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Valuing Interest Rate Contingent Claims: A Review of the Ho and Lee Model in the Irish Context

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Abstract

This paper describes and illustrates with examples the use of the Ho and Lee model for valuing interest rate contingent claims. Some difficulties with the model are pointed out and its application in the Irish context is discussed.

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Valuing Interest Rate Contingent Claims
A Review of the Ho and Lee Model in the Irish Context
Approaches

Introduction

Interest Rate Contingent Claims are any financial securities whose payoff depends upon future interest rates. For example:

Options on Government Gilts: an option on a government gilt is a contract which gives the purchaser either the right to buy (a call option) or to sell (a put option) a quantity of a specified government gilt at a fixed price on or before a particular date. The payoff to this option depends upon the price of the gilt when (if at all) the option is exercised.

Futures on Interest Rates: The IFOX 3 month DIBOR and Long Gilt contracts are examples of interest rate contingent claims. The settlement price of both these contracts depends upon prevailing interest rates.

Swaps: The value of an interest rate swap (floating/fixed) is given by the discounted sum of the swap payments, which in turn will be a function of prevailing interest rates and the past history of interest rates at the time of each swap payment [1].

Securitized Assets: The value of a securitized asset is the discounted sum of its payments. However its value is contingent on interest rates in at least two ways. If the assets which are securitized are fixed interest loans then there is the possibility of early loan redemption if future interest rates fall. In addition there is the possibility that, if the securitized assets are floating rate loans, a rise in interest rates in the future will lead to increasing levels of default.

Other examples of interest rate contingent claims are interest rate caps and collars, swaptions, options on interest rate futures (for example, an option on, say, the IFOX 3 month DIBOR contract), and so on.

Valuing Interest Rate Contingent Claims

Because the value of an IRCC today depends upon interest rates in the future we require, in order to value an IRCC, some beliefs about future interest rates. The models which are used in this context assume that the future is uncertain and thus in valuing IRCC we do not try to predict exactly what interest rates will hold in the future: rather, we set up a probability distribution of interest rates (or the path of interest rates). In other words, we assign a probability of occurence to each of a set of possible specific values of the interest rate(s) in question. Technically this means that we model the change in interest rates as a stochastic process.

In general in modelling interest rates, it is usually assumed that they follow what is called a diffusion process. The implication here is that we cannot predict interest rates. If the market had any knowledge of the likely future direction of interest rates this would already be absorbed in the rates themselves. Rates change because new information is received. But this new information is, by definition, unpredictable, and thus its arrival imparts a random shock to the time path of interest rates. Hence interest rates evolve as a stochastic process through time. Typically this is modelled by what is known, variously, as Brownian motion or a Wiener process. Interest rates may display some sort of long run drift: most often the assumption here is that the process is mean reverting. If interest rates get very high, they will tend to drift back down, and vice versa.

The Term Structure of Interest Rates

Thus far we have referred to interest rates in a general sense. In fact models used for valuing IRCC focus on a specific set of interest rates called the term structure. The yield curve for gilts relates the yield on a gilt to its maturity. The term structure is something slightly different: it is the yield curve for a set of pure discount bonds with varying maturities. That is to say, it is the yields to maturity on gilts which only repay their face value and do not pay any coupon during their life. In other words, the term structure is a set of interest rates or discount factors stretching all the way out to the maturity of the longest gilt in the market. This term structure is implicit in the set of prices in the gilts market at any particular time and can be derived from them. Figure 1, for example, shows the term structure derived (using a model we are in the process of developing at The Economic and Social Research Institute) from the prices of gilts at the close of business on Friday 13 October last and on Monday 16th October.

The term structure can be written as a set of rates, as shown here, or as a set of prices. The two are related via the simple expression:

$$r(T) = -\ln P(T)/T$$

i.e. today's continuously compounded spot rate for T years is equal to minus the log of the price of a T year pure discount bond divided by its maturity. Hence the term structure expressed in terms of prices of a £1 bond has a very simple form, being a (weak) monotonically declining curve starting at the value 1 for zero maturity bonds. The market price of gilts can be reassembled from the term structure of rates or prices since a gilt can be viewed as a set of pure discount bonds having different face values and maturities (i.e. each coupon payment is a separate discount bond and so is the principal repayment). A third way of depicting the term structure is in terms of forward rates or prices. The latter are defined as

$$F(t,T) = P(T)/P(t)$$

For example, the current forward price of a 3 month pure discount bond in three months time is the current price of a 6 month pure discount bond divided by the current price of a 3 month pure discount bond. The forward rate can be derived analogously.

