Aspects of the mathematical theory of water waves

by

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Summary

In this thesis we study various aspects of the mathematical theory of water waves.

In Chapter 2 some qualitative results for two recently-derived nonlinear models for shallow water waves are presented. In the first part of the chapter we examine the solutions of a family of nonlinear differential equations which initially have compact support, and investigate whether they retain this property over a nontrivial time interval. In the second section of Chapter 2 we particularise to the Degasperis-Procesi equation and conduct a more detailed analysis of the propagation speed of solutions for a larger class of initial data, achieving results on the persistence properties of solutions and the unique continuation of solutions.

In Chapter 3 we study a recently-derived linear model equation for edge waves. This model equation is a self-adjoint second order ordinary differential equation with coefficients that depend on the topography of the seabed, the longshore speed of the current and the celerity of the wave. We present a definition of edge-wave solutions for the equation, and rigorously establish the existence of edge-wave solutions using results for self-adjoint second order ordinary differential equations concerning the monotonicity and asymptotic behaviour of solutions from the literature. We conclude the chapter by proving the existence of a large family of edge-wave solutions for the model equation, given a wide range of coefficients for the model equation, the coefficients depending on the particular environmental data for the wave.

Finally in Chapter 4 we present results concerning particle trajectories in water waves. The first section is concerned with linear periodic capillary and capillary-gravity waves, in both the shallow water and the deep water settings. In the final part of the chapter we then work with the fully-nonlinear governing equations for the case of the deep water Stokes wave.

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Aspects of the mathematical theory of water waves

David Jarlath Patrick Henry

Abstract

In this thesis we study various aspects of the mathematical theory of water waves. In Chapters 2 and 3 we analyse certain qualitative properties of some model equations, while in Chapter 4 we work with the nonlinear governing equations and their linearisation. In Chapter 2 some qualitative results for two recently derived nonlinear models for shallow water waves are presented. In the study of edge waves in Chapter 3 we investigate a linear model equation and present some results relating to the existence of edge-wave solutions. Finally, in Chapter 4, we present results concerning particle trajectories in water waves, both in the linear setting as well as for the nonlinear governing equations.

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Chapter 1

Introduction

In this thesis we study certain aspects of the mathematical theory of water waves. The mathematical theory of water waves has a long history, dating as far back as Newton's attempts to construct a rigorous theory in 1687 [39]. We can view the mathematical theory of water waves as essentially consisting of two branches: in the first instance the aim is to construct model equations which represent the wave motion as accurately and, ideally, as elegantly as possible; in the second instance the task for the mathematician consists of trying to glean information, either qualitative or quantitative, from the model equations by using tools from pure or applied mathematics. In the best-case scenario we would obtain explicit solutions for the model equations.

It is inevitable in the course of constructing model equations that certain assumptions or simplifications must be utilised. In the work presented in this thesis the fluid is assumed to be a continuous medium, the so-called *continuum* hypothesis. We also adopt the simplifying assumptions that the water is *inviscid* (no internal friction forces) and *homogenous* (constant density). Under these assumptions the governing equations of hydrodynamics were derived in the 1750's [39] by Leonhard Euler and are known as Euler's equation.

Other than the nonlinear governing equations we analyse some recently-derived model equations for shallow water waves. Since Euler's equation is highly nonlinear it is appropriate as a first study to analyse the equations in a linear setting (see Chapter 4). However much of the recent work in the modelling of hydrodynamics has seen a departure from the various linearisations of Euler's equations and instead the focus has shifted to the study of various nonlinear model equations which approximate Euler's equations and which also have interesting structural properties (see Chapter 2).

The layout of this thesis is as follows. In Chapter 2 we present some qualitative aspects of two recently-derived nonlinear models for shallow water waves. The model equations are approximations of the full Euler equations and are also members of a general family of equations which have interesting properties inherent to their structure.

In Chapter 3 we present some results for edge waves. Edge waves are confined to the nearshore area of a sea or ocean and propagate in the longshore direction. Edge waves are commonly generated by sea-storms and it is believed that they have a significant role to play in the transport of seashore sediment. However it is difficult to observe edge waves and perhaps it is for this reason that they have been somewhat neglected in the mathematical literature on water waves. In Chapter 3 we investigate a linear model equation and present some results relating to the existence of edge-wave solutions.

Finally in Chapter 4 we present results concerning particle trajectories in water waves. We work both within the linear framework as well as with the nonlinear governing equations, and will deal with both deep water and shallow water waves.

Chapter 2

Qualitative aspects of some nonlinear water-wave models

2.1 Evolution of compactly supported solutions for the $b-$equations

2.1.1 The $b-$equations

In this section we study a parameterised-family of nonlinear partial differential equations — the so-called $b-$equations (2.1). Two members of this family model the unidirectional propagation of waves in shallow water over a flat bed. Each member of the family in general possess interesting structural properties. In this section we are interested in the evolution of solutions of the equations which are initially compactly supported. We relax this condition in the course of the analysis in section 2.2. Here we aim to ascertain if any nontrivial solutions u, m to equations (2.1) and (2.2) respectively retain the property of being compactly supported over time. The results we obtain enable us to provide concrete answers to this question with regard to solutions of (2.2) for any $b \in \mathbb{R}$, and for a restricted range of equations in the $b-$family for solutions of (2.1).

