  evaluates to over a range  $\Delta v$ . Repeated indices are summed over and  $\star$  denotes convolution (see appendix A for the precise definition) over the domain of the rapidity of the appropriate string, which we denote by  $\mathcal{I}_\alpha$ . The kernels  $K_{\alpha\beta}$  are defined by

$$K_{\alpha\beta} = \sigma_\alpha \mathcal{K}_{\alpha\beta}, \quad \mathcal{K}_{\alpha\beta}(v) = \frac{1}{2\pi i} \frac{d}{dv} \log S_{\alpha\beta}(v). \quad (3.8)$$

Since the counting functions are all defined to be increasing functions it follows that the densities of all particles and holes in the equations (3.7) are positive.

The equations for the densities can be used to determine the total number of particles and holes for each type of string, which we call the range of mode numbers. Indeed, integrating equations (3.7) and multiplying by  $L$  one gets the range

$$N_\alpha + \bar{N}_\alpha = L \frac{\Delta p_\alpha}{2\pi} + k_{\alpha\beta} N_\beta, \quad (3.9)$$

where  $\bar{N}_\alpha$  is the number of holes of type  $\alpha$  in the state, and  $\Delta p_\alpha = 1 \star \left| \frac{dp_\alpha}{dv} \right|$  for momentum carrying strings and  $\Delta p_\alpha = 0$  for strings that do not carry momentum. The  $k_{\alpha\beta}$ , defined as  $k_{\alpha\beta} \equiv 1 \star K_{\alpha\beta}$ , are constants. Note that all roots of the string Bethe equations are counted in the range, including those which correspond to  $v = \infty$ . Furthermore since  $\bar{N}_\alpha \geq 0$  eqs. (3.9) imply the following selection rules

$$N_\alpha \leq L \frac{\Delta p_\alpha}{2\pi} + k_{\alpha\beta} N_\beta \quad (3.10)$$

which restrict the allowed  $N_\alpha$  appearing in the string Bethe equations.



## Equilibrium state

To access the thermodynamic behaviour let us introduce the partition function of the model. In the grand canonical ensemble it is given by

$$Z = \text{tr} \left[ \exp \left( -\frac{\mathbf{H} - \mu_i \mathbf{N}_i}{T} \right) \right], \quad (3.11)$$

where  $T$  is the temperature. The partition function can be simplified by working directly with the Bethe ansatz eigenstates. In the thermodynamic limit the eigenvalues of the Hamiltonian  $\mathbf{H}$  and the conserved charges  $\mathbf{N}_i$  become

$$\begin{aligned} E &= \sum_a \sum_{j=1}^{N_a} \mathcal{E}_a(v_{a,j}) \rightarrow L \rho_\alpha \star \mathcal{E}_\alpha, \\ N_i &= \sum_a \sum_{j=1}^{N_a} w_\alpha^i \rightarrow w_\alpha^i N_\alpha. \end{aligned} \quad (3.12)$$

These provide us with a simplification as they depend only on the densities, which are macroscopic quantities, rather than on microscopic information such as the mode numbers of the Bethe strings composing the eigenstate. In particular, let us recall that for each interval  $\Delta v$  of rapidity of an  $\alpha$ -string there are  $L \rho_\alpha \Delta v$  strings with rapidity in  $\Delta v$  and  $L \bar{\rho}_\alpha \Delta v$  holes. Then there are

$$e^{\Delta S_\alpha} = \frac{(L(\rho_\alpha + \bar{\rho}_\alpha)\Delta v)!}{(L\rho_\alpha\Delta v)!(L\bar{\rho}_\alpha\Delta v)!} \quad (3.13)$$

possible reordering of the strings and holes of the  $\alpha$ -strings with rapidity in  $\Delta v$ , and the eigenvalues (3.12) are insensitive to this. As  $L$  is large we can use Stirling's formula to simplify this expression giving, to leading order,

$$\Delta S_\alpha \approx L \rho_\alpha \log \left( 1 + \frac{\bar{\rho}_\alpha}{\rho_\alpha} \right) + L \bar{\rho}_\alpha \log \left( 1 + \frac{\bar{\rho}_\alpha}{\rho_\alpha} \right). \quad (3.14)$$

The integral  $S_\alpha = 1 \star \Delta S_\alpha$  is the entropy of an  $\alpha$ -string and gives the corresponding weight in the partition function for a macroscopic state described through the densities. Thus on diagonalising the argument of the trace in the partition function (3.11) we get

$$Z = \int \mathcal{D}[\rho] e^{-\frac{tL}{T}}, \quad (3.15)$$

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where the trace is replaced by an appropriately normalised functional integral  $\int \mathcal{D}[\rho]$  which samples all sets of densities satisfying the selection rules (3.10). Here

$$f = e_\alpha \star \rho_\alpha - Ts \quad (3.16)$$

is the free energy,  $e_\alpha = \mathcal{E}_\alpha - \mu_i w_\alpha^i$  denotes the energy carried by an  $\alpha$ -string, and  $s = \sum_\alpha S_\alpha/L$  is the entropy density.

The equilibrium state is the dominant contribution to the partition function. In the limit  $L \rightarrow \infty$  this corresponds to the stationary phase  $\delta f = 0$  of (3.15). In the zero temperature limit the equilibrium state becomes the ground state of the model. Taking the variation of the equation for densities (3.7) gives

$$\delta\rho_\alpha + \delta\bar{\rho}_\alpha = \mathbf{K}_{\alpha\beta} \star \delta\rho_\beta, \quad (3.17)$$

and also the variation of the entropy density is

$$\delta s = \sum_\alpha 1 \star \left( \delta\rho_\alpha \log \frac{\bar{\rho}_\alpha}{\rho_\alpha} + (\delta\rho_\alpha + \delta\bar{\rho}_\alpha) \log \left( 1 + \frac{\bar{\rho}_\alpha}{\rho_\alpha} \right) \right). \quad (3.18)$$

Then imposing  $\delta f = 0$  requires

$$\log Y_\alpha = \frac{e_\alpha}{T} - \log \left( 1 + \frac{1}{Y_\beta} \right) \star \mathbf{K}_{\beta\alpha}. \quad (3.19)$$

Here we have introduced the functions  $Y_\alpha = \frac{\bar{\rho}_\alpha}{\rho_\alpha}$ , which we will refer to as Y-functions. These equations (3.19) are the conditions for equilibrium and are known as the thermodynamic Bethe ansatz (TBA) equations. With their solution one can further solve eq. (3.7) for the equilibrium densities. It is useful to also introduce the pseudo-energy defined as

$$\epsilon_\alpha = T \log Y_\alpha. \quad (3.20)$$

Both quantities  $Y_\alpha$  and  $\epsilon_\alpha$  will be useful and we will use them interchangeably.

We now obtain a rewriting of the expression for the free energy that involves the Y-functions instead of the densities. To this end we first rewrite the entropy densities, using eq. (3.7), as

$$s_\alpha = \frac{1}{2\pi} \left| \frac{dp_\alpha}{dv} \right| \star \log \left( 1 + \frac{1}{Y_\alpha} \right) + \rho_\alpha \star \log Y_\alpha + \log \left( 1 + \frac{1}{Y_\alpha} \right) \star \mathbf{K}_{\alpha\beta} \star \rho_\beta.$$



## 3.2 Excitations about equilibrium

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Expression (3.16) can then be simplified using the TBA equations (3.19) giving

$$f = -\frac{T}{2\pi} \left| \frac{dp_\alpha}{dv} \right| \star \log \left( 1 + \frac{1}{Y_\alpha} \right). \quad (3.21)$$

The calculation of the free energy is an important step in the exact solution of the model, as it gives access to all thermodynamic properties. To obtain it one needs to first solve the TBA equations (3.19). This is by no means a straightforward task as they are a large set, usually infinite in number, of non-linear integral equations, with dependence on temperature and the coupling constants and chemical potentials of the model. In general these cannot be solved analytically and one must resort to numerical methods. A useful approach to solving the TBA equations when they are infinite in number is outlined in [53].

The free energy gives access to the thermodynamic variables in the standard way. In particular, the charge densities  $n_i$ , the susceptibilities  $\chi_i$ , and the specific heat capacity  $c$  are given by the following derivatives

$$n_i = -\frac{\partial f}{\partial \mu_i}, \quad \chi_i = -\frac{\partial^2 f}{\partial \mu_i^2}, \quad c = -T \frac{\partial^2 f}{\partial T^2}. \quad (3.22)$$

We will discuss the calculation of these quantities in some detail in section 3.3. First however we turn our attention to analysis of excitations.

## 3.2 Excitations about equilibrium

Now we consider excitations about the equilibrium state outlined above. Let us restrict our attention to excitations where the numbers of excited roots are much smaller than the numbers of particles in the equilibrium state.

We first return to a point that was skipped above, the choice of the branch for the counting function (3.5). The choice of branch affects the formalism one obtains for the excitations and we present here what appears to be an optimal choice. In particular, in order to obtain a reasonable expression for the dressed momentum of a string it is necessary to keep track of the branch of each  $\log S_{\alpha\beta}$  term in eq. (3.5). Moreover, the choice of branch is guided by the behaviour of the pseudo-energies for the equilibrium state. The branch of  $\log S_{\alpha\beta}$  and the

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range of momentum  $p_\alpha$  are chosen so that the counting function is continuous about the minimum of  $\epsilon_\alpha$ .

Let us be specific about the strings we will consider. For strings with rapidity variable defined on  $\mathbb{R}$  we assume that the pseudo-energy is even, and monotonic on the interval  $(0, \infty)$ . Then there are two cases, and we give explicit expressions for the phase in each

- Type 1:  $\epsilon_\alpha(v)$  has a minimum at  $v = 0$ , and is increasing on the interval  $(0, \infty)$ ,

$$\frac{1}{i} \log S_{\alpha\beta}(v, t) = 2\pi b_{\alpha\beta} + \pi c_{\alpha\beta} + \Theta_{\alpha\beta}(v - t) = \phi_{\alpha\beta}(v, t), \quad (3.23)$$

and the range of  $p_\alpha(v)$  is chosen so that it is continuous for  $v$  along  $(-\infty, \infty)$ .

- Type 2:  $\epsilon_\alpha(v)$  has a minimum at  $v = \pm\infty$ , and is decreasing on the interval  $(0, \infty)$ ,

$$\frac{1}{i} \log S_{\alpha\beta}(v, t) = 2\pi b_{\alpha\beta} - \pi c_{\alpha\beta} \text{sign}(v) + \Theta_{\alpha\beta}(v - t) = \phi_{\alpha\beta}(v, t), \quad (3.24)$$

and the range of  $p_\alpha(v)$  is chosen so that  $p_\alpha(-\infty) = p_\alpha(+\infty)$ , and it is discontinuous only at  $v = 0$ .

Here

$$\Theta_{\alpha\beta}(v) = 2\pi \int_0^v dt \mathcal{K}_{\alpha\beta}(t), \quad \kappa_{\alpha\beta} = \frac{1}{\pi} \Theta_{\alpha\beta}(\infty) = 1 \star \mathcal{K}_{\alpha\beta}, \quad (3.25)$$

$b_{\alpha\beta}$  is an integer capturing the freedom in the choice of branch, and the  $c_{\alpha\beta}$  are defined such that  $c_{\alpha\beta} - \kappa_{\alpha\beta}$  is as close to zero as possible subject to the constraint that  $c_{\alpha\beta}$  is an even integer if  $S_{\alpha\beta}(0) = 1$  and  $c_{\alpha\beta}$  is an odd integer if  $S_{\alpha\beta}(0) = -1$ . Note that if  $S_{\alpha\beta}(\pm\infty) = 1$  (which is the case for rational S-matrices) then  $c_{\alpha\beta} = \kappa_{\alpha\beta}$ . For type 1 strings  $\phi_{\alpha\beta}(\pm\infty, t) = \pi(c_{\alpha\beta} \pm \kappa_{\alpha\beta})$  and the range of  $\phi_{\alpha\beta}$  is  $\phi_{\alpha\beta}(+\infty, t) - \phi_{\alpha\beta}(-\infty, t) = 2\pi\kappa_{\alpha\beta}$ . For type 2 strings it is worth stressing that the scattering phase  $\phi_{\alpha\beta}$  is no longer of a difference form with this choice of the branch of  $\log S_{\alpha\beta}$ . It is mildly broken so that, for models with  $c_{\alpha\beta} = \kappa_{\alpha\beta}$ , the counting function and scattering phases would be continuous everywhere but at  $v = 0$ , the maximum of  $\epsilon_\alpha$ . The jump discontinuity of the



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scattering phase at  $v = 0$  is equal to  $\phi_{\alpha\beta}(-0, t) - \phi_{\alpha\beta}(+0, t) = 2\pi c_{\alpha\beta}$ , which is equal to  $2\pi\kappa_{\alpha\beta}$  if  $S_{\alpha\beta}(\pm\infty) = 1$ .

These two cases capture the behaviour of most strings of Bethe ansatz solvable models. For strings that are not captured modifications of the subsequent analysis will have to be made<sup>1</sup>. In the next chapter we will see that the  $y$ -particles of the Hubbard-Shastry models must be considered carefully because their rapidity variable is not defined on  $\mathbb{R}$ .

With this technical point taken care of we now consider excitations above the equilibrium state. For definiteness we choose any one of the states which in the thermodynamic limit contribute to the equilibrium state as a *reference* state. Let us say that the reference state consists of  $N_\alpha$   $\alpha$ -strings with rapidities  $v_{\alpha,k}$ ,  $k = 1, \dots, N_\alpha$ . This choice of reference state is unimportant as we shall see that all quantities of interest depend only on macroscopic quantities such as the densities.

Now consider a general excited state and let us say that it consists of  $\tilde{N}_\alpha$   $\alpha$ -strings with rapidities  $\tilde{v}_{\alpha,k}$ ,  $k = 1, \dots, \tilde{N}_\alpha$ . These rapidities also satisfy the string Bethe equations (3.4)

$$(-1)^{\varphi_\alpha} = e^{iLp_\alpha(\tilde{v}_{\alpha,k})} \prod_{\beta} \prod_{j=1}^{\tilde{N}_\beta} S_{\alpha\beta}(\tilde{v}_{\alpha,k} - \tilde{v}_{\beta,j}). \quad (3.26)$$

The rapidities  $\tilde{v}_{\alpha,k}$  of the excited state can be divided into two groups. The first group consists of rapidities with mode numbers  $\tilde{I}_{\alpha,k}$  which coincide with some of the mode numbers of the particles of the reference state. They are close to the corresponding rapidities of the reference state, that is, the difference between the rapidities with the same mode number is of order  $1/L$ . We denote these rapidities as  $v'_{\alpha,k}$ ,  $k = 1, \dots, N'_\alpha$ . The second group consists of the remaining rapidities, those which have mode numbers not coinciding with any mode number of the particles of the reference state. There are  $N_\alpha^a = \tilde{N}_\alpha - N'_\alpha$  of strings of type  $\alpha$  in this group. These rapidities will be denoted as  $\tilde{v}_{\alpha_j}$ . The reference state also contains strings with mode numbers different from any  $\tilde{I}_{\alpha,k}$ , the mode numbers

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<sup>1</sup> For instance there are models where the counting function is not always monotonic. So long as it oscillates only a finite number of times however this should not affect the description of excitations above equilibrium.

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of the particles of the excited state. These correspond to holes of the excited state and their rapidities will be denoted as  $\tilde{v}_{\tau_j}$ . For a string of type  $\alpha$  there are  $N_\alpha^r = N_\alpha - N'_\alpha$  of them. One can think about the excited state as being obtained by adding  $N_\alpha^a$  strings to, and removing  $N_\alpha^r$  strings from, the reference state. Let  $N^a = \sum_\alpha N_\alpha^a$  be the total number of strings added to the reference state, and let  $N^r = \sum_\alpha N_\alpha^r$  be the total number of strings removed from the reference state. Thus the equations (3.26) can be rewritten in the form

$$(-1)^{\varphi_\alpha} = e^{iLp_\alpha(v'_{\alpha,k})} \frac{\prod_j^{N^a} S_{\alpha a_j}(v'_{\alpha,k} - \tilde{v}_{a_j})}{\prod_j^{N^r} S_{\alpha r_j}(v'_{\alpha,k} - \tilde{v}_{r_j})} \prod_\beta \prod_{j=1}^{N_\beta} S_{\alpha\beta}(v'_{\alpha,k} - v'_{\beta,j}), \quad (3.27)$$

$$(-1)^{\varphi_{a_k}} = e^{iLp_\alpha(\tilde{v}_{a_k})} \frac{\prod_j^{N^a} S_{a_k a_j}(\tilde{v}_{a_k} - \tilde{v}_{a_j})}{\prod_j^{N^r} S_{a_k r_j}(\tilde{v}_{a_k} - \tilde{v}_{r_j})} \prod_\beta \prod_{j=1}^{N_\beta} S_{a_k\beta}(\tilde{v}_{a_k} - v'_{\beta,j}), \quad (3.28)$$

where the product  $\prod_\beta \prod_{j=1}^{N_\beta} S_{\alpha\beta}(\cdot, v'_{\beta,j})$  includes the product  $\prod_j^{N^r} S_{\alpha r_j}(\cdot, \tilde{v}_{r_j})$ ; it is equal to  $\prod_\beta \prod_{j=1}^{N'_\beta} S_{\alpha\beta}(\cdot, v'_{\beta,j}) \prod_j^{N^r} S_{\alpha r_j}(\cdot, \tilde{v}_{r_j})$  and reduces to  $\prod_\beta \prod_{j=1}^{N_\beta} S_{\alpha\beta}(\cdot, v_{\beta,j})$  in the thermodynamic limit.

Here we have made an implicit assumption that the mode numbers of every string in the reference state are also mode numbers of the excited state. This is justified at non-zero temperature as one can always choose the reference state so that this is the case<sup>1</sup>. At zero temperature however, specifically at half-filling, it may happen that the mode number of a string in the reference state is not a mode number of the excited state, either due to a change in the range of mode numbers or an overall shift of the range of the counting function. For example if the range is decreased for an excitation and all mode numbers correspond to particles then some strings are necessarily removed, and moreover there is no excited state rapidity one can assign to them. We will return to discuss this when we treat the zero temperature limit.

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<sup>1</sup>Indeed the densities  $\rho(v)$ ,  $\bar{\rho}(v)$  are related to the numbers of particles and holes with rapidity in an interval  $dv$  about  $v$ . At non-zero temperature both the densities of particles and holes are non-trivial for all  $v$  and the particles and holes within each interval  $dv$  can be rearranged. Thus one can always choose a reference state at non-zero temperature such that a finite number of particles and holes have required mode numbers.



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Let us relate the rapidities of the reference state Bethe roots to those with corresponding mode number in the excited state through

$$\tilde{v}_{\alpha,k} - v_{\alpha,k} = \frac{\sigma_\alpha \zeta_\alpha(v_{\alpha,k})}{2\pi L}, \quad (3.29)$$

where we have introduced  $\zeta_\alpha$  which are of order 1. It is possible to obtain a closed equation for  $\zeta_\alpha$  by subtracting the logarithm of the Bethe equations of the ground state (3.4) from those of the excited state (3.27) for Bethe roots with the same mode number. Expanding (3.27) and taking the thermodynamic limit one obtains

$$\zeta_\alpha(\rho_\alpha + \bar{\rho}_\alpha) = \zeta_\beta \rho_\beta \star \mathbf{K}_{\beta\alpha} - \phi_{\alpha a} + \phi_{\alpha r}, \quad (3.30)$$

with the help of the equation for densities (3.7). Here we have taken into account that

$$\mathbf{S}_{\alpha\beta}(v)\mathbf{S}_{\beta\alpha}(-v) = 1 \Rightarrow \frac{\sigma_\beta}{2\pi i} \frac{d}{dt} \log \mathbf{S}_{\alpha\beta}(v-t) = -\mathbf{K}_{\beta\alpha}(t-v), \quad (3.31)$$

and introduced the notation

$$X_a \equiv \sum_{j=1}^{N^a} X_{a_j}(\tilde{v}_{a_j}), \quad X_r \equiv \sum_{j=1}^{N^r} X_{r_j}(v_{r_j}), \quad (3.32)$$

for any quantity  $X_\alpha$ , in particular

$$\phi_{\alpha a}(v) = \sum_{j=1}^{N^a} \phi_{\alpha a_j}(v - \tilde{v}_{a_j}), \quad \phi_{\alpha r}(v) = \sum_{j=1}^{N^r} \phi_{\alpha r_j}(v - v_{r_j}). \quad (3.33)$$

The function  $F_\alpha = -\zeta_\alpha(\rho_\alpha + \bar{\rho}_\alpha)$ , which we refer to as the shift function [54, 20], is an important object and it satisfies the following closed equation

$$F_\alpha = \frac{F_\beta}{1 + Y_\beta} \star \mathbf{K}_{\beta\alpha} + \phi_{\alpha a} - \phi_{\alpha r}. \quad (3.34)$$

Another form of this equation which will prove useful is

$$\zeta_\alpha \rho_\alpha = \zeta_\beta \rho_\beta \star \frac{\mathbf{K}_{\beta\alpha}}{1 + Y_\alpha} - \frac{\phi_{\alpha a} - \phi_{\alpha r}}{1 + Y_\alpha}. \quad (3.35)$$

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#### Dressed energy

The change in the energy of the excited state from the equilibrium state is given by

$$\Delta E = \sum_{\alpha} \left( \sum_{k=1}^{\tilde{N}_{\alpha}} e_{\alpha}(\tilde{v}_{\alpha,k}) - \sum_{k=1}^{N_{\alpha}} e_{\alpha}(v_{\alpha,k}) \right) \rightarrow e_a - e_r + \frac{\sigma_{\alpha}}{2\pi} \zeta_{\alpha} \rho_{\alpha} \star e'_{\alpha}, \quad (3.36)$$

where  $e'_{\alpha} = \frac{d}{dv} e_{\alpha}(v)$ , and summation over  $\alpha$  is assumed, and we have used the notation (3.32). Differentiating the TBA equations (3.19) and integrating by parts one gets

$$\frac{e'_{\alpha}}{T} = (\log Y_{\alpha})' - K_{\alpha\beta} \star \frac{(\log Y_{\beta})'}{1 + Y_{\beta}}, \quad (3.37)$$

and substituting into (3.36) gives

$$\Delta E = e_a - e_r + T \frac{\sigma_{\alpha}}{2\pi} \zeta_{\alpha} \rho_{\alpha} \star \left( (\log Y_{\alpha})' - \frac{K_{\alpha\beta}}{1 + Y_{\beta}} \star (\log Y_{\beta})' \right), \quad (3.38)$$

Dependence of  $\Delta E$  on  $\zeta_{\alpha}$  can be eliminated through eq. (3.35) yielding

$$\Delta E = e_a - e_r + \frac{T}{2\pi} \sigma_{\alpha} \left( \log \left( 1 + \frac{1}{Y_{\alpha}} \right) \right)' \star (\phi_{\alpha a} - \phi_{\alpha r}). \quad (3.39)$$

Then integrating by parts gives <sup>1</sup>

$$\Delta E = e_a - e_r - T \log \left( 1 + \frac{1}{Y_{\alpha}} \right) \star (K_{\alpha a} - K_{\alpha r}) + T \log \left( 1 + \frac{1}{Y_{\alpha}^{\max}} \right) (k_{\alpha a} - k_{\alpha r}), \quad (3.40)$$

where  $Y_{\alpha}^{\max} = Y_{\alpha}(v^{\max})$  is equal to  $Y_{\alpha}$  evaluated at the value of  $v$  corresponding to the maximum of the pseudo-energy, and recall  $k_{\alpha\beta} \equiv 1 \star K_{\alpha\beta} = \sigma_{\alpha} \kappa_{\alpha\beta}$ . Finally we use again the TBA equations (3.19) to obtain

$$\Delta E = \sum_{j=1}^{N_a} \left( \epsilon_{a_j} + T \log \left( 1 + \frac{1}{Y_{\beta}^{\max}} \right) k_{\beta a_j} \right) - \sum_{j=1}^{N_r} \left( \epsilon_{r_j} + T \log \left( 1 + \frac{1}{Y_{\beta}^{\max}} \right) k_{\beta r_j} \right). \quad (3.41)$$

Hence the dressed energy of an  $\alpha$ -string is

$$E_{\alpha}(v) = \epsilon_{\alpha}(v) + T \log \left( 1 + \frac{1}{Y_{\beta}^{\max}} \right) k_{\beta \alpha}. \quad (3.42)$$

---

<sup>1</sup>Let us remark that the jump discontinuity of  $\phi_{\alpha\beta}$  for type 2 strings is  $2\pi c_{\alpha\beta}$ , which is equal to  $2\pi k_{\alpha\beta}$  only if  $S_{\alpha\beta}(\pm\infty) = 1$ .



Note that the last term in the formula is rapidity independent and does not contribute to the total energy in a particle-hole excitation. In previous studies of excitations at non-zero temperature, see e.g. [20], only particle-hole excitations were considered and the dressed energies were given just by the pseudo-energies  $\epsilon_\alpha(v)$ , and the rapidity independent term was neglected. Let us stress however that this term is important, in particular so that in the limit of infinite temperature the dressed energies take their bare values  $E_\alpha = e_\alpha$  as is expected. This follows as the functions  $Y_\alpha$  become constant in the limit  $T \rightarrow \infty$  because the driving terms drop out of the TBA equations (3.19).

### Dressed momentum

In a similar way the change in the momentum of the excited state from the equilibrium state is given by

$$\Delta P = \sum_{\alpha} \left( \sum_{k=1}^{\tilde{N}_{\alpha}} \tilde{p}_{\alpha,k} - \sum_{k=1}^{N_{\alpha}} p_{\alpha,k} \right) \rightarrow p_a - p_r + \frac{1}{2\pi} \zeta_{\alpha} \rho_{\alpha} \star \left| \frac{dp_{\alpha}}{dv} \right|. \quad (3.43)$$

Let us first remark that the momentum of a state is defined modulo  $2\pi$  and thus, as the momentum of the reference state is fixed, the change in momentum is also defined modulo  $2\pi$ . To simplify expression (3.43) we substitute (3.7) into (3.30) and get

$$\frac{1}{2\pi} \zeta_{\alpha} \left| \frac{dp_{\alpha}}{dv} \right| + \zeta_{\alpha} \mathbf{K}_{\alpha\beta} \star \rho_{\beta} = \zeta_{\beta} \rho_{\beta} \star \mathbf{K}_{\beta\alpha} - \phi_{\alpha a} + \phi_{\alpha r}. \quad (3.44)$$

Multiplying by  $\rho_{\alpha}$ , integrating, and taking the sum over  $\alpha$  we find

$$\frac{1}{2\pi} \zeta_{\alpha} \rho_{\alpha} \star \left| \frac{dp_{\alpha}}{dv} \right| = -\rho_{\alpha} \star (\phi_{\alpha a} - \phi_{\alpha r}), \quad (3.45)$$

and thus

$$\Delta P = \sum_{j=1}^{N^a} (p_{a_j} - \rho_{\beta} \star \phi_{\beta a_j}) - \sum_{j=1}^{N^r} (p_{r_j} - \rho_{\beta} \star \phi_{\beta r_j}). \quad (3.46)$$

Hence we identify the dressed momentum of an added  $\alpha$ -string as

$$P_{\alpha} = p_{\alpha} - \rho_{\beta} \star \phi_{\beta\alpha}. \quad (3.47)$$

and a removed one with the opposite sign.

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To examine the range of dressed momentum it is useful to note that

$$\left| \frac{dP_\alpha}{dv} \right| = 2\pi(\rho_\alpha + \bar{\rho}_\alpha), \quad (3.48)$$

which is seen using eqs. (3.7) and (3.31). For  $\alpha$ -strings of type 1 the range of dressed momentum is over  $(P_\alpha(0) - \pi(n_\alpha + \bar{n}_\alpha), P_\alpha(0) + \pi(n_\alpha + \bar{n}_\alpha))$  where  $P_\alpha(0) = p_\alpha(0) + 2\pi n_\beta b_{\beta\alpha} + \pi n_\beta c_{\beta\alpha}$ ,  $n_\alpha = 1 \star \rho_\alpha$  and  $\bar{n}_\alpha = 1 \star \bar{\rho}_\alpha$ . On the other hand for strings of type 2 the range is split into two parts. Recall that the bare momentum of a type 2  $\alpha$ -string has a jump at  $v = 0$  and that it increases from  $p_\alpha(\sigma_\alpha 0)$  to  $p_\alpha(-\sigma_\alpha 0)$ . The range of dressed momentum in this case is thus over  $(P_\alpha(\sigma_\alpha 0), P_\alpha(\sigma_\alpha 0) + \pi(n_\alpha + \bar{n}_\alpha))$  and  $(P_\alpha(-\sigma_\alpha 0) - \pi(n_\alpha + \bar{n}_\alpha), P_\alpha(-\sigma_\alpha 0))$ . In general this may result in a gap in the dressed momentum.

Let us remark that the dressed momenta depend on the choice of branch of  $\log S_{\alpha\beta}$ , i.e. the  $b_{\alpha\beta}$  in eqs. (3.23), (3.24). Considering only strings of type 1 and 2 the dressed momenta (3.47) can be written as

$$P_\alpha = p_\alpha - \rho_\beta \star \Theta_{\beta\alpha} - 2\pi n_\beta b_{\beta\alpha} - \sum_{\beta \text{ of type 1}} \pi n_\beta c_{\beta\alpha}, \quad (3.49)$$

where one is free to choose the integers  $b_{\beta\alpha}$ . Terms for  $c_{\beta\alpha}$  with  $\beta$  of type 2 do not contribute as we restrict ourselves to parity invariant models and so the densities are even. It is of course possible to describe any excitation with a definite choice of  $b_{\beta\alpha}$ , e.g. one can set all  $b_{\beta\alpha} = 0$ . However in this case some excitations would have unnatural description which would require considering particle-hole excitations with zero energy contributing only to the total momentum. For  $n_\beta$  irrational one can achieve any value of dressed momentum by choosing  $b_{\beta\alpha}$  appropriately. Let us remark that one is free to choose  $b_{\beta\alpha}$  independently for each added and removed  $\alpha$ -string and one may refer to the set  $b_{\beta a_j}$ ,  $b_{\beta r_j}$ , where  $\beta$  runs over all strings that interact with the excited string, as the branch of the excitation.

### Dressed phase shift

Now we turn our attention to the scattering phase shift. Consider first the counting function for an  $\alpha$ -string of the excited state

$$L \sigma_\alpha \tilde{z}_\alpha(v) = \pi \varphi_\alpha + L p_\alpha(v) + \phi_{\alpha a}(v) - \phi_{\alpha r}(v) + \sum_{\beta} \sum_{j=1}^{N_\beta} \phi_{\alpha\beta}(v, v'_{\beta,j}), \quad (3.50)$$



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Expanding  $v'_{\beta,j}$  in the final term about its equilibrium value, replacing the sums by integrals, and noting equations (3.30, 3.47), one gets

$$L \sigma_\alpha \tilde{z}_\alpha = \pi \varphi_\alpha + L P_\alpha + F_\alpha. \quad (3.51)$$

Recall that  $F_\alpha$  here is the shift function which is determined through the closed set of equations (3.34). Exponentiating equation (3.51) and evaluating it at a rapidity  $v$  corresponding to a mode number of the excited state it takes the form

$$1 = e^{iL P_\alpha} e^{i(F_\alpha + \pi \varphi_\alpha)}. \quad (3.52)$$

An added  $\alpha$ -string with rapidity  $v$  has dressed momentum  $P_\alpha(v)$  and so its scattering phase shift is  $\delta_\alpha = F_\alpha(v) + \pi \varphi_\alpha$ . Similarly a removed  $\alpha$ -string with rapidity  $v$  has dressed momentum  $-P_\alpha(v)$  and so its scattering phase shift is  $\delta_\alpha = -F_\alpha(v) - \pi \varphi_\alpha$ . Clearly the phase shifts  $\delta$  are defined modulo  $2\pi$ .

As the equations (3.34) are linear it is natural to introduce the set of functions  $\Phi_{\alpha\beta}(v, t)$  satisfying the following system of equations

$$\Phi_{\alpha\beta} = \phi_{\alpha\beta} + \frac{\Phi_{\gamma\beta}}{1 + Y_\gamma} \star K_{\gamma\alpha}, \quad (3.53)$$

where it is understood that  $\left(\frac{\Phi_{\gamma\beta}}{1 + Y_\gamma} \star K_{\gamma\alpha}\right)(v, t) = \int dw \frac{\Phi_{\gamma\beta}(w, t)}{1 + Y_\gamma(w)} K_{\gamma\alpha}(w, v)$ . Then  $F_\alpha = \Phi_{\alpha a} - \Phi_{\alpha r}$  and hence we refer to the  $\Phi_{\alpha\beta}$  as dressed scattering phases. In terms of these functions equation (3.52) takes the following physically intuitive form

$$(-1)^{\varphi_\alpha} = e^{iP_\alpha(v)L} \prod_{j=1}^{N^a} e^{i\Phi_{\alpha a_j}(v, \tilde{v}_{a_j})} \prod_{j=1}^{N^r} e^{-i\Phi_{\alpha r_j}(v, \tilde{v}_{r_j})}. \quad (3.54)$$

It is worth mentioning that the dressed scattering phases are in general not of a difference form, and in particular  $\Phi_{\alpha\beta}(v, v) \neq 0$ . This is a reflection of the fact that an excitation has nontrivial scattering with the equilibrium state.