Modelling the Evolution of the Term Structure

In analysing interest rate changes, models focus on how the term structure so defined evolves over time, since this term structure is the most primitive representation of interest rates that we have. Broadly speaking there have been two approaches to modelling the term structure. One takes today's term structure to be endogenous: i.e. it is something which the model should explain. The other takes today's term structure to be exogenous, as the starting point of the analysis. It is the latter approach which we ddeal with here. But briefly, the other approach runs along the following sort of lines. Movements of all the rates in the term structure are held to be functions of changes in at most two of the rates. Typically these would be a very short and very long rate. In the simplest case, one interest rate (say the very shortest interest rate, sometimes called the instantaneous rate) is held to follow a diffusion process with drift, and all other (i.e. longer) rates are then viewed as a function of this very short rate plus some form of liquidity or risk premium that attaches to committing money for longer periods. [2]

The problems with this approach are various. Most importantly, perhaps, it may by no means be easy to get the model to generate a current term structure which is the same as the currently observed term structure. If it does not do this then, in theory at least, arbitrage will be possible between the model based rates or prices and those existing in the market. On the other hand, to get the model to agree with present reality requires the correct choice of both the parameters of the diffusion process characterising the short rate and the various liquidity premia.

The Ho and Lee Model of the Term Structure

The alternative approach is to take the current term structure as given and to develop a stochastic model which explains the evolution of the whole structure into the future. The first model to seek to do this was developed by Thomas Ho and Sang-Bin Lee and was published in *The Journal of Finance* in 1986. This paper has proved enormously influential in redirecting the whole approach to term structure modelling and has been widely used among market practitioners in the United States. At present no commercial package incorporating what has come to be known as the Ho and Lee model is available, but programming the model is not of itself difficult. The remainder of this paper outlines the Ho and Lee model, illustrates it with some simple examples, and discusses some of its strengths and shortcomings and how these might be overcome.

No Arbitrage Pricing Condition

A common feature of models of the term structure is that they are all based on what is called the 'no arbitrage condition'. This is an extremely powerful condition since it allows us to develop term structure and pricing models which do not have to take account of individual differences in attitudes towards the trade off between risk and return. What this condition says is the following. If we can form a portfolio of any two bonds which have different maturities such that this portfolio always yields a certain return over one period of time no matter what happens, then the return which this portfolio yields must be equal to the return on a bond whose maturity is one period. In other words, if we can construct a portfolio which yields a riskless return over a given period of time it must be worth exactly the same as a bond which matures at the end of that period of time. This assumption imposes limits on the admissable term structure movements in term structure models.

Ho and Lee's model begins by assuming that the whole term structure fluctuates over time according to a binomial process. This means that, if we think of time as being made up of discrete periods and that trading takes place only at the end of each period, then during each period the whole term structure can either move up or it can move down. Ho and Lee work with prices, so if the term structure moves up in their model rates are falling, and conversely if the structure moves down. Their model of the term structure's movement, then, has a very simple form. Given today's term structure, the possible term structures which will prevail at the end of next period are two - prices will either have gone up or down. At the end of two periods there are four possibilities - prices can have gone up then down, up then up again, down and up or down and down again. This is shown in Figure 2.

The Parameters of the Ho and Lee Model

At first sight this seems a highly implausible model, but on closer inspection this proves not to be the so. This is because the binomial evolution is what's termed a discrete time approximation to the Brownian motion process that was mentioned earlier. If the time intervals between movements (the periods of the model) are suitably short, then a binomial process of the Ho and Lee type will be a very good approximation to a term structure which is evolving randomly through time driven by a single Brownian motion.

If we agree that the idea of a binomial process for the term structure is reasonable, we can then progress to the stage of determining the parameters of the process. This is another way of saying: first, by how much does the term structure move up or down each time it jumps; and, second, what is the probability that, in a given period, it will move up rather than down? Ho and Lee derive the answers to these questions directly by virtue of the constraints they impose on the possible term structure movements. That is, given certain conditions or assumptions, the magnitudes and probabilities of the term structure's jumps follow automatically. One condition that Ho and Lee (in common with everyone else) impose is the no arbitrage condition, which we discussed earlier. A second condition is path independence. This means that, if the term structure over two periods moves up in the first period and down in the second, the result will be the same as if it had moved down in the first period and up in the second. This means that in the resulting binomial tree, as it is called, there are only three, rather four, possible outcomes after two periods, as shown in Figure 3. The tree shown in Figure 3 summarises the whole idea of the Ho and Lee model: today - at node zero - we can envisage the term structure taking any one of a number of possible paths in the future. Valuing interest rate contingent claims requires that we assign a probability of occurence to each possible path (each possible outcome) and value our contingent claim as some loosely defined 'average' of these possible outcomes.