The $b-$equations are the family of nonlinear partial differential equations of the form

$$u_t - u_{txx} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}, \quad x \in \mathbb{R},\ t \geq 0, \tag{2.1}$$

where $b \in \mathbb{R}$ parameterises the family. These $b-$equations can also be re-expressed in the form

$$m_t + bu_xm + um_x = 0, \quad x \in \mathbb{R},\ t > 0, \tag{2.2}$$

where $m = u - u_{xx}$. The $b-$equations (2.1) belong to the more general family of nonlinear dispersive partial differential equations

$$u_t - \gamma u_{xxx} - \alpha^2 u_{txx} = (c_1u^2 + c_2u_x^2 + c_3uu_{xx})_x$$

where γ, α, c_1, c_2 and c_3 are real constants. Degasperis and Procesi [43] found that, up to rescaling, there are only three equations that are integrable within this family: the Korteweg-de Vries (KdV) equation

$$u_t - uu_x + u_{xxx} = 0,$$

the Camassa-Holm equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{CH}$$

and the Degasperis-Procesi equation

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{DP}$$

The Camassa-Holm (CH) [9] equation arises in a variety of different contexts. It has been widely studied since 1993 when Camassa and Holm [9] proposed it as a model for the unidirectional propagation of shallow water waves over a flat bed. The CH equation has an interesting background. The derivation of the CH equation followed in [9] suppresses the role played by the vertical $y-$coordinate. Furthermore, the CH equation can not obtained via the standard asymptotic approach, cf. KdV. However, in the 2002 paper by Johnson a $y-$dependence is derived whereby, in the shallow water setting, the CH equation is a model equation where $u(x,t)$ represents the horizontal velocity component of the fluid motion at a certain depth over a flat bed in nondimensional variables [68, 66]. We note that the equation was originally derived in 1981 as a bi-Hamiltonian equation with infinitely many conservation laws by Fokas and Fuchssteiner [49]. Furthermore, the CH equation is also a re-expression of geodesic flow on the diffeomorphism group of the line [15, 75, 83] and it models axially symmetric waves in hyperelastic rods [40]. The CH equation is integrable [9, 16, 31, 49, 76], models wave breaking in the shallow water regime as well as the propagation of permanent waves [15, 22, 23, 84], and its solitary waves are stable solitons [9, 34]. There is in fact a whole hierarchy of integrable equations associated with the CH equation, cf. [52].

The Degasperis-Procesi (DP) equation was derived in 1999 by Degasperis and Procesi [43]. This equation has a physical derivation similar to the CH equation modelling the unidirectional propagation of shallow water waves over a flat bed c.f. [68, 66]. It is formally integrable [42] and has infinitely many conservation laws, however it has not currently been afforded any geometrical interpretation. Certain classical solutions of the DP equation exist for all times, whereas others blow up in finite time [48, 93, 94]— a situation which occurs for the CH equation [15, 22] but not for KdV (where all classical solutions are global).

This section is arranged as follows. Before proving results relating to solutions of the $b-$equations we firstly present in Section 2.1.2 the local well-posedness of (2.1) and of (2.2) for all $b \in \mathbb{R}$ based on Kato's semigroup approach [72]. In Section 2.1.3 we address our main point of our interest, that is we investigate for which values of $b \in \mathbb{R}$ the classical solutions m, u of (2.2) and (2.1) respectively will have compact support for all times t in a nontrivial interval if their initial data has this property. We prove that this holds for any $b \in \mathbb{R}$ throughout the maximal time interval of existence for classical solutions m of (2.2), whereas for b in the range $0 \leq b \leq 3$ any nontrivial classical solution u of (2.1) loses instantly the property of being compactly supported. Particularising to $b = 2$ and $b = 3$ in our results we recover the recent results obtained in [19, 56, 57, 84] for the CH and for the DP equation respectively, the two equations of importance which motivate our investigation.

2.1.2 Local Well-Posedness

To prove the local existence of solutions for (2.1) one can implement Kato's semigroup approach [72], following the example of the CH equation [23]. If $p(x) = \frac{1}{2}e^{-|x|}$ for all $x \in \mathbb{R}$, and $*$ denotes convolution, then for all $f \in L^2(\mathbb{R})$ we have

$$(1-\partial_x^2)\,(p*f)\,(x) = \tfrac{1}{2}(1-\partial_x^2)\left[\int_{-\infty}^{x} e^{y-x}f(y)\mathrm{d}y + \int_{x}^{\infty} e^{x-y}f(y)\mathrm{d}y\right] =$$
$$\tfrac{1}{2}\left[\int_{-\infty}^{x} e^{y-x}f(y)\mathrm{d}y + \int_{x}^{\infty} e^{x-y}f(y)\mathrm{d}y - \int_{-\infty}^{x} e^{y-x}f(y)\mathrm{d}y - \int_{x}^{\infty} e^{x-y}f(y)\mathrm{d}y + f(x) + f(x)\right]$$
$$= f(x),$$

and therefore $p * m = u$. Using this identity we rewrite (2.1) as

$$u_t + uu_x + \partial_x p * \left(\frac{3-b}{2}u_x^2 + \frac{b}{2}u^2\right) = 0, \quad x \in \mathbb{R},\ t \geq 0. \tag{2.3}$$

As was shown in Yin's paper [93], for any $b \in \mathbb{R}$ equation (2.3) is suitable for applying Kato's theory [72], using a framework similar to those provided in [23, 85] for the case of the CH equation — see also [54] for the general case. For $s \geq 0$, let $\mathbb{H}^s(\mathbb{R})$ be the Sobolev space of functions $f \in L^2(\mathbb{R})$ with the property that the Fourier transform $\hat{f}$ satisfies

$\int_{\mathbb{R}} (1+|\xi|)^{2s} |\hat{f}(\xi)|^2 \, d\xi < \infty$. Let us now state [93, Theorem 1], which we adapt to the present situation:

Theorem 2.1.1 *For all $b \in \mathbb{R}$, given $u_0 \in \mathbb{H}^s(\mathbb{R}), s > \frac{3}{2}$, there exists a maximal $T = T(b, u_0) > 0$, and a unique solution u to (2.1), such that*

$$u = u(\cdot, u_0) \in C([0,T), \mathbb{H}^s(\mathbb{R})) \cap C^1([0,T), \mathbb{H}^{s-1}(\mathbb{R})).$$

Moreover, the solution depends continuously on the initial data, and if $u_0 \in \mathbb{H}^4(\mathbb{R})$, then $u \in C^2([0,T), \mathbb{H}^2(\mathbb{R}))$.