### Dressed charge

Each  $\alpha$ -string carries a bare charge  $w_\alpha^i$  under each of the conserved operators  $N_i$ . These quantities too get dressed for excitations above the equilibrium state. Recall first that the bare charge carried by an  $\alpha$ -string is  $w_\alpha^i = -\frac{\partial \mathcal{E}_\alpha}{\partial \mu_i}$ . Now we

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introduce an object  $\omega_\alpha^i = -\frac{\partial \epsilon_\alpha}{\partial \mu_i}$ , which we call the pseudo-charge of an  $\alpha$ -string. It satisfies the following set of integral equations

$$\omega_\alpha^i = w_\alpha^i + \frac{\omega_\beta^i}{1 + Y_\beta} \star K_{\beta\alpha}. \quad (3.55)$$

Let us consider the dressing for a specific charge and so we momentarily suppress the index  $i$ . The change in the total charge of the excited state from the equilibrium state is

$$\Delta W = \sum_\alpha \left( \sum_{k=1}^{\tilde{N}_\alpha} w_\alpha(\tilde{v}_{\alpha,k}) - \sum_{k=1}^{N_\alpha} w_\alpha(v_{\alpha,k}) \right) \rightarrow w_a - w_r + \frac{\sigma_\alpha}{2\pi} \zeta_{\alpha\rho_\alpha} \star w'_\alpha, \quad (3.56)$$

where here we are being formal as  $w_\alpha$  has no rapidity dependence. Indeed the final term is zero, but let us further analyse it nevertheless. Recalling that  $\zeta_{\alpha\rho_\alpha} = -\frac{F_\alpha}{1+Y_\alpha}$ , we have

$$\begin{aligned} \zeta_{\alpha\rho_\alpha} \star w'_\alpha &= -\frac{F_\alpha}{1+Y_\alpha} \star w'_\alpha = -\frac{F_\alpha}{1+Y_\alpha} \star \omega'_\alpha + \frac{F_\alpha}{1+Y_\alpha} \star K_{\alpha\beta} \star \left( \frac{\omega_\beta}{1+Y_\beta} \right)' \\ &= -\frac{F_\alpha}{1+Y_\alpha} \star \omega'_\alpha + F_\alpha \star \left( \frac{\omega_\alpha}{1+Y_\alpha} \right)' - \phi_{\alpha a} \star \left( \frac{\omega_\alpha}{1+Y_\alpha} \right)' + \phi_{\alpha r} \star \left( \frac{\omega_\alpha}{1+Y_\alpha} \right)' \\ &= F_\alpha \omega_\alpha \star \left( \frac{1}{1+Y_\alpha} \right)' + \frac{2\pi\sigma_\alpha\omega_\alpha}{1+Y_\alpha} \star (K_{\alpha a} - K_{\alpha r}) - \frac{2\pi\sigma_\alpha\omega_\alpha^{\max}}{1+Y_\alpha^{\max}} (k_{\alpha a} - k_{\alpha r}). \end{aligned}$$

Here  $\omega_\alpha^{\max} = \omega_\alpha(v^{\max})$  is defined similarly to  $Y_\alpha^{\max}$ , both functions being evaluated at the value of  $v$  corresponding to the maximum of the pseudo-energy. This allows one to write the total change in charge as

$$\begin{aligned} \Delta W^i &= \sum_{j=1}^{N^a} \left( \omega_{a_j}^i(\tilde{v}_{a_j}) - \frac{\omega_\alpha^{i,\max}}{1+Y_\alpha^{\max}} k_{\alpha a_j} \right) - \sum_{j=1}^{N^r} \left( \omega_{r_j}^i(\tilde{v}_{r_j}) - \frac{\omega_\alpha^{i,\max}}{1+Y_\alpha^{\max}} k_{\alpha r_j} \right) \\ &\quad + \frac{\sigma_\alpha}{2\pi} F_\alpha \omega_\alpha^i \star \left( \frac{1}{1+Y_\alpha} \right)', \end{aligned}$$

where we have reinstated the index  $i$ . We would like to present an interpretation of this change as

$$\Delta W^i = \sum_{j=1}^{N^a} W_{a_j}^i(\tilde{v}_{a_j}) - \sum_{j=1}^{N^r} W_{r_j}^i(\tilde{v}_{r_j}) + \Delta W_{\text{ind}}^i. \quad (3.57)$$



### 3.2 Excitations about equilibrium

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Here the excited strings are assigned a dressed charge

$$W_\alpha^i(v) = \omega_\alpha^i(v) - \frac{\omega_\beta^{i, \max}}{1 + Y_\beta^{\max}} k_{\beta\alpha} = -\frac{\partial E_\alpha}{\partial \mu_i} \quad (3.58)$$

which they carry, while the final term

$$\Delta W_{\text{ind}}^i = \frac{\sigma_\alpha}{2\pi} F_\alpha \omega_\alpha^i \star \left( \frac{1}{1 + Y_\alpha} \right)' \quad (3.59)$$

is understood as an induced charge of the system. An interesting feature here is that the dressed charge carried by an excited string depends in general on the string's rapidity. In the limit of infinite temperature the functions  $Y_\alpha$  become constant and the dressed charges take their bare values while the induced charge goes to zero. The zero temperature limit will be discussed in section 3.4 where it is seen that the induced charge resides at the edge of the Fermi sea.

The above is the interpretation we shall adopt in this paper but let us mention that the final term in eq. (3.57) can be redistributed among the added and removed roots using  $F_\alpha = \Phi_{\alpha a} - \Phi_{\alpha r}$ . In particular, integrating by parts this final term one obtains back

$$\Delta W^i = \sum_{j=1}^{N^a} w_{a_j}^i - \sum_{j=1}^{N^r} w_{r_j}^i, \quad (3.60)$$

via the curious identity

$$\frac{\omega_\beta^i}{1 + Y_\beta} \star K_{\beta\alpha} = \frac{\sigma_\alpha}{2\pi} \frac{1}{1 + Y_\beta} \star (\omega_\beta^i \Phi_{\beta\alpha})'. \quad (3.61)$$

One may wonder why we insist on the interpretation of eq. (3.57) over that of eq. (3.60). These are two ways of interpreting  $\Delta W^i$  that imply different physics. That the change in charge can be split as in eq. (3.57) and that the dressed charge is related to the dressed energy as  $W^i = -\frac{dE}{d\mu_i}$  is quite convincing. An important factor also is that spin-charge separation has been observed experimentally [55] and to account for it requires an understanding of the dressing of charge that extends to non-zero temperatures. Equation (3.60) does not provide this.

### 3. EXACT SOLUTION

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#### Summary of the excited state formalism

Let us finish this detailed section with a summary of the principle features of the excited state formalism. Excitations of a finite number of strings above the equilibrium state were investigated by examining the resulting shifts of the string Bethe roots (3.29). These shifts satisfy a closed set of non-linear integral equations (3.30), and with the aid of these equations all the features of an excitation can be extracted.

Let us clear the notations we have been working with and parametrise an excited state by  $N^p$  added particles with rapidities  $v_{p_k}$ ,  $k = 1, \dots, N^p$ , and by  $N^h$  holes with rapidities  $v_{h_k}$ ,  $k = 1, \dots, N^h$ . Here the indices  $p_k$  and  $h_k$  include the information of the type of string. The excitation can be encoded in a set of Bethe equations for which the pseudo-vacuum is the equilibrium state

$$(-1)^{\varphi_a} = e^{iP_{p_k}(v_{p_k})L} \prod_{j=1}^{N^p} e^{i\Phi_{p_k p_j}(v_{p_k}, v_{p_j})} \prod_{j=1}^{N^h} e^{-i\Phi_{p_k h_j}(v_{p_k}, v_{h_j})}, \quad (3.62)$$

$$(-1)^{-\varphi_a} = e^{iP_{h_k}(v_{h_k})L} \prod_{j=1}^{N^h} e^{i\Phi_{h_k h_j}(v_{h_k}, v_{h_j})} \prod_{j=1}^{N^p} e^{-i\Phi_{h_k p_j}(v_{h_k}, v_{p_j})}. \quad (3.63)$$

Let us stress that these Bethe equations are only valid for large  $L$ . The momentum is given through eq. (3.47)

$$P_\alpha = p_\alpha - \rho_\beta \star \phi_{\beta\alpha} \quad \text{for particle excitations,} \quad (3.64)$$

$$P_\alpha = -p_\alpha + \rho_\beta \star \phi_{\beta\alpha} \quad \text{for hole excitations,} \quad (3.65)$$

and the scattering phases  $\Phi_{\alpha\beta}$  are determined through the closed set of equations (3.53). Due to interactions with the equilibrium state the energy of each excited root gets dressed (3.42)

$$E_\alpha = \epsilon_\alpha + T \log\left(1 + \frac{1}{Y_\beta^{\max}}\right) k_{\beta\alpha} \quad \text{for particle excitations,} \quad (3.66)$$

$$E_\alpha = -\epsilon_\alpha - T \log\left(1 + \frac{1}{Y_\beta^{\max}}\right) k_{\beta\alpha} \quad \text{for hole excitations.} \quad (3.67)$$



### 3.3 Thermodynamic variables

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Here  $\epsilon_\alpha = T \log Y_\alpha$  are the pseudo-energies and the constants  $Y_\alpha^{\max}$  and  $k_{\alpha\beta}$  are defined after eq. (3.40). Similarly, for each conserved operator  $\mathbf{N}_i$  the corresponding charge of the excited root gets dressed and is given by (3.58)

$$W_\alpha^i = \omega_\alpha^i - \frac{\omega_\beta^{i,\max}}{1 + Y_\beta^{\max}} k_{\beta\alpha} = -\frac{dE_\alpha}{d\mu_i} \quad \text{for particle excitations,} \quad (3.68)$$

$$W_\alpha^i = -\omega_\alpha^i + \frac{\omega_\beta^{i,\max}}{1 + Y_\beta^{\max}} k_{\beta\alpha} = -\frac{dE_\alpha}{d\mu_i} \quad \text{for hole excitations,} \quad (3.69)$$

where  $\omega_\alpha^i$  are the pseudo-charges determined through the closed set of equations (3.55) and the constant  $\omega_\alpha^{i,\max}$  is defined above eq. (3.57). The total change in charge  $\mathbf{N}_i$  for the excitation also has a contribution  $\Delta W_{\text{ind}}^i$  given in eq. (3.59). This is an induced charge of the system that is not carried by the excited roots, but rather is due to a back-reaction of the densities. In the limit of infinite temperature  $T \rightarrow \infty$  the dressed energy and dressed charge take their bare values. The limit of zero temperature requires special attention and is discussed in section 3.4.

### 3.3 Thermodynamic variables

Now we discuss the calculation of the thermodynamic variables. In particular we consider the charge densities  $n_i$ , the susceptibilities  $\chi_i$ , and the specific heat capacity  $c$ , which are defined through eq. (3.22).

Let first show that the definition of charge density as a derivative of the free energy is consistent with

$$n_i = w_\alpha^i \star \rho_\alpha. \quad (3.70)$$

Taking the derivative of the free energy (3.21) gives

$$\frac{\partial f}{\partial \mu_i} = \frac{T}{2\pi} \left| \frac{dp_\alpha}{dv} \right| \star \frac{\partial_{\mu_i} \log Y_\alpha}{1 + Y_\alpha}, \quad (3.71)$$

where we have used the shorthand  $\partial_{\mu_i} = \frac{\partial}{\partial \mu_i}$ . Substituting for  $\left| \frac{dp_\alpha}{dv} \right|$  through the equation for densities (3.7), and simplifying using the equation for dressed charge (3.55), this becomes  $\frac{\partial f}{\partial \mu_i} = -w_\alpha^i \star \rho_\alpha$ . Thus as expected we obtain (3.70).

### 3. EXACT SOLUTION

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There is an important observation to be made here. The formula (3.70) allows for the calculation of the charge density without the need to explicitly take a derivative of the free energy. As the free energy must generally be calculated numerically, so too must its derivatives. Calculating derivatives by taking finite differences is not however a reliable approach, as it may be difficult to obtain accurate results. This is especially true in regimes where there is noteworthy behaviour. Instead it is better to re-express the derivatives in terms of quantities that obey closed sets of integral equations [56], as is the case for the densities  $\rho_\alpha$  on the right hand side of (3.70).

Now we consider the calculation of the susceptibilities

$$\chi_i = -\frac{\partial^2 f}{\partial \mu_i^2} = \frac{\partial n_i}{\partial \mu_i} = w_\alpha^i \star \frac{\partial \rho_\alpha}{\partial \mu_i}. \quad (3.72)$$

Here we wish to rewrite  $\partial_{\mu_i} \rho_\alpha$  in terms of quantities that do not require derivatives to be taken numerically. This can be achieved by taking the derivative of the equation for densities (3.7)

$$\partial_{\mu_i} \rho_\alpha + \frac{\rho_\alpha \partial_{\mu_i} Y_\alpha}{1 + Y_\alpha} = \frac{1}{1 + Y_\alpha} K_{\alpha\beta} \star \partial_{\mu_i} \rho_\beta. \quad (3.73)$$

Convoluting both sides of this equation with the dressed charge  $\omega_\alpha^i$  and simplifying with eq. (3.55) one obtains

$$\begin{aligned} w_\alpha^i \star \partial_{\mu_i} \rho_\alpha &= -\omega_\alpha^i \star \frac{\rho_\alpha \partial_{\mu_i} Y_\alpha}{1 + Y_\alpha} \\ &= -(\omega_\alpha^i)^2 \star \frac{\rho_\alpha + \bar{\rho}_\alpha}{\epsilon'_\alpha} \left( \frac{1}{1 + Y_\alpha} \right)', \end{aligned} \quad (3.74)$$

where prime denotes the derivative with respect to the rapidity variable. Here the terms  $\epsilon_\alpha$  and  $Y_\alpha$  satisfy the set of integral equations eq. (3.37). Finally we can write the susceptibility as follows

$$\chi_i = w_\alpha^i \star \partial_{\mu_i} \rho_\alpha = -\frac{(\omega_\alpha^i)^2}{2\pi v_\alpha} \star \left( \frac{1}{1 + Y_\alpha} \right)', \quad (3.75)$$

where we have introduced the velocity  $v_\alpha = \frac{dE_\alpha}{dP_\alpha}$ .

Lastly we examine the specific heat. To simplify the calculation we note that the free energy factorises  $f = T \tilde{f}(T)$ , and so

$$c = -T \partial_T^2 f = -2T \partial_T \tilde{f} - T^2 \partial_T^2 \tilde{f}'', \quad (3.76)$$



where  $\partial_T^2 = \frac{\partial^2}{\partial T^2}$ . Here

$$\tilde{f} = -\frac{1}{2\pi} \left| \frac{dp_\alpha}{dv} \right| \star \log \left( 1 + \frac{1}{Y_\alpha} \right), \quad (3.77)$$

and so

$$\partial_T \tilde{f} = \frac{1}{2\pi} \left| \frac{dp_\alpha}{dv} \right| \star \frac{\partial_T \log Y_\alpha}{1 + Y_\alpha}. \quad (3.78)$$

This can be simplified through the temperature derivative of the TBA equations

$$T \partial_T \left( \frac{\epsilon_\alpha}{T} \right) = -\frac{e_\alpha}{T} + \frac{T \partial_T \left( \frac{\epsilon_\beta}{T} \right)}{1 + Y_\beta} \star K_{\beta\alpha}, \quad (3.79)$$

and the equation for densities (3.7) giving  $\partial_T \tilde{f} = -\frac{1}{T^2} e_\alpha \star \rho_\alpha$ , and thus we obtain

$$c = e_\alpha \star \partial_T \rho_\alpha = \frac{\partial E}{\partial T}. \quad (3.80)$$

Now in an analogous manner to our consideration above for the susceptibilities, we can rewrite the specific heat by convoluting the temperature derivative of the equation for densities with  $\partial_T \left( \frac{\epsilon_\alpha}{T} \right)$ , and simplifying using eq. (3.79)

$$\begin{aligned} c = e_\alpha \star \partial_T \rho_\alpha &= T^2 \partial_T \left( \frac{\epsilon_\alpha}{T} \right) \star \frac{\rho_\alpha \partial_T Y_\alpha}{1 + Y_\alpha} \\ &= -\frac{1}{2\pi v_\beta} \left( T \partial_T \left( \frac{\epsilon_\alpha}{T} \right) \right)^2 \star \left( \frac{1}{1 + Y_\alpha} \right)'. \end{aligned} \quad (3.81)$$

Here  $T \partial_T \left( \frac{\epsilon_\beta}{T} \right)$  is determined through the set of integral equations (3.79).

### 3.4 Zero temperature

The limit of zero temperature is interesting both physically and mathematically. From a physical point of view the limit takes the equilibrium state to the ground state. The absence of thermal fluctuations here allows for a clear picture of the behaviour of a model to be obtained. On the mathematical side this is manifest as a dramatic simplification the TBA equations. The equations become linear in the limit, and their solution here provides a platform for understanding the solution for general  $T$ . Chapters 5 and 6 provide detailed studies of the zero temperature limits of the Hubbard-Shastry A- and B-models, and in them much of what is indicated here is made precise.

### 3. EXACT SOLUTION

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Examining the TBA equations (3.19) in the zero temperature limit we see that it is better to work with the pseudo-energies  $\epsilon_\alpha = T \log Y_\alpha$  rather than with the functions  $Y_\alpha$  directly. Indeed, in the limit  $T \rightarrow 0$  we see that the functions  $Y_\alpha$  become singular

$$\begin{aligned} \lim_{T \rightarrow 0} \epsilon_\alpha(v) < 0 &\Leftrightarrow \lim_{T \rightarrow 0} Y_\alpha(v) = 0 &\Rightarrow \lim_{T \rightarrow 0} \bar{\rho}_\alpha(v) = 0, \\ \lim_{T \rightarrow 0} \epsilon_\alpha(v) > 0 &\Leftrightarrow \lim_{T \rightarrow 0} Y_\alpha(v) = \infty &\Rightarrow \lim_{T \rightarrow 0} \rho_\alpha(v) = 0. \end{aligned} \quad (3.82)$$

Let us remark here that  $\epsilon_\alpha(v) < 0$  implies that there are no holes for  $\alpha$ -strings with spectral parameter  $v$  in the ground-state whereas  $\epsilon_\alpha(v) > 0$  implies that there are no particles of  $\alpha$ -strings with spectral parameter  $v$  in the ground-state. For each  $\alpha$ -string let us define the following subintervals of  $\mathcal{I}_\alpha$

$$\begin{aligned} Q_\alpha &= \{v : \epsilon_\alpha(v) < 0\}, \\ \bar{Q}_\alpha &= \{v : \epsilon_\alpha(v) > 0\}. \end{aligned} \quad (3.83)$$

We say that an  $\alpha$ -string is at half-filling if  $Q_\alpha = \mathcal{I}_\alpha$ , which implies from (3.82) that there are no holes in the ground state for such strings. Let us next denote the boundaries between  $Q_\alpha$  and  $\bar{Q}_\alpha$ . For increasing  $v$  we label as  $q_\alpha^+$  the point where  $v$  goes from  $\bar{Q}_\alpha$  to  $Q_\alpha$ , and as  $q_\alpha^-$  the point where  $v$  goes from  $Q_\alpha$  to  $\bar{Q}_\alpha$ . Then in the zero temperature limit

$$\frac{1}{1 + Y_\alpha(v)} \rightarrow \begin{cases} 1 & \text{if } v \in Q_\alpha \\ 0 & \text{if } v \in \bar{Q}_\alpha \end{cases}, \quad (3.84)$$

and

$$\left(\frac{1}{1 + Y_\alpha}\right)'(v) \rightarrow \delta(v - q_\alpha^+) - \delta(v - q_\alpha^-) \quad (3.85)$$

where  $\delta$  is the Dirac delta function. These two formulae lead to many simplifications of the equations in the preceding sections of this chapter. The relations (3.82) also lead to following zero temperature limit of the TBA equations (3.19)

$$\epsilon_\alpha = e_\alpha + \epsilon_\beta \star_{Q_\beta} K_{\beta\alpha}. \quad (3.86)$$

Let us remark that in addition to linearity of the TBA equations at zero temperature there are often further simplifications. If  $\epsilon_\beta(v) > 0$  for all  $v$  for some  $\beta$ -strings, then the respective pseudo-energies  $\epsilon_\beta$  do not contribute on the right hand side of the equations (3.86). This can represent a notable further simplification of the equations.



## Ground state phase diagram

Due to the sharp behaviour (3.82) of the densities at zero temperature it is possible to construct a phase diagram. The various phases are determined by identifying which string types occupy the ground state. A relevant phase space is that generated by the various chemical potentials and coupling constants that appear in the model. For each  $\alpha$ -string that can exist in the ground state useful phase boundaries are the lines along which it enters the ground state and lines along which the  $\alpha$ -string reaches half-filling. The first set are determined by the condition  $\min_v \epsilon_\alpha(v) = 0$ , while the second set are determined by  $\max_v \epsilon_\alpha(v) = 0$ .

## Excitations at zero temperature

Now we turn our attention to excitations at zero temperature. First consider the situation where all strings are away from half-filling. Here the problem mentioned in the paragraph above eqs. (3.27), (3.28) does not arise and the zero temperature limit of the formalism for excitations above equilibrium is straightforward. The total change in energy for an excitation, given by (3.41), reduces to

$$\Delta E = \sum_{j=1}^{N^a} \epsilon_{a_j}(\tilde{v}_{a_j}) - \sum_{j=1}^{N^r} \epsilon_{r_j}(\tilde{v}_{r_j}), \quad (3.87)$$

as for each string at less than half-filling  $\epsilon_\alpha^{\max} > 0$  implies  $Y_\alpha^{\max} = \infty$ . This is the familiar picture in which the pseudo-energies play the role of the dressed energies.

Similarly the change in charge (3.57) becomes

$$\Delta W^i = \sum_{j=1}^{N^a} \omega_{a_j}^i(\tilde{v}_{a_j}) - \sum_{j=1}^{N^r} \omega_{r_j}^i(\tilde{v}_{r_j}) + \Delta W_{\text{ind}}^i. \quad (3.88)$$

The limit of the induced charge can be taken using eq. (3.85) giving

$$\Delta W_{\text{ind}}^i = \frac{\sigma_\alpha}{2\pi} \zeta_\alpha(q_\alpha^-) \rho_\alpha(q_\alpha^-) \omega_\alpha^i(q_\alpha^-) - \frac{\sigma_\alpha}{2\pi} \zeta_\alpha(q_\alpha^+) \rho_\alpha(q_\alpha^+) \omega_\alpha^i(q_\alpha^+). \quad (3.89)$$

We thus see that the induced charge is due to the shift of the rapidities at the boundaries of the intervals  $Q_\alpha$ . This can be understood as a back-reaction of the density, which here at zero temperature occurs at the edge of the Fermi sea.

### 3. EXACT SOLUTION

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Now we turn to the situation of having some strings in the ground state at half-filling, let us say that  $\epsilon_\gamma^{\max} \leq 0$  for some  $\gamma$ -strings. Here one must be careful to only consider excitations for which the  $\tilde{N}_\gamma$  satisfy the selection rules (3.10) as there are no holes for  $\gamma$ -strings. Put another way, some of the  $\gamma$ -strings in the ground state may have no corresponding mode number in the excited state, due to a decrease in the range of mode numbers, and are thus necessarily removed. On the other hand an increase in the range of mode numbers will mean that there are some holes in the excited state that do not correspond to removed strings and are thus not dynamical. This situation requires one to reconsider the nature of the excitations.

Let us outline a convenient prescription for dealing with excitations that change the range of mode numbers of strings which are at half-filling. If the range increases we choose to consider only excitations for which all the extra mode numbers are filled. In our terminology this means that in such an excitation these extra mode numbers always correspond to added strings and thus all holes of the excited state correspond to removed strings. Obviously if the range decreases in an excitation then the removed mode numbers always correspond to removed strings. We refer to such added and removed strings as singular strings. Such singular strings have rapidities that approach  $v^{\max}$  in the limit  $L \rightarrow \infty$  because they correspond to mode numbers at the edges of the range. We refer to the remainder of the added and removed strings as physical strings. Note that this prescription does not limit the freedom to capture all possible excitations. Indeed any excitation for which not all the extra mode numbers are filled can be considered as a limit of an allowed excitation where the rapidities of the necessary number of physical removed strings approach  $v^{\max}$ .

An excitation could also result in an overall shift of the mode numbers. This would correspond to the removal of some singular strings at one end of the range and the addition of singular strings at the other. It can be seen however, that for each of the quantities of interest to us, that this transfer of singular strings is not important.

Let us thus break the added and removed strings into two types, physical and singular

$$N_\gamma^a = N_\gamma^{\text{pa}} + N_\gamma^{\text{sa}}, \quad N_\gamma^r = N_\gamma^{\text{pr}} + N_\gamma^{\text{sr}}, \quad (3.90)$$



### 3.4 Zero temperature

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where we use p and s to denote physical and singular respectively. Let us further denote the changes in numbers of physical and singular strings as

$$\delta N_\gamma^p = N_\gamma^{pa} - N_\gamma^{pr}, \quad \delta N_\gamma^s = N_\gamma^{sa} - N_\gamma^{sr}. \quad (3.91)$$

Then our prescription is that

$$\delta N_\gamma^s = k_{\gamma\beta} \delta N_\beta^p + k_{\gamma\gamma'} \delta N_{\gamma'}^s, \quad (3.92)$$

where the right hand side here is the change in the range of mode numbers of  $\gamma$ -strings found from eq. (3.9), and we use  $\gamma'$  as a dummy index to make it clear that the sum is only over strings which are at half-filling. Let us remark that there may be a restriction on the number of physical roots one can excite as only solutions to eq. (3.92) for which  $\delta N_\gamma^s$  is an integer for each half-filled string are allowed.

Now consider again the change in energy formula (3.41) which here takes the form

$$\Delta E = \sum_{j=1}^{N^a} (\epsilon_{a_j}(\tilde{v}_{a_j}) - \epsilon_\gamma^{\max} k_{\gamma a_j}) - \sum_{j=1}^{N^r} (\epsilon_{r_j}(\tilde{v}_{r_j}) - \epsilon_\gamma^{\max} k_{\gamma r_j}). \quad (3.93)$$

Splitting the strings between their physical and singular subsets this becomes

$$\begin{aligned} \Delta E &= \sum_{j=1}^{N^{pa}} \epsilon_{a_j}(\tilde{v}_{a_j}) - \sum_{j=1}^{N^{pr}} \epsilon_{r_j}(\tilde{v}_{r_j}) - \epsilon_\gamma^{\max} k_{\gamma\alpha} \delta N_\alpha^p + \epsilon_\gamma^{\max} \delta N_\gamma^s - \epsilon_\gamma^{\max} k_{\gamma\gamma'} \delta N_{\gamma'}^s \\ &= \sum_{j=1}^{N^{pa}} \epsilon_{a_j}(\tilde{v}_{a_j}) - \sum_{j=1}^{N^{pr}} \epsilon_{r_j}(\tilde{v}_{r_j}), \end{aligned} \quad (3.94)$$

where all the constant terms have cancelled due to (3.92). The singular strings may also have non-zero dressed momentum and non-trivial dressed scattering. These can be redistributed among the physical strings according to the solution of (3.92). For example, if the solution to eq. (3.92) is  $\delta N_\gamma^s = f_{\gamma\alpha} \delta N_\alpha^p$  then the dressed momentum and dressed scattering take the following form for the half-filled phase

$$P_\alpha^{\text{h.f.}} = P_\alpha + P_\gamma(v^{\max}) f_{\gamma\alpha}, \quad \Phi_{\alpha\beta}^{\text{h.f.}}(v, t) = \Phi_{\alpha\beta}(v, t) + \Phi_{\alpha\gamma}(v, v^{\max}) f_{\gamma\beta}. \quad (3.95)$$

### 3. EXACT SOLUTION

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Finally let us consider again the change in charge. As for the energy, the contributions of the singular roots cancel all constant terms appearing in the dressed charge (3.58). Also the derivative in (3.59) is zero at zero temperature for half-filled strings and so such strings do not give rise to an induced charge of the system. The formula for the change in charge thus takes the form

$$\Delta W^i = \sum_{j=1}^{N_{\text{Pa}}} \omega_{\mathbf{a}_j}^i(\tilde{v}_{\mathbf{a}_j}) - \sum_{j=1}^{N_{\text{Pr}}} \omega_{\mathbf{r}_j}^i(\tilde{v}_{\mathbf{r}_j}) + \Delta W_{\text{ind,ahf}}^i,$$

where  $\Delta W_{\text{ind,ahf}}^i$  denotes the induced charge due to the back-reaction of the strings which are away from half-filling. Let us conclude by remarking that regardless of whether some strings are at half-filling the dressed energy and dressed charge of excited roots take their pseudo values in the zero temperature limit.

#### Thermodynamic variables at zero temperature

Let us describe the zero temperature limit of the susceptibilities. Using (3.85) to take the limit of eq. (3.75) we obtain

$$\begin{aligned} \lim_{T \rightarrow 0} \chi_i &= \sum_{\alpha} \left( \frac{\omega_{\alpha}^i(q_{\alpha}^{-})^2}{2\pi \mathbf{v}_{\alpha}(q_{\alpha}^{-})} - \frac{\omega_{\alpha}^i(q_{\alpha}^{+})^2}{2\pi \mathbf{v}_{\alpha}(q_{\alpha}^{+})} \right) \\ &= \sum_{\alpha} \frac{\omega_{\alpha}^i(q_{\alpha}^{-})^2}{\pi \mathbf{v}_{\alpha}^f}, \end{aligned} \tag{3.96}$$

where to get to the second line we used that  $q_{\alpha}^{-} = -q_{\alpha}^{+}$  for the equilibrium state of parity invariant models, and we write  $\mathbf{v}_{\beta}^f = \mathbf{v}_{\alpha}(q_{\alpha}^{-})$ . Let us remark that only  $\alpha$ -strings that exist in the ground state contribute to the zero temperature susceptibilities, as  $q_{\alpha}$  is only defined for such strings.

Finally let us examine the zero temperature limit of the specific heat. Taking the zero temperature limit of eq. (3.79) and comparing it with eq. (3.86) we observe that

$$\lim_{T \rightarrow 0} T^2 \partial_T \left( \frac{\epsilon_{\alpha}}{T} \right) = -\epsilon_{\alpha}. \tag{3.97}$$



### 3.4 Zero temperature

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Substituting this into the eq. (3.81) for the specific heat and using (3.85) to take the limit we find

$$\begin{aligned}\lim_{T \rightarrow 0} c &= \lim_{T \rightarrow 0} \sum_{\alpha} \frac{1}{T^2} \left( \frac{\epsilon_{\alpha}(q_{\alpha}^{-})^2}{2\pi v_{\alpha}(q_{\alpha}^{-})} - \frac{\epsilon_{\alpha}(q_{\alpha}^{+})^2}{2\pi v_{\alpha}(q_{\alpha}^{+})} \right) \\ &= \lim_{T \rightarrow 0} \sum_{\alpha} \frac{1}{T^2} \frac{\epsilon_{\alpha}(q_{\alpha}^{-})^2}{\pi v_{\alpha}^f}.\end{aligned}\tag{3.98}$$

The expression on the right hand side here is not well defined, as the pseudo-energies go to zero at  $v = q_{\alpha}^{\pm}$ . Thus it appears that the zero temperature limit of the specific heat must be taken more carefully.

### 3. EXACT SOLUTION

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# Chapter 4

## Thermodynamics of the Hubbard-Shastry models

In the previous chapter we outlined a general method for the exact solution of a Bethe ansatz solvable model. Now we focus in on the Hubbard-Shastry models, and in particular on the parity invariant A- and B-models identified in (2.43). This is not a straightforward application of the general formalism as these models have Bethe roots with a non-trivial rapidity structure. The derivation of the TBA equations for the models is thus done in some detail. The equations are then simplified and analysed in various limits. In this chapter we deal mainly with the technical details of the Bethe ansatz solution. The results obtained here are used to examine the zero temperature properties of the A- and B-models in the following two chapters.

### 4.1 Bethe equations and the string hypothesis

Let us begin by recalling the Bethe equations for the Hubbard-Shastry models

$$\begin{aligned} 1 &= e^{iLp(v_k)} \prod_{j=1}^M \frac{v_k - w_j - i\mathbf{u}}{v_k - w_j + i\mathbf{u}}, & k = 1, \dots, N \leq L, \\ -1 &= \prod_{j=1}^N \frac{w_k - v_j - i\mathbf{u}}{w_k - v_j + i\mathbf{u}} \prod_{l=1}^M \frac{w_k - w_l + 2i\mathbf{u}}{w_k - w_l - 2i\mathbf{u}}, & k = 1, \dots, M \leq \frac{N}{2}. \end{aligned} \tag{4.1}$$

The momentum  $p$  is in defined through eq. (2.39). For each of the A-, B- and Hubbard models  $e^{ip(v)}$  is given in Table 4.1, along with the dispersion relations

#### 4. THERMODYNAMICS OF THE HS MODELS

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$\mathcal{E}(p)$ . The rapidity variable  $v$  of momentum carrying roots is related to the momentum through  $y$  with  $v = \frac{1}{2}(y + 1/y)$ , and so  $e^{ip(v)}$  is a double valued function of  $v$ . We refer to these roots as  $y$ -particles.

	A-model	B-model	Hubbard
$e^{ip(v)}$	$i \frac{1+yx^+}{y-x^+}$	$\frac{y+x^+}{y-x^+}$	$iy$
$\mathcal{E}(p)$	$-2 \cos p - 2\sqrt{1+u^2}$	$-2 \cos p$	$-2 \cos p - 2u$

**Table 4.1:** Momenta and dispersion relations of Hubbard-Shastry models with  $v = \frac{1}{2}(y + 1/y)$ , and  $x^+ = i(u + \sqrt{1+u^2})$ .

The conserved charges for the models are  $N_c = N$  and  $N_s = S_z$  and their corresponding chemical potentials are respectively  $\mu_c = \mu$  and  $\mu_s = 2B$ , see eq. (2.8). The eigenvalues of these charges are related to  $N$  and  $M$  as

$$N_c = N, \quad N_s = \frac{N - 2M}{2}, \quad (4.2)$$

and so the  $y$  and  $w$  Bethe roots are charged as

$$w_y^c = 1, \quad w_y^s = \frac{1}{2}, \quad w_w^c = 0, \quad w_w^s = -1. \quad (4.3)$$

It is convenient to introduce the following functions

$$x_A(v) = v + v\sqrt{1 - \frac{1}{v^2}}, \quad |x_A(v)| \geq 1, \quad v \in \mathbb{C}, \quad (4.4)$$

with a cut  $\mathcal{I}^A = (-1, 1)$ , and

$$x_B(v) = v + i\sqrt{1 - v^2}, \quad \text{Im}(x_B(v)) \geq 0, \quad v \in \mathbb{C}, \quad (4.5)$$

with a cut  $\mathcal{I}^B = (-\infty, -1) \cup (1, \infty)$ . Let us also define  $\mathcal{I}_+^{A,B}$  and  $\mathcal{I}_-^{A,B}$  to be respectively the upper and lower edges of  $\mathcal{I}^{A,B}$ . For values of  $v$  on the cuts we define  $x(v) = x(v + i0)$ . Both functions solve the equation

$$x(v) + \frac{1}{x(v)} = 2v,$$



## 4.1 Bethe equations and the string hypothesis

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and the parameters  $x^\pm$  of the A- and B-models can be uniformly written as  $x^+ = x(i\mathbf{u})$ ,  $x^- = 1/x(-i\mathbf{u})$  where  $x = x_A$  and  $x = x_B$  for the A- and the B-model, respectively.<sup>1</sup>

Since  $y + 1/y = 2v$  for any given  $v$  one has two possible corresponding  $y$ -roots and they can be parametrized as

$$y_+ = x(v), \quad y_- = \frac{1}{x(v)}, \quad (4.6)$$

so that the set of complex  $y$ -roots is divided into these two subsets. Here and in what follows  $x(v) = x_A(v)$  for the Hubbard and A-models, and  $x(v) = x_B(v)$  for the B-model. The two types of  $y$ -roots  $y_\pm$  in turn lead to two branches  $p_\pm(v)$  and  $\mathcal{E}_\pm(v)$  of the momentum  $p$  and the energy  $\mathcal{E}$  as functions of  $v$ .