The magnitudes of the up or down jumps in prices are given as follows. Suppose that there was no uncertainty attached to the future path of interest rates: in that case today's forward rates would be tomorrow's spot rates. Thus Ho and Lee model tomorrow's spot prices as equal to today's forward prices multiplied by a 'disturbance term' to model the uncertainty attached to future rates and prices. That is

$$P(t+1,T) = \frac{P(t,T)}{P(t,t+1)} h(T-t-1)$$

or

$$P(t+1,T) = \frac{P(t,T)}{P(t,t+1)} h*(T-t-1)$$

These two correspond to upward or downward jumps, respectively. These expressions say that the price tomorrow of a discount bond which matures at T is equal to today's one period forward price for a bond which matures at T multiplied by a disturbance factor which pushes the price up (h) or by a disturbance factor which pushes the price down (h*). The specific value taken by h and h* depends on the maturity of the bond. Ho and Lee provide only a cursory discussion of how one might estimate h and h*, but it is easy to demonstrate that they are functions of the historical variance of forward prices (see appendix for details). Typically, for a bond of one period maturity in the Ho and Lee model, h will be very marginally greater than one, and h* will be marginally less than one. For bonds of greater maturity h and h* will move correspondingly further away from unity.

The other paramater which must be estimated is the probability of an upward movement, p. Here a subtle distinction has to be made. If we assume that the binomial model is a good depiction of the real world, then there will exist real world values of p which will directly reflect the rate of return offered by a discount bond. But we cannot know these in advance. What Ho and Lee demonstrate, however, is the applicability of a standard argument in the pricing of contingent claims, which goes as follows. Although the true probability of an up and down movement cannot be known, in order to value contingent claims in an arbitrage free environment we must set p equal to the so-called 'risk neutral' probability. One way of expressing this constraint is that p must be such as to satisfy the equation:

$$p\,h(T-t) + (1-p)h^*(T-t) = 1$$

for all values of t and T (i.e. at all times and for disturbance terms applicable to bonds of all maturities).

A Very Simple Term Structure Example

To make this discussion a little more concrete, assume that, from our term structure model, we have a set of interest rates for 1 year, 2 years, out to 5 years, and that all trading takes place once a year. This means that we are calling one period of the Ho and Lee model one year. *Of course, in reality we should set one period of the model to be a very much shorter length of time.*

Let's suppose that the rates for our small term structure are as shown in Table 1. The rates are low by today's standards, but that is of no consequence. From these we can estimate the prices (per pound nominal) of pure discount bonds maturing at the end of each year. Note that the bond with zero time to maturity is worth its face value, £1.

In Figure 4 we show the evolution of the term structure of prices over two years. We have set the size of the up and down jumps according to an assumed variance in the forward rates of around .015 per cent (see appendix for details). The whole term structure moves up or down once per year and today's two year bond becomes tomorrow's one year maturity bond, and so the number of prices in the model declines by one each year. In reality, of course, we would have prices going out to the maturity date of the longest gilt in the market. Figure 4, then, shows the possible term structures over the next two years, given the current term structure.

As an example of how to read Figure 4 consider today's price of a pure discount bond with four years to maturity. Its price is 82 pence per pound face value. In a year's time if interest rates fall (and thus prices rise) this same bond (which will now have three years to maturity) will be worth 88.91 pence. If prices fall (rates rise) it will be worth 82.41. In two years time, when it is a two period bond, it will be worth one of 85.66, 90.11 or 94.78 pence. Another way of using this model is to ask what happens to a particular interest rate over this period. From Figure 4 we can calculate the possible future values of, say, the one year interest rate, and these are shown in Figure 5.

Note that we have kept Figure 5 the same way up as Figure 4, so that increases in the rate arise from downward moves in the term structure.

Using the Model to Value Contingent Claims: 4 examples.

The risk neutral probability for use with this particular model is p=.6.

(a) A two year European call option on a five year pure discount bond with an exercise price of 85 pence per pound nominal.

From Figure 4 we can immediately read off the possible prices of the five year bond at the end of two years (remember that at the end of two years this will have become a three year bond). These prices are .9075, .8411, and .7796. For each of the three possible prices the value of the option will be

$$\max\,(\text{bond price}-\text{exercise price};0) \qquad (1)$$

This will be zero for two of the three possible bond prices. We then take the single non-zero value, multiply it by its risk neutral probability of being realised (in this case .36) and discount its value by the shortest rate in the market In our case this is the one year rate whose schedules are given in Figure 5. The result is

$$.0575 \times .36 \times \exp\,(-.0441-.0513) = .01882$$

Thus the price of the option is £1.88 per £100 nominal.