Remark The lower bound 3/2 which is stated for s in Theorem 2.1.1 is the optimal lower bound resulting from Kato's theory. In the case where we require a higher level of differentiability of our solution u we will assume $u_0 \in \mathbb{H}^s(\mathbb{R})$ for a higher value of s.

Given the Sobolev embedding $\mathbb{H}^{r+1}(\mathbb{R}) \subset C^r(\mathbb{R})$ for $r \geq 0$, we have, in view of Theorem 2.1.1, that $u \in C([0,T), C^3(\mathbb{R})) \cap C^1([0,T), C^2(\mathbb{R}))$ if $u_0 \in \mathbb{H}^4(\mathbb{R})$. Thus, if $u_0 \in \mathbb{H}^4(\mathbb{R})$, we get a corresponding classical solution $m \in C^1([0,T) \times \mathbb{R}, \mathbb{R})$ of (2.2) with initial data $m_0 = u_0 - u_{0,xx}$.

2.1.3 Main Results

Knowing the existence of classical solutions to (2.2) for all $b \in \mathbb{R}$, we now set about examining the propagation speed of these solutions. The following proposition says that, for any $b \in \mathbb{R}$, a classical solution m of (2.2) with compactly supported initial data m_0 will remain compactly supported on some finite time interval.

Proposition 2.1.2 *Assume that $u_0 \in \mathbb{H}^4(\mathbb{R})$ is such that $m_0 = u_0 - u_{0,xx}$ has compact support. If $T = T(b, u_0) > 0$ is the maximal existence time of the unique solution $u(x,t)$ to (2.1) with initial data $u_0(x)$, then for any $t \in [0,T)$ the C^1 function $x \mapsto m(x,t)$ has compact support.*

Proof We associate with the solution m the family $\{\varphi(\cdot,t)\}_{t\in[0,T)}$ of increasing C^2 diffeomorphisms of the line defined by

$$\varphi_t(x,t) = u(\varphi(x,t),t), \qquad t\in[0,T), \tag{2.4}$$

with

$$\varphi(x,0) = x, \qquad x\in\mathbb{R}. \tag{2.5}$$

The claimed smoothness of the functions φ follows from classical results on the dependence on parameters of the solutions of differential equations [1].

Using (2.2) and (2.4)–(2.5), the fact that $\varphi_x > 0$ (see (2.7)), and then differentiating with respect to t, we can check that the following identity holds:

$$m(\varphi(x,t),t)\cdot\varphi_x^b(x,t) = m(x,0), \qquad x\in\mathbb{R}, t\in[0,T). \tag{2.6}$$

Note that if we are dealing with the case $b=0$, then (2.6) gives

$$m(\cdot,t) = m_0 \qquad x\in\mathbb{R},\ t\in[0,T),$$

and so $m(\cdot,t)$ is automatically compactly supported. For the other cases, we infer from (2.4)–(2.5) that

$$\varphi_x(x,t) = \exp\left(\int_0^t u_x(\varphi(x,s),s)\,ds\right), \qquad x\in\mathbb{R}, t\in[0,T). \tag{2.7}$$

It follows that if m_0 is supported in the compact interval $[\alpha,\beta]$, then since $\varphi_x(x,t)>0$ on $\mathbb{R}\times[0,T)$ from (2.7), we can conclude from (2.6) that $m(\cdot,t)$ has its support in the interval $[\varphi(\alpha,t),\varphi(\beta,t)]$. ∎

It is interesting to note that this result holds for all $b\in\mathbb{R}$. We now show that the propagation speed of a classical solution u of (2.1) is infinite for b in the range $0\le b\le 3$.

Theorem 2.1.3 *Fix $0\le b\le 3$. Assume that the function $u_0\in\mathbb{H}^4(\mathbb{R})$ has compact support. Let $T>0$ be the maximal existence time of the unique solution $u(x,t)$ to (2.1) with initial data $u_0(x)$. If at every $t\in[0,T)$ the C^2 function $x\mapsto u(x,t)$ has compact support, then u must be identically zero.*

For $(x, t) \in \mathbb{R} \times [0, T)$, let $m(x, t) = (1 - \partial_x^2)u(x, t)$. Clearly $m(\cdot, 0)$ has compact support since u_0 does. By Proposition 2.1.2 the C^1 function $x \mapsto m(x, t)$ has compact support for all $t \in [0, T)$. By means of taking Fourier transforms we arrive at the following explicit formula for u in terms of m,

$$u(x, t) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} m(y, t) \, dy,$$

that is,

$$2u(x, t) = e^{-x} \int_{-\infty}^{x} e^{y} m(y, t) \, dy + e^{x} \int_{x}^{\infty} e^{-y} m(y, t) \, dy. \tag{2.8}$$

In order to prove Theorem 2.1.3 we use the following result [56] which holds true for all of the $b-$equations (2.1).

Proposition 2.1.4 *Let $u \in C^2(\mathbb{R}) \cap \mathbb{H}^2(\mathbb{R})$ be such that $m = u - u_{xx}$ has compact support. Then u has compact support if and only if*

$$\int_{\mathbb{R}} e^{x} m(x) \, dx = \int_{\mathbb{R}} e^{-x} m(x) \, dx = 0. \tag{2.9}$$

Proof Assume that u has compact support. Let $0 < N_m \in \mathbb{R}$ be a positive constant large enough that $m(x) \equiv 0$ for all $|x| > N_m$, and let N_u be defined accordingly with respect to the compactly supported function u. Let $N = \max\{N_m, N_u\}$ and consider firstly the case $x > N$. Writing (2.8) as the sum of three integrals as follows

$$2u(x) = e^{-x} \int_{-\infty}^{-x} e^{y} m(y) \, dy + \; e^{-x} \int_{-x}^{x} e^{y} m(y) \, dy + \; e^{x} \int_{x}^{\infty} e^{-y} m(y) \, dy,$$

it is obvious that the first and third integrals are both identically zero since $m(y) = 0$ for $|y| \geq |x| > N$. It follows that $u(x) = 0$ for all $x > N$ if and only if $e^{-x} \int_{-x}^{x} e^{y} m(y) \, dy = 0$ for all $x > N$, that is, if and only if $\int_{-x}^{x} e^{y} m(y) \, dy = 0$ for all $x > N$. Since $u(x)$ has compact support we infer that

$$\lim_{x \to \infty} \int_{-x}^{x} e^{y} m(y) \, dy = \int_{\mathbb{R}} e^{y} m(y) \, dy = 0.$$

Upon considering the case $-x > N$ it follows similarly that $\int_{\mathbb{R}} e^{-x} m(x) \, dx = 0$ if u has compact support, proving the second relation in (2.9).