The string hypothesis states that each root of the Bethe equations in the thermodynamic limit is a member of the following types of Bethe strings:

1.  $y$ -particle:  $w_y^c = 1$ ,  $w_y^s = 1/2$ , a single charge spin-up excitation parametrized by its real momentum  $p$ . The reality of  $p$  implies that

$$\begin{aligned} |y| = 1, \quad |v| \leq 1, \quad v \in \mathbb{R} \quad &\text{for Hubbard and A-models,} \\ y \in \mathbb{R}, \quad |v| \geq 1, \quad v \in \mathbb{R} \quad &\text{for B-model.} \end{aligned} \quad (4.7)$$

Since these  $v$  are on the cut of the corresponding function  $x$ , the  $y_\pm$  roots can be written also as

$$y_\pm = x(v \pm i0). \quad (4.8)$$

This formula implies that the momentum and energy  $p_\pm(v)$  and  $\mathcal{E}_\pm(v)$  can be thought of as the values of the functions  $p_y(v) = p_+(v)$  and  $\mathcal{E}_y = \mathcal{E}_+(v)$  on the upper and lower edges of the cut of  $x(v)$ . One finds for all the models that moving around the cut of  $x(v)$  in the counter-clockwise direction increases the momentum  $p_y$ , and therefore  $\frac{dp_-}{dv} > 0$  and  $\frac{dp_+}{dv} < 0$  for  $v$  on the cut.

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<sup>1</sup> The parameters  $x^\pm$  of the two families of Hermitian (but in general not parity-invariant) Hubbard-Shastry models can be written as  $x^+ = x(\lambda + i\mathbf{u})$ ,  $x^- = 1/x(\lambda - i\mathbf{u})$  where  $\lambda$  is an arbitrary real number. For the models with  $x^+x^-$  being a phase  $x = x_A$ , and for the ones with  $x^+/x^-$  being a phase  $x = x_B$ . Then the Hubbard and the  $\mathfrak{su}(2|2)$  spin chain model's parameters correspond to  $\lambda = +\infty$ .

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2.  $M|vw$ -string:  $w_{M|vw}^c = 2M$ ,  $w_{M|vw}^s = 0$ , a spinless charge  $2M$  bound state composed of  $2M$  roots  $y_j$  and  $M$  roots  $w_j$  and parametrised by  $v \in \mathbb{R}$

$$v_j = v + (M + 2 - 2j) u i, \quad v_{-j} = v - (M + 2 - 2j) u i, \quad (4.9)$$

$$w_j = v + (M + 1 - 2j) u i, \quad j = 1, \dots, M, \quad (4.10)$$

where the roots  $y_j$  are expressed through  $v_j$  as

$$y_j = x(v_j), \quad y_{-j} = 1/x(v_{-j}), \quad j = 1, \dots, M. \quad (4.11)$$

As a result, the momentum and the dispersion relation for the  $M|vw$ -string are

$$p_{M|vw}(v) = \sum_{j=1}^M [p_+(v_j) + p_-(v_{-j})], \quad \mathcal{E}_{M|vw}(v) = \sum_{j=1}^M [\mathcal{E}_+(v_j) + \mathcal{E}_-(v_{-j})]. \quad (4.12)$$

One can check for all the models that  $\frac{dp_{M|vw}}{dv} < 0$  for any real  $v$ .

3.  $M|w$ -string:  $w_{M|w}^c = 0$ ,  $w_{M|w}^s = -M$ , a chargeless spin  $-M$  bound state composed of  $M$  roots  $w_j$  and parametrised by  $w \in \mathbb{R}$

$$w_j = w + (M + 1 - 2j) u i, \quad j = 1, \dots, M. \quad (4.13)$$

This family includes the  $1|w$ -string which has a single real root  $w$ .

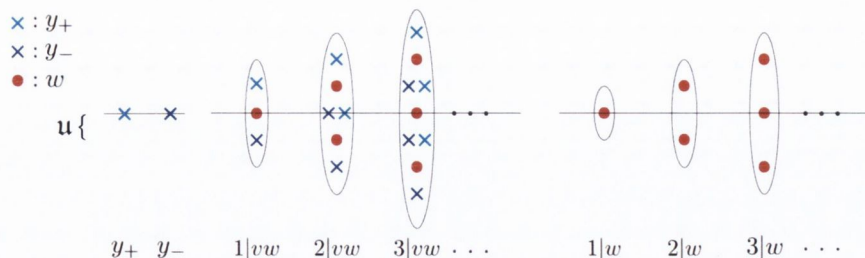
This has been formulated for the Hubbard model in some detail in [48, 57, 22]. We present a summary illustrating the root configurations in figure 4.1. The string hypothesis for the Hubbard and A-models differs from the one for the B-model only by the location of the rapidity of the  $y$ -particle. This is accommodated by the functions  $x_{A,B}(v)$ , and in terms of these all the formulae look the same.

The string Bethe equations are then constructed by fusing together the Bethe equations for the string configurations

$$\begin{aligned} 1 &= e^{iL p_y(v_{y,k})} \prod_{M=1}^{\infty} \prod_{j=1}^{N_{M|vw}} S_M(v_{y,k} - v_{M,j}) \prod_{N=1}^{\infty} \prod_{l=1}^{N_{N|w}} S_M(v_{y,k} - w_{N,l}), \\ -1 &= e^{iL p_{M|vw}(v_{M,k})} \prod_{j=1}^{N_y} S_M(v_{M,k} - v_{y,j}) \prod_{N=1}^{\infty} \prod_{l=1}^{N_{N|vw}} S_{MN}(v_{M,k} - v_{N,l}), \\ -1 &= \prod_{j=1}^{N_y} S_M(w_{M,k} - v_{y,j}) \prod_{N=1}^{\infty} \prod_{l=1}^{N_{N|w}} \frac{1}{S_{MN}(w_{M,k} - w_{N,l})}. \end{aligned} \quad (4.14)$$



## 4.2 Free energy and the TBA equations



**Figure 4.1:** An illustration of string hypothesis for the Hubbard-Shastry models. The horizontal line represents the real axis while the imaginary direction extends vertically. The  $y_{\pm}$  refer to the two branches of the  $y$ -particle and the  $\times$  mark the corresponding value of the  $v$ -rapidity variable, which take values on  $(-1, 1)$  for the A-model, and on  $(-\infty, -1) \cup (1, \infty)$  for the B-model. The circles mark the rapidities of  $w$  roots. The  $M|vw$  and  $M|w$  strings have real centres that take values on  $\mathbb{R}$ . The  $M|vw$ -strings are momentum carrying and their momentum  $p_{M|vw}$  and dispersion  $\mathcal{E}_{M|vw}$  are obtained by summing the contributions of the roots of which they are composed.

The S-matrices here are given by

$$S_M(v) = \frac{v - i \mathbf{u} M}{v + i \mathbf{u} M}, \quad S_{MN}(v) = S_{M+N}(v) S_{N-M}(v) \prod_{j=1}^{M-1} S_{N-M+2j}(v)^2 = S_{NM}(v),$$

and are related to the S-matrices  $S_{\alpha\beta}$ , used predominantly in chapter 3. Let us note the relationship between the S-matrices  $S$ , used predominantly in chapter 3, and  $S$  through the examples

$$S_{M|vw, N|vw} = S_{MN}, \quad S_{M|w, N|w} = S_{MN}^{-1}. \quad (4.15)$$

## 4.2 Free energy and the TBA equations

To take the thermodynamic limit we first introduce the counting function, as this allows us to define particles and holes. The counting functions for the  $M|vw$  and  $M|w$ -strings are defined as outlined in chapter 3 but the definition of the counting function of the  $y$ -particle requires special attention. The rapidity of the  $y$ -particle is defined on a closed contour and so the counting function is too. Recall from section 3.2 however that the counting function is only allowed to have one discontinuity, at the maximum of the pseudo-energy. This is not enough to define

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a counting function for the  $y$ -particle that is monotonously increasing, and so we must be careful about how the densities are defined in order to ensure that they are positive. Let us restrict ourselves to the parity invariant Hubbard-Shastry models and let us assume without loss of generality<sup>1</sup> that the maximum of  $\epsilon_y(v)$  is on the edge  $\mathcal{I}_-^{A,B}$ . Then we define the counting function as

$$L \sigma_+ z_y(v) = \pi \varphi_y + L p_y(v) + \sum_{\beta} \sum_{n=1}^{N_{\beta}} \phi_{y\beta}(v, v_{\beta,n}), \quad (4.16)$$

where the range of  $p$  is chosen to be continuous everywhere along  $\mathcal{I}_{\pm}^{A,B}$  except at the value  $v = v_{\max} \in \mathcal{I}_-^{A,B}$ , the value of  $v$  corresponding to the maximum of  $\epsilon_y(v)$ , and the scattering phases  $\phi_{y\beta}$  are defined as in eq. (3.23) for the case  $y_+ = x_A$  and as in eq. (3.24) for the case  $y_+ = x_B$ . For the Hubbard-Shastry models  $\sigma_+ = \text{sign}(\frac{dp_+}{dv}) = -1$  and  $\sigma_- = \text{sign}(\frac{dp_-}{dv}) = 1$  and so  $z_y$  is an increasing function of  $v$  for  $v \in \mathcal{I}_+^{A,B}$  and it is a decreasing function of  $v$  for  $v \in \mathcal{I}_-^{A,B}$ . This ensures that the counting function is increasing on the contour that is clockwise around the cut of  $x_{A,B}(v)$  (i.e. that goes along  $\mathcal{I}_+^{A,B}$  on the upper side and oppositely along  $\mathcal{I}_-^{A,B}$  on the other), with the exception of the point  $v_{\max} \in \mathcal{I}_-^{A,B}$  at which it is discontinuous.

Now we introduce the densities of particles and holes. Again the densities for the  $M|vw$  and  $M|w$ -strings are defined in the standard way, but when introducing the densities for the  $y$ -particles it is necessary to include the factor  $\sigma_{\pm}$  in their definition so that they are positive on the edges  $\mathcal{I}_{\pm}^{A,B}$  of the cut. From equation (4.16) we then have

$$\rho_{\pm} + \bar{\rho}_{\pm} = \frac{1}{2\pi} \left| \frac{dp_{\pm}}{dv} \right| + \mathbf{K}_{\pm\beta} \star \rho_{\beta}, \quad (4.17)$$

where

$$\mathbf{K}_{\pm\beta} = \sigma_{\pm} \mathcal{K}_{\pm\beta}, \quad \mathcal{K}_{\pm\beta}(v) = \frac{1}{2\pi i} \frac{d}{dv} \log \mathbf{S}_{\pm\beta}(v), \quad (4.18)$$

and

$$\rho_y(v) = \begin{cases} \rho_+(v) & \text{if } v \in \mathcal{I}_+ \\ \rho_-(v) & \text{if } v \in \mathcal{I}_- \end{cases}, \quad \bar{\rho}_y(v) = \begin{cases} \bar{\rho}_+(v) & \text{if } v \in \mathcal{I}_+ \\ \bar{\rho}_-(v) & \text{if } v \in \mathcal{I}_- \end{cases}. \quad (4.19)$$

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<sup>1</sup>This is the case for both the A- and B-models. The alternative possibility applies to the Hubbard model and proceeds in exactly the same manner under interchange of + and -.



## 4.2 Free energy and the TBA equations

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Let us remark that  $K_{+\beta} = -K_{-\beta}$  as  $S_{+\beta} = S_{-\beta}$ , and so

$$\mathbf{k}_{y\beta} = 1 \otimes K_{y\beta} = 0, \quad (4.20)$$

for all  $\beta$ -strings. The equations for densities then take the form

$$\begin{aligned} \rho_+ + \bar{\rho}_+ &= \frac{1}{2\pi} \left| \frac{dp_+}{dv} \right| - K_M \star (\rho_{M|vw} + \rho_{M|w}), \\ \rho_- + \bar{\rho}_- &= \frac{1}{2\pi} \left| \frac{dp_-}{dv} \right| + K_M \star (\rho_{M|vw} + \rho_{M|w}), \\ \rho_{M|vw} + \bar{\rho}_{M|vw} &= -\frac{1}{2\pi} \left| \frac{dp_{M|vw}}{dv} \right| - K_M \bar{\star} (\rho_+ + \rho_-) - K_{MN} \star \rho_{N|vw}, \\ \rho_{M|w} + \bar{\rho}_{M|w} &= K_M \bar{\star} (\rho_+ + \rho_-) - K_{MN} \star \rho_{N|w}, \end{aligned} \quad (4.21)$$

where the kernels  $K = \frac{1}{2\pi i} \frac{d}{dv} \log S$  are given explicitly in equations (A.3), (A.4) of appendix A, and repeated  $M$  and  $N$  are summed from 1 to  $\infty$ . Our definitions and conventions for convolutions are also outlined in appendix A.

Next we introduce the  $Y$ -functions  $Y_\alpha \equiv \bar{\rho}_\alpha / \rho_\alpha$  and pseudo-energies  $\epsilon_\alpha \equiv T \log Y_\alpha$ . A remark is in order here. The densities  $\rho_\pm$  are not to be regarded as two branches of the one function as we were forced to distinguish them by including the factor of  $\sigma_\pm$  in their definition. In the functions  $Y_\pm$  these factors cancel however and so they can be viewed as the values of a function  $Y_y(v)$  on the upper and lower edges of its square-root branch cut coinciding with the cut of  $x(v)$

$$Y_+(v) = Y_y(v + i0), \quad Y_-(v) = Y_y(v - i0). \quad (4.22)$$

Now we can obtain the conditions for equilibrium by minimising the free energy. As outlined in chapter 3 the resulting TBA equations are

$$\begin{aligned} \log Y_y &= \frac{\mathcal{E}_y - \mu - B}{T} + \log \frac{1 + \frac{1}{Y_{N|vw}}}{1 + \frac{1}{Y_{N|w}}} \star K_N, \\ \log Y_{M|vw} &= \frac{\mathcal{E}_{M|vw} - 2M\mu}{T} + \log \left( 1 + \frac{1}{Y_{N|vw}} \right) \star K_{NM} - \log \left( 1 + \frac{1}{Y_y} \right) \otimes K_M, \\ \log Y_{M|w} &= \frac{2MB}{T} + \log \left( 1 + \frac{1}{Y_{N|w}} \right) \star K_{NM} - \log \left( 1 + \frac{1}{Y_y} \right) \otimes K_M. \end{aligned} \quad (4.23)$$

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The minimized free energy is then

$$f = -\frac{T}{2\pi} \log \left( 1 + \frac{1}{Y_y} \right) \circledast \frac{dp_y}{dv} + \frac{T}{2\pi} \log \left( 1 + \frac{1}{Y_{M|vw}} \right) \star \frac{dp_{M|vw}}{dv}. \quad (4.24)$$

Let us observe that, due to the square-root branch cut of  $\mathcal{E}_y$  and the singular nature of the kernels  $K_M$  at  $v = \pm iMu$ , the  $Y$ -functions have infinitely-many square-root branch points on the  $v$ -plane. To be precise,  $Y(v)$  has branch points located at  $v = \pm 1 + 2kui$ ,  $k \in \mathbf{Z}$ , while  $Y_{M|vw}$  and  $Y_{M|w}$  have branch points located at  $v = \pm 1 + Mui + 2kui$  and  $v = \pm 1 - Mui - 2kui$ ,  $k \in \mathbf{N}$ .

### 4.3 Simplification of the TBA equations

We refer to the TBA equations written in the form (4.23) as the *canonical* ones. They can however be rewritten in a simpler “local” form which eliminates the infinite sums over the  $Y$ -functions. To achieve this we introduce the following useful kernels

$$(K + \delta)_{MN}^{-1}(v) = \delta_{MN}\delta(v) - I_{MN}s(v), \quad s(v) = \frac{1}{4u \cosh \frac{\pi v}{2u}}, \quad (4.25)$$

with  $I_{MN} = \delta_{M+1,N} + \delta_{M-1,N}$ ,  $M \geq 2$  and  $I_{1N} = \delta_{2N}$ . Here  $(K + \delta)_{MN}^{-1}$  is the kernel that is inverse to  $K_{NQ} + \delta_{NQ}$ :

$$\sum_{N=1}^{\infty} (K + \delta)_{MN}^{-1} \star (K_{NQ} + \delta_{NQ}) = \sum_{N=1}^{\infty} (K_{QN} + \delta_{QN}) \star (K + \delta)_{NM}^{-1} = \delta_{MQ}, \quad (4.26)$$

which can be also written in the following convenient form

$$\sum_{N=1}^{\infty} (K + \delta)_{MN}^{-1} \star K_{NQ} = \sum_{N=1}^{\infty} K_{QN} \star (K + \delta)_{NM}^{-1} = I_{MQ}s. \quad (4.27)$$

Here and throughout this section we use the convention  $\delta_{MN}(v) \equiv \delta_{MN}\delta(v)$  when this will not cause confusion. The inverse kernel obeys the useful identities

$$\sum_{N=1}^{\infty} (K + \delta)_{MN}^{-1} \star K_N = \sum_{N=1}^{\infty} K_N \star (K + \delta)_{NM}^{-1} = \delta_{M1}s, \quad (4.28)$$

$$\mathcal{E}_{N|vw} \star (K + \delta)_{NM}^{-1} = -\delta_{1M} \mathcal{E}_y \circledast s = \delta_{1M} (\mathcal{E}_+ - \mathcal{E}_-) \bar{\star} s. \quad (4.29)$$



### 4.3 Simplification of the TBA equations

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Now we can proceed with simplifying the canonical TBA equations. Subtracting the TBA equations (4.23) for  $y_{\pm}$ -particles gives

$$\log \frac{Y_+}{Y_-} = \frac{\mathcal{E}_+ - \mathcal{E}_-}{T}, \quad (4.30)$$

due to the analyticity of the kernel  $s(t-v)$  for real  $v$ . Then applying the inverse kernel to the equations for the  $vw$ - and  $w$ -strings, and using eq. (4.30) the identities above, gives the simplified TBA equations

$$\log Y_{M|vw} = I_{MN} \log(1 + Y_{N|vw}) \star s - \delta_{M1} \log(1 + Y_y) \circledast s, \quad (4.31)$$

$$\log Y_{M|w} = I_{MN} \log(1 + Y_{N|w}) \star s - \delta_{M1} \log\left(1 + \frac{1}{Y_y}\right) \circledast s. \quad (4.32)$$

Here the kernel  $(K + \delta)^{-1}$  has annihilated the  $\mu$  and  $B$  dependent terms. Note however that the large  $M$  asymptotics of the  $Y$ -functions are not fixed by the simplified equations, whereas it follows from the canonical TBA equations (4.23) that

$$\lim_{M \rightarrow \infty} \frac{\log Y_{M|vw}}{M} = -\frac{2\mu}{T}, \quad \lim_{M \rightarrow \infty} \frac{\log Y_{M|w}}{M} = \frac{2B}{T}. \quad (4.33)$$

The dependence of the  $Y$ -functions on  $\mu$  and  $B$  is thus obtained by imposing (4.33) on solutions of the simplified equations (4.31), (4.32).

The simplified TBA equations can be used to compute the infinite sums appearing in the canonical equations for  $y$ -particles. We show how to do this on the example of  $vw$ -strings. Since a similar sum appears in expression (4.24) for the free energy we do the computation by assuming that we are given a kernel  $\mathcal{K}_M$  satisfying the identity<sup>1</sup>

$$\sum_{N=1}^{\infty} (K + \delta)_{MN}^{-1} \star \mathcal{K}_N = \mathcal{K}_M - I_{MNS} \star \mathcal{K}_N = \delta_{M1} \delta \mathcal{K}, \quad (4.34)$$

where  $\delta \mathcal{K}$  is any kernel. We want to compute the following sum

$$\sum_{M=1}^{\infty} \log\left(1 + \frac{1}{Y_{M|vw}}\right) \star \mathcal{K}_M. \quad (4.35)$$

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<sup>1</sup> This method was applied in [58] to more general kernels but under an assumption of  $Y_M$ -functions approaching 1 at large  $M$ .

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To this end (4.31) is rewritten in the form

$$\begin{aligned} \log Y_{M|vw}^{reg} - I_{MN} \log Y_{N|vw}^{reg} \star s &= I_{MN} \log \left( 1 + \frac{1}{Y_{N|vw}} \right) \star s \\ &\quad - \delta_{M1} \left( \frac{\mathcal{E}_y}{T} + \log \left( 1 + \frac{1}{Y_y} \right) \right) \otimes s, \end{aligned}$$

where on the left hand side of this equation we replaced  $Y_{M|vw}$  by  $Y_{M|vw}^{reg} = Y_{M|vw} e^{2M\mu/T}$  to make sure it asymptotes to 1 at large  $M$ . This does not change the equation because the kernel  $s$  integrates to  $1/2$ , that is  $1 \star s = 1/2$ . This replacement is necessary because at the next step we multiply this equation by  $\mathcal{K}_M$ , take the sum over  $M$ , and use the identities (4.29) and (4.34) to get

$$\begin{aligned} \log Y_{1|vw}^{reg} \star \delta \mathcal{K} &= \log \left( 1 + \frac{1}{Y_{M|vw}} \right) \star \mathcal{K}_M - \log \left( 1 + \frac{1}{Y_{1|vw}} \right) \star \delta \mathcal{K} \\ &\quad + \frac{\mathcal{E}_{1|vw} \star \delta \mathcal{K}}{T} - \log \left( 1 + \frac{1}{Y_y} \right) \otimes s \star \mathcal{K}_1. \end{aligned}$$

From this equation and the definition of  $Y_{M|vw}^{reg}$  we immediately obtain

$$\begin{aligned} \log \left( 1 + \frac{1}{Y_{M|vw}} \right) \star \mathcal{K}_M &= \log (1 + Y_{1|vw}) \star \delta \mathcal{K} - \frac{\mathcal{E}_{1|vw} - 2\mu}{T} \star \delta \mathcal{K} \\ &\quad + \log \left( 1 + \frac{1}{Y_y} \right) \otimes s \star \mathcal{K}_1. \end{aligned} \tag{4.36}$$

A similar formula can be derived for the infinite sum with  $Y_{M|w}$ -functions

$$\log \left( 1 + \frac{1}{Y_{M|w}} \right) \star \mathcal{K}_M = \log (1 + Y_{1|w}) \star \delta \mathcal{K} - \frac{2B}{T} \star \delta \mathcal{K} + \log \left( 1 + \frac{1}{Y_y} \right) \otimes s \star \mathcal{K}_1.$$

In our case  $\mathcal{K}_M = K_M$ ,  $\delta \mathcal{K} = s$ , and subtracting these two equations gives

$$\log \frac{1 + \frac{1}{Y_{M|vw}}}{1 + \frac{1}{Y_{M|w}}} \star K_M = \log \frac{1 + Y_{1|vw}}{1 + Y_{1|w}} \star s + \frac{\mu + B}{T} - \frac{\mathcal{E}_{1|vw}}{T} \star s.$$

The infinite sum is substituted for in the TBA equation (4.23) for  $y$ -particles to give the simplified equation for  $Y_+ Y_-$

$$\log Y_+ Y_- = \frac{\mathcal{E}_+ + \mathcal{E}_- - 2\mathcal{E}_{1|vw} \star s}{T} + 2 \log \frac{1 + Y_{1|vw}}{1 + Y_{1|w}} \star s.$$



### 4.3 Simplification of the TBA equations

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Combining the equations for  $Y_+/Y_-$  and  $Y_+Y_-$  gives

$$\log Y_y = \frac{\mathcal{E}_y - \mathcal{E}_{1|vw} \star s}{T} + \log \frac{1 + Y_{1|vw}}{1 + Y_{1|w}} \star s. \quad (4.37)$$

It is also possible to simplify the sum appearing in the free energy expression (4.24) using eq. (4.36). First we rewrite (4.36) as

$$\log \left( 1 + \frac{1}{Y_{M|vw}} \right) \star \mathcal{K}_M = \log (1 + Y_{1|vw}) \star \delta \mathcal{K} + \frac{2\mu}{T} \star \delta \mathcal{K} + \log (1 + Y_y) \circledast s \star \mathcal{K}_1. \quad (4.38)$$

Then we notice that

$$(K + 1)_{MN}^{-1} \star \frac{d p_{N|vw}}{d v} = -\delta_{M1} s \circledast \frac{d p_y}{d v} = \delta_{M1} s \bar{\star} \left( \frac{d p_+}{d v} - \frac{d p_-}{d v} \right),$$

and therefore in this case  $\mathcal{K}_M = \frac{d p_{M|vw}}{d v}$ ,  $\delta \mathcal{K} = -s \circledast \frac{d p}{d v}$ . Thus eq.(4.38) becomes

$$\begin{aligned} \log \left( 1 + \frac{1}{Y_{M|vw}} \right) \star \frac{d p_{M|vw}}{d v} &= -\log (1 + Y_{1|vw}) \star s \circledast \frac{d p_y}{d v} - \frac{2\mu}{T} \star s \circledast \frac{d p_y}{d v} \\ &\quad + \log (1 + Y_y) \circledast s \star \frac{d p_{1|vw}}{d v}. \end{aligned} \quad (4.39)$$

Taking into account that  $1 \star s \bar{\star} \left( \frac{d p_+}{d v} - \frac{d p_-}{d v} \right) = -\pi$ , one gets

$$\begin{aligned} f &= -\mu - \frac{T}{2\pi} \log \left( 1 + \frac{1}{Y_y} \right) \circledast \frac{d p_y}{d v} + \frac{T}{2\pi} \log (1 + Y_y) \circledast s \star \frac{d p_{1|vw}}{d v} \\ &\quad - \frac{T}{2\pi} \log (1 + Y_{1|vw}) \star s \circledast \frac{d p_y}{d v}. \end{aligned} \quad (4.40)$$

Finally using eq. (4.37) and the identity

$$\mathcal{E}_y \circledast s \star \frac{d p_{1|vw}}{d v} = \mathcal{E}_{1|vw} \star s \circledast \frac{d p_y}{d v}, \quad (4.41)$$

the free energy can be rewritten as

$$\begin{aligned} f &= -\mu + \frac{1}{2\pi} \mathcal{E}_{1|vw} \star s \circledast \frac{d p_y}{d v} - \frac{T}{2\pi} \log \left( 1 + \frac{1}{Y_y} \right) \circledast \left( \frac{d p_y}{d v} - s \star \frac{d p_{1|vw}}{d v} \right) \\ &\quad - \frac{T}{2\pi} \log (1 + Y_{1|vw}) \star s \circledast \frac{d p_y}{d v}, \end{aligned} \quad (4.42)$$

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or with dependence on the  $1|w$ -string in the form

$$f = -\mu + \frac{1}{2\pi}(\mathcal{E}_y - \mathcal{E}_{1|vw} \star s) \otimes \frac{dp_y}{dv} - \frac{T}{2\pi} \log(1 + Y_y) \otimes \left( \frac{dp_y}{dv} - s \star \frac{dp_{1|vw}}{dv} \right) - \frac{T}{2\pi} \log(1 + Y_{1|w}) \star s \otimes \frac{dp_y}{dv}. \quad (4.43)$$

Let us collect together the simplified TBA equations

$$\log Y_y(v) = \frac{\mathcal{E}_y - \mathcal{E}_{1|vw} \star s}{T} + \log \frac{1 + Y_{1|vw}}{1 + Y_{1|w}} \star s, \quad (4.44)$$

$$\log Y_{M|vw}(v) = I_{MN} \log(1 + Y_{N|vw}) \star s - \delta_{M1} \log(1 + Y_y) \otimes s, \quad (4.45)$$

$$\log Y_{M|w}(v) = I_{MN} \log(1 + Y_{N|w}) \star s - \delta_{M1} \log\left(1 + \frac{1}{Y_y}\right) \otimes s. \quad (4.46)$$

These must be supplemented with the large  $M$  asymptotics (4.33) that capture the  $\mu$  and  $B$  dependence of the solutions. In the case of the Hubbard model one can easily check that the TBA equations and the free energy (4.43) match exactly the ones in [9]. To be precise one finds the relations

$$e_0 = \mathbf{u} + \frac{1}{2\pi}(\mathcal{E}_y - \mathcal{E}_{1|vw} \star s) \otimes \frac{dp_y}{dv}, \quad \sigma_0(v) = \frac{1}{2\pi} s \otimes \frac{dp_y}{dv}, \quad (4.47)$$

$$\rho_0(p_y(v)) \frac{dp_y}{dv} = \frac{1}{2\pi} \left( \frac{dp_y}{dv} - s \star \frac{dp_{1|vw}}{dv} \right), \quad (4.48)$$

where  $e_0$ ,  $\sigma_0$  and  $\rho_0$  are the ground state energy per site, the density of  $w$ -strings, and the density of roots of single charge excitations for the half filled repulsive Hubbard model, respectively, see eqs.(5.69) and (5.70) of [9].

It will happen that the sign of the final terms in equations (4.45), (4.46) play an important role in understanding the physics of the models. Through equation (4.30) these signs follow from the relative magnitude of  $\mathcal{E}_+$  and  $\mathcal{E}_-$ . Let us thus note that

$$\begin{aligned} \mathcal{E}_- - \mathcal{E}_+ &= -4\sqrt{1-v^2} \leq 0 && \text{for the Hubbard model,} \\ \mathcal{E}_- - \mathcal{E}_+ &= \frac{4\mathbf{u}^2\sqrt{1-v^2}}{\mathbf{u}^2+v^2} \geq 0 && \text{for the A model,} \\ \mathcal{E}_- - \mathcal{E}_+ &= \frac{4v^2\sqrt{1-\frac{1}{v^2}}}{\mathbf{u}^2+v^2} \geq 0 && \text{for the B model.} \end{aligned} \quad (4.49)$$



## 4.4 Various limits

The free energy (4.43) together with the TBA equations (4.44)-(4.46) and their large  $M$  asymptotics (4.33) contain all the information about the equilibrium states of the Hubbard, A- and B-models in the thermodynamic limit. In general one needs to numerically solve the infinitely many coupled non-linear integral TBA equations to extract this information. In various limits however the equations simplify and one can make analytic progress.

It is convenient here to introduce the following simplifying notations

$$\beta = \frac{1}{T}, \quad \beta_c = \frac{\mu}{T}, \quad \beta_s = \frac{B}{T}. \quad (4.50)$$

### 4.4.1 Limits of temperature

#### Zero temperature limit

Let us first consider the important limit of zero temperature. Here we lay the foundation for the studies of the A- and B-models in this limit in the following two chapters.

We saw in section 3.4 that the TBA equations simplify in the  $T \rightarrow 0$  limit due to the singular behaviour (3.82) of the densities  $\rho_\alpha(v)$ ,  $\bar{\rho}_\alpha(v)$ . In particular let us recall that there are no  $\alpha$ -strings in the ground state if  $\epsilon_\alpha(v) > 0$  for all  $v$ . With this fact one can rule a lot of strings out of the ground state by considering the simplified TBA equations (4.44)-(4.46). From the equations for  $M|vw$  and  $M|w$  strings we see that

$$\epsilon_{M|vw} > 0 \quad \text{and} \quad \epsilon_{M|w} > 0 \quad \text{for} \quad M \geq 2,$$

as the right hand sides of these equations are strictly positive. Moreover, the sign of the extra term in the TBA equations for the case  $M = 1$  follows from the inequalities (4.49). For the A- and B-models one has that  $Y_{1|w} > 1$ , while  $Y_{1|vw} > 1$  for the Hubbard model.

With this insight we turn to the canonical TBA equations (4.23), as they are more convenient to work with since they contain  $\mu$  and  $B$  explicitly. Restricting

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ourselves to the A- and B-models, and writing the equations in terms of the pseudo-energies, they take the form

$$\begin{aligned}\epsilon_y &= \mathcal{E}_y - \mu - B - \epsilon_{1|vw} \star_{Q_{1|vw}} K_1, \\ \epsilon_{M|vw} &= \mathcal{E}_{M|vw} - 2M\mu + \epsilon_y \circledast_{Q_y} K_M - \epsilon_{1|vw} \star_{Q_{1|vw}} K_{1M}, \\ \epsilon_{M|w} &= 2MB + \epsilon_y \circledast_{Q_y} K_M.\end{aligned}\tag{4.51}$$

Here we see that there are just two functions,  $\epsilon_y$  and  $\epsilon_{1|vw}$ , that determine all the others. This represents a major simplification compared to the infinitely many functions that one has to solve for at non-zero temperature.

### Infinite temperature limit

In the infinite temperature limit one expects many of the details of a system to get hidden behind thermal fluctuations. We find that in this limit all the Hubbard-Shastry models become identical. First let us consider the limit  $T \rightarrow \infty$  with  $\beta_c$  and  $\beta_s$  fixed. Here the driving terms of the TBA equations (4.44)-(4.46) become rapidly independent, and thus so do the  $Y$ -functions and  $Y_+ = Y_- = Y_y$ . The TBA equations then become recursion relations and their solution is given by

$$Y_y = \frac{\cosh \beta_c}{\cosh \beta_s}, \quad Y_{M|vw} = \left[ \frac{\sinh(M+1)\beta_c}{\sinh \beta_c} \right]^2 - 1, \quad Y_{M|w} = \left[ \frac{\sinh(M+1)\beta_s}{\sinh \beta_s} \right]^2 - 1,$$

and the free energy is

$$\begin{aligned}f &= -\mu - T \log(1 + Y_y) - \frac{T}{2} \log(1 + Y_{1|vw}) \\ &= -T \log(2e^{\beta_c} \cosh \beta_s + e^{2\beta_c} + 1).\end{aligned}$$

In the strict infinite temperature limit however  $\beta_c \rightarrow 0$  and  $\beta_s \rightarrow 0$ . Thus  $f \rightarrow -T \log 4$  as expected for a model with four degrees of freedom per site.

### 4.4.2 Limits of magnetic field and chemical potential

#### Strong magnetic field limit $B \rightarrow \infty$

In the limit  $B \rightarrow \infty$  the  $Y$ -functions for  $M|w$  strings diverge. This can be seen from the canonical TBA equations (4.23). Note that the final term on the right hand side of the equation for  $M|w$ -strings behaves as

$$\log\left(1 + \frac{1}{Y_y}\right) \circledast K_M \rightarrow -\mathcal{E}_y \circledast K_M,$$



in the limit  $B \rightarrow \infty$  because  $Y \rightarrow 0$  from its TBA equation. Therefore

$$Y_{M|w} \rightarrow e^{2M\beta_s},$$

and the free energy simplifies to

$$f = -\mu - B + \frac{1}{2\pi} (\mathcal{E}_y - \mathcal{E}_{1|vw} \star s) \otimes \frac{dp_y}{dv}.$$

The electron density is one and the magnetisation per site is one-half, and thus the ground state is full of spin-up electrons as expected.