(b) A two year European call option on a five year gilt paying an annual coupon of 6 per cent with an exercise price of par.

The only differences between this example and the previous one are, first, that the set of possible values of the five year gilt in two years' time will be different because of the annual coupon payments, and, second, the exercise price is now set at 100. The set of possible values of the gilt (in pounds per £100 nominal) in this case are: £107.731; £100.266; £93.337. Applying equation (1) and discounting the resulting two positive values by their probability of occurence and by the short term interest rates gives a value for this option of £2.65 per £100 nominal.

(c) Futures price for a contract on a notional three year pure discount bond with contract closing date in two years time.

In this case, whereas the options we valued were assumed to be written on specific bonds (whose maturity shortened as time passed) here we assume a notional bond whose maturity does not change. Hence the terminal values which we use here are the values of a three year discount bond - which are in fact the same set of values we used in example one. But since this is a different type of contingent claim, we use a different function (different from equation (1)) to value the future. Without going into the details, the current fair futures price should be .9046 according to the model. This differs from the forward price which we calculate as .9111. The difference arises, of course, because of the marking to market of the futures contract.

Having valued the future it is, of course, very easy to value an option on the future.

(d) Valuing a securitized asset allowing for early repayment.

Securitized assets are generally bundles of debt to which the rights to the interest streams are sold. In general these are debts, such as house mortgages, which, individually, would not find a ready market. Equally, high risk debt, such as junk bonds, can be securitized by forming a portfolio of such bonds. Investors then purchase rights to a share in the coupon payments which accrue to the portfolio. To illustrate how securitized assets' values are contingent on interest rates, and how the Ho and Lee model might be used to value them, consider the following example. Suppose that we have a portfolio of 100 bonds which pay annual coupons as follows: 25 pay interest at 3 per cent; 25 at 4 per cent; and 50 at 5 per cent, each on a debt of £1000 principal. Any borrower has the option to repay his loan at any time and issue new debt with a coupon fixed equal to the current one year rate of interest plus one quarter of one per cent [3]. Finally we assume that the bonds have just gone ex-dividend and they have two more years to live. Given the term structure model we have already developed we can estimate the remaining value of the debt portfolio.

Suppose that the possibility of prepayment of the debt were not an issue: then the securitized portfolio would pay an aggregate coupon of £4250 each year for the next two years. Using today's term structure (given in Table 1) this is currently worth £7863. But clearly, prepayment is an issue: it will be profitable for our bond issuers to repay their debt under certain circumstances. And if we turn again to Figure 5 - which shows the time path of the one year rate, which is the benchmark for the debt issuers' cost of borrowing - we can see that in a year's time, if rates fall, those 50 individuals who have issued debt at 5 per cent will repay their debt (and replace it with debt paying a coupon of 4.75 per cent). The value of our portfolio will, under these circumstances be less, though how much less will depend on the treatment of early prepayment in the securitization agreement. If we suppose that parties to the agreement receive the present value of the early prepayment (i.e. interest on £1000 for one year at the prevailing one year rate) then the value of the portfolio, under these circumstances, would fall to £7678. Thus under the Ho and Lee model (in other words, taking account of the 'probability' of early prepayment) the value of the portfolio of securitized assets today is £7728. Note that the difference between this value and the value if no prepayment were possible is the sum of the values of (a) the option to prepay which was retained by the issuers of the debt; plus (b) the risk premium that the issuers must pay over the market rate of interest should they exercise their option to refinance. The former is worth £1.36, the latter £1.32. Multiplying these by 50 gives the difference between the uncorrected and corrected estimates of the portfolio's value.

Summary and Conclusions

The Ho and Lee model is nowadays quite widely used to value contingent claims, and we have given four very simple examples of how this might be done. Clearly, however, the potential applications of a model of this kind are very numerous indeed. Not least is the possibility of integrating this model with conventional valuation models (of, say, equity options) which rest upon the assumption of unchanging interest rates (at least in the short term). We might also extend this model to examine the question of default risk. Here we would need to model both the time path of interest rates and the time path of the net wealth of the possible defaulting party: hence we would need some estimate of the covariance between these two in order to complete our model.

The Ho and Lee model, in the basic form in which it has been presented here, has a number of technical problems associated with it. First, the model makes assumptions about the variance - or variability - of interest rates that may not be correct. These assumptions are that this variance is constant across time for any particular rate, and that the variance of different rates are related in a very simple fashion (see appendix for details). Neither of these assumptions are necessarily correct. In particular the variability of interest rates may very well fluctuate quite markedly over time. Secondly, the model can yield estimates of negative forward interest rates. Technically this follows from the assumptions about the variance of rates. This is an obviously undesirable feature of the model. Thirdly, the model only allows for one source of random variation in the evolution of the term structure. This means that a diversified portfolio of gilts is no less risky than a portfolio made up of just one gilt - which again is a counter-intuitive result which is an undesirable feature of any term structure model.