To prove the converse we know that m has compact support, which means that there is a constant $N > 0$ such that $m(x) = 0$ for all $|x| > N$. Assume that (2.9) holds. Therefore

$$\int_{\mathbb{R}} e^x m(x)\,\mathrm{d}x = \int_{-N}^{N} e^x m(x)\,\mathrm{d}x = 0 \tag{A}$$

$$\int_{\mathbb{R}} e^{-x} m(x)\mathrm{d}x = \int_{-N}^{N} e^{-x} m(x)\,\mathrm{d}x = 0. \tag{B}$$

Pick $x > N$. We can now write equation (2.8) as

$$2u(x) = e^{-x}\int_{-N}^{N} e^y m(y)\,\mathrm{d}y + e^x \int_x^{\infty} e^{-y} m(y)\,\mathrm{d}y = 0,$$

if we take into account (A) and the fact that $m(y) = 0$ for all $y \geq x > N$.

Similarly, for $x < -N$, equation (2.8) becomes

$$2u(x) = e^{-x}\int_{-\infty}^{x} e^y m(y)\,\mathrm{d}y + e^x \int_{-N}^{N} e^{-y} m(y)\,\mathrm{d}y = 0.$$

So u has compact support. ∎

Proof of Theorem 2.1.3 We assume u has compact support and show that this implies $u \equiv 0$. Using (2.2) and differentiating (2.9) with respect to t we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} e^x m(x,t)\,\mathrm{d}x = \int_{\mathbb{R}} e^x m_t\,\mathrm{d}x = -b\int_{\mathbb{R}} e^x m u_x\,\mathrm{d}x - \int_{\mathbb{R}} e^x m_x u\,\mathrm{d}x$$

$$\begin{aligned}
&= -b\int_{\mathbb{R}} e^x m u_x\,\mathrm{d}x + \int_{\mathbb{R}} e^x m u_x\,\mathrm{d}x + \int_{\mathbb{R}} e^x m u\,\mathrm{d}x \\
&= (1-b)\int_{\mathbb{R}} e^x m u_x\,\mathrm{d}x + \int_{\mathbb{R}} e^x m u\,\mathrm{d}x \\
&= (1-b)\int_{\mathbb{R}} e^x u u_x\,\mathrm{d}x - (1-b)\int_{\mathbb{R}} e^x u_x u_{xx}\,\mathrm{d}x + \int_{\mathbb{R}} e^x u^2\,\mathrm{d}x - \int_{\mathbb{R}} e^x u u_{xx}\,\mathrm{d}x \\
&= (1-b)\int_{\mathbb{R}} e^x u u_x\,\mathrm{d}x + \frac{1-b}{2}\int_{\mathbb{R}} e^x u_x^2\,\mathrm{d}x + \int_{\mathbb{R}} e^x u^2\,\mathrm{d}x + \int_{\mathbb{R}} e^x u_x(u+u_x)\,\mathrm{d}x \\
&= \frac{b}{2}\int_{\mathbb{R}} e^x u^2\,\mathrm{d}x + \frac{3-b}{2}\int_{\mathbb{R}} e^x u_x^2\,\mathrm{d}x,
\end{aligned}$$

where all boundary terms after integration by parts vanish since both $m(\cdot,t)$ and, by assumption, $u(\cdot,t)$ have compact support for all $t \in [0,T)$. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} e^x m(x,t)\,\mathrm{d}x = \int_{\mathbb{R}} e^x \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right)\,\mathrm{d}x, \quad t \in [0,T). \tag{2.10}$$

Similarly, for the sake of completeness we note that upon differentiating the second integral in (2.9), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} e^{-x} m(x,t)\,\mathrm{d}x = \int_{\mathbb{R}} e^{-x}\left(\frac{-b}{2}u^2 + \frac{b-3}{2}u_x^2\right)\,\mathrm{d}x, \quad t \in [0,T). \tag{2.11}$$

We will use relation (2.10) to complete the proof — the same result can be proved by following a similar reasoning with (2.11). Relations (2.10) and (2.11) are equivalently useful for our purposes. Since $u(x,t)$ has compact support, relation (2.9) implies that (2.10) $\equiv$ (2.11) $\equiv$0. In particular for fixed b in the range $0 \leq b \leq 3$ the condition that the integral on the right hand side of (2.10) must be identically zero implies that either u or u_x is identically zero. Since u is compactly supported both of these situations imply $u \equiv 0$. The proof is complete. ∎

Remark We note that $0 \leq b \leq 3$ includes the two main cases of our interest, namely $b = 2$ (CH) and $b = 3$ (DP). However the relation derived above in equation (2.10) seems to offer us no results relating to any values of b outside of this range, which suggests that perhaps another approach is necessary in order to prove Theorem 2.1.3 for the entire family of $b-$equations.

Remark If $u_0 \not\equiv 0$ is a function in $\mathbb{H}^4(\mathbb{R})$ with compact support, then for b in the range $0 \leq b \leq 3$ the classical solution $u(\cdot,t)$ of (2.1) loses instantly the property of having compact support. To see this we go through the same argument as above, this time restricting our attention to an arbitrarily small time interval $[0,\varepsilon)$.