### Large chemical potential limit $\mu \rightarrow -\infty$

The analysis of the limit  $\mu \rightarrow -\infty$  is similar to the limit  $B \rightarrow \infty$ . Here the  $Y$ -functions for  $M|vw$ -strings behave as

$$Y_{M|vw} \rightarrow e^{-2M\beta_c}.$$

The free energy becomes

$$f = \frac{1}{2\pi} \mathcal{E}_{1|vw} \star s \otimes \frac{dp_y}{dv},$$

and, as this has no dependence on  $\mu$ , the ground state is empty. This is as expected, since a large chemical potential means a large energy cost for having a particle in the system.

### 4.4.3 Limits of the coupling constant

Now we examine the weak and strong coupling limits of the TBA equations for the Hubbard-Shastry models of interest to us. We will work primarily with the simplified version of the TBA equations (4.44)-(4.46). As the kernels  $s(v)$  and  $K(v)$  have explicit dependence on  $\mathbf{u}$ , it will be useful to rescale the variable  $v$  to  $v/\mathbf{u}$  and introduce a notation for functions that take a rescaled argument

$$\mathcal{Y}_k(v) = Y_k(\mathbf{u}v), \quad \mathfrak{s}(v) = \frac{1}{4 \cosh \frac{\pi v}{2}}, \quad \mathfrak{K}_M(v) = \frac{1}{\pi} \frac{M}{M^2 + v^2}. \quad (4.52)$$

Here we have also altered  $s(v)$  and  $K_M(v)$  by a factor of  $\mathbf{u}$  for convenience as they generally appear together with  $dv$ . Note that after rescaling the variable in the TBA equations they take the same form with  $Y_k$  replaced by  $\mathcal{Y}_k$  and  $s$  by  $\mathfrak{s}$ .

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### Weak coupling limit of the Hubbard and B-models: free fermions

From the Hamiltonians of the Hubbard model (2.3) and the B-model (2.48), (6.1) it is clear that they reduce to free fermions in the weak coupling  $\mathbf{u} \rightarrow 0$  limit. Here the limit is analysed on the level of the TBA equations. We find as expected that TBA equations can be solved and that an explicit expression for the free energy can be obtained.

Let us consider the  $\mathbf{u} \rightarrow 0$  limit without rescaling the variable  $v$ .<sup>1</sup> First note that  $\lim_{\mathbf{u} \rightarrow 0} s(v) = \frac{1}{2}\delta(v)$ . Next for  $v$  on the intervals  $\mathcal{I}^A = (-1, 1)$  and  $\mathcal{I}^B = (-\infty, -1) \cup (1, \infty)$ , for the Hubbard and B-model respectively, one finds the following limits

$$\begin{aligned} \lim_{\mathbf{u} \rightarrow 0} \mathcal{E}_+(v) &= \Delta, & \lim_{\mathbf{u} \rightarrow 0} \mathcal{E}_-(v) &= -\Delta, & \lim_{\mathbf{u} \rightarrow 0} \mathcal{E}_{1|vw}(v) &= 2\Delta, & (4.53) \\ \lim_{\mathbf{u} \rightarrow 0} \frac{dp_+}{dv}(v) &= -\frac{2}{\Delta}, & \lim_{\mathbf{u} \rightarrow 0} \frac{dp_-}{dv}(v) &= \frac{2}{\Delta}, & \lim_{\mathbf{u} \rightarrow 0} \frac{dp_{1|vw}}{dv}(v) &= -\frac{4}{\Delta}, \end{aligned}$$

where for the Hubbard model  $\Delta = 2\sqrt{1-v^2}$  and for the B-model  $\Delta = -2\sqrt{1-\frac{1}{v^2}}$ . Then in this limit the TBA equations become a set of algebraic equations

$$\begin{aligned} \log Y_+ &= \frac{1}{2} \log \frac{1 + Y_{1|vw}}{1 + Y_{1|w}}, & \log \frac{Y_+}{Y_-} &= \frac{2\Delta}{T}, \\ \log Y_{M|vw} &= \frac{1}{2} I_{MN} \log(1 + Y_{N|vw}) + \frac{1}{2} \delta_{M1} \log \frac{1 + Y_+}{1 + Y_-}, & (4.54) \\ \log Y_{M|w} &= \frac{1}{2} I_{MN} \log(1 + Y_{N|w}) + \frac{1}{2} \delta_{M1} \log \frac{1 + \frac{1}{Y_+}}{1 + \frac{1}{Y_-}}. \end{aligned}$$

The general solution to these algebraic TBA equations is well-known and is given in appendix D.

In the limit  $\mathbf{u} \rightarrow 0$  the free energy (4.43) simplifies dramatically

$$f = -\mu - \frac{2}{\pi} \int_{\mathcal{I}} dv - \frac{T}{\pi} \int_{\mathcal{I}} \frac{dv}{\Delta} \log \left( (1 + Y_-) \sqrt{1 + Y_{1|w}} \right),$$

---

<sup>1</sup>It is worth pointing out that if in the Hubbard case one would rescale  $v$  then the naive  $\mathbf{u} \rightarrow 0$  limit would lead to a constant solution of the TBA equations and to divergent integrals in the free energy expression (4.43), while for the B-model one would lose all information about  $Y_{\pm}$ -functions.



and substituting the solution (D.1) one obtains

$$f = -T \int_{-\pi}^{\pi} \frac{dp}{2\pi} \log \left( (1 + e^{\frac{2 \cos p + \mu + B}{T}}) (1 + e^{\frac{2 \cos p + \mu - B}{T}}) \right).$$

This is the expected result for free electrons [22].

### Weak coupling limit of the A-model: $\mathfrak{su}(2|2)$ spin chain

The weak coupling limit of the A-model is studied in detail in section 7.2 in the more general context of the Bethe equations of the extended models. Here we point out how the limit can be taken at the level of the TBA equations. To take the limit it is necessary to first rescale  $v$  to  $v/u$  as otherwise one finds that most of the functions in (4.53) exhibit singular behaviour at  $v = 0$ . On rescaling the rapidity the the cut  $(-1, 1) \rightarrow (-\infty, \infty)$  as  $u \rightarrow 0$  and so  $\mathcal{Y}_{\pm}(v)$  can be considered as distinct functions defined on independent complex planes. In this way the TBA equations (7.11) with  $\mathfrak{g} = 0$  can be obtained without going back to the Bethe ansatz for the  $\mathfrak{su}(2|2)$  spin chain.

### Strong coupling limit of the Hubbard and A-model: $\mathfrak{su}(2)$ spin chain

Here we consider the models in the limit of infinite coupling. The analysis is quite similar for the Hubbard and A-models and so these are examined together.

It is well known that the strong coupling limit of the less than half-filled Hubbard model is the  $t$ - $J$  model (see appendix E for a discussion), which at half-filling takes the form of the antiferromagnetic XXX spin chain

$$\mathbf{H} = -2u - J \sum_{j=1}^L \left( \vec{\mathbf{S}}_j \cdot \vec{\mathbf{S}}_{j+1} - \frac{1}{4} \right) - 2B \mathbf{S}^z, \quad (4.55)$$

where for convenience we use  $J = -\frac{1}{u}$ . The free energy and TBA equations for this Hamiltonian are known to be [22]:

$$f = -2u + J \log 2 - T \int_{-\infty}^{\infty} dv \log(1 + \mathcal{Y}_1(v)) \mathfrak{s}(v),$$

$$\log \mathcal{Y}_M(v) = I_{MN} \log(1 + \mathcal{Y}_N) \star \mathfrak{s}(v) + \delta_{M1} \frac{2\pi J \mathfrak{s}(v)}{T}, \quad \lim_{M \rightarrow \infty} \frac{\log \mathcal{Y}_M}{M} = \frac{2B}{T}. \quad (4.56)$$

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The Hubbard free energy and TBA equations reduce to these as  $u \rightarrow \infty$ .

Likewise we will see that in the strong coupling limit the free energy and TBA equations of the A-model reduce to (4.56) with  $J = \frac{1}{u}$ . Thus it is natural to conclude that the strong coupling limit of the half-filled A-model is the ferromagnetic XXX spin chain with Hamiltonian given by (4.55) with  $J = \frac{1}{u}$ . Indeed in appendix E we show this to be the case.

First let us observe that  $\mathcal{E}_y^H \rightarrow \mathcal{E}_y^A \rightarrow -2u$ ,  $\mathcal{E}_{1|vw}^H \star s \rightarrow \frac{\log 2}{u}$  and  $\mathcal{E}_{1|vw}^A \star s \rightarrow -\frac{\log 2}{u}$  as  $u \rightarrow \infty$ . Then from TBA equation (4.44) it follows that  $Y_y \rightarrow 0$ , and so

$$\log\left(1 + \frac{1}{Y_y}\right) \otimes s = \log \frac{1 + \frac{1}{Y_-}}{1 + \frac{1}{Y_+}} \hat{\star} s \rightarrow \frac{1}{T}(\mathcal{E}_+ - \mathcal{E}_-) \hat{\star} s \quad (4.57)$$

through equation (4.30). The free energy and TBA equations for the Hubbard and A-models thus simplify to

$$f = -\mu - 2u + \kappa \frac{\log 2}{u} - \frac{T}{2\pi} \log(1 + Y_{1|w}) \star s \otimes \frac{dp_y}{dv}, \quad (4.58)$$

$$\log Y_{M|w}(v) = I_{MN} \log(1 + Y_{N|w}) \star s - \frac{\delta_{M1}}{T}(\mathcal{E}_+ - \mathcal{E}_-) \hat{\star} s, \quad (4.59)$$

with  $\kappa = -1$  for the Hubbard model and  $\kappa = 1$  for the A-model. Since  $\mu$  enters only the first term of the free energy it follows that the electron density is one, i.e. the limit takes the model to half-filling. These resemble quite closely the corresponding equations for the XXX spin chain (4.56). Before the exact equivalence is shown let us remark that the sign of the final term in (4.59) gives an indication of whether one should expect an antiferromagnetic or ferromagnetic spin chain. From (4.49) we thus expect the Hubbard model to exhibit antiferromagnetic behaviour and the A-model to exhibit ferromagnetic behaviour.

Next note the strong coupling limits

$$\int_{-1}^1 dt (\mathcal{E}_+(t) - \mathcal{E}_-(t))s(t, v) \rightarrow \begin{cases} 2\pi s(v) & \text{for the Hubbard model,} \\ -2\pi s(v) & \text{for the A-model.} \end{cases}$$

Rescaling the kernel  $s$  and  $Y$ -functions as in (4.52), eq.(4.59) takes the form

$$\mathcal{Y}_{M|w}(v) = I_{MN} \log(1 + \mathcal{Y}_{N|w}) \star \mathfrak{s}(v) + \kappa \delta_{M1} \frac{2\pi \mathfrak{s}(v)}{uT}.$$



To take into account the rescaling in the final term of equation (4.58) note

$$\log(1 + Y_{1|w}) \star s \otimes \frac{dp_y}{dv} = \int_{-\infty}^{\infty} dt \log(1 + \mathcal{Y}_{1|w}(t)) \int_{-1}^1 dv \mathfrak{s}(t, uv) \left( \frac{dp_-}{dv}(v) - \frac{dp_+}{dv}(v) \right),$$

$$\int_{-1}^1 dv \mathfrak{s}(t, uv) \left( \frac{dp_-}{dv}(v) - \frac{dp_+}{dv}(v) \right) \xrightarrow{\text{large } u} 2\pi \mathfrak{s}(t).$$

Hence under the identification  $\mathcal{Y}_{M|w} \equiv \mathcal{Y}_M$  the free energy and TBA equations for the Hubbard and A models do indeed reduce to those for the XXX spin chain (4.56) with  $J = -\frac{1}{u}$ ,  $J = \frac{1}{u}$  respectively.

Let us comment on the attractive versions of the models. Here  $\tilde{\mathcal{E}}_y^H \rightarrow \tilde{\mathcal{E}}_y^A \rightarrow 2u$ ,  $\tilde{\mathcal{E}}_{1|vw}^H \star s \rightarrow -\frac{\log 2}{u}$ ,  $\tilde{\mathcal{E}}_{1|vw}^A \star s \rightarrow \frac{\log 2}{u}$  and so the TBA equation (4.44) implies  $Y_y \rightarrow \infty$ . It follows that the free energy (using the form (4.42)) and TBA equations simplify to

$$f = -\mu + \kappa \frac{\log 2}{u} - \frac{T}{2\pi} \log(1 + Y_{1|vw}) \star s \otimes \frac{dp_y}{dv},$$

$$\log Y_{M|vw}(v) = I_{MN} \log(1 + Y_{N|vw}) \star s + \frac{\delta_{M1}}{T} (\mathcal{E}_+ - \mathcal{E}_-) \hat{\star} s,$$

with  $\kappa = -1$  for the attractive Hubbard model and  $\kappa = 1$  for the attractive A-model. Since there is no  $B$  dependence in the equations the magnetisation vanishes. As the inequalities in (4.49) are reversed in going to the opposite models one sees that in the strong coupling limit the attractive Hubbard model is an antiferromagnetic XXX chain while the attractive A-model is a ferromagnetic XXX chain. Note that these are charge  $\mathfrak{su}(2)$  chains, governed by  $\boldsymbol{\eta}$  of (2.6), as opposed to spin chains. For the strongly attractive A-model one thus expects a discontinuity of the density at  $T = 0$ , with the model being empty for  $\mu < 0$  and having all sites doubly occupied for  $\mu > 0$ .

### Strong coupling limit of the B-model: extended $\mathfrak{su}(2|2)$ spin chain

The strong coupling limit of the B-model is studied in detail in section 7.2. The analysis is similar to that for the weak coupling limit of the A-model above. Note that if  $v$  is rescaled to  $v/u$  then the cut  $(-\infty, -1) \cup (1, \infty) \rightarrow (-\infty, \infty)$  as  $u \rightarrow \infty$ . Again  $\mathcal{Y}_{\pm}(v)$  can be considered as distinct functions defined on their own complex planes, and one obtains the TBA equations (7.11) for the extended B-model with  $\mathfrak{g} = 0$ .

## 4.5 TBA for the opposite models

In section 2.3 we discussed the relationship between a model and its opposite, the model defined by reversing the sign of the Hamiltonian. In this appendix we discuss a relationship between the Hubbard-Shastry models and their opposites at the level of the TBA equations. In particular we show how the TBA equations and free energy for the attractive Hubbard-Shastry models can be straightforwardly obtained from those for the repulsive models, which are those identified in (2.46). Our results match with the derivation of the TBA equations for the attractive Hubbard model in [59].

As the Hamiltonians of the opposite models differ only by their sign the models share the same set of Bethe equations. The only change to the thermodynamic Bethe ansatz analysis is that the sign of the dispersion relation is reversed. Hence the TBA equations for the attractive models are given simply by

$$\log \tilde{Y}_y = -\frac{\mathcal{E}_y - \mathcal{E}_{1|vw} \star s}{T} + \log \frac{1 + \tilde{Y}_{1|vw}}{1 + \tilde{Y}_{1|w}} \star s, \quad (4.60)$$

$$\log \tilde{Y}_{M|vw}(v) = I_{MN} \log \left( 1 + \tilde{Y}_{N|vw} \right) \star s - \delta_{M1} \log(1 + \tilde{Y}_y) \otimes s, \quad (4.61)$$

$$\log \tilde{Y}_{M|w}(v) = I_{MN} \log \left( 1 + \tilde{Y}_{N|w} \right) \star s - \delta_{M1} \log\left(1 + \frac{1}{\tilde{Y}_y}\right) \otimes s, \quad (4.62)$$

and they are supplemented with the large  $M$  asymptotics

$$\lim_{M \rightarrow \infty} \frac{\log \tilde{Y}_{M|vw}}{M} = -\frac{2\tilde{\mu}}{T}, \quad \lim_{M \rightarrow \infty} \frac{\log \tilde{Y}_{M|w}}{M} = \frac{2\tilde{B}}{T}. \quad (4.63)$$

Here and in what follows tilded quantities refer to the attractive models, and untilded ones to the corresponding repulsive ones. The free energy of the attractive models are given by either (4.42) or (4.43) with the replacement  $\mathcal{E}_y \rightarrow -\mathcal{E}_y$ ,  $\mathcal{E}_{1|vw} \rightarrow -\mathcal{E}_{1|vw}$ ,  $Y_y \rightarrow \tilde{Y}_y$ ,  $Y_{1|w} \rightarrow \tilde{Y}_{1|w}$  and  $Y_{1|vw} \rightarrow \tilde{Y}_{1|vw}$ .

Let us note that these TBA equations (4.60 - 4.62) are identical to those of the repulsive models (4.44 - 4.46) under the identification

$$\tilde{Y}_y = \frac{1}{Y_y}, \quad \tilde{Y}_{M|vw} = Y_{M|w}, \quad \tilde{Y}_{M|w} = Y_{M|vw}, \quad \tilde{\mu} = -B, \quad \tilde{B} = -\mu \quad (4.64)$$

By using this identification one can relate the free energy of the attractive models to those of the repulsive one. For convenience let us write  $\mathcal{E}_y \otimes \frac{dp_y}{dv} = \mathcal{E}_{\pi/2}$ , where



## 4.5 TBA for the opposite models

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$\mathcal{E}_{\pi/2}$  is an electron energy at  $p = \pi/2$ . From (2.46),  $\mathcal{E}_{\pi/2} = -2\mathbf{u}$  for the Hubbard model,  $\mathcal{E}_{\pi/2} = -2\sqrt{1 + \mathbf{u}^2}$  for the A-model and  $\mathcal{E}_{\pi/2} = 0$  for the B-model. Then taking into account the identity (4.41) one finds

$$\tilde{f}(\tilde{\mu}, \tilde{B}, T) = f_{\{\mu \rightarrow -\tilde{B}, B \rightarrow -\tilde{\mu}\}} - \tilde{\mu} - \tilde{B} - \mathcal{E}_{\pi/2}. \quad (4.65)$$

Since  $\mathcal{E}_{\pi/2}$  depends only on  $\mathbf{u}$  this will not affect the physical properties as they are related to derivatives of  $f$ . In fact  $\mathcal{E}_{\pi/2}$  can be removed by redefining the Hubbard interaction potential (2.5) as follows

$$\mathbf{V}_{j,k}^H \rightarrow \mathbf{V}_{j,k}^H + \frac{1}{4} = \frac{1}{2}(\mathbf{n}_{j,\uparrow} - \frac{1}{2})(\mathbf{n}_{j,\downarrow} - \frac{1}{2}) + \frac{1}{2}(\mathbf{n}_{k,\uparrow} - \frac{1}{2})(\mathbf{n}_{k,\downarrow} - \frac{1}{2}), \quad (4.66)$$

which is the potential used in [9].

The same relation between the free energies of the attractive and repulsive models can be also found by using the Woynarovich transformation [60] which is a superposition of the parity transformation and the partial particle-hole transformation of the spin-up electrons, see [9] for detail. Under the transformation the Hubbard potential (2.5) transforms as  $\mathbf{V}_{j,k}^H \rightarrow -\mathbf{V}_{j,k}^H - \frac{1}{2}$ , and all the other kinetic and potential terms just change their signs. The relation (4.65) then follows from the definition of free energy and that  $\mathbf{N} \rightarrow -2\mathbf{S}^z + 1$  and  $\mathbf{S}^z \rightarrow -\frac{1}{2}\mathbf{N} + \frac{1}{2}$  under the Woynarovich transformation.

#### 4. THERMODYNAMICS OF THE HS MODELS

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# Chapter 5

## The A-model at zero temperature

In this chapter we present our detailed investigation of the A-model. A summary of our findings can be found in the introduction 1.1.

### 5.1 The model

The A-model Hamiltonian is given through the parameters (2.47) with  $\mathbf{u} = \sinh \nu$

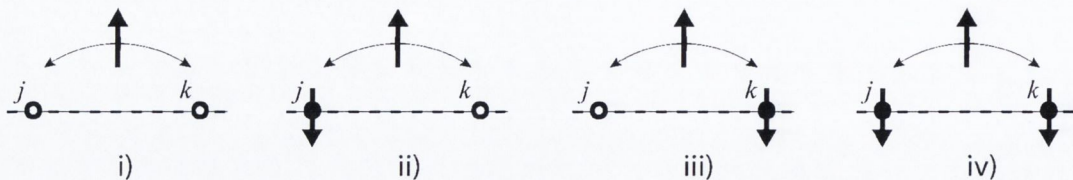
$$\begin{aligned} \mathbf{H}^A &= \sum_{j=1}^L \left( \mathbf{T}_{j,j+1}^A + \frac{2 \cosh 2\nu}{\cosh \nu} \mathbf{V}_{j,j+1}^H + \frac{2}{\cosh \nu} (\mathbf{V}_{j,j+1}^{CC} - \mathbf{V}_{j,j+1}^{SS} + \mathbf{V}_{j,j+1}^{PH}) \right), \\ \mathbf{T}_{j,k}^A &= - \sum_{\sigma} \left[ (\mathbf{c}_{j,\sigma}^{\dagger} \mathbf{c}_{k,\sigma} + \mathbf{c}_{k,\sigma}^{\dagger} \mathbf{c}_{j,\sigma}) (1 - \mathbf{n}_{j,-\sigma} - \mathbf{n}_{k,-\sigma}) \right. \\ &\quad \left. - i \tanh \nu (\mathbf{c}_{j,\sigma}^{\dagger} \mathbf{c}_{k,\sigma} - \mathbf{c}_{k,\sigma}^{\dagger} \mathbf{c}_{j,\sigma}) (\mathbf{n}_{j,-\sigma} - \mathbf{n}_{k,-\sigma})^2 \right]. \end{aligned} \quad (5.1)$$

It is easy to check that this hermitian but the issue of parity invariance is somewhat subtle. The transformation

$$\mathbf{c}_{j,\sigma}^{\dagger} \rightarrow \mathbf{c}_{L-j+1,\sigma}^{\dagger}, \quad \mathbf{c}_{j,\sigma} \rightarrow \mathbf{c}_{L-j+1,\sigma}, \quad (5.2)$$

is not a symmetry of the Hamiltonian but is instead equivalent to a replacement  $\nu \rightarrow -\nu$ . This replacement can however also be realised as a unitary transformation. More generally, the Hamiltonian (5.1) is unitary equivalent to the following

## 5. THE A-MODEL AT ZERO TEMPERATURE



**Figure 5.1:** The four cases of an electron hopping between sites  $j$  and  $k$ : i) no electrons of the opposite spin on either site, ii) an electron of the opposite spin on site  $j$  only, iii) an electron of the opposite spin on site  $k$  only, iv) electrons of the opposite spin on both sites.

one

$$\mathbf{H}^A(\alpha) \equiv \mathbf{U}_2^\dagger(\alpha) \mathbf{H}^A \mathbf{U}_2(\alpha), \quad \mathbf{U}_2(\alpha) \equiv \exp\left(i\alpha \sum_{j=1}^L \mathbf{n}_{j,\uparrow} \mathbf{n}_{j,\downarrow}\right), \quad (5.3)$$

$$\mathbf{H}^A(\alpha) = \sum_{j=1}^L \left( \mathbf{T}_{j,j+1}^A(\alpha) + \frac{2 \cosh 2\nu}{\cosh \nu} \mathbf{V}_{j,j+1}^H + \frac{2}{\cosh \nu} (\mathbf{V}_{j,j+1}^{CC} - \mathbf{V}_{j,j+1}^{SS} + \mathbf{V}_{j,j+1}^{PH}) \right),$$

$$\begin{aligned} \mathbf{T}_{j,k}^A(\alpha) = & - \sum_{\sigma} \left[ (\mathbf{c}_{j,\sigma}^\dagger \mathbf{c}_{k,\sigma} + \mathbf{c}_{k,\sigma}^\dagger \mathbf{c}_{j,\sigma}) (1 - \mathbf{n}_{j,-\sigma} - \mathbf{n}_{k,-\sigma}) \right. \\ & - \sin \alpha \tanh \nu (\mathbf{c}_{j,\sigma}^\dagger \mathbf{c}_{k,\sigma} + \mathbf{c}_{k,\sigma}^\dagger \mathbf{c}_{j,\sigma}) (\mathbf{n}_{j,-\sigma} - \mathbf{n}_{k,-\sigma}) \\ & \left. - i \cos \alpha \tanh \nu (\mathbf{c}_{j,\sigma}^\dagger \mathbf{c}_{k,\sigma} - \mathbf{c}_{k,\sigma}^\dagger \mathbf{c}_{j,\sigma}) (\mathbf{n}_{j,-\sigma} - \mathbf{n}_{k,-\sigma})^2 \right], \end{aligned}$$

where  $\alpha$  is an free real parameter. If one chooses  $\alpha = \pi$  one gets

$$\mathbf{H}^A(\pi) = \mathbf{H}^A \quad \text{with } \nu \rightarrow -\nu,$$

and so the parity transformation (5.2) should be supplemented with the change of basis generated by  $\mathbf{U}_2(\pi)$  in order to have an invariance of the model. Let us remark that with the choice  $\alpha = \pm\pi/2$  one removes the imaginary term from the Hamiltonian.

The kinetic part of the Hamiltonian is of a complicated correlated hopping type. It is instructive to see how it acts in various cases. For an electron of a given spin to hop between two neighbouring sites, the sites cannot contain other electrons of that spin. They can however contain electrons of the opposite and there are four cases to consider, as illustrated in Figure 5.1. The kinetic density  $\mathbf{T}_{j,k}^A(\alpha)$  acting on each of the cases takes the form



- i)  $-\mathbf{c}_{j,\sigma}^\dagger \mathbf{c}_{k,\sigma} - \mathbf{c}_{k,\sigma}^\dagger \mathbf{c}_{j,\sigma}$ ,
- ii)  $-ie^{-i\alpha} \tanh \nu \mathbf{c}_{j,\sigma}^\dagger \mathbf{c}_{k,\sigma} + ie^{i\alpha} \tanh \nu \mathbf{c}_{k,\sigma}^\dagger \mathbf{c}_{j,\sigma}$ ,
- iii)  $-ie^{i\alpha} \tanh \nu \mathbf{c}_{j,\sigma}^\dagger \mathbf{c}_{k,\sigma} + ie^{-i\alpha} \tanh \nu \mathbf{c}_{k,\sigma}^\dagger \mathbf{c}_{j,\sigma}$ ,
- iv)  $\mathbf{c}_{j,\sigma}^\dagger \mathbf{c}_{k,\sigma} + \mathbf{c}_{k,\sigma}^\dagger \mathbf{c}_{j,\sigma}$ .

Note the different sign between cases i) and iv), while for free electrons all the cases would be the same as case i). This reflects the strong electron correlation in the A-model which cannot be reduced to free electrons for any coupling.

The A-model inherits spin and charge  $\mathfrak{su}(2)$  symmetries from the  $\mathfrak{su}(2|2)_c$  structure. The spin  $\mathfrak{su}(2)$  generators take the same form as those for the Hubbard model (2.6) whereas the charge  $\mathfrak{su}(2)$  generators take the following untwisted form

$$\boldsymbol{\eta}^+ = \sum_{j=1}^L \boldsymbol{\eta}_j^+, \quad \boldsymbol{\eta}^- = \sum_{j=1}^L \boldsymbol{\eta}_j^-, \quad \boldsymbol{\eta}^z = \sum_{j=1}^L \boldsymbol{\eta}_j^z. \quad (5.4)$$

Next we present yet another way to write the Hamiltonian of the A-model. If we apply the unitary transformation

$$\mathbf{U}_3 \equiv \exp \left( i \pi \sum_{j=1}^L j \mathbf{n}_{j,\uparrow} \mathbf{n}_{j,\downarrow} \right), \quad (5.5)$$

to the Hamiltonian (5.3) with  $\alpha = \pi/2$  we obtain

$$\begin{aligned} \mathbf{H}^{\text{A}_3} &= \sum_{j=1}^L \left( \mathbf{T}_{j,j+1} + \frac{2 \cosh 2\nu}{\cosh \nu} \mathbf{V}_{j,j+1}^H - \frac{2}{\cosh \nu} (\mathbf{V}_{j,j+1}^{SS} - \mathbf{V}_{j,j+1}^{CC} + \mathbf{V}_{j,j+1}^{PH}) \right), \\ \mathbf{T}_{j,k} &= - \sum_{\sigma} (\mathbf{c}_{j,\sigma}^\dagger \mathbf{c}_{k,\sigma} + \mathbf{c}_{k,\sigma}^\dagger \mathbf{c}_{j,\sigma}) \left( 1 - (1 - (-1)^j \tanh \nu) (\mathbf{n}_{j,-\sigma} - \mathbf{n}_{k,-\sigma})^2 \right), \end{aligned} \quad (5.6)$$

In this form the correlated hopping term has a contribution that alternates along alternating bonds of the lattice. The unitary transformation generated by (5.5) can be understood as shifting the momenta of doubly occupied sites by  $\pi$  while leaving the momenta of singly occupied sites unchanged. Acting on the charge  $\mathfrak{su}(2)$  generators (5.4) the transformation has the effect of twisting the generators

## 5. THE A-MODEL AT ZERO TEMPERATURE

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to the form (2.6). Let us stress however that (5.5) is not diagonal on the level of the Bethe ansatz solution.

In the weak coupling limit the A-model reduces to the  $\mathfrak{su}(2|2)$  spin chain. This has received much study due to an interesting generalisation identified in [27]. A key observation is that  $\mathbf{V}^H$  commutes with  $\mathbf{H}^A$  in this limit and so one can extend the model. We will return to a discussion of the  $\mathfrak{su}(2|2)$  spin chain and its extension in chapter 7. In the strong coupling limit the Hubbard interaction term dominates and the model thus shares the strict strong coupling limit of the Hubbard model. The strong coupling limit of the less than half-filled model is analysed in appendix E and is found to be a ferromagnetic  $t$ - $J$  model, in contrast with the antiferromagnetic behaviour of the Hubbard model.

Let us now turn to the diagonal description of the model provided by Bethe ansatz. The Bethe equations are given in (2.40) with the momentum  $p$  related to the rapidity variable through

$$e^{ip(v)} = \frac{i - x_A(v)(\mathbf{u} + \sqrt{1 + \mathbf{u}^2})}{x_A(v) - i(\mathbf{u} + \sqrt{1 + \mathbf{u}^2})}, \quad (5.7)$$

where  $x_A(v)$  defined in (4.4). The corresponding dispersion relation is given by

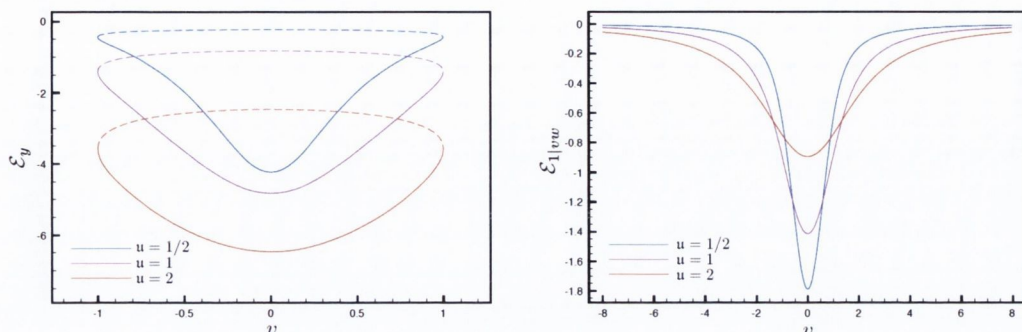
$$\mathcal{E} = -2 \cos p - 2\sqrt{1 + \mathbf{u}^2}, \quad (5.8)$$

and it is clear that it is invariant under the parity transformation  $p \rightarrow -p$ . Indeed, this property was used to identify the model in section 2.6. With formulae (5.7) and (5.8) the results of chapter 4 can be used to investigate the thermodynamics of the A-model. For the readers convenience we present plots of  $\mathcal{E}_y(v) = \mathcal{E}(v)$  and  $\mathcal{E}_{1|vw} = \mathcal{E}(v + i\mathbf{u}) + \mathcal{E}(v - i\mathbf{u})$  in Figure 5.2.

From the Bethe equations we can again see the symmetries of the model. First we note that to any set of roots that solve the Bethe equations we can add a  $w$ -root at  $\infty$ , and the resulting set will also be a solution. Adding the root decreases the spin of the corresponding eigenstate by one without changing its energy, momentum or charge. It thus has the same affect as acting with the spin lowering operator  $\mathbf{S}^-$  on the eigenstate, and reflects the spin  $\mathfrak{su}(2)$  symmetry of the model. Next we observe that we cannot add a single  $v$ -root at  $\infty$  as its values are restricted to the cut  $\mathcal{I}^A$ . This reflects the fact that the A-model does



## 5.2 Ground state phase diagram



**Figure 5.2:** A-model: Plots of  $\mathcal{E}_y(v)$  and  $\mathcal{E}_{1|vw}(v)$  for  $u = 1/2, 1, 2$ . In the plots of  $\mathcal{E}_y(v)$  the  $y_+$  branch is represented by a solid line and the  $y_-$  branch is represented by a dashed line.

not have a fermionic symmetry. One can however add two complex conjugate  $v$ -roots, whose real parts take their value at  $\infty$ . As we already understand the spin symmetry it makes sense to add a  $w$ -root at  $\infty$  also, so that the addition of the three leaves the spin of the state unchanged. That such a combination can be added to any solution of the Bethe equations is made possible by the freedom to choose the imaginary parts of the  $v$ -roots. The combination will increase the charge of the state by two, while from the plot of  $\mathcal{E}_{1|vw}$  in Figure 5.2 we can infer that it will not change the energy of the state, and through (5.7) we can see that it will not affect the momentum either. Adding such a combination thus corresponds to acting with the charge raising operator  $\eta^+$  and reflects the untwisted charge  $\mathfrak{su}(2)$  symmetry of the A-model, see eq. (5.4).

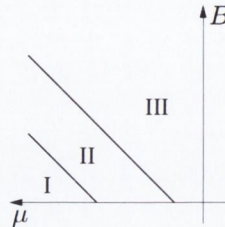
## 5.2 Ground state phase diagram

The ground state of the A-model is determined by the zero temperature TBA equations (4.51). Examining the equations numerically we see that  $\epsilon_{1|vw} < 0$  is not consistent with  $B > 0$ . There are thus no  $1|vw$ -strings in the zero temperature ground state and so we can remove the contributions of  $1|vw$ -strings from the right hand sides of (4.51). The equations then reduce to

$$\begin{aligned}
 \epsilon_y &= \mathcal{E}_y - \mu - B, \\
 \epsilon_{M|vw} &= \mathcal{E}_{M|vw} - 2M\mu + \epsilon_y \circledast_{Q_y} K_M, \\
 \epsilon_{M|w} &= 2MB + \epsilon_y \circledast_{Q_y} K_M.
 \end{aligned}
 \tag{5.9}$$

## 5. THE A-MODEL AT ZERO TEMPERATURE

**Figure 5.3:** A-model: Zero temperature phase diagram in the  $\mu B$ -plane. The phases identified are: I) empty band, II) partially filled and spin polarised band, III) half-filled and spin-polarised band. The line separating phases I and II is  $\mu + B = 2 - 2\sqrt{1 + u^2}$ , and the line separating phases II and III is  $\mu + B = -2 - 2\sqrt{1 + u^2}$ .