Against this, however, is the flexibility of the Ho and Lee framework. This is such that it is quite easy to make adjustments to the basic model in order to overcome these difficulties. Different assumptions about the variances of rates can overcome some of these problems, while it is a straightforward matter to generalise the model to permit two or more sources of random variation to enter the term structure's evolution. In practice this means moving from a binomial to a multinomial process which is computationally more burdensome but theoretically tractable.

What about the Ho and Lee model in the Irish context? In order to apply this sort of model - or, indeed, any of the formal term structure models in the literature - one needs estimates of the term structure of the kind I showed earlier. The ability to estimate such term structures is, in itself, immensely valuable. Among other uses it permits the pricing of new debt issue - by government or companies - of any form whatsoever (e.g. semi-annual or annual coupons, zero coupons, term repayments, etc.) and it prices conventional debt issue much more accurately than the more familiar yield curve methods.

Given estimates of the term structure, the next step would be to decide on the appropriate form of Ho and Lee model to use. In particular how should one go about modelling the future variance of interest rates; and how many random disturbance factors should be in the model - should it be one, as in the basic Ho and Lee model, or more?

What might be the applications of such a model in the Irish context? The market for derivative instruments in Ireland is in its infancy: so far as I am aware only one bank is writing options for customers, for example, and IFOX (the Irish Futures and Options Exchange) has been in operation for less than a year. However, there are a number of areas where the Ho and Lee model might be used. Obvious examples are for the valuing of futures and swaps. Less obvious would be applications of the model to value callable gilts, of which a number currently trade in the market, and to value company debt which has complex call and/or conversion provisions. In the longer term it might be applied to value interest rate options on the part of, say, banks who wished to write such options over the counter, and on the part of IFOX members, assuming that the exchange will, at some point, begin to trade options on either gilts or on its existing interest rate futures contracts, or both. And finally, of course, it is a useful tool in valuing securitised assets.

FOOTNOTES

[1] This is because the floating rate is often defined as some form of weighted sum of previous interest rates.

[2] For example, Vasicek (1976) writes the T period interest rate as equal to the expectation of the mean value of the integral of the instantaneous rate over the period 0 to T, plus a risk premium term.

[3] More precisely, we assume that borrowers can issue new debt at a rate of one quarter of a per cent over the annually compounded equivalent of the continuously compounded rate shown in Figure 5. So, if rates fall, those issuers of debt with a 5 per cent annual coupon will refinance at an annual coupon of

$$\exp(.0441) - 1 + .0025 = .04759$$

REFERENCES

1. Ho, T.S., and S-B. Lee, 'Term Structure Movements and Pricing Interest Rate Contingent Claims,' *The Journal of Finance,* 41, 5, December 1986, pp. 1011-1029.

2. Vasicek, O., 'An Equilibrium Characterization of the Term Structure,' *Journal of Financial Economics,* 5, 1977, pp. 177-188.

Appendix: The Parameters of the Ho and Lee Model

The Ho and Lee model depends on two parameters: π, the risk neutral probability of an upward jump in prices; and δ, which determines the magnitude of the up (h) and down (h^*) jump parameters. In this appendix we show that δ is a function of the variance of rates. This is significant insofar as it means that the parameter can be estimated from historical data, in contrast to Ho and Lee, who suggest that δ and π should be determined from the value of contingent claims already in the market, in a manner analogous to that in which the implicit variance of the underlying asset can be extracted from a sample of option prices. In the Irish context this is particularly relevant given the dearth of marketed contingent claims.

Consider the price for a one period bond in one period's time under the Ho and Lee model. Its expected value is the current one period forward price, F (see equation 6 of Ho and Lee). The variance of the price is

$$\pi(1-\pi)\ [F(h-h^*)]^2$$

The variance of the equivalent rate is:

$$\begin{aligned}&\pi(1-\pi)\ [-\ln\ [Fh]\ -\ (-\ln\ [Fh^*])]^2\\ =\ &\pi(1-\pi)\ [\ln\ (h^*/h)]^2\end{aligned}$$