2.2 Persistence properties for the Degasperis-Procesi equation

In this section we focus on the Degasperis-Procesi (DP) equation with the intention of improving on the results obtained in the previous Section 2.1. In Section 2.1 we showed that if a solution of the DP equation was initially supported on a compact interval then it would instantly lose this property [57]. In the theorems that follow we admit solutions which belong to a less restrictive class (the initial data decays exponentially or faster at infinity). We derive results concerning the persistence of these asymptotic properties as the solutions evolve. Analogous results for the Camassa-Holm equation were previously obtained in [63]. The Degasperis-Procesi (DP) equation takes the form

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad t \geq 0, \; x \in \mathbb{R}, \tag{2.12}$$

or equivalently,

$$m_t + 3u_x m + um_x = 0, \quad x \in \mathbb{R}, \; t > 0, \tag{2.13}$$

where $m = u - u_{xx}$. If $p(x) = \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$, then $(1 - \partial_x^2)^{-1} f = p * f$ for all $f \in L^2(\mathbb{R})$ and so $p * m = u$, where $*$ denotes convolution. Using this identity we rewrite (2.12) as

$$u_t + uu_x + \frac{3}{2}\partial_x p * u^2 = 0, \quad x \in \mathbb{R}, \; t \geq 0, \tag{2.14}$$

this form of the DP equation will be convenient for what is to follow.

Notation

(a) We write $|f(x)| \sim O(g(x))$ as $x \to \infty$ if $\lim_{x\to\infty} \frac{|f(x)|}{g(x)} = C$, for a constant C.

(b) We write $|f(x)| \sim o(g(x))$ as $x \to \infty$ if $\lim_{x\to\infty} \frac{|f(x)|}{g(x)} = 0$.

We will allow the constant C to be zero, in which case $(a) \iff (b)$. For $s \geq 0$ and $p \geq 1$ we denote by $\|\cdot\|_{\mathbb{H}^s}$ the norm in the Sobolev space $\mathbb{H}^s$ and by $\|\cdot\|_p$ the norm in L^p. Furthermore, in what follows we employ subscripts to represent the partial differentiation of strong solutions with respect to the variables, while partial differentiation in the sense of distributions is represented by the $\partial_{\text{“}variable\text{''}}$ operator.

2.2.1 Propagation speed of the Degasperis-Procesi equation

If we assume that $u_0 \in \mathbb{H}^4(\mathbb{R})$ then (see Section 2.1.2 and [94]) there is a maximal time $T = T(u_0) > 0$ such that (2.12) has a unique solution with

$$u \in C([0,T), \mathbb{H}^4) \cap C^1([0,T), \mathbb{H}^3) \cap C^2([0,T), \mathbb{H}^2). \tag{2.15}$$

Accordingly,

$$m \in C([0,T), \mathbb{H}^2) \cap C^1([0,T), \mathbb{H}^1) \cap C^2([0,T), L^2). \tag{2.16}$$

Given the Sobolev embedding $\mathbb{H}^{k+1}(\mathbb{R}) \subset C^k(\mathbb{R})$ for $k \geq 0$, we have that $u \in C^2([0,T) \times \mathbb{R}, \mathbb{R})$ and $m \in C^1([0,T) \times \mathbb{R}, \mathbb{R})$, thus ensuring the local existence of strong solutions to the DP equation (2.12). If we express the DP equation as (2.14), then it was shown in [93] (see Section 2.1.2) that applying Kato's theory ensures the local existence of strong solutions to equation (2.14) for $u_0 \in \mathbb{H}^s$, with s having the strict lower bound $s > 3/2$. In general these are weak solutions to the DP equation (2.12). Throughout the remainder of this section we assume the local existence of strong solutions to equation (2.14) for $T > 0$. It is known (see [84, 93]) that $T = \infty$ or $T < \infty$. The latter situation occurs if the solution "blows-up", in which case the solution remains bounded but the spatial-derivative becomes infinite, that is,

$$\lim_{t \to T} \sup \|u_x(t)\|_\infty = \infty. \tag{2.17}$$

Our first theorem tells us that a solution u of (2.14) which initially has a weighted-exponential decay rate at infinity will retain the same decay rate at all later times of its existence. Compactly supported functions are obviously included in this class of initial data, and we saw in Section 2.1 that nontrivial solutions of (2.14) which initially have compact support lose instantaneously the property of being compactly supported: they have an infinite propagation speed. The following theorem provides some information on the asymptotic behaviour of these solutions throughout their evolution.

Theorem 2.2.1 *For $s > 3/2$, let $T > 0$ be the maximal existence time of the strong solution $u \in C([0,T), \mathbb{H}^s(\mathbb{R}))$ to equation (2.14) with initial data $u_0 = u(x,0)$. Suppose* *there is a $\theta \in (0,1)$ such that both*

$$|u_0(x)| \text{ and } |u_{0,x}(x)| \sim O(e^{-\theta|x|}) \text{ as } |x| \to \infty. \tag{2.18}$$

Then both

$$|u(x,t)| \text{ and } |u_x(x,t)| \sim O(e^{-\theta|x|}) \text{ as } |x| \to \infty, \tag{2.19}$$

uniformly in $[0, T-\epsilon]$ with $\epsilon \in (0,T)$.