Hence the  $y$ -particles are the only strings which have a non-zero density and so the ground state is spin polarised. As  $y$ -particles have charge 1 and spin  $\frac{1}{2}$  the magnetisation of the ground state is half of the density. Subbing the dispersion relation (5.8) into the equation for  $y$ -particles it becomes

$$\epsilon_y = -2 \cos p - 2\sqrt{1 + u^2} - \mu - B. \quad (5.10)$$

The determination of the phase diagram is now a straightforward process. The ground state is empty for  $\mu < -2 - 2\sqrt{1 + u^2} - B$ , then partially filled up to  $\mu = 2 - 2\sqrt{1 + u^2} - B$ , and saturated with  $y$ -particles thereafter. The phase diagram for the  $A$ -model for  $\mu \leq 0$ ,  $B \geq 0$  is presented in Figure 5.3. Let us remark that the line  $B = 0$  is singular at  $T = 0$  as the magnetisation has a discontinuity there.

### 5.3 Excitations

Now we examine the zero temperature excitations of the  $A$ -model. To do so we need to first identify which strings are of which type, as discussed at the beginning of section 3.2. Analysing the TBA equations (5.9) we see that the  $M|vw$ -strings are type 1 strings while the  $M|w$ -strings are of type 2. The  $y$ -particles should be treated as type 1 strings from the discussion at the beginning of section 4.2. Then the phase shifts are given by

$$\phi_{y,M|vw}(v, t) = \pi b_{y,M|vw} + \Theta_M(v - t), \quad \phi_{y,M|w}(v, t) = \pi b_{y,M|w} + \Theta_M(v - t),$$

and the dressed momenta by

$$\begin{aligned} P_y = p_y, \quad P_{M|vw} &= p_{M|vw} - \frac{dp_y}{dv} \otimes_{Q_y} \Theta_M - \pi n_y (2b_{y,M|vw} + 1), \\ P_{M|w} &= -\frac{dp_y}{dv} \otimes_{Q_y} \Theta_M - \pi n_y (2b_{y,M|w} + 1), \end{aligned} \quad (5.11)$$



where  $b_{y,M|w}$  and  $b_{y,M|vw}$  are the integers that determine the branch of the excitation.

Let us first consider excitations above the half-filled phase III. One does not have to introduce singular strings through eq. (3.92) as the range of mode numbers for  $y$ -particles  $N_y + \bar{N}_y = L$  does not change for an excitation. That  $k_{y\beta} = 0$  for all  $\beta$ -strings, see eq. (4.20), is a reflection of this. Moreover the dressed energies can be written explicitly as

$$\epsilon_y = \mathcal{E}_y - \mu - B, \quad \epsilon_{M|vw} = -2M\mu, \quad \epsilon_{M|w} = -\mathcal{E}_{M|vw} + 2MB, \quad (5.12)$$

while the dressed momenta are branch independent and simplify to

$$P_y = p_y, \quad P_{M|vw} = 0, \quad P_{M|w} = -p_{M|vw}. \quad (5.13)$$

Here the identities (A.9) have been used. The dressed charge and spin are equal to their bare values. Thus the  $y$ -particles behave as electrons and are gapped, and so there is an energy cost to remove them from the ground state. Their dressed energy and momentum take their bare values and so are related as

$$\epsilon_y = -2 \cos P_y - 2\sqrt{1+u^2} - \mu - B. \quad (5.14)$$

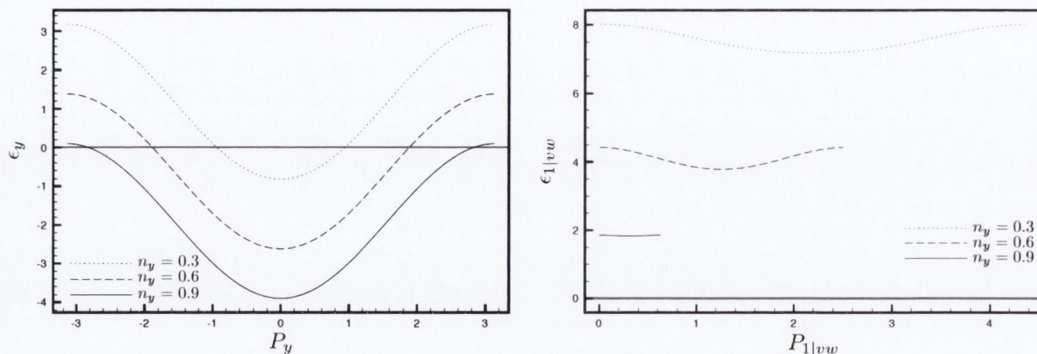
The  $1|w$ -string is the magnon and the  $M|w$ -strings are their bound states. In  $B = 0$  magnetic field they have quadratic dispersion at low energies,  $\epsilon_{M|w} \sim \frac{\sqrt{1+u^2}-u}{M} P_{M|w}^2$ , while in a  $B > 0$  magnetic field they are gapped. The  $M|vw$ -strings are not dynamical. In the strong coupling  $u \rightarrow \infty$  limit the energy gap for removing an electron goes to infinity and the physics becomes that of the ferromagnetic spin chain<sup>1</sup>.

Now we consider excitations above phase II where the filling ranges between 0 and 1. The dressed energies are given by (5.9) and the dressed momentum by (5.11), and here the choice of branch becomes important. For convenience we choose  $b_{y,M|vw} = 0$ ,  $b_{y,M|w} = -1$  for all  $M$ . The range of dressed momentum is then  $(-\pi, \pi)$  for  $y$ -particles,  $(0, 2\pi - 2\pi n_y)$  for  $M|vw$ -strings and  $(0, 2\pi n_y)$  for  $M|w$ -strings. These should be considered modulo  $2\pi$  and it is convenient to

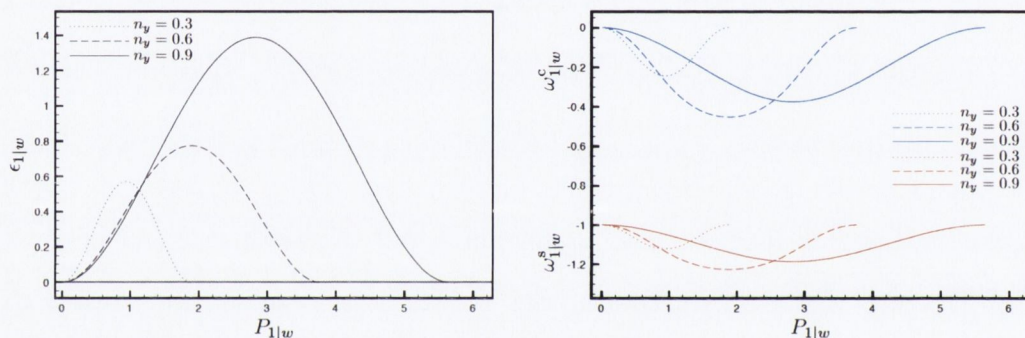
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<sup>1</sup> The ferromagnetic spin-chain Hamiltonian appears at order  $\frac{1}{u}$ , see appendix E, and so it is necessary to rescale the energies appropriately.

## 5. THE A-MODEL AT ZERO TEMPERATURE



**Figure 5.4:** A-model: Plots of  $\epsilon_y(P_y)$  and  $\epsilon_{1|vw}(P_{1|vw})$  at  $u = 1$  and  $B = 0$ .



**Figure 5.5:** A-model: Plots of  $\epsilon_{1|w}(P_{1|w})$ ,  $\omega_{1|w}^c(P_{1|w})$  and  $\omega_{1|w}^s(P_{1|w})$  at  $u = 1$  and  $B = 0$ .

plot the dressed momenta of the  $M|vw$ -strings and the  $M|w$ -strings in the range  $(0, 2\pi)$ . It should be kept in mind that the other branches of the excitations are obtained by shifts of  $2\pi n_y$ . Plots of dressed energy as functions of dressed momentum for  $y$ -particles,  $M|w$ -strings and  $M|vw$ -strings for various filling at  $B = 0$  and  $u = 1$  are given in Figures 5.4 and 5.5. The velocity can easily be read from the plots as the derivative of dressed energy with respect to dressed momentum. Away from half-filling the dressed charge and spin of some strings gain dependence on rapidity. They are given by

$$\begin{aligned} \omega_y^c &= 1, & \omega_{M|vw}^c &= 2M - \mathcal{W}_M, & \omega_{M|w}^c &= -\mathcal{W}_M, \\ \omega_y^s &= \frac{1}{2}, & \omega_{M|vw}^s &= -\frac{1}{2}\mathcal{W}_M, & \omega_{M|w}^s &= -M - \frac{1}{2}\mathcal{W}_M, \end{aligned} \quad (5.15)$$

where  $\mathcal{W}_M(v) = -1 \otimes_{Q_y} K_M$  is a non-negative function that goes to zero both at half-filling and zero filling. We see that the  $y$ -particle behaves as an electron



in this phase also, but that it is now gapless. In  $B = 0$  magnetic field the  $M|w$ -strings retain their quadratic dispersion away from half-filling but they are no longer pure spin, they gain charge opposite to that of an electron as their energy increases. The dressed spin and charge of a  $1|w$ -string is plotted as a function of its dressed momentum for various fillings at  $B = 0$  and  $\mathbf{u} = 1$  in Figure 5.5. In a  $B > 0$  magnetic field the  $M|w$ -strings become gapped. The  $M|vw$ -strings become dynamical in phase II but they are gapped throughout.

## 5. THE A-MODEL AT ZERO TEMPERATURE

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# Chapter 6

## The B-model at zero temperature

In this chapter we present our detailed investigation of the B-model. A summary of our findings can be found in the introduction 1.2.

### 6.1 The model

The B-model Hamiltonian is given through the parameters (2.48) with  $\mathbf{u} = \sinh \nu$

$$\begin{aligned} \mathbf{H}^{\text{B}} &= \sum_{j=1}^L \left( \mathbf{T}_{j,j+1}^{\text{B}} + 2 \tanh \nu (\mathbf{V}_{j,j+1}^{\text{H}} - \mathbf{V}_{j,j+1}^{\text{CC}} + \mathbf{V}_{j,j+1}^{\text{SS}} + \mathbf{V}_{j,j+1}^{\text{PH}}) \right), \quad (6.1) \\ \mathbf{T}_{j,k}^{\text{B}} &= - \sum_{\sigma} (\mathbf{c}_{j,\sigma}^{\dagger} \mathbf{c}_{k,\sigma} + \mathbf{c}_{k,\sigma}^{\dagger} \mathbf{c}_{j,\sigma}) \left( 1 - \left( 1 - \frac{1}{\cosh \nu} \right) (\mathbf{n}_{j,-\sigma} - \mathbf{n}_{k,-\sigma})^2 \right). \end{aligned}$$

Like for the A-model, the kinetic term is of a correlated hopping type. Let analyse it by considering how it acts on the four cases illustrated in Figure 5.1. The model is clearly parity invariant and so cases ii) and iii) are identical. The kinetic density  $\mathbf{T}_{j,k}^{\text{B}}(\alpha)$  acting on each of the cases takes the form

$$\begin{aligned} \text{i), iv)} & \quad -\mathbf{c}_{j,\sigma}^{\dagger} \mathbf{c}_{k,\sigma} - \mathbf{c}_{k,\sigma}^{\dagger} \mathbf{c}_{j,\sigma}, \\ \text{ii), iii)} & \quad -\frac{1}{\cosh \nu} (\mathbf{c}_{j,\sigma}^{\dagger} \mathbf{c}_{k,\sigma} + \mathbf{c}_{k,\sigma}^{\dagger} \mathbf{c}_{j,\sigma}). \end{aligned}$$

Both cases i) and iv) preserve the number of doubly occupied sites, while for hopping in cases ii) and iii) the number will be either increased or decreased by 1. We conclude that hopping processes that create or destroy doubly occupied sites are suppressed.

## 6. THE B-MODEL AT ZERO TEMPERATURE

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The B-model possesses a rich symmetry structure. It has spin and charge  $\mathfrak{su}(2)$  symmetries and the generators take the same form as for the Hubbard model (2.6). Moreover the model has a dynamical  $\mathfrak{su}(2|2)$  symmetry for all values of the coupling constant  $\nu$ . In addition to the bosonic  $\mathfrak{su}(2)$  symmetries the algebra  $\mathfrak{su}(2|2)$  includes fermionic symmetries. Let us first define the following onsite operators

$$\begin{aligned} \mathcal{Q}_{j,\sigma} &= \left(\frac{\beta}{2} + \frac{1}{2\beta}\right) \mathbf{c}_{j,\sigma} - \beta \mathbf{n}_{j,-\sigma} \mathbf{c}_{j,\sigma}, & \mathcal{Q}_{j,\sigma}^\dagger &= (\mathcal{Q}_{j,\sigma})^\dagger, \\ \mathcal{Q}_{j,\sigma} &= \left(\frac{\beta}{2} - \frac{1}{2\beta}\right) \mathbf{c}_{j,\sigma} - \beta \mathbf{n}_{j,-\sigma} \mathbf{c}_{j,\sigma}, & \mathcal{Q}_{j,\sigma}^\dagger &= (\mathcal{Q}_{j,\sigma})^\dagger, \end{aligned} \quad (6.2)$$

where  $\beta = \sqrt{\tanh \nu/2}$ . The fermionic supercharges are then given by

$$\mathcal{Q}_\sigma = \sum_{j=1}^L \mathcal{Q}_{j,\sigma}, \quad \mathcal{Q}_\sigma^\dagger = \sum_{j=1}^L \mathcal{Q}_{j,\sigma}^\dagger, \quad \mathcal{Q}_\sigma = \sum_{j=1}^L (-1)^j \mathcal{Q}_{j,\sigma}, \quad \mathcal{Q}_\sigma^\dagger = \sum_{j=1}^L (-1)^j \mathcal{Q}_{j,\sigma}^\dagger.$$

We will soon see that the four  $\mathcal{Q}_{0,\sigma}^\dagger, \mathcal{Q}_{0,\sigma}$  create/annihilate an electron with momentum  $p = 0$ , while the other four  $\mathcal{Q}_{\pi,\sigma}^\dagger, \mathcal{Q}_{\pi,\sigma}$  create/annihilate an electron with momentum  $p = \pi$ . Let us thus write the eight supercharges equivalently as

$$\mathbf{Q}_{0,\sigma} = \mathcal{Q}_\sigma, \quad \mathbf{Q}_{0,\sigma}^\dagger = \mathcal{Q}_\sigma^\dagger, \quad \mathbf{Q}_{\pi,\sigma} = \mathcal{Q}_\sigma, \quad \mathbf{Q}_{\pi,\sigma}^\dagger = \mathcal{Q}_\sigma^\dagger. \quad (6.3)$$

They satisfy the  $\mathfrak{su}(2|2)$  algebra with central charge  $\coth \nu$ :

$$\begin{aligned} \{\mathbf{Q}_{0,\downarrow}, \mathbf{Q}_{0,\uparrow}^\dagger\} &= -\{\mathbf{Q}_{\pi,\downarrow}, \mathbf{Q}_{\pi,\uparrow}^\dagger\} = \mathbf{S}^+, & \{\mathbf{Q}_{0,\uparrow}, \mathbf{Q}_{0,\downarrow}^\dagger\} &= -\{\mathbf{Q}_{\pi,\uparrow}, \mathbf{Q}_{\pi,\downarrow}^\dagger\} = \mathbf{S}^-, \\ \{\mathbf{Q}_{0,\downarrow}^\dagger, \mathbf{Q}_{\pi,\uparrow}^\dagger\} &= -\{\mathbf{Q}_{0,\uparrow}^\dagger, \mathbf{Q}_{\pi,\downarrow}^\dagger\} = \boldsymbol{\eta}^+, & \{\mathbf{Q}_{0,\downarrow}, \mathbf{Q}_{\pi,\uparrow}\} &= -\{\mathbf{Q}_{0,\uparrow}, \mathbf{Q}_{\pi,\downarrow}\} = \boldsymbol{\eta}^-, \\ \{\mathbf{Q}_{0,\uparrow}, \mathbf{Q}_{0,\uparrow}^\dagger\} &= \mathbf{S}^z - \boldsymbol{\eta}^z + \coth \nu, & \{\mathbf{Q}_{0,\downarrow}, \mathbf{Q}_{0,\downarrow}^\dagger\} &= -\mathbf{S}^z - \boldsymbol{\eta}^z + \coth \nu, \\ \{\mathbf{Q}_{\pi,\uparrow}, \mathbf{Q}_{\pi,\uparrow}^\dagger\} &= -\mathbf{S}^z + \boldsymbol{\eta}^z + \coth \nu, & \{\mathbf{Q}_{\pi,\downarrow}, \mathbf{Q}_{\pi,\downarrow}^\dagger\} &= \mathbf{S}^z + \boldsymbol{\eta}^z + \coth \nu, \end{aligned} \quad (6.4)$$

and all other anti-commutators vanish. The fermionic generators  $\mathbf{Q}$  do not commute with the Hamiltonian but instead

$$[\mathbf{H}^B, \mathbf{Q}] = h_Q \mathbf{Q}, \quad [\mathbf{N}, \mathbf{Q}] = n_Q \mathbf{Q}, \quad [\mathbf{S}^z, \mathbf{Q}] = s_Q \mathbf{Q},$$

with  $h_Q, n_Q$  and  $s_Q$  given in Table 6.1, and so one can introduce a corresponding set of operators  $\tilde{\mathbf{Q}}(t) = e^{-ih_Q t} \mathbf{Q}$  that are conserved. It is not difficult to check that the  $\tilde{\mathbf{Q}}(t)$  also satisfy the  $\mathfrak{su}(2|2)$  algebra (6.4) and so represent a dynamical



	$\mathbf{Q}_{0,\uparrow}$	$\mathbf{Q}_{0,\downarrow}$	$\mathbf{Q}_{\pi,\uparrow}$	$\mathbf{Q}_{\pi,\downarrow}$	$\mathbf{Q}_{0,\uparrow}^\dagger$	$\mathbf{Q}_{0,\downarrow}^\dagger$	$\mathbf{Q}_{\pi,\uparrow}^\dagger$	$\mathbf{Q}_{\pi,\downarrow}^\dagger$
$h_Q$	2	2	-2	-2	-2	-2	2	2
$n_Q$	-1	-1	-1	-1	1	1	1	1
$s_Q$	-1/2	1/2	-1/2	1/2	1/2	-1/2	1/2	-1/2

**Table 6.1:** Commutators of supercharges of the B-model with  $\mathbf{H}$ ,  $\mathbf{N}$  and  $\mathbf{S}^z$ .

symmetry of the model, see section 2.4. Let us recall that dynamical symmetry leads to exact relationships between the energies of eigenstates rather than to degeneracies in the spectrum. For example if  $|\psi\rangle$  is an eigenstate with energy  $E_\psi$ , then the energy of the state  $\mathbf{Q}|\psi\rangle$  is given through  $\mathbf{H}^B\mathbf{Q}|\psi\rangle = (E_\psi + h_Q)\mathbf{Q}|\psi\rangle$ . In particular it follows that the ground state of  $\mathbf{H}^B$  is in the kernel of  $\mathbf{Q}_{0,\uparrow}^\dagger$ ,  $\mathbf{Q}_{0,\downarrow}^\dagger$ ,  $\mathbf{Q}_{\pi,\uparrow}$  and  $\mathbf{Q}_{\pi,\downarrow}$ .

The operators (6.2) allow us to rewrite the kinetic term of the B-model in a particularly simple fashion. One can easily check that

$$\mathbf{T}_{j,j+1}^B = -\tanh\nu \sum_{\sigma} \left[ \mathcal{Q}_{j,\sigma}^\dagger \mathcal{Q}_{j+1,\sigma} + \mathcal{Q}_{j+1,\sigma}^\dagger \mathcal{Q}_{j,\sigma} + \mathcal{Q}_{j,\sigma}^\dagger \mathcal{Q}_{j+1,\sigma} + \mathcal{Q}_{j+1,\sigma}^\dagger \mathcal{Q}_{j,\sigma} \right].$$

Let us observe that with this form  $\tanh\nu$  completely factorises out of the Hamiltonian (6.1), and the coupling constant only appears implicitly through the operators (6.2).

Next we discuss the limits of coupling. In the weak coupling limit the Hamiltonian reduces to that of free electrons on the lattice. The dynamical symmetry of the model becomes singular here, as the central charge of the algebra diverges:  $\coth\nu \rightarrow \infty$ . In the strong coupling limit note that  $[\mathbf{H}^B, \mathbf{V}^H] = 0$ , as for the weak coupling limit of the A-model. This limit maps onto the EKS model [27] and will be discussed in detail in chapter 7.

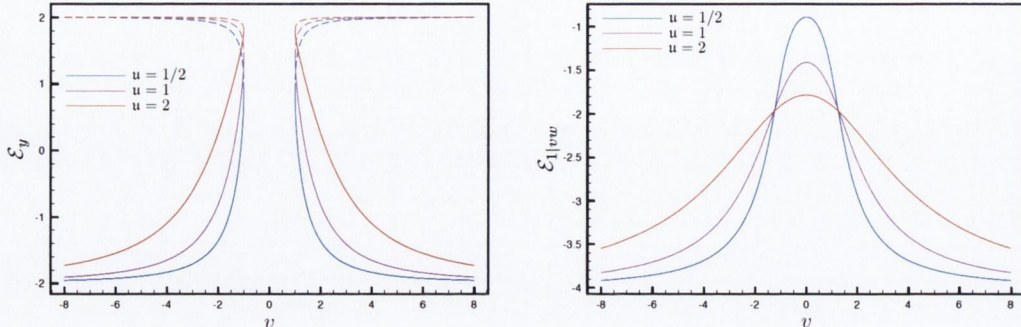
Now we turn to the diagonal description. The Bethe equations are given by (2.40) and the momentum  $p$  is related to the rapidity variable as

$$e^{ip(v)} = \frac{x_B(v) + i(\mathbf{u} + \sqrt{1 + \mathbf{u}^2})}{x_B(v) - i(\mathbf{u} + \sqrt{1 + \mathbf{u}^2})}, \quad (6.5)$$

where  $x_B(v)$  defined in (4.5). The model's dispersion relation is given simply by

$$\mathcal{E} = -2 \cos p. \quad (6.6)$$

## 6. THE B-MODEL AT ZERO TEMPERATURE



**Figure 6.1:** Plots of  $\mathcal{E}_y(v)$  and  $\mathcal{E}_{1|vw}(v)$  for the B-model for  $u = 1/2, 1, 2$ . In the plots of  $\mathcal{E}_y(v)$  the  $y_+$  branch is represented by a solid line and the  $y_-$  branch is represented by a dashed line.

As the function  $x_B(v)$  has two branches so does  $\mathcal{E}_y(v) = \mathcal{E}(v)$ , and for the readers convenience we present plots of  $\mathcal{E}_y(v)$  and  $\mathcal{E}_{1|vw} = \mathcal{E}(v + iu) + \mathcal{E}(v - iu)$  in Figure 6.1.

We can now argue the physical interpretation of the supercharges (6.3). From the dispersion relation (6.6) we see that an electron that decreases the energy by 2 has momentum zero, while an electron that increases the energy by 2 has momentum  $\pi$ . As  $\mathbf{Q}_{0,\uparrow}^\dagger$  and  $\mathbf{Q}_{0,\downarrow}^\dagger$  add charge 1 and spin  $+\frac{1}{2}$  and  $-\frac{1}{2}$  respectively to a state while decreasing its energy by two, they must correspond to adding an electron with momentum zero and spin, respectively up and down, to the state. Similarly  $\mathbf{Q}_{0,\sigma}$  corresponds to removing an electron with momentum zero, while  $\mathbf{Q}_{\pi,\sigma}^\dagger$  and  $\mathbf{Q}_{\pi,\sigma}$  correspond respectively to adding and removing an electron with momentum  $\pi$ .

Finally let us examine the symmetries of the model through the Bethe equations. As we saw for the A-model, it is possible to add a  $w$ -root to any solution and it has the effect of acting with the spin  $\mathfrak{su}(2)$  lowering operator  $\mathbf{S}^-$  on the corresponding eigenstate. Unlike the A-model however, here we can also add a single  $v$ -root at  $\infty$ . Indeed one can be added at either  $v = \infty \pm i0$ , i.e. at either side of the cut  $\mathcal{I}^B$ . From equation (6.5) we see that

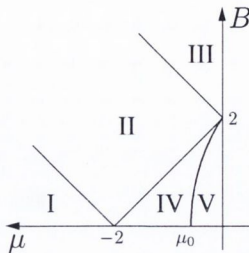
$$p(\infty + i0) = 0, \quad p(\infty - i0) = \pi, \quad (6.7)$$

and so from equation (6.6) (see also the plot of  $\mathcal{E}_y$  in Figure 6.1) we get

$$\mathcal{E}(\infty + i0) = -2, \quad \mathcal{E}(\infty - i0) = 2. \quad (6.8)$$



## 6.2 Ground state phase diagram



**Figure 6.2:** B-model: Zero temperature phase diagram in the  $\mu B$ -plane. The phases identified are: I) empty band, II) partially filled and spin polarised band, III) half-filled and spin-polarised band, IV) partially filled and partially spin polarised band, V) half-filled and partially spin polarised band. The value  $\mu_0$  ranges between 0 in the limit of weak coupling, and  $-2 + 2 \log 2 \approx -0.6137$  in the limit of strong coupling.

Adding the roots increase the charge of the state by 1 while decreasing the spin by  $1/2$ , and so correspond to the action of  $\mathbf{Q}_{0,\sigma}^\dagger$  and  $\mathbf{Q}_{\pi,\sigma}^\dagger$  for  $v = \infty + i0$  and  $v = \infty - i0$  respectively. The fermionic operators of the opposite spin are realised by adding a  $w$ -root at  $\infty$  in addition to a  $v$ -root. Moreover adding a  $w$ -root and two  $v$ -roots, one  $\infty + i0$  and one  $\infty - i0$ , increases the charge by two while leaving the spin unchanged. The combination does not change the energy of the state but it does shift its momentum by  $\pi$ . This then corresponds to the twisted charge  $\mathfrak{su}(2)$  raising operator  $\boldsymbol{\eta}^+$  of (2.6), and completes the symmetries of the model.

## 6.2 Ground state phase diagram

The ground state of the B-model is determined by the set of zero temperature TBA equations (4.51). Only  $y$ -particles and  $1|vw$ -strings can have non-zero densities. Let us construct the phase diagram on a case by case basis. The result is presented in Figure 6.2.

First we consider the question of when  $1|vw$ -strings begin to appear in the ground state, this determines the boundary between phase IV and phases I, II and III. When there are no  $1|vw$ -strings then  $\epsilon_{1|vw}(v) \geq 0$  for all  $v$ . Thus the terms with  $\epsilon_{1|vw}$  do not contribute to the right hand side of the TBA equations (4.51) and they become

$$\epsilon_y = \mathcal{E}_y - \mu - B, \quad \epsilon_{1|vw} = \mathcal{E}_{1|vw} - 2\mu + \epsilon_y \circledast_{Q_y} K_1. \quad (6.9)$$

The phase boundary is determined by the condition that  $\min_v \epsilon_{1|vw}(v) = 0$ . Note that  $\epsilon_y(v)$  is even and a monotonically increasing function of  $v$  around the cut  $(1, \infty)$  from the minimum of  $\mathcal{E}_y(v)$  at  $v = \infty + i0$  to the maximum of  $\mathcal{E}_y(v)$  at  $v = \infty - i0$ , that is  $\epsilon_+(v)$  and  $\epsilon_-(v)$  are decreasing and increasing for positive

## 6. THE B-MODEL AT ZERO TEMPERATURE

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$v$  respectively. Also  $\epsilon_{1|vw}(v)$  is even and a monotonically decreasing function<sup>1</sup> for positive  $v$ . Thus the minimum of  $\epsilon_{1|vw}(v)$  is at  $v = \pm\infty$ . By evaluating the pseudo-energies at  $\pm\infty$  one can see that they have finite asymptotics, which we denote by  $\epsilon_\alpha^\infty$  for an  $\alpha$ -string. In this regard let us remark that for a function  $f(v)$  which asymptotes to  $f^\infty$  one has

$$\lim_{v \rightarrow \infty} \int_\Lambda^\infty dt f(t) K_1(t, v) = \lim_{v \rightarrow \infty} \int_{\Lambda-v}^\infty dt f(t+v) K_1(t, 0) = f^\infty,$$

for any finite  $\Lambda$ . The asymptotics can be used to determine the phase boundaries, let us consider separately the following cases: i)  $Q_y = \emptyset$ , the boundary between phases I and IV, ii)  $Q_y \subset \mathcal{I}_+^B \cup \mathcal{I}_-^B$ , a proper subset, the boundary between phases II and IV, iii)  $Q_y = \mathcal{I}_+^B \cup \mathcal{I}_-^B$ , the boundary between phases III and IV. We have

i) Evaluated at  $v = \infty$  the TBA equations (6.9) reduce to the linear equations

$$\epsilon_+^\infty = -2 - \mu - B, \quad \epsilon_-^\infty = 2 - \mu - B, \quad \epsilon_{1|vw}^\infty = -4 - 2\mu.$$

From the conditions  $\epsilon_+^\infty \geq 0$  and  $\epsilon_{1|vw}^\infty = 0$  it follows that the boundary of phases I and IV is the point  $(\mu, B) = (-2, 0)$ .

ii) Evaluated at  $v = \infty$  the TBA equations (6.9) reduce to the linear equations

$$\epsilon_+^\infty = -2 - \mu - B, \quad \epsilon_-^\infty = 2 - \mu - B, \quad \epsilon_{1|vw}^\infty = -4 - 2\mu - \epsilon_+^\infty.$$

From the condition  $\epsilon_{1|vw}^\infty = 0$  it follows that the boundary of phases II and IV is the line  $B = 2 + \mu$ .

iii) Evaluated at  $v = \infty$  the TBA equations (6.9) reduce to the linear equations

$$\epsilon_+^\infty = -2 - \mu - B, \quad \epsilon_-^\infty = 2 - \mu - B, \quad \epsilon_{1|vw}^\infty = -4 - 2\mu + 4.$$

From the conditions  $\epsilon_-^\infty \leq 0$  and  $\epsilon_{1|vw}^\infty = 0$  it follows that the boundary of phases III and IV is the point  $(\mu, B) = (0, 2)$ .

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<sup>1</sup>This second property is seen via the identity  $\mathcal{E}_{1|vw} = -\mathcal{E}_y \otimes K_1 = -\epsilon_y \otimes K_1$ , where the first equality follows from shifting the contours of integration and the second from the fact that the constant terms in  $\epsilon_y$  cancel when integrated over both edges of the cut. This implies that  $\epsilon_{1|vw} = -2\mu - \epsilon_y \otimes \bar{Q}_y K_1$ , where  $\bar{Q}_y$  is the complement of  $Q_y$  in the contour  $\mathcal{I}_+^B \cup \mathcal{I}_-^B$  around the cut, and it follows that the derivative of  $\epsilon_{1|vw}(v)$  is negative for  $v \in (0, \infty)$ .



## 6.2 Ground state phase diagram

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Next let us determine the phase boundaries for  $y$ -particles. Neglecting for a moment the contribution from  $1|vw$ -strings, the relevant TBA equation takes the form

$$\epsilon_y = \mathcal{E}_y - \mu - B.$$

Recalling the dispersion relation (6.6) one can see that the line  $\mu = -2 - B$  separates phases I and II and the line  $\mu = 2 - B$  separates phases II and III. Note that neither of these lines enter the region, located in the previous paragraph, for which  $1|vw$ -strings appear in the ground state. Thus neglecting the  $1|vw$ -strings in the determination of the  $y$ -particle phase boundaries turns out to be self-consistent.

Finally we consider when the ground state becomes half-filled. Let us begin by establishing that this happens when  $Q_{1|vw} = \mathbb{R}$ . Half-filling corresponds to  $n_c = 1$ , where  $n_c = n_y + 2n_{1|vw}$  is the charge density,  $n_y = 1 \check{\star}_{Q_+} \rho_+ + 1 \check{\star}_{Q_-} \rho_-$  is the average number of  $y$ -particles per site and  $n_{1|vw} = 1 \star_{Q_{1|vw}} \rho_{1|vw}$  is the average number of  $1|vw$ -strings per site. The equation for the density of  $1|vw$ -strings in the zero temperature limit provides a useful relation

$$\rho_{1|vw} = \frac{1}{2\pi} \left| \frac{dp_{1|vw}}{dv} \right| - K_1 \check{\star}_{Q_+} \rho_+ - K_1 \check{\star}_{Q_-} \rho_- - K_2 \star_{Q_{1|vw}} \rho_{1|vw}. \quad (6.10)$$

Integrating over  $\mathbb{R}$  this gives

$$n_{1|vw} + 1 \star_{\bar{Q}_{1|vw}} \rho_{1|vw} = 1 - n_y - n_{1|vw}, \quad (6.11)$$

and so  $Q_{1|vw} = \mathbb{R}$  implies  $\bar{Q}_{1|vw} = \emptyset$  and thus  $n_c = 1$ . Moreover, it is clear that  $Q_{1|vw} \subset \mathbb{R}$ , a proper subset, implies  $n_c < 1$ . Thus the boundary between phases IV and V is determined by setting  $Q_{1|vw} = \mathbb{R}$  and varying  $Q_y$ . First consider the case  $Q_y = \emptyset$ . Here  $\epsilon_+ \geq 0$  and evaluating the TBA equations (4.51) at  $v = \infty$  one finds that  $\epsilon_+^\infty = -B$ , and thus  $B = 0$ . Therefore the boundary is the point  $(\mu, B) = (\mu_0, 0)$  where  $\mu_0$  is the solution to the equation  $\epsilon_{1|vw}(0) = 0$ , with  $\epsilon_{1|vw}$  determined by the integral equation

$$\epsilon_{1|vw} = \mathcal{E}_{1|vw} - 2\mu - \epsilon_{1|vw} \star K_2.$$

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Using the identities  $s + K_2 \star s = K_1$  and  $\mathcal{E}_{1|vw} = -\mathcal{E}_y \circledast K_1$  this equation can be inverted

$$\mu_0 = -(\mathcal{E}_y \circledast s)(0) = - \int_{|t|>1} dt \frac{4t^2 \sqrt{1 - \frac{1}{t^2}}}{u^2 + t^2} s(t). \quad (6.12)$$

Evaluating this integral one finds that  $\mu_0$  ranges from  $\mu_0 = 0$  at weak coupling to  $\mu_c = 2 \log 2 - 2 \approx -0.6137$  at strong coupling. Next for the case  $Q_y = \mathcal{I}_+^B \cup \mathcal{I}_-^B$ , examining the asymptotics of the TBA equations as above yields that the boundary is the point  $(\mu, B) = (0, 2)$ . Thus phases II, III, IV and V all meet here. Furthermore it follows that at this point  $\epsilon_{1|vw}(v) = 0$  for all  $v$  and hence there are neither particles nor holes of  $1|vw$  strings in the ground state here. To fill in the picture the line joining  $(\mu_0, 0)$  and  $(0, 2)$  must be determined numerically from the coupled integral equations

$$\epsilon_y = \mathcal{E}_y - \mu - B - \epsilon_{1|vw} \star K_1, \quad \epsilon_{1|vw} = \mathcal{E}_{1|vw} - 2\mu + \epsilon_y \circledast_{Q_y} K_1 - \epsilon_{1|vw} \star K_2,$$

parametrised by the interval  $Q_y \subset \mathcal{I}_+^B \cup \mathcal{I}_-^B$ . A practical way to achieve this is to introduce the derivative with respect to  $v$  of these equations

$$\epsilon'_y = \mathcal{E}'_y - \epsilon'_{1|vw} \star K_1, \quad \epsilon'_{1|vw} = \mathcal{E}'_{1|vw} + \epsilon'_y \circledast_{Q_y} K_1 - \epsilon'_{1|vw} \star K_2,$$

where here we have used  $\frac{\partial}{\partial v} K_M(t-v) = -\frac{\partial}{\partial t} K_M(t-v)$ , integrated by parts, and used that  $\epsilon_y(v) = 0$  on the boundary of  $Q_y$ . Then for a given interval  $Q_y$  the corresponding point  $(\mu, B)$  is found by identifying the asymptotics  $\epsilon_+^\infty = -2B$ ,  $\epsilon_{1|vw}^\infty = -2 - \mu + B$  with

$$\epsilon_+^\infty = \int_{Q^+, t>0} dt \epsilon'_+(t) - \int_{Q^-, t>0} dt \epsilon'_-(t), \quad \epsilon_{1|vw}^\infty = \int_0^\infty dt \epsilon'_{1|vw}(t).$$

### 6.3 Excitations

Now we investigate the excitations at zero temperature. First we observe that the  $M|vw$ -strings are type 2 strings, the  $M|w$ -strings are of type 1, and the  $y$ -particles should be treated as type 2 strings from the discussion at beginning of section 4.2.