From and Ho and Lee equations 19 and 20, $\delta = h^*/h$. So the variance of the one period rate one period hence is

$$\pi(1-\pi)\ [\ln\ (\delta)]^2 \qquad (1)$$

In order to calculate δ, then, we require the historical variance of the one period forward rates or prices: i.e. the variance estimated from the realised rate at period n compared with the forward rate at n-1 over a sample of periods. Call this σ^2 on an annualised basis. If Δ is the number of model periods per year then define

$$v^2 = \sigma^2/\Delta$$

and thus

$$\delta = \exp\ [-(v^2/\pi(1-\pi))^{1/2}\ ]$$

In the example in the text the annualised variance of the one period forward rate has been set at .015 per cent. This yields (given that π = .6)

$$\delta = \exp\ [-(.00015/.24)^{1/2}] = .975$$

The variance of the one period rate n periods forward is

$$n\pi(1-\pi)\ [\ln\ (\delta)]^2$$

So far we have discussed the case of the one period rate. However, the variances of all rates one period forward are the same as for the one period rate. Let F_2 be the one period forward price for the two period bond. Then the variance of the forward price is:

$$\pi(1-\pi)\ [F_2(h(2)-h^*(2)]^2$$

For rates the variance is

$$\pi(1-\pi)1/2\ [-\ln\ [F_2 h(2)]\ -\ (-\ln\ [F_2 h^*(2)]])]^2$$

and since h(2) is as given by Ho and Lee equations 19 and 20 this reduces to equation (1) above. Thus under the Ho and Lee model all n period forward rates have the same variance regardless of their maturity.

TABLES

Table 1: Sample Term Structure in Rates and Prices

Year	Rate %	Price of Pure Discount Bond per £1 face value in £s
0	0	1
1	5.13	.95
2	5.27	.90
3	5.03	.86
4	4.96	.82
5	5.23	.77