Proof We begin by multiplying equation (2.14) by u^{2q-1} for any $q \in \mathbb{Z}^+$. Then integrating the result in the $x-$variable we obtain

$$\int_{-\infty}^{\infty} u^{2q-1} u_t \mathrm{d}x + \int_{-\infty}^{\infty} u^{2q-1} u u_x \mathrm{d}x + \frac{3}{2}\int_{-\infty}^{\infty} u^{2q-1} \partial_x p * u^2 \mathrm{d}x = 0, \tag{2.20}$$

and so it follows that

$$\int_{-\infty}^{\infty} u^{2q-1} u_t \mathrm{d}x \leq \left|\int_{-\infty}^{\infty} u^{2q-1} u u_x \mathrm{d}x\right| + \frac{3}{2}\left|\int_{-\infty}^{\infty} u^{2q-1} \partial_x p * u^2 \mathrm{d}x\right|. \tag{2.21}$$

Using the identity

$$\int_{-\infty}^{\infty} u^{2q-1} u_t \mathrm{d}x = \frac{1}{2q}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{2q}^{2q} = \|u(t)\|_{2q}^{2q-1}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{2q}, \tag{2.22}$$

together with the estimates

$$\left|\int_{-\infty}^{\infty} u^{2q-1} u u_x \mathrm{d}x\right| \leq \|u(t)\|_{2q}^{2q}\|u_x(t)\|_{\infty}, \tag{2.23}$$

$$\left|\int_{-\infty}^{\infty} u^{2q-1} \partial_x p * u^2 \mathrm{d}x\right| \overset{\left(\substack{Hölder's \\ inequality}\right)}{\leq} \|u(t)\|_{2q}^{2q-1}\left\|\partial_x p * u^2(t)\right\|_{2q}, \tag{2.24}$$

we obtain from (2.21) the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{2q} \leq \|u_x(t)\|_{\infty}\|u(t)\|_{2q} + \frac{3}{2}\left\|\partial_x p * u^2(t)\right\|_{2q}. \tag{2.25}$$

Multiplying both sides of (2.25) by $e^{-\int_0^t \|u_x(\tau)\|_\infty \mathrm{d}\tau}$ and integrating we get

$$\|u(t)\|_{2q} \leq \left(\|u(0)\|_{2q} + \frac{3}{2}\int_0^t \left\|\partial_x p * u^2(\tau)\right\|_{2q}\mathrm{d}\tau\right) e^{\int_0^t \|u_x(\tau)\|_\infty \mathrm{d}\tau}, \quad t \in [0, T-\epsilon], \tag{2.26}$$

where if $T = \infty$ we define $[0, T-\epsilon] := [0, \infty)$. We remark that any function $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is contained in $L^r(\mathbb{R})$ for all $1 \le r \le \infty$ and furthermore

$$\lim_{r \to \infty} \|f\|_r = \|f\|_\infty .$$

Since $\partial_x p \in L^1$ and $u^2 \in L^1 \cap L^\infty$, we have by Young's inequality that $\partial_x p * u^2(\tau) \in L^1 \cap L^\infty$. Thus we can let $q \to \infty$ in the norms in (2.26) to get

$$\|u(t)\|_\infty \le \left(\|u(0)\|_\infty + \frac{3}{2} \int_0^t \left\|\partial_x p * u^2(\tau)\right\|_\infty \mathrm{d}\tau \right) e^{\int_0^t \|u_x(\tau)\|_\infty \mathrm{d}\tau}, \quad t \in [0, T-\epsilon]. \tag{2.27}$$

On the other hand, if we differentiate equation (2.14) with respect to the $x-$variable we get

$$u_{tx} + u u_{xx} + u_x^2 + \frac{3}{2} \partial_x^2 p * u^2 = 0. \tag{2.28}$$

We work with this equation in a similar manner as in steps (2.20)-(2.27), the only difference being that we multiply equation (2.28) by u_x^{2q-1} before we integrate in the $x-$variable. Applying integration by parts in the second term and then taking estimates we finally arrive at the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u_x(t)\|_{2q} \le 2 \|u_x(t)\|_\infty \|u_x(t)\|_{2q} + \frac{3}{2} \left\|\partial_x^2 p * u^2(t)\right\|_{2q} . \tag{2.29}$$

As above we get

$$\|u_x(t)\|_{2q} \le \left(\|u_x(0)\|_{2q} + \frac{3}{2} \int_0^t \left\|\partial_x^2 p * u^2(\tau)\right\|_{2q} \mathrm{d}\tau \right) e^{2\int_0^t \|u_x(\tau)\|_\infty \mathrm{d}\tau}, \quad t \in [0, T-\epsilon]. \tag{2.30}$$

Because $\partial_x^2 p * u^2 = \partial_x p * \partial_x u^2$ and $\partial_x u^2 = 2uu_x \in L^1 \cap L^\infty$ we can take limits of the norm on each side to get

$$\|u_x(t)\|_\infty \le \left(\|u_x(0)\|_\infty + \frac{3}{2} \int_0^t \left\|\partial_x^2 p * u^2(\tau)\right\|_\infty \mathrm{d}\tau \right) e^{2\int_0^t \|u_x(\tau)\|_\infty \mathrm{d}\tau}, \quad t \in [0, T-\epsilon]. \tag{2.31}$$

For each $N \in \mathbb{Z}^+$ we define the weight function, for $\theta \in (0,1)$, by

$$\phi_N(x) = \begin{cases} 1, & x \le 0, \\ e^{\theta x}, & x \in (0, N), \\ e^{\theta N}, & x \ge N. \end{cases} \tag{2.32}$$

It follows from direct examination that for each $N \in \mathbb{Z}^+$ we have

$$0 \leq \phi_N'(x) \leq \phi_N(x) \quad \textit{for all } x \in \mathbb{R}\backslash\{0, N\}. \tag{2.33}$$

Upon multiplying equation (2.14) and equation (2.28) by ϕ_N we get

$$\begin{aligned}(u\phi_N)_t + (u\phi_N)u_x + \frac{3}{2}\phi_N \partial_x p * u^2 &= 0,\\ (u_x\phi_N)_t + uu_{xx}\phi_N + (u_x\phi_N)u_x + \frac{3}{2}\phi_N \partial_x^2 p * u^2 &= 0.\end{aligned} \tag{2.34}$$