### 6.3 Excitations

First we focus on excitations over the ground state when it is half-filled and has zero magnetisation. This is the subregion of phase V where  $\mu_0 \leq \mu \leq 0$  and  $B = 0$ . Here the TBA equations 6.9 can be solved explicitly with solution

$$\epsilon_{1|vw} = -\mathcal{E}_y \circledast s - \mu, \quad \epsilon_{M \geq 2|vw} = -2(M-1)\mu, \quad \epsilon_y = \mathcal{E}_y - \mathcal{E}_{1|vw} \star s, \quad \epsilon_{M|w} = 0. \quad (6.13)$$

The  $M|vw$ -strings for  $M \geq 2$  and the  $M|w$ -strings are not dynamical. The  $1|vw$ -strings are half-filled and so we use the prescription (3.92) for handling excitations which change the range of mode numbers and get

$$\delta N_{1|vw}^s = -\frac{1}{2}\delta N_{1|vw}^p - \frac{1}{2}\delta N_y^p - \sum_{M=2}^{\infty} \delta N_{M|vw}^p. \quad (6.14)$$

Thus only excitations with  $\delta N_{1|vw}^p + \delta N_y^p$  even are allowed, as  $\delta N_{1|vw}^s$  must be an integer.

To calculate the dressed momenta let us make the branch choice  $b_{1|vw, M|vw} = 1$  and  $b_{1|vw, y} = 0$ . Then we get

$$\begin{aligned} P_{1|vw} &= -\pi + p_{1|vw} - \frac{1}{2\pi} \Upsilon \star \frac{dp_{1|vw}}{dv} = -p_y \circledast s, \quad P_{M \geq 2|vw} = (M-1)\pi \pmod{2\pi}, \\ P_y &= -\Psi + p_y - p_{1|vw} \star s, \quad P_{M|w} = 0, \end{aligned} \quad (6.15)$$

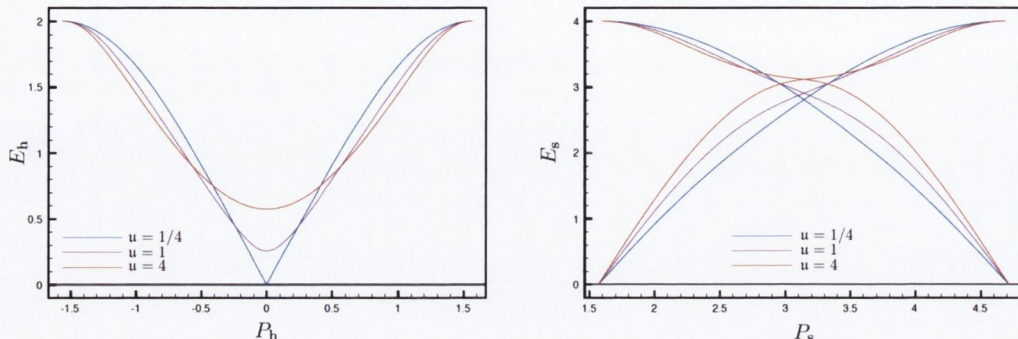
where we have introduced the useful functions

$$\begin{aligned} \Upsilon(v) &= \Theta_1 \star s(v) = i \log \left[ \frac{\Gamma(\frac{1}{2} + i\frac{v}{4u})\Gamma(1 - i\frac{v}{4u})}{\Gamma(\frac{1}{2} - i\frac{v}{4u})\Gamma(1 + i\frac{v}{4u})} \right], \\ \Psi(v) &= \Theta_2 \star s(v) - \Theta_1(v) = \frac{\pi}{2} - 2 \arctan \left[ \exp\left(\frac{\pi v}{2u}\right) \right]. \end{aligned} \quad (6.16)$$

The range of  $P_{1|vw}$  is  $(-\frac{\pi}{2}, \frac{\pi}{2})$  while the range of  $P_y$  is  $(-\frac{3\pi}{2}, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{2})$ . Let us remark that when taken modulo  $2\pi$  the range of  $P_y$  will have an overlap. The singular  $1|vw$ -strings appearing through eq. (6.14) have rapidity  $v^{\max} = 0$  and so carry momentum 0. The dressed charge and spin are

$$\begin{aligned} \omega_y^c &= 0, \quad \omega_{1|vw}^c = 1, \quad \omega_{M \geq 2|vw}^c = 2M - 2, \quad \omega_{M|w}^c = 0, \\ \omega_y^s &= \frac{1}{2}, \quad \omega_{M|vw}^s = 0, \quad \omega_{M|w}^s = -M, \end{aligned} \quad (6.17)$$

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**Figure 6.3:** Plots of  $E_h(P_h)$  and  $E_s(P_s)$  for the half-filled B-model at  $\mu = 0$ ,  $B = 0$ .

and we observe that the excitations are spin-charge separated in this phase. The removed  $1|vw$ -strings get dressed as holons while added  $y$ -particles get dressed as spinons. The energies and momenta of the holons, antiholons and spinons are

$$\begin{aligned}
 E_h &= -\epsilon_{1|vw} = \mathcal{E}_y \otimes s + \mu, & P_h &= -P_{1|vw} = p_y \otimes s, \\
 E_{\bar{h}} &= -\epsilon_{1|vw} + \epsilon_{2|vw} = \mathcal{E}_y \otimes s - \mu, & P_{\bar{h}} &= -P_{1|vw} + P_{2|vw} = -\pi \text{sign} + p_y \otimes s, \\
 E_s &= E_{\bar{s}} = \epsilon_y = \mathcal{E}_y - \mathcal{E}_{1|vw} \star s, & P_s &= P_{\bar{s}} = P_y = -\Psi + p_y - p_{1|vw} \star s.
 \end{aligned}
 \tag{6.18}$$

Here an antiholon is identified as a composite excitation of a holon and a  $2|vw$ -string because a  $2|vw$ -string is not dynamical. Let us remark that through eqs. (6.13), (6.15) and (6.17) it can be seen that its addition can also be regarded as the action of the charge  $\mathfrak{su}(2)$  raising operator on a state. Similarly the spinon  $\bar{s}$  is a composite of a spinon  $s$  and a  $1|w$ -string. Plots of  $E_h(P_h)$  and  $E_s(P_s)$  are given in Figure 6.3 for various values of  $u$ . The velocities can again easily be read from the slopes. The holons are gapped for  $\mu > \mu_0$ . The gap goes to zero in the weak coupling  $u \rightarrow 0$  limit, while the gap has a maximal value of  $2 - 2\log 2 \approx 0.6137$  at  $\mu = 0$  in the strong coupling  $u \rightarrow \infty$  limit. The spinons are gapless and display an “hourglass” dispersion. The similarity to experimental data on spinon scattering in some cuprate materials, say Fig. 2 of [25], cannot go unremarked. In the strong coupling  $u \rightarrow \infty$  limit the two lower wings join differentiably at  $\pi$  and the upper wings can be understood as the contributions of dressed electrons on doubly occupied sites, see chapter 7.



Next let us calculate the dressed scattering phases to examine the scattering of the holons and spinons. First we present explicitly the bare scattering phases

$$\begin{aligned}\phi_{1|vw,1|vw}(v,t) &= 2\pi - \pi \operatorname{sign}(v) + \Theta_2(v-t), \\ \phi_{1|vw,M \geq 2|vw}(v,t) &= 2\pi - 2\pi \operatorname{sign}(v) + \Theta_{1M}(v-t), \\ \phi_{1|vw,y}(v,t) &= \phi_{y,1|vw}(v,t) = -\pi \operatorname{sign}(v) + \Theta_1(v-t), \\ \phi_{1|vw,M|w}(v,t) &= 0, \quad \phi_{y,y}(v,t) = 0, \quad \phi_{y,M|w}(v,t) = -\pi \operatorname{sign}(v) + \Theta_M(v-t).\end{aligned}$$

To calculate the dressed scattering it will be necessary to redistribute the contributions of the singular strings as in eq. (3.95). Solving eqs. (3.53) for  $\Phi_{1|vw,1|vw}$ ,  $\Phi_{M|vw,1|vw}$  and  $\Phi_{y,1|vw}$  we get

$$\begin{aligned}\Phi_{1|vw,1|vw}(v,t) &= \pi - \pi \operatorname{sign}(v) + \Upsilon(v) + \Upsilon(v-t), \\ \Phi_{M \geq 2|vw,1|vw}(v,t) &= -2\pi \operatorname{sign}(v) + \Theta_{M-1}(v) + \Theta_{M-1}(v-t), \\ \Phi_{y,1|vw}(v,t) &= -\pi - \pi \operatorname{sign}(v) - \Psi(v) - \Psi(v-t),\end{aligned}\tag{6.19}$$

where the identity  $\pi \operatorname{sign} \star K_1 = \Theta_1$  has been used. As we are interested in the scattering phase shift, which is defined modulo  $2\pi$ ,  $\operatorname{sign}(v)$  can be dropped from (6.19). Taking into account the scattering phases of singular strings the dressed scattering phases are

$$\begin{aligned}\Phi_{1|vw,1|vw}^{\text{h.f.}}(v,t) &= \Upsilon(v-t), & \Phi_{y,y}^{\text{h.f.}}(v,t) &= \pi - \Upsilon(v-t), \\ \Phi_{\alpha\beta}^{\text{h.f.}}(v,t) &= \Phi_{\beta\alpha}^{\text{h.f.}}(v,t), & \Phi_{y,1|vw}^{\text{h.f.}}(v,t) &= \pi + \Psi(v-t), \\ \Phi_{1|vw,M \geq 2|vw}^{\text{h.f.}}(v,t) &= \pi + \Theta_{M-1}(v-t), & \Phi_{y,M|w}^{\text{h.f.}}(v,t) &= \pi + \Theta_M(v-t), \\ \Phi_{M|vw,N|w}^{\text{h.f.}}(v,t) &= 0, & \Phi_{y,M \geq 2|vw}^{\text{h.f.}}(v,t) &= 0, \\ \Phi_{M \geq 2|vw,N \geq 2|vw}^{\text{h.f.}}(v,t) &= \delta_{MN}\pi + \Theta_{MN}(v-t) - \Theta_{M+N}(v-t) - \Theta_{M+N-2}(v-t), \\ \Phi_{M|w,N|w}^{\text{h.f.}}(v,t) &= \delta_{MN}\pi - \Theta_{MN}(v-t).\end{aligned}$$

Let us compute explicitly the phase shifts for the charge triplet and charge singlet excitations.

*Charge triplet:* holon-holon scattering. Here two  $1|vw$ -strings with rapidities  $v_1$  and  $v_2$  are removed. Let us say that  $v_1$  has a greater velocity  $\frac{d\epsilon_h}{dP_h}$  than  $v_2$ , and let us denote this as  $v_1 \succ v_2$ . Then

$$F_{1|vw}(v) = -\Phi_{1|vw,1|vw}^{\text{h.f.}}(v, v_1) - \Phi_{1|vw,1|vw}^{\text{h.f.}}(v, v_2)$$

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and the phase shift is

$$\delta_{\text{CT}} = \pi + \Upsilon(v_1 - v_2). \quad (6.20)$$

*Charge singlet:* holon-antiholon scattering. Here two  $1|vw$ -strings with rapidities  $v_1 \succ v_2$  are removed and a  $2|vw$ -string with rapidity  $\tilde{v}$  is added. The rapidity  $\tilde{v}$  of the added  $2|vw$ -string can be fixed through eq. (3.52) using  $P_{2|vw} = 0$ ,

$$\begin{aligned} F_{2|vw} &= -\Phi_{2|vw,1|vw}^{\text{h.f.}}(v, v_1) - \Phi_{2|vw,1|vw}^{\text{h.f.}}(v - v_2) + \Phi_{2|vw,2|vw}^{\text{h.f.}}(v - \tilde{v}) \\ &= \pi - \Theta_1(v - v_1) - \Theta_1(v - v_2) + \Theta_2(v - \tilde{v}), \end{aligned}$$

and so  $F_{2|vw}(\tilde{v}) = \pi$  gives  $\tilde{v} = \frac{v_1 + v_2}{2}$ . Here

$$F_{1|vw}(v) = -\Phi_{1|vw,1|vw}^{\text{h.f.}}(v, v_1) - \Phi_{1|vw,1|vw}^{\text{h.f.}}(v, v_2) + \Phi_{1|vw,2|vw}^{\text{h.f.}}\left(v, \frac{v_1 + v_2}{2}\right)$$

and thus

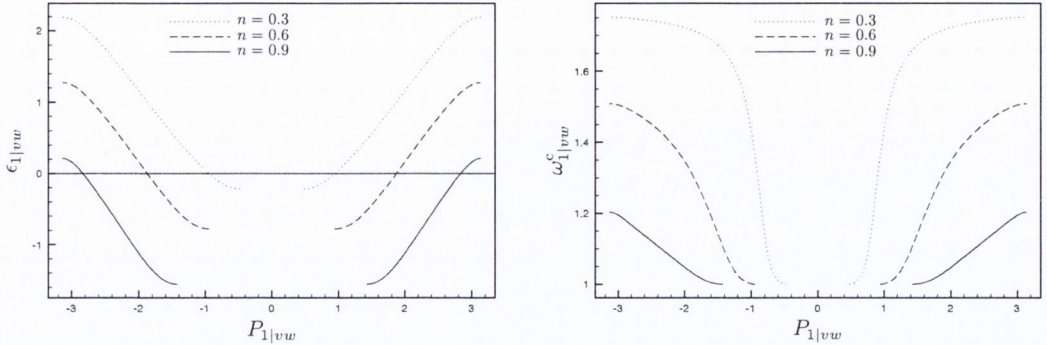
$$\delta_{\text{CS}} = \Upsilon(v_1 - v_2) - \Theta_1\left(\frac{v_1 - v_2}{2}\right). \quad (6.21)$$

Let us remark that these results as functions of the rapidity are the same as those of the half-filled Hubbard model [61], see e.g. eqs. (7.124) and (7.126) of [9]. The scattering shifts for the spin triplet, singlet and spin-charge excitations can be computed similarly and also agree with those of the Hubbard model, eqs. (7.139), (7.141) and eq. (7.145) of [9].

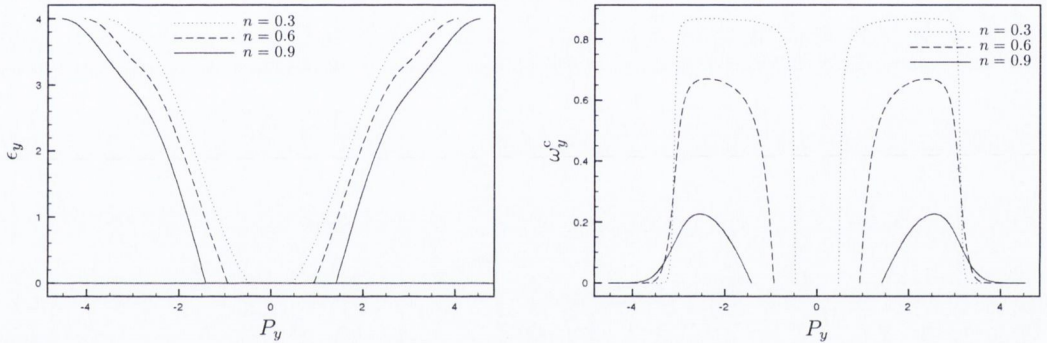
Now let us consider the less than half-filled phase while still keeping  $B = 0$ . In Figure 6.2 this is the portion of phase IV along the  $\mu$ -axis. The magnetisation is zero and the filling is  $2n_{1|vw}$ . Here again  $\epsilon_{M|w} = 0$  and  $P_{M|w} = 0$  for  $M|w$ -strings but to find the dressed energies and momenta of  $M|vw$ -strings and  $y$ -particles one must solve the TBA equations (6.9) numerically. Taking all  $b_{\alpha\beta} = 0$ , the dressed momentum for  $y$ -particles takes values in  $\pi n_{1|vw} < |P_y| < \pi + \pi n_{1|vw}$ , for  $1|vw$ -strings in  $\pi n_{1|vw} < |P_{1|vw}| < \pi$ , for  $M|vw$ -strings with  $M \geq 3$  odd in  $2\pi n_{1|vw} < |P_{M|vw}| < \pi$ , and for  $M|vw$ -strings with  $M$  even in  $\pi + 2\pi n_{1|vw} < |P_{M|vw}| < 2\pi$ . These ranges should be considered modulo  $2\pi$  but it is more convenient in plots to use the ranges specified here. The other branches are obtained by shifts of  $2\pi n_{1|vw}$ . The dressed spins of the excited strings take their bare values while the equations for dressed charge are

$$\omega_y^c = 1 - \omega_{1|vw}^c \star_{Q_{1|vw}} K_1, \quad \omega_{M|vw}^c = 2M - \omega_{1|vw}^c \star_{Q_{1|vw}} K_{1M}, \quad \omega_{M|w}^c = 0. \quad (6.22)$$





**Figure 6.4:** B-model: Plots of dressed energy  $\epsilon_{1|vw}(P_{1|vw})$  and dressed charge  $\omega_{1|vw}^c(P_{1|vw})$  at  $u = 1$  and  $B = 0$  for various fillings.



**Figure 6.5:** B-model: Plots of dressed energy  $\epsilon_y(P_y)$  and dressed charge  $\omega_y^c(P_y)$  at  $u = 1$  and  $B = 0$  for various fillings.

These are rapidity dependent for the  $y$ -particles and  $M|vw$ -strings. Let us remark however that at  $v = \pm\infty$  the dressed charges take the values they have at half-filling (6.17)

$$\omega_y^c(\pm\infty) = 0, \quad \omega_{1|vw}^c(\pm\infty) = 1, \quad \omega_{M \geq 2|vw}^c(\pm\infty) = 2M - 2. \quad (6.23)$$

In Figure 6.4 the dressed energy and dressed charge of a  $1|vw$ -string as a function of its dressed momentum is plotted for various fillings at  $u = 1$ . The corresponding plots for  $y$ -particles are given in Figure 6.5 and the ‘‘hourglass’’ behaviour about  $\pi$  is seen again. Both excitations are gapless. The  $M|vw$ -strings with  $M \geq 2$  are dynamical but have a gap of  $-2(M - 1)\mu$ . The  $1|vw$ -strings remain spinless but  $y$ -particles with non-zero energy here have dressed charge in addition to their spin. At low energies the charge carried by a  $y$ -particle scales with its energy, and moreover the magnitude of the charge carried increases sharply as the filling

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is decreased. Thus the excitations are not spin-charge separated at non-zero energies away from half-filling. Nevertheless, in the limit of zero energy the excitations carry either spin or charge and so this is compatible with spin-charge separated wave-like excitations that one may expect in the continuum limit, as in the Luttinger liquid. Let us remark that, as can be seen from Figure 6.4, the charge of a zero-energy  $1|vw$ -string becomes greater than 1 at less than half-filling and thus we expect that the charge carried by a charge-wave gets increased at reduced filling. Moreover we note that the magnitude of the charge carried by a particle excitation is greater than that carried by a hole excitation.

It is noteworthy that at  $B = 0$  one can clearly see that the  $1|vw$ -string is a spin-singlet bound state. Let us show this. The spin singlet excitation is achieved by adding two  $y$ -particles with rapidities  $v_1$  and  $v_2$ , and adding a  $1|w$ -string with rapidity  $w$  which we initially take to be arbitrary. The relevant dressed phase shifts are

$$\begin{aligned}\Phi_{y,y}(v,t) &= -(K_1 \star_{Q_1|vw} \Phi_{1|vw,y})(v,t), \\ \Phi_{1|vw,y}(v,t) &= -\pi \operatorname{sign}(v) + \Theta_1(v,t) - (K_2 \star_{Q_1|vw} \Phi_{1|vw,y})(v,t), \\ \Phi_{y,1|w}(v,t) &= \pi + \Theta_1(v-t), \quad \Phi_{1|w,y}(v,t) = \pi + \Theta_1(v-t), \quad \Phi_{1|w,1|w}(v,t) = \pi - \Theta_2(v-t).\end{aligned}$$

The rapidity  $w$  is fixed to  $w = \frac{v_1+v_2}{2}$  by eq. (3.52) as  $P_{1|w} = 0$ . The scattering phase shift is

$$F_y(v_1) = \Phi_{y,y}(v_1, v_1) + \Phi_{y,y}(v_1, v_2) + \pi + \Theta_1\left(\frac{v_1 - v_2}{2}\right). \quad (6.24)$$

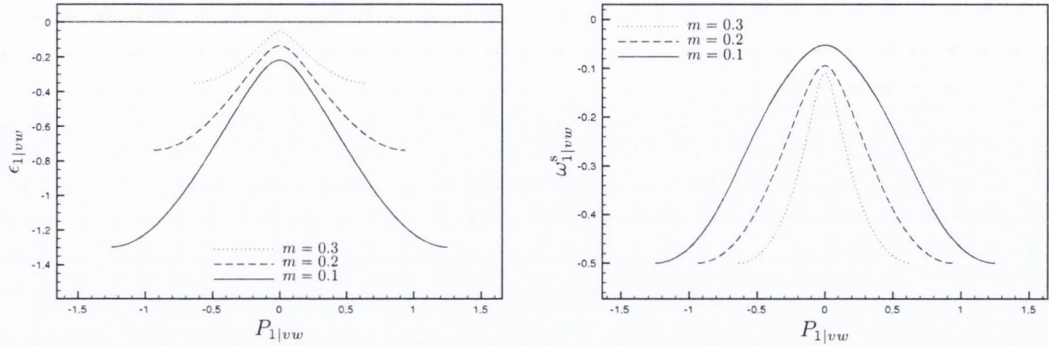
Note that  $\Theta_1(\pm i\mathbf{u}) = \pm i\infty$  and so the final term gives rise to a pole of the S-matrix at  $v_1 = v_2 - 2i\mathbf{u}$ . Although the first two terms cannot be obtained explicitly it can be seen analytically that they cancel the pole through the term  $\Theta_1$  in  $\Phi_{1|vw,y}$  for  $v \in Q_{1|vw}$ , while for  $v \notin Q_{1|vw}$  the pole remains. The pole corresponds to a bound state of a  $y_-$ - and a  $y_+$ -particle as  $\operatorname{Im} P_-(v) > 0$  for  $\operatorname{Im} v \neq 0$  and  $\operatorname{Im} P_+(v) < 0$  for  $\operatorname{Im} v \neq 0$ . The bound state, with rapidity  $v$ , thus corresponds to a spin singlet excitation with

$$v_1 = v - i\mathbf{u}, \quad v_2 = v + i\mathbf{u}, \quad \tilde{w} = v. \quad (6.25)$$

and the changes of energy and momentum are

$$\Delta E = \epsilon_{1|vw}(v), \quad \Delta P = P_{1|vw}(v) \quad \text{for } v \notin Q_{1|vw}. \quad (6.26)$$





**Figure 6.6:** B-model: Plots of dressed energy  $\epsilon_{1|vw}(P_{1|vw})$  and dressed charge  $\omega_{1|vw}^s(P_{1|vw})$  at  $u = 1$  and  $\mu = 0$  for various values of magnetisation  $m$ .

Here the identity  $K_1(v + iu - i0) + K_1(v - iu + i0) = \delta(v) + K_2(v)$  was used to obtain the energy and eq. (3.47) was used to obtain the momentum. Hence the bound state is indeed an added  $1|vw$ -string. This indicates that spin-spin interactions are responsible for the pairing of the electrons into  $1|vw$ -strings.

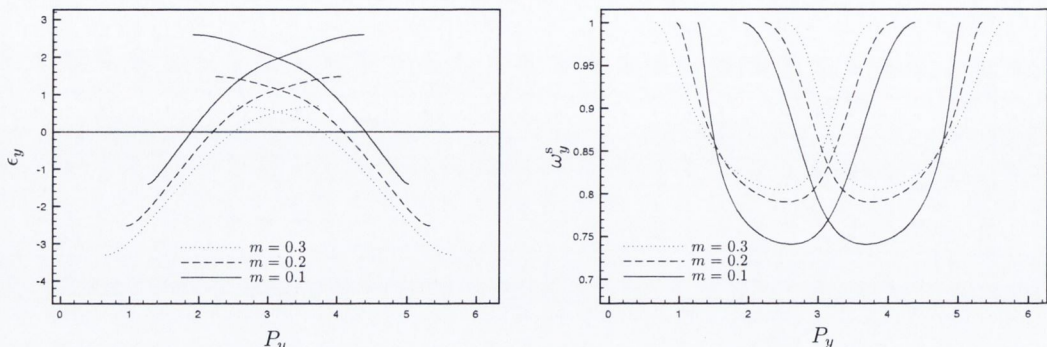
Let us consider briefly the effect of a  $B > 0$  magnetic field at half-filling, the interior of phase V in Figure 6.2. Here there are both  $1|vw$ -strings and  $y$ -particles in the ground state and it has a magnetisation between 0 and  $1/2$ . The  $1|vw$ -strings are at half-filling<sup>1</sup> and so excitations must satisfy (6.14), from which it follows that only excitations with  $\delta N_{1|vw}^p + \delta N_y^p$  even are allowed. The  $1|vw$ -strings are gapped and have dressed charge 1 while the  $y$ -particles have dressed charge zero and are gapless. The  $M \geq 2|vw$ -strings are non-dynamical while the  $M|w$ -strings are dynamical but gapped. The spin dressing equations for  $y$ -particles and  $1|vw$ -strings are

$$\omega_y^s = \frac{1}{2} - \omega_{1|vw}^s \star K_1, \quad \omega_{1|vw}^s = \omega_y^s \circledast_{Q_y} K_1 - \omega_{1|vw}^s \star_{Q_{1|vw}} K_{1M}. \quad (6.27)$$

Let us remark that  $\omega_y^s(\pm\infty) = 1$ ,  $\omega_{1|vw}^s(\pm\infty) = -\frac{1}{2}$  and thus at zero temperature the dressed spin jumps as soon as a magnetic field is introduced. This is true for any filling. Plots of the dressed energy and dressed spin of a  $1|vw$ -string are given in Figure 6.6 for various magnetisations at  $u = 1$  and  $\mu = 0$ . The corresponding plots for  $y$ -particles are given in Figure 6.7.

<sup>1</sup>There are no holes for  $1|vw$ -strings here. There are less of them than there are at  $B = 0$  as their range of mode numbers is decreased by the presence of the  $y$ -particles.

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**Figure 6.7:** B-model: Plots of dressed energy  $\epsilon_y(P_y)$  and dressed charge  $\omega_y^s(P_y)$  at  $u = 1$  and  $\mu = 0$  for various values of magnetisation  $m$ .

We conclude by comparing our findings to studies, respectively [62] and [63], of related models the supersymmetric  $t$ - $J$  model [26, 64, 65] and EKS model [27, 63]. These share similar phase diagrams, indeed in the limit of strong coupling the B-model reduces to the EKS-model with coupling  $U = 4$ , and the supersymmetric  $t$ - $J$  model also shares their common ground state. Thus the excitations are expected to be very similar and this is the case. The dressing of spin and charge of excitations, which we describe in detail, is suggested at in [62] and is not discussed in [63]. In particular we do not observe the existence of an electronic excitation carrying charge 1 and spin  $\frac{1}{2}$  for any filling as claimed in [63]. In our language the excitation they consider is understood as a  $y_-$ -particle and it carries this charge and spin only at zero filling where it is undressed. An interesting feature of the EKS model is the presence of gapless excitations for  $U < 4 \ln 2 \sim 2.77$ . They call these excitations localons, and in our language they correspond to  $M|v$ -strings with  $M \geq 2$ . These are gapped throughout the B-model however, as is to be expected due to the presence of a charge gap at half-filling. Let us comment on a difference between the B-model and its strong coupling limit related to the hourglass-like dispersion of the  $y$ -particles, see Figures 6.3 and 6.5. In the  $u \rightarrow \infty$  limit the dispersion curves split into two branches, one upper one lower, that touch tangentially at one point. These correspond to the  $y_+$  and  $y_-$  branches of the  $y$ -particle respectively. For the supersymmetric  $t$ - $J$  model excitations corresponding to the upper branch do not appear. For finite  $u$  however the structure of the excitation is no longer of an upper and lower branch but rather of left and right moving excitations. Finally we should comment on the



advantages of our formalism over those of say [62, 63]. By working directly with the Bethe strings we obtain a clear description of excitations over the pseudo-vacuum reference state instead of the somewhat unnatural reference state which is the preferred choice of [62, 63] because it makes it easier to work with Bethe roots. Furthermore, overcoming the need to deal directly with mode numbers and using dressing equations (3.55) to determine the dressing of spin and charge allowed us to straightforwardly identify the nature of the excitations.

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# Chapter 7

## Integrable extensions of the Hubbard-Shastry models

In this chapter we discuss the integrable extensions of the Hubbard-Shastry models. These correspond to decoupling the Hubbard interaction for the weak coupling limit of the A-model and the strong coupling limit of the B-model, and so they have one free coupling constant. We find that the two resulting models are equivalent to one another, and also equivalent to the Essler-Korepin-Schoutens (EKS) model [27]. We investigate this at the level of both the Hamiltonian and the Bethe equations. The thermodynamics are considered and a simplification of the TBA equations for the EKS model is obtained. The ground state phase diagram is also examined.

### 7.1 The Hamiltonians

Let us present the integrable extended models. The construction follows that of [27] and the model  $\mathbf{H}^{\tilde{A}_0}$  below is indeed the one they studied. The key observation is that if  $[\mathbf{H}, \mathbf{V}^H] = 0$  for a given model then  $\mathbf{H}$  and  $\mathbf{V}^H$  can be diagonalised simultaneously and it is natural to add a term  $4\mathbf{g}\mathbf{V}^H$  to the Hamiltonian, where  $\mathbf{g}$  is a new coupling constant. This allows one to decouple the term capturing onsite Coulomb repulsion from the other terms in the Hamiltonian. Since the number of doubly occupied sites is conserved if  $[\mathbf{H}, \mathbf{N}] = 0$ , then  $\mathbf{g}$  can be thought of as a chemical potential for doubly occupied sites. Furthermore if the original

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model is integrable and  $\mathbf{V}^H$  commutes with the transfer matrix then the resulting model is integrable too.

This is the case for the weak coupling limit of the A-model and the strong coupling limit of the B-model. We consider also the extensions of the opposite models as we can choose  $\mathfrak{g}$  so that the coefficient of the Hubbard interaction is positive for these also.

The extended Hamiltonians for the weak coupling limit of the A-model and its opposite model (see equation (2.15)) are

$$\mathbf{H}^{\mathbf{A}_0}(\mathfrak{g}) = \sum_{j=1}^L \left( \mathbf{T}_{j,j+1}^{\mathbf{A}_0} + 2((2\mathfrak{g} + 1) \mathbf{V}_{j,j+1}^H + \mathbf{V}_{j,j+1}^{CC} - \mathbf{V}_{j,j+1}^{SS} + \mathbf{V}_{j,j+1}^{PH}) \right) \quad (7.1)$$

$$\mathbf{H}^{\tilde{\mathbf{A}}_0}(\mathfrak{g}) = \sum_{j=1}^L \left( \mathbf{T}_{j,j+1}^{\tilde{\mathbf{A}}_0} + 2((2\mathfrak{g} - 1) \mathbf{V}_{j,j+1}^H - \mathbf{V}_{j,j+1}^{CC} + \mathbf{V}_{j,j+1}^{SS} - \mathbf{V}_{j,j+1}^{PH}) \right) \quad (7.2)$$

$$\mathbf{T}_{j,k}^{\mathbf{A}_0} = - \sum_{\sigma} (\mathbf{c}_{j,\sigma}^{\dagger} \mathbf{c}_{k,\sigma} + \mathbf{c}_{k,\sigma}^{\dagger} \mathbf{c}_{j,\sigma}) (1 - \mathbf{n}_{j,-\sigma} - \mathbf{n}_{k,-\sigma}) .$$

The Hamiltonian  $\mathbf{H}^{\tilde{\mathbf{A}}_0}(\mathfrak{g})$  is the one for the EKS model (with  $\mathfrak{g} = U/4$ ), and eq. (7.2) shows that for  $\mathfrak{g} < 1/2$  it should be thought of as a model with effective attractive on-site Coulomb interaction. The models have a dynamical  $\mathfrak{su}(2|2)$  symmetry. When  $\mathfrak{g} = 0$  the models reduce to the  $\mathfrak{su}(2|2)$  spin chain and the symmetry becomes exact, and the Hamiltonians commute with the supercharges

$$\mathbf{Q}_{s,\sigma} = \sum_{j=1}^L (\mathbf{c}_{j,\sigma} - \mathbf{n}_{j,-\sigma} \mathbf{c}_{j,\sigma}), \quad \mathbf{Q}_{d,\sigma} = \sum_{j=1}^L \mathbf{n}_{j,-\sigma} \mathbf{c}_{j,\sigma}, \quad (7.3)$$

and their hermitian conjugates. These have the clear physical interpretation that they annihilate (and their hermitian conjugates create) electrons on singly occupied and doubly occupied sites respectively. Computing the commutators

$$[\mathbf{V}^H, \mathbf{Q}_{s,\sigma}] = \frac{1}{2} \mathbf{Q}_{s,\sigma}, \quad [\mathbf{V}^H, \mathbf{Q}_{d,\sigma}] = -\frac{1}{2} \mathbf{Q}_{d,\sigma}, \quad (7.4)$$

one finds

$$[\mathbf{H}^{\mathbf{A}_0}(\mathfrak{g}), \mathbf{Q}_{s,\sigma}] = 2\mathfrak{g} \mathbf{Q}_{s,\sigma}, \quad [\mathbf{H}^{\mathbf{A}_0}(\mathfrak{g}), \mathbf{Q}_{d,\sigma}] = -2\mathfrak{g} \mathbf{Q}_{d,\sigma}, \quad (7.5)$$

and thus the  $\mathfrak{su}(2|2)$  symmetry is dynamically broken for  $\mathfrak{g} \neq 0$ .