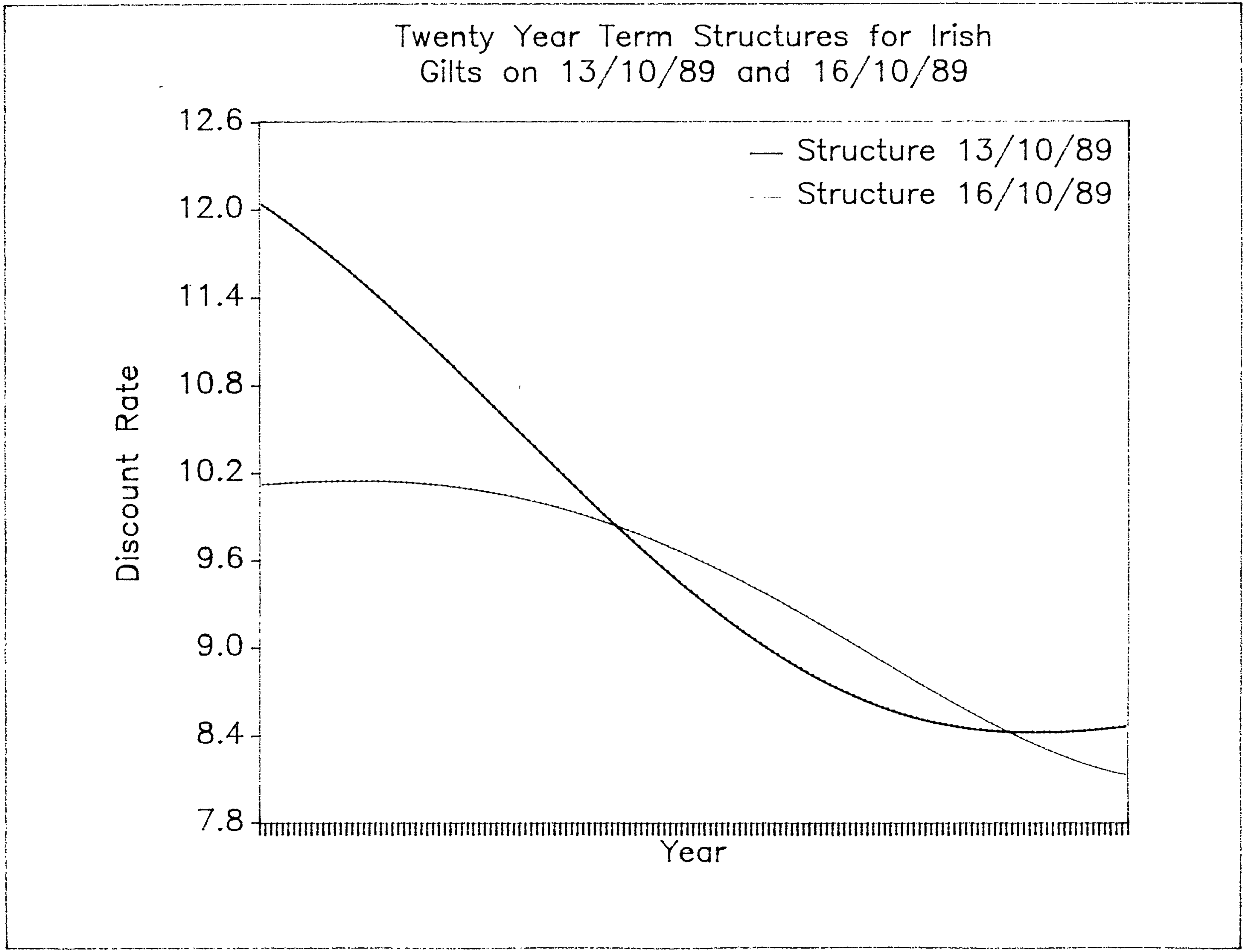

FIGURE 1: TERM STRUCTURES

FIGURE 2: TERM STRUCTURE MOVEMENTS IN HO AND LEE MODEL

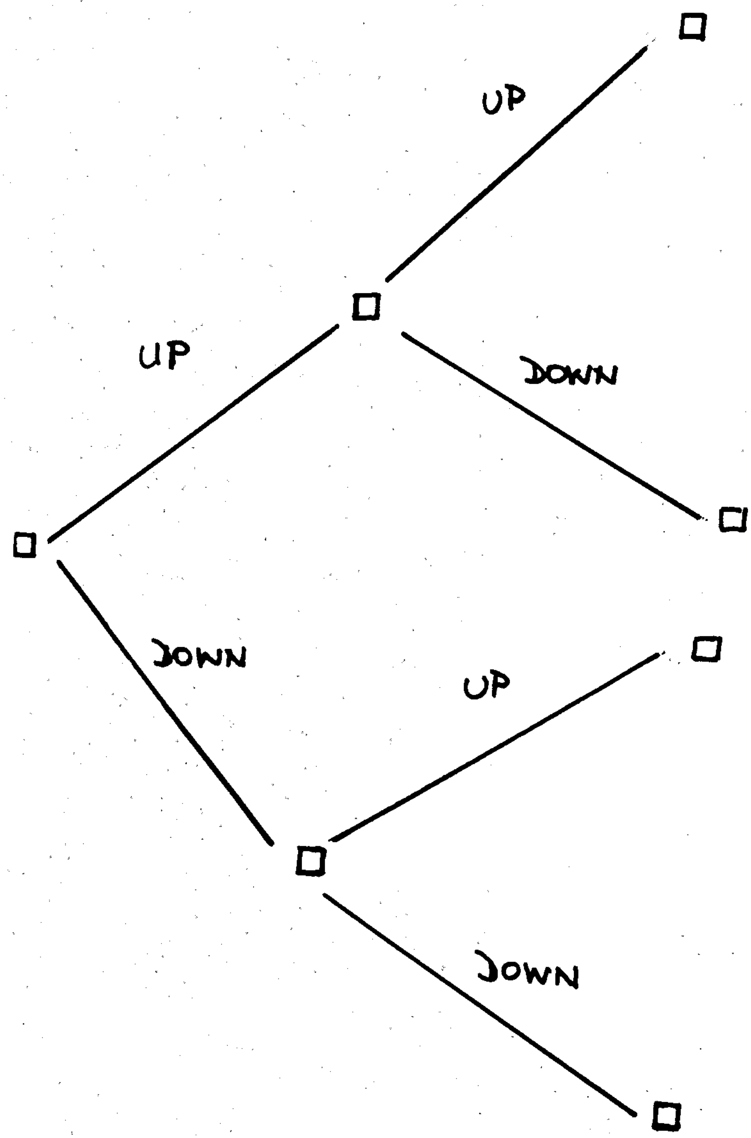

FIGURE 3: BINOMIAL LATTICE