Let us multiply the second equation in (2.34) by $(u_x\phi_N)^{2q-1}$ and integrate with respect to the $x-$variable. The second term of the resulting expression, after integration by parts and using (2.33), can be estimated as follows

$$\begin{aligned}\left|\int_{-\infty}^{\infty} uu_{xx}\phi_N(u_x\phi_N)^{2q-1}\mathrm{d}x\right| &= \left|\int_{-\infty}^{\infty} u(u_x\phi_N)^{2q-1}\left((u_x\phi_N)_x - u_x\phi_N'\right)\mathrm{d}x\right|\\ &= \left|\int_{-\infty}^{\infty} u\left\{\left(\frac{(u_x\phi_N)^{2q}}{2q}\right)_x - u_x\phi_N'(u_x\phi_N)^{2q-1}\right\}\mathrm{d}x\right|\\ &\leq C\left(\|u(t)\|_\infty + \|u_x(t)\|_\infty\right)\|u_x\phi_N\|_{2q}^{2q}.\end{aligned} \tag{2.35}$$

Working as before, after adding up the equations resulting from (2.34), we eventually arrive at the following inequality for the weighted function:

$$\begin{aligned}\|u(t)\phi_N\|_\infty + \|u_x(t)\phi_N\|_\infty \leq \Big\{ &\|u(0)\phi_N\|_\infty + \|u_x(0)\phi_N\|_\infty +\\ &+ \frac{3}{2}\int_0^t \left(\left\|\phi_N\partial_x p * u^2(\tau)\right\|_\infty + \left\|\phi_N \partial_x^2 p * u^2(\tau)\right\|_\infty\right)\mathrm{d}\tau\Big\}\\ &\times e^{2\int_0^t(\|u_x(\tau)\|_\infty + \|u(\tau)\|_\infty)\mathrm{d}\tau}, \quad t \in [0, T-\epsilon].\end{aligned} \tag{2.36}$$

This expression requires further manipulation before we are in a position to apply Gronwall's inequality, and we will use the following result.

Lemma 2.2.2 *Given any $x \in \mathbb{R}$ and any $0 \leq \theta < 1$, we have*

$$I \equiv \phi_N(x)\int_{-\infty}^{\infty} \frac{e^{-|x-y|}}{\phi_N(y)}\mathrm{d}y \leq 2 + \frac{\theta}{1-\theta} = C > 0. \tag{2.37}$$

Proof We express the integral as

$$I = e^{-x}\phi_N(x)\int_{-\infty}^{x}\frac{e^y}{\phi_N(y)}\mathrm{d}y + e^x\phi_N(x)\int_x^{\infty}\frac{e^{-y}}{\phi_N(y)}\mathrm{d}y \equiv I_1 + I_2.$$

Since ϕ_N is increasing, the second integral is bounded by

$$I_2 \le e^x\int_x^{\infty} e^{-y}\mathrm{d}y = 1.$$

To estimate I_1, we integrate by parts to get

$$I_1 = e^{-x}\phi_N(x)\left[e^x\phi_N^{-1}(x) + \int_{-\infty}^{x} e^y\phi_N^{-2}(y)\phi_N'(y)\mathrm{d}y\right].$$

When $x \le 0$, this gives $I_1 = 1$ and we are done. For $x > 0$, then

$$\begin{aligned}
I_1 &= 1 + \theta e^{-x}\phi_N(x)\int_0^{\min(x,N)} e^{(1-\theta)y}\mathrm{d}y \\
&= 1 + \theta e^{-x}e^{\theta\min(x,N)}\left[\frac{e^{(1-\theta)\min(x,N)} - 1}{1-\theta}\right] \\
&\le 1 + \frac{\theta e^{\min(x,N)-x}}{1-\theta} \\
&\le 1 + \frac{\theta}{1-\theta}.
\end{aligned}$$

∎

We now use Lemma 2.2.2 to derive estimates for the terms in (2.36) which involve convolutions. For a solution u of (2.14) we have

$$\begin{aligned}
\left|\phi_N\partial_x p * u^2(x)\right| &= \left|\frac{1}{2}\phi_N(x)\int_{-\infty}^{\infty} sgn(x-y)e^{-|x-y|}u^2(y)\mathrm{d}y\right| \\
&\le \frac{1}{2}\phi_N(x)\int_{-\infty}^{\infty} e^{-|x-y|}\frac{1}{\phi_N(y)}\phi_N(y)u(y)u(y)\mathrm{d}y \\
&\le \frac{1}{2}\left(\phi_N(x)\int_{-\infty}^{\infty} e^{-|x-y|}\frac{1}{\phi_N(y)}\mathrm{d}y\right)\|\phi_N u\|_\infty\|u\|_\infty \\
&\le C\|\phi_N u\|_\infty\|u\|_\infty, \qquad (2.38)
\end{aligned}$$

with the constant C given by equation (2.37). By the definition of convolution it follows that

$$\partial_x^2 p * u^2 = \partial_x p * (u^2)_x = 2\partial_x p * (uu_x),$$

so following the method of (2.38) we get

$$\left|\phi_N \partial_x^2 p * u^2(x)\right| \leq 2C\|\phi_N u_x\|_\infty \|u\|_\infty. \tag{2.39}$$

Using the relations (2.38) and (2.39) in (2.36) it follows that

$$\|u(t)\phi_N\|_\infty + \|u_x(t)\phi_N\|_\infty \leq \Bigg\{ \|u(0)\phi_N\|_\infty + \|u_x(0)\phi_N\|_\infty \tag{2.40}$$

$$+ 3C \int_0^t \|u\|_\infty \left(\|u(\tau)\phi_N\|_\infty + \|u_x(\tau)\phi_N\|_\infty\right) \mathrm{d}\tau \Bigg\} e^{2\int_0^t (\|u_x(\tau)\|_\infty + \|u(\tau)\|_\infty)\mathrm{d}\tau}, \quad t \in [0, T-\epsilon].$$