The extended Hamiltonians for the strong coupling limit of the B-model and its opposite model are

$$\mathbf{H}^{\mathbf{B}\infty}(\mathfrak{g}) = \sum_{j=1}^L \left( \mathbf{T}_{j,j+1}^{\mathbf{B}\infty} + 2((2\mathfrak{g} + 1) \mathbf{V}_{j,j+1}^H - \mathbf{V}_{j,j+1}^{CC} + \mathbf{V}_{j,j+1}^{SS} + \mathbf{V}_{j,j+1}^{PH}) \right) \quad (7.6)$$

$$\mathbf{H}^{\bar{\mathbf{B}}\infty}(\mathfrak{g}) = \sum_{j=1}^L \left( \mathbf{T}_{j,j+1}^{\mathbf{B}\infty} + 2((2\mathfrak{g} - 1) \mathbf{V}_{j,j+1}^H + \mathbf{V}_{j,j+1}^{CC} - \mathbf{V}_{j,j+1}^{SS} - \mathbf{V}_{j,j+1}^{PH}) \right) \quad (7.7)$$

$$\mathbf{T}_{j,k}^{\mathbf{B}\infty} = - \sum_{\sigma} (\mathbf{c}_{j,\sigma}^{\dagger} \mathbf{c}_{k,\sigma} + \mathbf{c}_{k,\sigma}^{\dagger} \mathbf{c}_{j,\sigma}) (1 - (\mathbf{n}_{j,-\sigma} - \mathbf{n}_{k,-\sigma})^2).$$

These too have a dynamical  $\mathfrak{su}(2|2)$  symmetry. Taking the limit  $\mathfrak{u} \rightarrow \infty$  for the supercharges (6.3) of the B-model they become

$$\mathbf{Q}_{0,\sigma} = \sum_{j=1}^L (\mathbf{c}_{j,\sigma} - \mathbf{n}_{j,-\sigma} \mathbf{c}_{j,\sigma}), \quad \mathbf{Q}_{\pi,\sigma} = \sum_{j=1}^L (-1)^j \mathbf{n}_{j,-\sigma} \mathbf{c}_{j,\sigma}. \quad (7.8)$$

These generate a dynamical symmetry for  $\mathfrak{g} = 0$  as discussed in section 6.1, and also provide a dynamical symmetry for the extended B-models as can be seen through

$$[\mathbf{H}^{\mathbf{B}\infty}(\mathfrak{g}), \mathbf{Q}_{0,\sigma}] = 2(\mathfrak{g} + 1) \mathbf{Q}_{0,\sigma}, \quad [\mathbf{H}^{\mathbf{B}\infty}(\mathfrak{g}), \mathbf{Q}_{\pi,\sigma}] = -2(\mathfrak{g} + 1) \mathbf{Q}_{\pi,\sigma}. \quad (7.9)$$

The Hamiltonians and symmetries of the extended A- and B-models look somewhat different, but they are in fact equivalent. The models are related by the unitary transformation (5.5) as

$$\mathbf{H}^{\mathbf{B}\infty}(\mathfrak{g}) = \mathbf{U}_3^{\dagger} \mathbf{H}^{\bar{\mathbf{A}}_0}(\mathfrak{g} - 1) \mathbf{U}_3, \quad \mathbf{H}^{\bar{\mathbf{B}}\infty}(\mathfrak{g}) = \mathbf{U}_3^{\dagger} \mathbf{H}^{\mathbf{A}_0}(\mathfrak{g} + 1) \mathbf{U}_3. \quad (7.10)$$

It follows in particular that the EKS model and the extended B-model are equivalent up to a shift of the coupling  $\mathfrak{g}$  by 1. Let us remark that the unitary transformation untwists the supercharges (7.8) of the extended B-model but leaves those in (7.3) unchanged, and it so matches the two sets.

## 7.2 Bethe equations

Let us first consider the Bethe equations for the models without the extension, i.e. with  $\mathfrak{g} = 0$ . In order to take the limit of the Bethe equations (2.40) some care is required. As discussed in section 4.4.3 it is best to first rescale  $v \rightarrow v/\mathbf{u}$ ,  $w \rightarrow w/\mathbf{u}$  and then take the appropriate limits of  $\mathbf{u}$ . Then the resulting Bethe equations are<sup>1</sup>

$$\begin{aligned}
 e^{ip_{+,k}L} &= \left( \alpha \frac{z_{+,k} + i}{z_{+,k} - i} \right)^L = \prod_{j=1}^M \frac{z_{+,k} - w_j + i}{z_{+,k} - w_j - i}, & k = 1, \dots, N_+, \\
 \prod_{j=1}^{N_+} \frac{w_k - z_{+,j} + i}{w_k - z_{+,j} - i} \prod_{j=1}^{N_-} \frac{w_k - z_{-,j} + i}{w_k - z_{-,j} - i} &= \prod_{j=1, j \neq k}^M \frac{w_k - w_j + 2i}{w_k - w_j - 2i}, & k = 1, \dots, M, \\
 e^{ip_{-,k}L} &= (-1)^L = \prod_{j=1}^M \frac{z_{-,k} - w_j + i}{z_{-,k} - w_j - i}, & k = 1, \dots, N_-,
 \end{aligned}$$

where  $\alpha = -1$  for weak coupling limit of the A- and opposite A models, and  $\alpha = 1$  for the strong coupling limit of the B- and opposite B models. The dispersion relations for the models are

$$\begin{aligned}
 \mathcal{E}_+^{\text{A}_0}(v) &= -2 \cos p_+(v) - 2 = -4\pi \mathfrak{K}_1(v), & \mathcal{E}_-^{\text{A}_0}(v) &= -2 \cos p_-(v) - 2 = 0, \\
 \mathcal{E}_+^{\text{B}_\infty}(v) &= -2 \cos p_+(v) = 4\pi \mathfrak{K}_1(v) - 2, & \mathcal{E}_-^{\text{B}_\infty}(v) &= -2 \cos p_-(v) = 2,
 \end{aligned}$$

and the respective dispersion relations for the opposite models have the opposite sign. In the limiting process the  $y$ -roots decouple into two sets

$$\begin{aligned}
 p_+ : & \quad \left\{ y_+ = x(\mathbf{u}v) : \text{Im}(v) \geq 0 \right\} \cup \left\{ y_- = \frac{1}{x(\mathbf{u}v)} : \text{Im}(v) < 0 \right\}, \\
 p_- : & \quad \left\{ y_- = \frac{1}{x(\mathbf{u}v)} : \text{Im}(v) \geq 0 \right\} \cup \left\{ y_+ = x(\mathbf{u}v) : \text{Im}(v) < 0 \right\},
 \end{aligned}$$

with the momenta as presented in the Bethe equations above. The corresponding  $v$ -roots are labelled  $z_+$  and  $z_-$ . Note that the  $z_-$ -roots have been frozen, while the  $z_+$ -roots retain non-trivial momentum dependence.

The decoupling of the  $y$ -roots and the emergence of a third level to the Bethe ansatz equations is indicative that the models have an integrable extension. The

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<sup>1</sup>These Bethe equations coincide with those derived in [66] with the BFFB grading.



conserved number  $N$  has split into two independently conserved numbers  $N_+$  and  $N_-$ . Recall that the conservation of the Hubbard interaction implies that the total number of doubly occupied sites is conserved. It is clear that this must be  $N_-$ , the number of frozen roots. Indeed this is understood as  $z_+$ -roots being a first level of electrons that occupy empty sites, and that  $z_-$ -roots are an extra level that combine with a  $z_+$ -root and a  $w$ -root to make a site doubly occupied. As a consistency check note that there are at most  $M$   $z_-$ -roots as each must satisfy the same polynomial of degree  $M$ , and as  $2M \leq N_+ + N_-$ , that for each  $z_-$  root there exists a corresponding  $z_+$ -root and  $w$ -root.

Next we make the observation that  $M|vw$ -strings are composed of  $M + 1$   $z_+$ -roots and  $M - 1$   $z_-$ -roots. Let us illustrate this for the case of the  $1|vw$ -string. Recalling equation (4.11),  $y_1 = x(v + i)$  and  $y_{-1} = \frac{1}{x(v-i)}$ , we see that the two  $y$ -roots are of  $p_+$  type. For longer strings the logic extends naturally. This has an important implication. The  $1|vw$ -string here corresponds to a bound pair of a spin-up electron and a spin-down electron, each on a singly occupied site. Indeed, the Hubbard interaction favours singly occupied sites and thus extending the models can favour electron pairing.

Now consider how the coupling constant  $\mathbf{g}$  enters the analysis. Since  $z_+$  corresponds to a singly occupied site the Hubbard term shifts its dispersion relation as  $\mathcal{E}_+ \rightarrow \mathcal{E}_+ - 2\mathbf{g}$ . Similarly the dispersion relation for  $z_-$  is shifted as  $\mathcal{E}_- \rightarrow \mathcal{E}_- + 2\mathbf{g}$  because it corresponds to a doubly occupied site. An  $M|vw$ -string is composed of  $M + 1$   $z_+$ -roots and  $M - 1$   $z_-$ -roots and so its dispersion relation is shifted as  $\mathcal{E}_{M|vw} \rightarrow \mathcal{E}_{M|vw} - 4\mathbf{g}$ , while an  $M|w$ -string does not contain  $z_{\pm}$ -roots and so is unaffected.

Now we can see the equivalence of the models on the level of the Bethe equations. Indeed the two sets corresponding to  $\alpha = \pm 1$  are identical for lattices of even length  $L$ . Moreover shifting  $\mathbf{g} \rightarrow \mathbf{g} - 1$  takes the dispersion relations for the B-model to the dispersion relations for the opposite A-model. Similarly, shifting  $\mathbf{g} \rightarrow \mathbf{g} + 1$  takes the dispersion relations for the opposite B-model to the dispersion relations for the A-model. Thus the spectra of the related models are identical for lattices of even length.

### 7.3 TBA equations

To access the thermodynamics let us consider the free energy

$$f(\mu, B, T) = \int dv(\mathcal{E}_+ - \mu - B - 2\mathfrak{g})\rho_+ + \int dv(\mathcal{E}_- - \mu - B + 2\mathfrak{g})\rho_- \\ + \sum_{M=1}^{\infty} \int dv(\mathcal{E}_{M|vw} - 2M\mu - 4\mathfrak{g})\rho_{M|vw} + \sum_{M=1}^{\infty} \int dv 2MB\rho_{M|w} - Ts,$$

where the dispersion relations for  $M|vw$ -strings are given by

$$\mathcal{E}_{M|vw}^{A_0}(v) = -4\pi\mathfrak{K}_{M+1}(v), \quad \mathcal{E}_{M|vw}^{B_\infty}(v) = 4\pi\mathfrak{K}_{M+1}(v) - 4,$$

with the signs being opposite for the opposite models. The effect of  $\mathfrak{g}$  is to shift the chemical potential, differently for different string types. The canonical TBA equations follow straightforwardly

$$\log \mathcal{Y}_+ = \frac{\mathcal{E}_+ - \mu - B - 2\mathfrak{g}}{T} + \log \frac{1 + \frac{1}{\mathcal{Y}_{N|vw}}}{1 + \frac{1}{\mathcal{Y}_{N|w}}} \star \mathfrak{K}_N, \\ \log \mathcal{Y}_- = \frac{\mathcal{E}_- - \mu - B + 2\mathfrak{g}}{T} + \log \frac{1 + \frac{1}{\mathcal{Y}_{N|vw}}}{1 + \frac{1}{\mathcal{Y}_{N|w}}} \star \mathfrak{K}_N, \\ \log \mathcal{Y}_{M|vw} = \frac{\mathcal{E}_{M|vw} - 2M\mu - 4\mathfrak{g}}{T} + \log \left(1 + \frac{1}{\mathcal{Y}_{N|vw}}\right) \star \mathfrak{K}_{NM} + \log \frac{1 + \frac{1}{\mathcal{Y}_+}}{1 + \frac{1}{\mathcal{Y}_-}} \star \mathfrak{K}_M, \\ \log \mathcal{Y}_{M|w} = \frac{2MB}{T} + \log \left(1 + \frac{1}{\mathcal{Y}_{N|w}}\right) \star \mathfrak{K}_{NM} + \log \frac{1 + \frac{1}{\mathcal{Y}_+}}{1 + \frac{1}{\mathcal{Y}_-}} \star \mathfrak{K}_M. \quad (7.11)$$

Simplifying these equations as in section 4.3 they become

$$\log \mathcal{Y}_+ = \frac{\mathcal{E}_+ - \mathcal{E}_{1|vw} \star \mathfrak{s}}{T} + \log \frac{1 + \mathcal{Y}_{1|vw}}{1 + \mathcal{Y}_{1|w}} \star \mathfrak{s}, \\ \log \mathcal{Y}_- = \frac{\mathcal{E}_- - \mathcal{E}_{1|vw} \star \mathfrak{s} + 4\mathfrak{g}}{T} + \log \frac{1 + \mathcal{Y}_{1|vw}}{1 + \mathcal{Y}_{1|w}} \star \mathfrak{s}, \\ \log \mathcal{Y}_{M|vw} = I_{MN} \log (1 + \mathcal{Y}_{N|vw}) \star \mathfrak{s} + \delta_{M1} \log \frac{1 + \mathcal{Y}_+}{1 + \mathcal{Y}_-} \star \mathfrak{s}, \\ \log \mathcal{Y}_{M|w} = I_{MN} \log (1 + \mathcal{Y}_{N|w}) \star \mathfrak{s} + \delta_{M1} \log \frac{1 + \frac{1}{\mathcal{Y}_+}}{1 + \frac{1}{\mathcal{Y}_-}} \star \mathfrak{s}, \quad (7.12)$$



### 7.3 TBA equations

	A <sub>0</sub>	$\tilde{A}_0$	B <sub>∞</sub>	$\tilde{B}_∞$
$\mathcal{E}_+$	$-4\pi\mathfrak{K}_1$	$4\pi\mathfrak{K}_1$	$4\pi\mathfrak{K}_1 - 2$	$2 - 4\pi\mathfrak{K}_1$
$\mathcal{E}_-$	0	0	2	-2
$\mathcal{E}_{1 vw}$	$-4\pi\mathfrak{K}_2$	$4\pi\mathfrak{K}_2$	$4\pi\mathfrak{K}_2 - 4$	$4 - 4\pi\mathfrak{K}_2$
$\mathcal{E}_+ - \mathcal{E}_{1 vw} \star \mathfrak{s}$	$-4\pi \mathfrak{s}$	$4\pi \mathfrak{s}$	$4\pi \mathfrak{s}$	$-4\pi \mathfrak{s}$
$\mathcal{E}_- - \mathcal{E}_{1 vw} \star \mathfrak{s}$	$4\pi\mathfrak{K}_1 - 4\pi \mathfrak{s}$	$4\pi \mathfrak{s} - 4\pi\mathfrak{K}_1$	$4\pi \mathfrak{s} - 4\pi\mathfrak{K}_1 + 4$	$4\pi\mathfrak{K}_1 - 4\pi \mathfrak{s} - 4$
$(J, c_A)$	$(2, 1)$	$(-2, -1)$	$(-2, 0)$	$(2, 0)$

**Table 7.1:** Driving terms of the TBA equations (7.12), and constants for the free energy.

with the familiar large  $M$  asymptotics

$$\lim_{M \rightarrow \infty} \frac{\log \mathcal{Y}_{M|vw}}{M} = -\frac{2\mu}{T}, \quad \lim_{M \rightarrow \infty} \frac{\log \mathcal{Y}_{M|w}}{M} = \frac{2B}{T}.$$

The driving terms in these equations are simplified and presented explicitly in Table 7.1.

Taking into account that

$$\frac{dp_+}{dv} = -2\pi\mathfrak{K}_1(v), \quad \frac{dp_-}{dv} = 0, \quad \frac{dp_{M|vw}}{dv} = -2\pi\mathfrak{K}_{M+1}(v),$$

one finds that the minimized free energy is

$$f = -T \log \left( 1 + \frac{1}{\mathcal{Y}_+} \right) \star \mathfrak{K}_1 - T \log \left( 1 + \frac{1}{\mathcal{Y}_{M|vw}} \right) \star \mathfrak{K}_{M+1}. \quad (7.13)$$

Computing the infinite sum in the second term as in section 4.3 one gets

$$\begin{aligned} f = & -\mu - 2\mathfrak{g} - 2c_A + J \log 2 \\ & - T \left( \log \left( 1 + \frac{1}{\mathcal{Y}_+} \right) \star \mathfrak{s} + \log \left( 1 + \frac{1}{\mathcal{Y}_-} \right) \star \mathfrak{K}_2 \star \mathfrak{s} + \log \left( 1 + \mathcal{Y}_{1|vw} \right) \star \mathfrak{K}_1 \star \mathfrak{s} \right), \end{aligned} \quad (7.14)$$

where  $J$  and  $c_A$  are given in Table 7.1 and they follow from the identities

$$2\pi \mathfrak{s} \star \mathfrak{K}_1 = \log 2, \quad 2\pi \mathfrak{s} \star \mathfrak{K}_3 = 1 - \log 2. \quad (7.15)$$

The free energy can be also written with dependence on the  $1|w$ -string

$$f = -\mu + J \log 2 - T \left( \log \left( 1 + \mathcal{Y}_+ \right) \star \mathfrak{s} + \log \left( 1 + \mathcal{Y}_- \right) \star \mathfrak{K}_2 \star \mathfrak{s} + \log \left( 1 + \mathcal{Y}_{1|w} \right) \star \mathfrak{K}_1 \star \mathfrak{s} \right).$$

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Since the extended opposite A-model is the EKS model the TBA equations and free energy above describe its equilibrium states in the thermodynamic limit. Our canonical TBA equations look very different from those derived in [63], because we used the BFFB grading for the Bethe equations while [63] used the BBFF grading. The sets of equations nevertheless are equivalent. Identifying the functions  $\alpha, \beta, \gamma$  used there with our Y-functions as

$$\alpha_1 = 1/\mathcal{Y}_+, \quad \beta_2 = \mathcal{Y}_-, \quad \alpha_{M+1} = \mathcal{Y}_{M|w}, \quad \beta_1 = 1/\mathcal{Y}_{1|vw}, \quad \gamma_{M-1} = \mathcal{Y}_{M|vw}, \quad (7.16)$$

we find that our simplified equations for  $\mathcal{Y}_+, \mathcal{Y}_{M|w}, M \geq 1$  and  $\mathcal{Y}_{M|vw}, M \geq 2$  agree with the equations (3.13) from [63], and our equation for  $\mathcal{Y}_+/\mathcal{Y}_-$  agrees with their eq. (3.10) combined with eq. (3.8) for  $n = 1$ . Finally the canonical equation for  $\mathcal{Y}_{1|vw}$  follows from the combination  $K_1 \star \log(1 + \alpha_1) + (\delta + K_2) \star \log \beta_1$  and their eqs. (3.8, 3.9).

Let us also mention that the simplified equations for  $\mathcal{Y}_-$  and  $\mathcal{Y}_{1|vw}$  are presented in [67] in eq. (5.7), and we match our equation for  $\mathcal{Y}_{1|vw}$  with theirs. The equation for  $\mathcal{Y}_-$  however in [67] does not contain the crucial coupling dependent term  $4\mathfrak{g}/T$ .

### Strong coupling limit

There are two natural ways to take the large  $\mathfrak{g}$  limit of the extended model TBA equations (7.11). Taking the strict  $\mathfrak{g} \rightarrow \infty$  limit with all other parameters fixed will lead to a half-filled model, due to the dominance of the Hubbard term favouring singly occupied sites.

Alternatively one can take  $\mu \rightarrow -\infty$  simultaneously with  $\mathfrak{g} \rightarrow \infty$  and retain the possibility for empty sites in the Hilbert space. In particular, if one redefines the chemical potential as  $\tilde{\mu}$  through

$$\tilde{\mu} = \mu + 2\mathfrak{g}, \quad (7.17)$$

the effect is that the models have been extended with the term  $4\mathfrak{g} \sum_{j=1}^L \mathbf{n}_{j,\uparrow} \mathbf{n}_{j,\downarrow}$ , rather than the term  $4\mathfrak{g} \mathbf{V}^H$ . Then the extension does not give a cost to empty sites. Let us proceed with the discussion of this limit with the redefinition of the chemical potential and return afterwards to the strict  $\mathfrak{g} \rightarrow \infty$  limit. Note



then that  $\mathcal{Y}_- \rightarrow \infty$ , and  $\mathcal{Y}_{M|vw} \rightarrow \infty$  for  $M \geq 2$ , as  $\mathfrak{g} \rightarrow \infty$  with  $\tilde{\mu}$  fixed. The equations (7.11) thus take the following simplified form

$$\begin{aligned}\log \mathcal{Y}_+ &= \frac{\mathcal{E}_+ - \mathcal{E}_{1|vw} \star \mathfrak{s}}{T} + \log \frac{1 + \mathcal{Y}_{1|vw}}{1 + \mathcal{Y}_{1|w}} \star \mathfrak{s}, \\ \log \mathcal{Y}_{1|vw} &= \frac{\mathcal{E}_{1|vw} - 2\tilde{\mu}}{T} + \log \left(1 + \frac{1}{\mathcal{Y}_{1|vw}}\right) \star \mathfrak{K}_2 + \log \left(1 + \frac{1}{\mathcal{Y}_+}\right) \star \mathfrak{K}_1, \\ \log \mathcal{Y}_{M|w} &= I_{MN} \log (1 + \mathcal{Y}_{N|w}) \star \mathfrak{s} + \delta_{M1} \log \left(1 + \frac{1}{\mathcal{Y}_+}\right) \star \mathfrak{s},\end{aligned}$$

which are the TBA equations of the supersymmetric  $t$ - $J$  model [65]. The expression for free energy (7.13) simplifies to

$$f = -T \log \left(1 + \frac{1}{\mathcal{Y}_+}\right) \star \mathfrak{K}_1 - T \log \left(1 + \frac{1}{\mathcal{Y}_{1|vw}}\right) \star \mathfrak{K}_2. \quad (7.18)$$

Now consider the  $\mathfrak{g} \rightarrow \infty$  limit with  $\mu$  fixed. From the TBA equations (7.11) and (7.12) note that  $\mathcal{Y}_- \rightarrow \infty$  and  $\mathcal{Y}_{1|vw} \rightarrow 0$ , and thus the simplified equations become

$$\begin{aligned}\log \mathcal{Y}_+ &= \frac{\mathcal{E}_+ - \mathcal{E}_{1|vw} \star \mathfrak{s}}{T} - \log (1 + \mathcal{Y}_{1|w}) \star \mathfrak{s}, \\ \log \mathcal{Y}_{M|w} &= I_{MN} \log (1 + \mathcal{Y}_{N|w}) \star \mathfrak{s} + \delta_{M1} \log \left(1 + \frac{1}{\mathcal{Y}_+}\right) \star \mathfrak{s},\end{aligned}$$

where  $\mathcal{Y}_{M|vw}$  have decoupled. These can be simplified further by relabelling:  $\mathcal{Y}_1 = 1/\mathcal{Y}_+$  and  $\mathcal{Y}_M = \mathcal{Y}_{M-1|w}$  for  $M \geq 2$ . Then the Y-functions form one set

$$\log \mathcal{Y}_M = I_{MN} \log (1 + \mathcal{Y}_N) \star \mathfrak{s} + \delta_{M1} \frac{2\pi J \mathfrak{s}}{T},$$

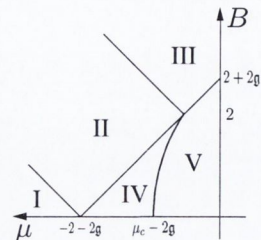
and the free energy is given by

$$f = -\mu - 2\mathfrak{g} - 2c_A + J \log 2 - T \log (1 + \mathcal{Y}_1) \star \mathfrak{s}. \quad (7.19)$$

This is none other than the set of TBA equations for the XXX-spin chain (4.56). Comparing the values of  $J$  from Table 7.1 it follows that the  $g \rightarrow \infty$  limit of the A-model and opposite B-model is a ferromagnetic spin chain, while the B-model and opposite A-model become the antiferromagnetic spin chain.

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**Figure 7.1:** Extended B-model: Zero temperature phase diagram in the  $\mu B$ -plane. The phases identified are: I) empty band, II) partially filled and spin polarised band, III) half-filled and spin-polarised band, IV) partially filled and partially spin polarised band, V) half-filled and partially spin polarised band. Here  $\mathfrak{g}$  parametrises the strength of the Hubbard interaction and  $\mu_c = -2 + 2 \log 2 \approx -0.6137$ .



### 7.4 Ground state phase diagram

Now we examine the ground state phase diagram. As the models are equivalent we choose to focus on the extended B-model. This allows us to understand the effect of varying the strength of the Hubbard interaction for the B-model. The result is presented in Figure 7.1.

Let us break the analysis into two parts. First we consider the case  $\mathfrak{g} > 0$ . As in section 4.4.1 there are no  $M|vw$ - or  $M|w$ -strings for  $M \geq 2$  in the ground state at zero temperature. There are also no  $1|w$ -strings as  $\mathcal{Y}_+ < \mathcal{Y}_-$  either, and from this there are no  $z_-$ -particles either. The TBA equations for the dressed energies are then

$$\begin{aligned} \epsilon_+ &= \mathcal{E}_+^{\text{B}\infty} - 2\mathfrak{g} - \mu - B - \epsilon_{1|vw} \star_{Q_{1|vw}} \mathfrak{K}_1, \\ \epsilon_{1|vw} &= \mathcal{E}_{1|vw}^{\text{B}\infty} - 4\mathfrak{g} - 2\mu - \epsilon_+ \star_{Q_y} \mathfrak{K}_1 - \epsilon_{1|vw} \star_{Q_{1|vw}} \mathfrak{K}_2, \end{aligned} \quad (7.20)$$

which differ from equations (4.51) only by a shift of  $\mu$  by  $2\mathfrak{g}$ . Thus the phase diagram will be that of the strong coupling limit of the B-model with  $\mu$  shifted as  $\mu \rightarrow \mu - 2\mathfrak{g}$ . The remaining issue then is how phases III and V are separated for the range  $\mu \in (-2\mathfrak{g}, 0)$ . Recall that the half-filled phase is determined by the condition that  $\epsilon_{1|vw}(v) \leq 0$  for all  $v \in \mathbb{R}$ , and the transition to the half-filled spin polarised phase is when  $\epsilon_+(v) \leq 0$  and  $\epsilon_{1|vw}(v) = 0$  for all  $v \in \mathbb{R}$ . From the large  $v$  asymptotics of (7.20) it follows that the phase boundary is the line  $B = \mu + 2 + 2\mathfrak{g}$ .

Now consider  $\mathfrak{g} < 0$ . Here one can no longer say that  $\mathcal{Y}_+(v) < \mathcal{Y}_-(v)$  for all  $v$  and so there is the possibility of  $1|w$ -strings and  $z_-$ -particles entering the ground state. Recall however that there can be no  $w$ -particles, and consequently no  $z_-$ -particles, unless there are  $z_+$ -particles. If however we assume that  $\epsilon_+ < 0$  we find this puts restrictions on  $\mu$  and  $B$  that result in  $\epsilon_{1|w} > 0$  and  $\epsilon_- > 0$ . Thus



## 7.4 Ground state phase diagram

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equations (7.20) govern the dressed energies for  $\mathfrak{g} < 0$  also, and again the phase diagram is that of the strong coupling limit of the B-model shifted by  $\mu \rightarrow \mu - 2\mathfrak{g}$ . Let us remark that phase V will disappear when  $\mathfrak{g} = -1 + \log 2$  and that phase IV will disappear when  $\mathfrak{g} = -1$ .

The phase diagram for the EKS model (extended opposite A-model) is obtained from the one for the extended B-model by shifting the coupling  $\mathfrak{g}$  above by 1, see (7.10). For  $B = 0$  our results agree with [27]. The corresponding phase diagrams for the extended A- and opposite B-models are obtained by using the relationships between opposite models discussed in section 4.5.

We will not examine the zero temperature excitations because the above analysis shows that they can be naturally inferred from those in section 6.3. A study for the EKS model at  $B = 0$  was performed in [63] and we compare our findings to theirs at the end of section 6.3.

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# Chapter 8

## Outlook

In this thesis we have begun the study of a new family of exactly solvable models for strongly correlated electrons: the Hubbard-Shastry models. Given the wealth of work that has been done for the Hubbard model, there are many directions for future studies. Let us discuss some of those which arise naturally from the investigations carried out in this thesis.

To better understand the physical properties of the models, it would be desirable to study the correlations they exhibit. The investigation of finite size corrections at zero temperature would be a first step in this direction, see for example [9, 20]. This however is not a straightforward task.

In this thesis we have accessed the thermodynamics of the models through the Thermodynamic Bethe Ansatz (TBA). While this approach provides a convenient and physically attractive description of the thermodynamics, it does have some drawbacks. For non-zero temperature, the TBA equations involve an infinite number of unknown functions and so it is a non-trivial problem to obtain and control solutions. Even more troubling however, the approach relies on a string hypothesis for the behaviour of Bethe roots in the limit of large lattice length  $L$ , and in general this cannot be proven. Moreover it is known for some models that the string configurations receive corrections of order  $1/L$  [68]. These corrections are to the imaginary parts of the Bethe roots, and do not cause problems at order  $1/L$  since they naturally cancel for all quantities of interest. They do however cause serious difficulties for the study of effects which appear at order  $1/L^2$  [69].

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In particular, they hinder the investigation of finite size corrections for models which contain Bethe strings in their ground state.

There is another approach to the thermodynamics of integrable lattice models that overcomes the above difficulties. It is based on an observation that the free energy per site is given by the largest eigenvalue of the Quantum Transfer Matrix (QTM). The diagonalization of the QTM is achieved by the algebraic Bethe ansatz and is used to derive a finite set of non-linear integral equations which determine the free energy. Moreover the QTM approach naturally encodes the finite size behaviour of Bethe roots, and allows direct access to finite size properties [70]. The QTM approach was carried out for the Hubbard model in [71, 72, 9], and the physical quantities computed by solving numerically the finite set of non-linear integral equations were shown to agree with those of the TBA with high precision [53]. For some models, see [67] and reference therein, one can set up a fusion hierarchy with the QTM and derive the TBA equations, and thus prove the equivalence of the two approaches. It would clearly be useful apply the QTM approach to the Hubbard-Shastry models.

Another worthwhile direction would be to investigate the models numerically. Beyond providing an important check of the exact results, these would allow one to examine the models as one moves away from integrability. In this regard let us mention the Density Matrix Renormalisation Group method [73] and related variational techniques, see the reviews [74, 75]. These provide efficient methods for simulating one-dimensional lattice models with local Hamiltonian densities, and would allow for a different perspective on the physics of the models.

Indeed, it would be interesting to examine effect of breaking the integrability of the models. In section 1.3 of the introduction we considered the effect of decoupling the interactions of the models, and numerical simulations would allow much scope to study this. Moreover, it would be highly desirable to investigate the behaviour models in two-dimensions. This represents a huge challenge however, due to a lack of efficient methods available here. Indeed even though the Hubbard model has received much study over many years, there is as yet no firm consensus on its properties in two dimensions.



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In addition to the further investigation of the models we examine in this thesis, there are also related sets of models that may be worthy of study. For example, it would be of interest to analyse the general parity-breaking Hubbard-Shastry models. The formalism that we have developed can straightforwardly be adapted, and in our general considerations we have taken care to mention wherever parity invariance is assumed.

Another interesting direction is provided by a set of integrable models constructed by Alcaraz and Bariev [36]. Recently it was realised that these can be constructed from a quantum deformation of Shastry's R-matrix [37], and so provide anisotropic deformations of the symmetries of the Hubbard-Shastry models. From a physics point of view, this corresponds to a freedom to introduce an effective spin-orbit coupling to the models. There has been no study of the properties of these model as of yet, but some progress has been made on formulation of a string hypothesis [76, 77].

Finally, let us consider models for the Kondo lattice [78, 79, 80]. Here Kondo does not refer to the geometry of the lattice, but instead indicates that the Hilbert space at each site is identical to that of the impurity site of the Kondo chain. In addition to describing a single band of electrons, there is a localised spin included at each site, and so the number of states is doubled to 8. This can be conveniently represented by assigning a spin to the pseudo-vacuum at each site, e.g.  $|\uparrow\rangle, |\downarrow\rangle$ . Kondo lattice models are relevant for the study of heavy fermion systems, where the localised spin arises from an effective description of a two-band theory.

Integrable models for the Kondo lattice arise in a natural way within the framework we have presented for the Hubbard-Shastry models. They can be realised on the 8-dimensional atypical representation of  $\mathfrak{su}(2|2)_c$ , whereas the Hubbard-Shastry models were realised on the fundamental four dimensional representation. The R-matrix for the 8-dimensional representation was constructed in [81], and it can be used to generate integrable models through the quantum inverse scattering method as outlined in section 2.5. The symmetries of the 8-dimensional representation match exactly the symmetries of the 8 states of the Kondo lattice model, and so it would be interesting to investigate the physics that these models exhibit.

8. OUTLOOK

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# Appendix A

## Conventions, definitions and notations

### Matching the notations and conventions of the Bethe ansatz analysis

Our notations and conventions come from [58], and here we compare them to those of [9].

In the Bethe ansatz we denote particles momenta as  $p_j$  and auxiliary roots as  $w_j$ , so they are related to the ones in [9] as  $p_j \leftrightarrow k_j$ ,  $w_j \leftrightarrow \Lambda_j$ .

In the string hypothesis a  $M|w$ -string is a  $\Lambda$  string of length  $M$ , a  $w$ -particle is a  $\Lambda$ -string of length 1, a  $M|vw$ -string is a  $k$ - $\Lambda$  string of length  $M$ , and  $y$ -particles could have been called  $k$ -particles.