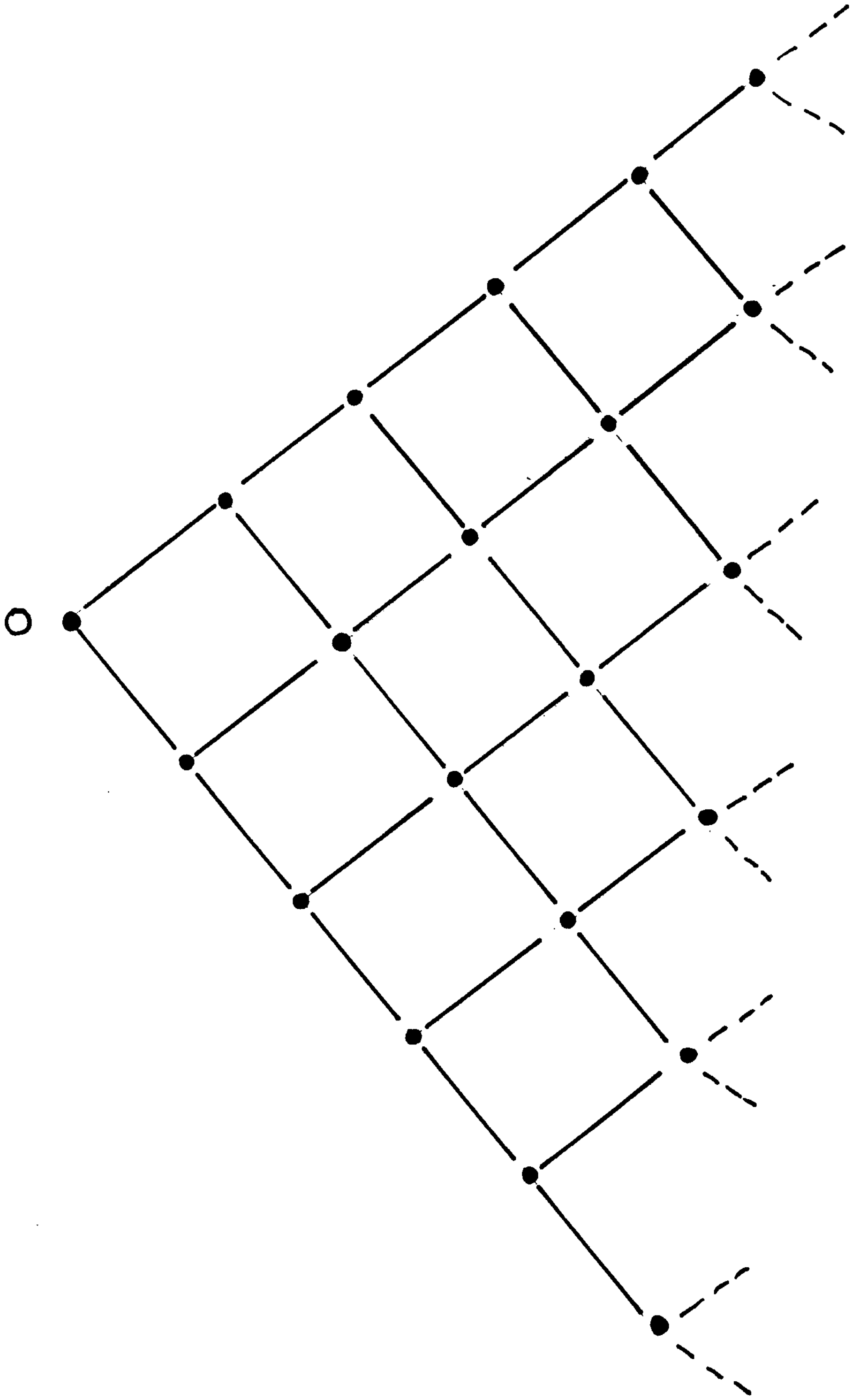

Figure 4: Evolution of the Term Structure in Prices

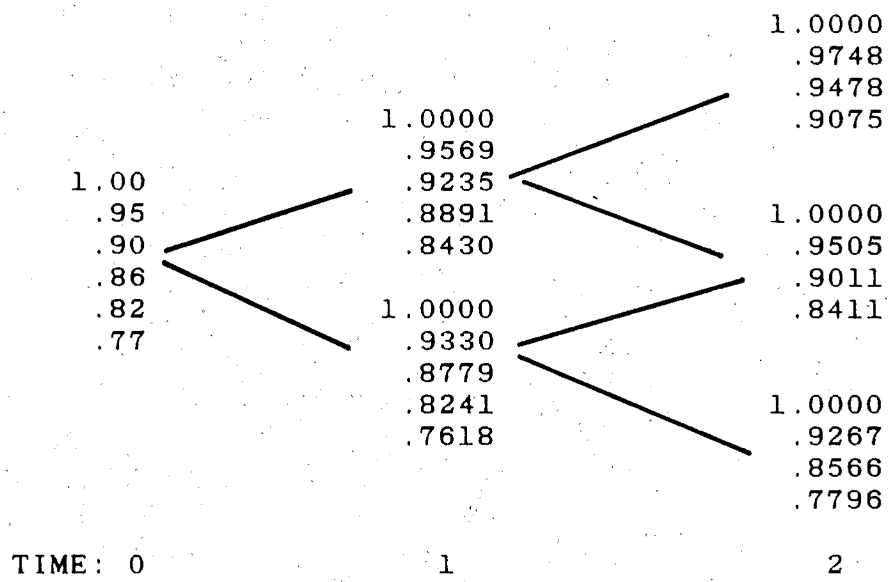

Figure 5: The Evolution of the One Year Interest Rate