In the TBA equations the  $Y$ -functions are related to the ones in [9] as  $Y_{M|w} \leftrightarrow \eta_M$ ,  $Y_{M|vw} \leftrightarrow \eta'_M$ ,  $Y_-(\sin(k)) \leftrightarrow \zeta(k)$ ,  $|k| \leq \pi/2$  and  $Y_+(\sin(k)) \leftrightarrow \zeta(k)$ ,  $|k| \geq \pi/2$ .

### Convolutions

The symbol  $\star$  denotes the following “convolution”

$$g \star h \equiv \int_{-\infty}^{\infty} dt g(u, t) h(t, v), \quad (\text{A.1})$$

If  $g$  (or  $h$ ) is a function of a single variable then one just drops  $u$  (or  $v$  or both), e.g. if  $g = g(t)$  then  $g \star h \equiv \int_{-\infty}^{\infty} dt g(t) h(t, v)$ . However, if  $g$  or  $h$  is a kernel defined through a function of one variable then it should be understood as  $g(t, v) \equiv g(t - v)$ .

## A. CONVENTIONS, DEFINITIONS AND NOTATIONS

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When we meet expressions of the form  $g_{\alpha\beta} \star h_{\beta\gamma}$ , the convolution is understood to be over the domain of the rapidity of the  $\beta$ -string. The  $y$ -particles of the Hubbard-Shastry models have a non-trivial domain of rapidity and so we introduce some special notations for them. The symbol  $\circledast$  is used to denote a contour integral in the counter-clockwise direction around the branch cuts of  $x_A(v)$  and  $x_B(v)$ . Explicitly, for the Hubbard and A-models one has

$$g_{\alpha y} \circledast h_{y\gamma} = \int_{|t| \leq 1} dt (g_{\alpha-}(u, t) h_{-\gamma}(t, v) - g_{\alpha+}(u, t) h_{+\gamma}(t, v)) = g_{\alpha-} \hat{\star} h_{-\gamma} - g_{\alpha+} \hat{\star} h_{+\gamma},$$

while for the B-model

$$g_{\alpha y} \circledast h_{y\gamma} = \int_{|t| \geq 1} dt (g_{\alpha-}(u, t) h_{-\gamma}(t, v) - g_{\alpha+}(u, t) h_{+\gamma}(t, v)) = g_{\alpha-} \check{\star} h_{-\gamma} - g_{\alpha+} \check{\star} h_{+\gamma},$$

where

$$g_{\alpha\pm}(u, t) \equiv g_{\alpha y}(u, t \pm i0), \quad h_{\pm\gamma}(t, v) \equiv h_{y\gamma}(t \pm i0, v),$$

and  $\hat{\star}$  and  $\check{\star}$  denote convolutions with the integration over  $|t| \leq 1$  and  $|t| \geq 1$  respectively. For generality we use the convolution  $\bar{\star}$  defined as

$$\begin{aligned} g \bar{\star} h &= g \hat{\star} h \quad \text{for the Hubbard and A - models,} \\ g \bar{\star} h &= g \check{\star} h \quad \text{for the B - model.} \end{aligned} \tag{A.2}$$

### Kernels and S-matrices

The Bethe equations and TBA equations of the Hubbard-Shastry models involve several S-matrices and kernels which we list here

$$K_M(v) = \frac{1}{2\pi i} \frac{d}{dv} \log S_M(v) = \frac{1}{\pi} \frac{\mathbf{u} M}{v^2 + \mathbf{u}^2 M^2}, \quad S_M(v) = \frac{v - i \mathbf{u} M}{v + i \mathbf{u} M}, \tag{A.3}$$

$$K_{MN}(v) = \frac{1}{2\pi i} \frac{d}{dv} \log S_{MN}(v) = K_{M+N}(v) + K_{N-M}(v) + 2 \sum_{j=1}^{M-1} K_{N-M+2j}(v), \tag{A.4}$$

$$S_{MN}(v) = S_{M+N}(v) S_{N-M}(v) \prod_{j=1}^{M-1} S_{N-M+2j}(v)^2 = S_{NM}(v), \tag{A.5}$$

$$s(v) = \frac{1}{2\pi i} \frac{d}{dv} \log S(v) = \frac{1}{4\mathbf{u} \cosh \frac{\pi v}{2\mathbf{u}}}, \quad S(v) = -\tanh\left(\frac{\pi v}{4\mathbf{u}} - \frac{i\pi}{4}\right), \tag{A.6}$$



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Also useful are the functions

$$\begin{aligned}\Theta_M(v) &= 2 \arctan\left(\frac{v}{uM}\right), \\ \Theta_{MN}(v) &= \Theta_{M+N}(v) + \Theta_{N-M}(v) + 2 \sum_{j=1}^{M-1} \Theta_{N-M+2j}(v).\end{aligned}\tag{A.7}$$

Some useful identities involving kernels

$$\begin{aligned}1 \star K_M &= 1, \quad K_M \star K_N = K_{M+N}, \quad 1 \star s = \frac{1}{2}, \\ K_1 - s \star K_2 &= s, \quad K_{M+1} - s \star K_M - s \star K_{M+2} = 0.\end{aligned}\tag{A.8}$$

$$\frac{dp_y}{dv} \otimes K_M = -\frac{dp_{M|vw}}{dv}, \quad \mathcal{E}_y \otimes K_M = -\mathcal{E}_{M|vw}.\tag{A.9}$$

$$\pi \text{sign} \star K_M = \Theta_M.\tag{A.10}$$

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# Appendix B

## Graded vector spaces

In this thesis adopt the conventions for graded vector spaces given in [29, 9]. Let us summarise them in this appendix.

A graded vector space can be decomposed as  $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$  and is equipped with the structure of parity. The subspaces  $\mathcal{V}_i$  are called homogeneous and we call  $\vec{v}_0 \in \mathcal{V}_0$  even and  $\vec{v}_1 \in \mathcal{V}_1$  odd. Parity is a function  $\epsilon : \mathcal{V}_i \rightarrow \mathbb{Z}_2$  defined on the homogeneous components of  $\mathcal{V}$ :

$$\epsilon(\vec{v}_i) = i, \quad i = 0, 1 \text{ for } \vec{v}_i \in \mathcal{V}_i. \quad (\text{B.1})$$

Thus if  $\dim \mathcal{V}_0 = m$  and  $\dim \mathcal{V}_1 = n$ , a basis  $\{e_1, \dots, e_m, e_{m+1}, \dots, e_{m+n}\}$  for  $\mathcal{V}$  can be chosen such that the first  $m$  elements are even and the remaining  $n$  elements are odd. We define  $\epsilon_\alpha = \epsilon(e_\alpha)$ .

The definition of parity can be extended to elements in  $\text{End}(\mathcal{V})$ . First we  $\mathbf{e}_\alpha^\beta \in \text{End}(\mathcal{V})$  such that  $\mathbf{e}_\alpha^\beta e_\gamma = \delta_\gamma^\beta e_\alpha$ . Then the set  $\{\mathbf{e}_\alpha^\beta : \alpha, \beta = 1, \dots, m+n\}$  is a basis of  $\text{End}(\mathcal{V})$  and we define  $\epsilon(\mathbf{e}_\alpha^\beta) = \epsilon_\alpha + \epsilon_\beta$ . An element  $\mathbf{A} = A_\beta^\alpha \mathbf{e}_\alpha^\beta \in \text{End}(\mathcal{V})$  is called homogeneous with some parity  $\epsilon_A = \epsilon(\mathbf{A})$  if and only if

$$(-1)^{\epsilon_\alpha + \epsilon_\beta} A_\beta^\alpha = (-1)^{\epsilon_A} A_\beta^\alpha \quad \text{for all } \alpha, \beta. \quad (\text{B.2})$$

Of particular interest is  $\text{End}(\mathcal{H}) = \text{End}(\mathcal{V}^{\otimes L})$ . This is the space of operators acting on the Hilbert space  $\mathcal{H}$ . A seemingly natural basis for this space is  $\{\mathbf{e}_{\alpha_1}^{\beta_1} \otimes \dots \otimes \mathbf{e}_{\alpha_L}^{\beta_L} : \alpha_i, \beta_i = 1, \dots, m+n \text{ for } i = 1, \dots, L\}$ , but taking account of the

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operators grading with respect to such a basis is however non-trivial. To do this it is we define the graded local projection operators:

$$\mathbf{E}_{j\alpha}^\beta = (-1)^{(\epsilon_\alpha + \epsilon_\beta) \sum_{i=1}^{j-1} \epsilon_{\alpha_i}} \mathbf{e}_{\alpha_1}^{\alpha_1} \otimes \dots \otimes \mathbf{e}_{\alpha_{j-1}}^{\alpha_{j-1}} \otimes \mathbf{e}_\alpha^\beta \otimes \mathbf{e}_{\alpha_{j+1}}^{\alpha_{j+1}} \otimes \dots \otimes \mathbf{e}_{\alpha_L}^{\alpha_L} \quad (\text{B.3})$$

These obey the following properties which follow straight from their definition

$$\begin{aligned} \epsilon(\mathbf{E}_{j\alpha}^\beta) &= \epsilon_\alpha + \epsilon_\beta, \\ \mathbf{E}_{j\alpha}^\beta \mathbf{E}_{j\gamma}^\delta &= \delta_\gamma^\beta \mathbf{E}_{j\alpha}^\delta, \\ \mathbf{E}_{j\alpha}^\beta \mathbf{E}_{k\gamma}^\delta &= (-1)^{(\epsilon_\alpha + \epsilon_\beta)(\epsilon_\gamma + \epsilon_\delta)} \mathbf{E}_{k\gamma}^\delta \mathbf{E}_{j\alpha}^\beta. \end{aligned} \quad (\text{B.4})$$

We represent a homogeneous operator  $\mathbf{A} \in \text{End}(\mathcal{V})$  acting on the  $j$ -th site of  $\mathcal{H}$  as  $\mathbf{A}_j = A_\beta^\alpha \mathbf{E}_{j\alpha}^\beta$ . We call an operator  $\mathbf{B} = B_{\beta\delta}^{\alpha\gamma} \mathbf{e}_\alpha^\beta \otimes \mathbf{e}_\gamma^\delta \in \text{End}(\mathcal{V} \otimes \mathcal{V})$  homogeneous if

$$(-1)^{\epsilon_\alpha + \epsilon_\beta + \epsilon_\gamma + \epsilon_\delta} B_{\beta\delta}^{\alpha\gamma} = (-1)^{\epsilon_B} B_{\beta\delta}^{\alpha\gamma} \quad \text{for all } \alpha, \beta, \gamma, \delta, \quad (\text{B.5})$$

and when acting on sites of  $\mathcal{H}$  we write it as

$$\mathbf{B}_{ij} = (-1)^{\epsilon_\beta(\epsilon_\gamma + \epsilon_\delta)} B_{\beta\delta}^{\alpha\gamma} \mathbf{E}_{i\alpha}^\beta \mathbf{E}_{j\gamma}^\delta. \quad (\text{B.6})$$

In particular the graded permutation operator is given by  $\mathbf{P} = (-1)^{\epsilon_\alpha \epsilon_\gamma} \delta_\delta^\alpha \delta_\beta^\gamma \mathbf{e}_\alpha^\beta \otimes \mathbf{e}_\gamma^\delta$  and takes the form

$$\mathbf{P}_{ij} = (-1)^{\epsilon_\alpha} \mathbf{E}_{i\beta}^\alpha \mathbf{E}_{j\alpha}^\beta. \quad (\text{B.7})$$

This satisfies the following identities

$$\mathbf{P}_{ij} = \mathbf{P}_{ji}, \quad (\text{B.8})$$

$$\mathbf{P}_{jj} = \dim(\mathcal{V}) \cdot \text{id}, \quad (\text{B.9})$$

$$\mathbf{P}_{ij}^2 = \text{id}, \quad j \neq k, \quad (\text{B.10})$$

$$\mathbf{P}_{ij} \mathbf{E}_{j\alpha}^\beta = \mathbf{E}_{i\alpha}^\beta \mathbf{P}_{ij}, \quad (\text{B.11})$$

$$\mathbf{P}_{ij} \mathbf{E}_{k\alpha}^\beta = \mathbf{E}_{k\alpha}^\beta \mathbf{P}_{ij}, \quad l \neq j, k. \quad (\text{B.12})$$

Finally the super trace is defined as

$$\text{str}(\mathbf{A}) = (-1)^{\epsilon_\alpha} A_\alpha^\alpha. \quad (\text{B.13})$$

This is cyclic for graded operators, just as the trace is for ungraded operators.



# Appendix C

## Shastry's R-matrix

Shastry's R-matrix underlies the integrability of the Hubbard-Shastry models. Originally it was constructed to as the R-matrix for the Hubbard model [14], but was later found to be equivalent to the  $\mathfrak{su}(2|2)_C$  invariant R-matrix [41, 31]. In this appendix we present the  $\mathfrak{su}(2|2)_C$  symmetry algebra and give the explicit expression for Shastry's R-matrix that we use to construct the Hubbard-Shastry models.

### The $\mathfrak{su}(2|2)_C$ algebra

Following the conventions<sup>1</sup> of [42], the centrally extended  $\mathfrak{su}(2|2)$  algebra, written concisely as  $\mathfrak{su}(2|2)_C$ , is generated by two sets of (bosonic)  $\mathfrak{su}(2)$  rotation generators  $\mathbf{L}_a^b$ ,  $\mathbf{R}_\alpha^\beta$ , supersymmetry generators  $\mathbf{Q}_\alpha^a$ ,  $\mathbf{Q}_a^{\dagger\alpha}$ , and three central elements  $\mathbf{H}$ ,  $\mathbf{C}$  and  $\mathbf{C}^\dagger$  subject to the following relations

$$\begin{aligned}
 [\mathbf{L}_a^b, \mathbf{J}_c] &= \delta_c^b \mathbf{J}_a - \frac{1}{2} \delta_a^b \mathbf{J}_c, & [\mathbf{R}_\alpha^\beta, \mathbf{J}_\gamma] &= \delta_\gamma^\beta \mathbf{J}_\alpha - \frac{1}{2} \delta_\alpha^\beta \mathbf{J}_\gamma, \\
 [\mathbf{L}_a^b, \mathbf{J}^c] &= -\delta_a^c \mathbf{J}^b + \frac{1}{2} \delta_a^b \mathbf{J}^c, & [\mathbf{R}_\alpha^\beta, \mathbf{J}^\gamma] &= -\delta_\alpha^\gamma \mathbf{J}^\beta + \frac{1}{2} \delta_\alpha^\beta \mathbf{J}^\gamma, \\
 \{\mathbf{Q}_\alpha^a, \mathbf{Q}_b^{\dagger\beta}\} &= \delta_b^a \mathbf{R}_\alpha^\beta + \delta_\alpha^\beta \mathbf{L}_b^a + \frac{1}{2} \delta_b^a \delta_\alpha^\beta \mathbf{H}, \\
 \{\mathbf{Q}_\alpha^a, \mathbf{Q}_\beta^b\} &= \epsilon_{\alpha\beta} \epsilon^{ab} \mathbf{C}, & \{\mathbf{Q}_a^{\dagger\alpha}, \mathbf{Q}_b^{\dagger\beta}\} &= \epsilon_{ab} \epsilon^{\alpha\beta} \mathbf{C}^\dagger.
 \end{aligned} \tag{C.1}$$

Here  $\mathbf{J}$  represents any generator carrying an index. The first two lines indicate how the indices  $c$  and  $\gamma$  of a generator transform under the action of  $\mathbf{L}_a^b$  and  $\mathbf{R}_\alpha^\beta$ .

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<sup>1</sup>Their coupling constant  $g$  is related to our  $u$  through  $u = \frac{1}{2g}$ .

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Both the latin and greek indices take two values. The generators  $\mathbf{C}$  and  $\mathbf{C}^\dagger$  are called the centrally extended elements as the algebra reduces to that of  $\mathfrak{su}(2|2)$  when  $\mathbf{C} = \mathbf{C}^\dagger = 0$ . Let us remark that the freedom to introduce the  $\mathbf{C}$  and  $\mathbf{C}^\dagger$  for  $\mathfrak{su}(m|n)$  is special to the case  $m = n = 2$  as only here are the anti-symmetric tensors  $\epsilon_{ab}$  and  $\epsilon_{\alpha\beta}$  defined. In the classification the Lie superalgebras  $\mathfrak{su}(2|2)_C$  can be found as the limit  $\alpha \rightarrow 0$  of the exceptional algebra  $\mathfrak{d}(2, 1; \alpha)$  [82].

The algebra  $\mathfrak{su}(2|2)_C$  has a rich representation theory. Let us describe the fundamental four dimensional unitary representations that we will identify with  $\mathcal{V}$  for the lattice model. As the generators  $\mathbf{H}$ ,  $\mathbf{C}$ , and  $\mathbf{C}^\dagger$  are central they have well defined eigenvalues  $H$ ,  $C$ ,  $\bar{C}$ , and these will parametrise the representations. Introducing a four dimensional basis  $\{|e_1\rangle, |e_2\rangle, |e_3\rangle, |e_4\rangle\}$  the generators can be realised as

$$\begin{aligned}
 \mathbf{L}_a^b |e_c\rangle &= \delta_c^b |e_a\rangle - \frac{1}{2} \delta_a^b |e_c\rangle & \mathbf{R}_{\alpha}^{\beta} |e_a\rangle &= 0 \\
 \mathbf{L}_a^b |e_\alpha\rangle &= 0 & \mathbf{R}_{\alpha}^{\beta} |e_\gamma\rangle &= \delta_\gamma^\beta |e_\alpha\rangle - \frac{1}{2} \delta_\alpha^\beta |e_\gamma\rangle \\
 \mathbf{Q}_\alpha^a |e_b\rangle &= a \delta_a^b |e_\alpha\rangle & \mathbf{Q}_a^\dagger |e_b\rangle &= c \epsilon_{ab} \epsilon^{\alpha\beta} |e_\beta\rangle \\
 \mathbf{Q}_\alpha^a |e_\beta\rangle &= b \epsilon_{\alpha\beta} \epsilon^{ab} |e_b\rangle & \mathbf{Q}_a^\dagger |e_\beta\rangle &= d \delta_\beta^\alpha |e_a\rangle,
 \end{aligned} \tag{C.2}$$

where  $a$ ,  $b$ ,  $c$  and  $d$  are complex numbers constrained by (C.1) to satisfy

$$ad - bc = 1, \tag{C.3}$$

and for unitary representations to further satisfy  $d^* = a$  and  $c^* = b$ . The eigenvalues of the central elements are

$$H = ad + bc, \quad C = ab, \quad \bar{C} = cd, \tag{C.4}$$

and so for unitary representations  $H$  is real and  $C$  and  $\bar{C}$  are complex conjugate. From eqs. (C.3), (C.4) the central charges are constrained through

$$H^2 - 4C\bar{C} = 1. \tag{C.5}$$

A convenient parametrization of  $a$ ,  $b$ ,  $c$  and  $d$  is given by

$$a = \frac{\eta}{2\sqrt{u}}, \quad b = \frac{i\zeta}{2\eta\sqrt{u}} \left( \frac{x^+}{x^-} - 1 \right), \quad c = -\frac{\eta}{2\zeta x^+ \sqrt{u}}, \quad d = \frac{x^+}{2i\eta\sqrt{u}} \left( 1 - \frac{x^-}{x^+} \right).$$



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where the parameter  $\eta$  reflects a freedom in the choice of basis vectors  $\{|e_a\rangle, |e_\alpha\rangle\}$  and the parameters  $x^\pm$  satisfy the constraint

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = 4i\mathbf{u} \quad (\text{C.6})$$

which follows from eq. (C.3). The parameter  $\zeta$  is a phase for unitary representations. In this parametrisation the central charges take the form

$$\begin{aligned} H &= 1 + \frac{i}{2\mathbf{u}} \left( \frac{1}{x^+} - \frac{1}{x^-} \right) = \frac{i}{2\mathbf{u}} (x^- - x^+) - 1, \\ C &= \frac{i\zeta}{4\mathbf{u}} \left( \frac{x^+}{x^-} - 1 \right), \quad \bar{C} = \frac{1}{4i\zeta\mathbf{u}} \left( \frac{x^-}{x^+} - 1 \right). \end{aligned} \quad (\text{C.7})$$

### The $\mathfrak{su}(2|2)_C$ invariant R-matrix

Invariance of an R-matrix under the action of the  $\mathfrak{su}(2|2)_C$  completely fixes it up to an overall scalar factor. We refer the reader to [31, 41, 42] for details and here we present the resulting R-matrix. Following [42] we write the R-matrix as

$$\mathbf{R}(x_1^\pm, x_2^\pm) = \sum_{k=1}^{10} a_k \mathbf{\Lambda}_k, \quad (\text{C.8})$$

where  $\mathbf{\Lambda}_1, \dots, \mathbf{\Lambda}_{10}$  form a basis of  $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$  invariant matrices acting in the tensor product  $\mathcal{V}(x_1^\pm) \otimes \mathcal{V}(x_2^\pm)$

$$\mathbf{\Lambda}_1 = \mathbf{e}_{1111} + \frac{1}{2}\mathbf{e}_{1122} + \frac{1}{2}\mathbf{e}_{1221} + \frac{1}{2}\mathbf{e}_{2112} + \frac{1}{2}\mathbf{e}_{2211} + \mathbf{e}_{2222},$$

$$\mathbf{\Lambda}_2 = \frac{1}{2}\mathbf{e}_{1122} - \frac{1}{2}\mathbf{e}_{1221} - \frac{1}{2}\mathbf{e}_{2112} + \frac{1}{2}\mathbf{e}_{2211},$$

$$\mathbf{\Lambda}_3 = \mathbf{e}_{3333} + \frac{1}{2}\mathbf{e}_{3344} + \frac{1}{2}\mathbf{e}_{3443} + \frac{1}{2}\mathbf{e}_{4334} + \frac{1}{2}\mathbf{e}_{4433} + \mathbf{e}_{4444},$$

$$\mathbf{\Lambda}_4 = \frac{1}{2}\mathbf{e}_{3344} - \frac{1}{2}\mathbf{e}_{3443} - \frac{1}{2}\mathbf{e}_{4334} + \frac{1}{2}\mathbf{e}_{4433},$$

$$\mathbf{\Lambda}_5 = \mathbf{e}_{1133} + \mathbf{e}_{1144} + \mathbf{e}_{2233} + \mathbf{e}_{2244},$$

$$\mathbf{\Lambda}_6 = \mathbf{e}_{3311} + \mathbf{e}_{3322} + \mathbf{e}_{4411} + \mathbf{e}_{4422},$$

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$$\Lambda_7 = \mathbf{e}_{1324} - \mathbf{e}_{1423} - \mathbf{e}_{2314} + \mathbf{e}_{2413},$$

$$\Lambda_8 = \mathbf{e}_{3142} - \mathbf{e}_{3241} - \mathbf{e}_{4132} + \mathbf{e}_{4231},$$

$$\Lambda_9 = \mathbf{e}_{1331} + \mathbf{e}_{1441} + \mathbf{e}_{2332} + \mathbf{e}_{2442},$$

$$\Lambda_{10} = \mathbf{e}_{3113} + \mathbf{e}_{3223} + \mathbf{e}_{4114} + \mathbf{e}_{4224}.$$

Here the symbols  $\mathbf{e}_{\alpha\beta\gamma\delta}$  are equal to  $\mathbf{e}_\alpha^\beta \otimes \mathbf{e}_\gamma^\delta$ . The coefficients  $a_k$  are given by

$$a_1 = 1,$$

$$a_2 = 2 \frac{(x_1^- - x_2^-)(x_1^+ x_2^- - 1)x_2^+}{(x_1^- - x_2^+)(x_1^+ x_2^+ - 1)x_2^-} - 1,$$

$$a_3 = \frac{x_1^+ - x_2^-}{x_1^- - x_2^+} \frac{\tilde{\eta}_1 \tilde{\eta}_2}{\eta_1 \eta_2},$$

$$a_4 = -\frac{x_1^+ - x_2^-}{x_1^- - x_2^+} \frac{\tilde{\eta}_1 \tilde{\eta}_2}{\eta_1 \eta_2} + 2 \frac{(x_1^- - x_2^-)(x_1^- x_2^+ - 1)x_1^+}{(x_1^- - x_2^+)(x_1^+ x_2^+ - 1)x_1^-} \frac{\tilde{\eta}_1 \tilde{\eta}_2}{\eta_1 \eta_2},$$

$$a_5 = \frac{x_1^+ - x_2^+}{x_1^- - x_2^+} \frac{\tilde{\eta}_2}{\eta_2},$$

$$a_6 = \frac{x_1^- - x_2^-}{x_1^- - x_2^+} \frac{\tilde{\eta}_1}{\eta_1},$$

$$a_7 = \frac{1}{4iu} \frac{(x_1^- - x_1^+)(x_1^+ - x_2^+)(x_2^- - x_2^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - 1)} \frac{1}{\eta_1 \eta_2},$$

$$a_8 = 4iu \frac{(x_1^- - x_2^-)}{(x_1^- - x_2^+)(x_1^+ x_2^+ - 1)} \tilde{\eta}_1 \tilde{\eta}_2,$$

$$a_9 = \frac{x_1^- - x_1^+}{x_1^- - x_2^+} \frac{\tilde{\eta}_2}{\eta_1},$$

$$a_{10} = \frac{x_2^- - x_2^+}{x_1^- - x_2^+} \frac{\tilde{\eta}_1}{\eta_2},$$

with

$$\begin{aligned} \tilde{\eta}_1 &= \left(\frac{x_1^+}{x_1^-}\right)^{\frac{1}{4}} \sqrt{i(x_1^- - x_1^+)}, & \eta_1 &= \tilde{\eta}_1 \sqrt{\frac{x_1^+}{x_1^-}}, \\ \eta_2 &= \left(\frac{x_2^+}{x_2^-}\right)^{\frac{1}{4}} \sqrt{i(x_2^- - x_2^+)}, & \tilde{\eta}_1 &= \eta_2 \sqrt{\frac{x_2^+}{x_2^-}}. \end{aligned}$$



# Appendix D

## Algebraic limit of the TBA equations

In this appendix we present the solution to an algebraic limit of the TBA equations (4.54). These equations appear in the weak coupling limit for the Hubbard and B-models, as was discussed in section 4.4.3. They also appear, with  $\Delta$  replaced by  $-2$ , as the large  $v$  asymptotics of the simplified TBA equations (4.44-4.46) for the B-model for any  $u$ . The general solution to the equations (4.54) is well-known, see e.g. [22], and can be written in the form

$$\begin{aligned}
 1 + Y_{M|vw} &= \left( \frac{\sinh(f_{vw} + M)c_{vw}}{\sinh c_{vw}} \right)^2, & 1 + Y_{M|w} &= \left( \frac{\sinh(f_w + M)c_w}{\sinh c_w} \right)^2, \\
 \frac{1 + Y_+}{1 + Y_-} &= \left( \frac{\sinh f_{vw}c_{vw}}{\sinh c_{vw}} \right)^2, & \frac{1 + \frac{1}{Y_+}}{1 + \frac{1}{Y_-}} &= \left( \frac{\sinh f_w c_w}{\sinh c_w} \right)^2.
 \end{aligned}$$

The constants  $c_{vw}$  and  $c_w$  are fixed by the large  $M$  asymptotics (4.33) of Y-functions  $c_{vw} = -\frac{\mu}{T} = \beta_c$ ,  $c_w = \frac{B}{T} = \beta_s$ , and the two remaining functions are then found from equations (4.54) for  $y_{\pm}$ -particles

$$\begin{aligned}
 \sinh f_{vw}\beta_c &= \frac{e^{\beta\Delta} \sinh \beta_c (\cosh \beta_c + e^{-\beta\Delta} \cosh \beta_s)}{\sqrt{2 \cosh \beta_c \cosh \beta_s \cosh \beta_{\Delta} + \cosh^2 \beta_c + \cosh^2 \beta_s + \sinh^2 \beta_{\Delta}}}, \\
 \sinh f_w\beta_s &= \frac{\sinh \beta_s (\cosh \beta_c + e^{-\beta\Delta} \cosh \beta_s)}{\sqrt{2 \cosh \beta_c \cosh \beta_s \cosh \beta_{\Delta} + \cosh^2 \beta_c + \cosh^2 \beta_s + \sinh^2 \beta_{\Delta}}},
 \end{aligned}$$

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with  $\beta_\Delta = \frac{\Delta}{T}$ . The explicit solution to the algebraic equations (4.54) is then

$$\begin{aligned}
 1 + Y_{M|vw} &= \frac{e^{2\beta_\Delta} (2e^{-\beta_\Delta} \cosh \beta_s \sinh(M\beta_c + \beta_c) + e^{-2\beta_\Delta} \sinh M\beta_c + \sinh(M\beta_c + 2\beta_c))^2}{4 \sinh^2 \beta_c (2 \cosh \beta_c \cosh \beta_s \cosh \beta_\Delta + \cosh^2 \beta_c + \cosh^2 \beta_s + \sinh^2 \beta_\Delta)}, \\
 1 + Y_{M|w} &= \frac{e^{2\beta_\Delta} (2e^{-\beta_\Delta} \cosh \beta_c \sinh(M\beta_s + \beta_s) + e^{-2\beta_\Delta} \sinh(M\beta_s + 2\beta_s) + \sinh M\beta_s)^2}{4 \sinh^2 \beta_s (2 \cosh \beta_c \cosh \beta_s \cosh \beta_\Delta + \cosh^2 \beta_c + \cosh^2 \beta_s + \sinh^2 \beta_\Delta)}, \\
 Y_+ &= \frac{4e^{-\beta_\Delta} \cosh \beta_c \cosh \beta_s + 2 \cosh 2\beta_c + e^{-2\beta_\Delta} + 1}{4e^{-\beta_\Delta} \cosh \beta_c \cosh \beta_s + e^{-2\beta_\Delta} (2 \cosh 2\beta_s + 1) + 1}, \\
 Y_- &= e^{-2\beta_\Delta} Y_+. \tag{D.1}
 \end{aligned}$$

In addition these formulae determine the large  $v$  asymptotics of the extended A- and B-models. Here  $\Delta = -2\mathfrak{g}$  for the A- and opposite A-models, and  $\Delta = -2-2\mathfrak{g}$  for the B-model and  $\Delta = 2 - 2\mathfrak{g}$  for the opposite B-model.



## Appendix E

# The $t$ - $J$ model: the limit of strong Coulomb repulsion

In this appendix we consider the effect of large on-site Coulomb on the following general Hamiltonian

$$\mathbf{H}(\gamma, \zeta, \kappa) = \sum_{j=1}^L \left( \mathbf{T}_{j,j+1} + 4\gamma \mathbf{V}_{j,j+1}^H + \zeta (\mathbf{V}_{j,j+1}^H + \mathbf{V}_{j,j+1}^{SS} - \mathbf{V}_{j,j+1}^{CC} + \mathbf{V}_{j,j+1}^{PH}) \right),$$

$$\mathbf{T}_{j,k} = - \sum_{\sigma} (\mathbf{c}_{j,\sigma}^{\dagger} \mathbf{c}_{k,\sigma} + \mathbf{c}_{k,\sigma}^{\dagger} \mathbf{c}_{j,\sigma}) \left( 1 - (1 - \kappa_j) (\mathbf{n}_{j,-\sigma} - \mathbf{n}_{k,-\sigma})^2 \right). \quad (\text{E.1})$$

As interesting special cases this includes the A-model for which  $\gamma = \cosh \nu$ ,  $\zeta = -2/\cosh \nu$  and  $\kappa_j = (-1)^j \tanh \nu$ , and the extended B-model for which  $\gamma = \mathbf{g}$ ,  $\zeta = 2 \tanh \nu$  and  $\kappa_j = 1/\cosh \nu$ .

In the large  $\gamma$  limit the Hubbard term dominates and the models are half-filled. It is useful to shift the chemical potential  $\mu \rightarrow \mu + 2\gamma$  to study the less than half-filled models, as this will remove the term proportional to  $\mathbf{N}$  from the Hubbard interaction (2.9). Then let us write the Hamiltonian (E.1) as  $\mathbf{H}(\gamma, \zeta, \kappa) = 4\gamma \sum_{j=1}^L \mathbf{n}_{j,\uparrow} \mathbf{n}_{j,\downarrow} + \mathbf{H}(0, \zeta, \kappa)$  and note that in the strict  $\gamma \rightarrow \infty$  limit any state that does not contain doubly occupied sites is a ground state and there is huge degeneracy. Taking the sub-leading terms into account in  $\frac{1}{\gamma}$  perturbation theory in a standard way, see e.g. section 2.A and particularly equations (2.A.26, 2.A.30) of the book [9], the effective large  $\gamma$  Hamiltonian is

$$\mathbf{H}^{t-J}(\gamma, \zeta, \kappa) = \mathbf{P}_0 \mathbf{H}(0, \zeta, \kappa) \mathbf{P}_0 + \frac{1}{4\gamma} \sum_{j=1}^L \mathbf{P}_0 \mathbf{H}(0, \zeta, \kappa) \mathbf{n}_{j,\uparrow} \mathbf{n}_{j,\downarrow} \mathbf{H}(0, \zeta, \kappa) \mathbf{P}_0,$$

## E. THE $T$ - $J$ MODEL: THE LIMIT OF STRONG COULOMB REPULSION

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where  $\mathbf{P}_0 = \prod_{j=1}^L (1 - \mathbf{n}_{j,\uparrow} \mathbf{n}_{j,\downarrow})$  is the projector onto the model's Hilbert space of dimension  $3^L$  where there are no doubly occupied sites. As happens for the similar analysis of the Hubbard model, the second term in the above expansion gives rise to three-site terms as well as two-site terms. The three site-terms however will be suppressed in the physically interesting regime where the model is close to half-filling and so we will ignore them here, as is generally done.

The effective Hamiltonian is thus

$$\mathbf{H}^{t-J}(\gamma, \zeta, \kappa) = \mathbf{P}_0 \sum_{j=1}^L \left( -\mathbf{c}_{j,\sigma}^\dagger \mathbf{c}_{j+1,\sigma} - \mathbf{c}_{j+1,\sigma}^\dagger \mathbf{c}_{j,\sigma} + \left( \zeta + \frac{\kappa^2}{\gamma} \right) \left( \mathbf{V}_{j,j+1}^{SS} - \frac{\mathbf{n}_j \mathbf{n}_{j+1}}{4} \right) \right) \mathbf{P}_0$$

This is the  $t$ - $J$  model with coupling  $J = \zeta + \frac{\kappa^2}{\gamma}$ . For the A-model  $J \sim -1/\mathbf{u}$  for large  $\mathbf{u}$ , and it behaves as a ferromagnetic  $t$ - $J$  model. For the B-model  $J \sim 2 \tanh \nu + \frac{1}{\mathbf{g} \cosh^2 \nu}$  for large  $\mathbf{g}$ , and at  $\nu = \infty$  the leading contribution is the supersymmetric  $t$ - $J$  model.



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