Taking into account the boundedness of $\|u(\tau)\|_\infty$ and $\|u_x(\tau)\|_\infty$ on the interval $[0, T-\epsilon]$ (see the blow-up pattern described by (2.17)), and applying Gronwall's inequality, we have

$$\begin{aligned} \|u(t)\phi_N\|_\infty + \|u_x(t)\phi_N\|_\infty &\leq C^\dagger \left(\|u(0)\phi_N\|_\infty + \|u_x(0)\phi_N\|_\infty\right) \\ &\leq C^\dagger \left(\left\|u(0)\max(1, e^{\theta x})\right\|_\infty + \left\|u_x(0)\max(1, e^{\theta x})\right\|_\infty\right), \end{aligned} \tag{2.41}$$

where $C^\dagger$ is a constant depending on ϵ. This obviously holds for any $N \in \mathbb{Z}^+$ so we let $N \to \infty$. The proof of the theorem now follows for $x \to \infty$ since we deduce that, for all $t \in [0, T-\epsilon]$,

$$\left(|u(x,t)e^{\theta x}| + |u_x(x,t)e^{\theta x}|\right) \leq C^\dagger \left(\|u(0)\max(1, e^{\theta x})\|_\infty + \|u_x(0)\max(1, e^{\theta x})\|_\infty\right).$$

The proof for $x \to -\infty$ follows in a similar fashion. We define for each $-N \in \mathbb{Z}^+$, and for $\theta \in (0,1)$, the auxiliary weighted function

$$\hat{\phi}_N(x) = \begin{cases} 1, & x \geq 0, \\ e^{-\theta x}, & x \in (0, N), \\ e^{-\theta N}, & x \geq N. \end{cases} \tag{2.42}$$

This auxiliary function $\hat{\phi}$ is the reflection of ϕ in the $x-$axis, and following steps analogous to (2.33)-(2.40), with some minor modifications, we arrive at the inequality

$$\left(|u(x,t)e^{-\theta x}| + |u_x(x,t)e^{-\theta x}|\right) \leq C^\ddagger \left(\|u(0)\max(1, e^{-\theta x})\|_\infty + \|u_x(0)\max(1, e^{-\theta x})\|_\infty\right),$$

where $C^\ddagger$ is a constant depending on ϵ. The proof of the theorem now follows. ∎

The next theorem generalises Theorem 2.1.3 and shows that nontrivial solution of (2.14) which initially decay faster than exponentially at infinity cannot, at any later time in its existence, have a similarly rapid decay rate.

Theorem 2.2.3 *For $s > 3/2$, let $T > 0$ be the maximal existence time of the strong solution $u \in C([0,T), \mathbb{H}^s(\mathbb{R}))$ to equation (2.14) with initial data $u_0 = u(x,0)$. Suppose there is a $\delta \in (1/2, 1)$ such that*

$$|u_0(x)| \sim o(e^{-|x|}) \text{ and } |u_{0,x}(x)| \sim O(e^{-\delta|x|}) \text{ as } |x| \to \infty. \tag{2.43}$$

If there is a $t_1 \in (0,T)$ such that

$$|u(x,t_1)| \sim o(e^{-|x|}) \text{ as } |x| \to \infty, \tag{2.44}$$

then $u \equiv 0$.

Proof We prove this result by examining the asymptotic order of the terms in the equation

$$u(x,t_1) - u(x,0) + \int_0^{t_1} uu_x(x,\tau)\mathrm{d}\tau + 3/2\int_0^{t_1} \partial_x p * u^2 \mathrm{d}\tau = 0, \tag{2.45}$$

as $|x| \to \infty$: this equation follows simply from equation (2.14) after integration on $[0, t_1]$. Without loss of generality, let us focus on the limit $x \to \infty$. We first note that from (2.43) and (2.44) we have

$$u(x,t_1) - u(x,0) \sim o(e^{-x}) \text{ as } x \to \infty. \tag{2.46}$$

Furthermore, relation (2.43) combined with the result of Theorem 2.2.1, imply that

$$\int_0^{t_1} uu_x(x,\tau)\mathrm{d}\tau \sim O(e^{-2\delta x}) \text{ as } x \to \infty, \tag{2.47}$$

which means

$$\int_0^{t_1} uu_x(x,\tau)\mathrm{d}\tau \sim o(e^{-x}) \text{ as } x \to \infty. \tag{2.48}$$

We now claim that if $u \not\equiv 0$ then the last integral in (2.45) does not decay faster than Ce^{-x} for a constant $C \neq 0$, which gives us a contradiction. If we define $\mathcal{I}(x) = \int_0^{t_1} u^2(x,\tau)\mathrm{d}\tau$ then we have

$$\int_0^{t_1} \partial_x p * u^2 \mathrm{d}\tau = \partial_x p * \mathcal{I}(x). \tag{2.49}$$

As in relation (2.47), (2.43) together with Theorem 2.2.1 imply that, as $x \to \infty$,

$$0 \leq \mathcal{I}(x) \sim O(e^{-2\delta x}). \tag{2.50}$$

Expanding we get

$$\partial_x p * \mathcal{I}(x) = -\frac{1}{2}e^{-x} \int_{-\infty}^{x} e^{y}\mathcal{I}(y)\mathrm{d}y + \frac{1}{2}e^{x} \int_{x}^{\infty} e^{-y}\mathcal{I}(y)\mathrm{d}y, \tag{2.51}$$

with the second integral being of the order $o(e^{-x})$, as $x \to \infty$, by (2.50) and so we must look at the contribution of the first integral. Suppose $u \not\equiv 0$, then $I(y) \not\equiv 0$ and therefore

$$\int_{-\infty}^{x} e^{y}\mathcal{I}(y)\mathrm{d}y \geq K > 0, \qquad \text{for large } x,$$

where K is a constant. Thus for large x we have

$$-\partial_x p * \mathcal{I}(x) \geq \frac{K}{2}e^{-x}, \tag{2.52}$$

and so the last term in (2.45) is of order $O(e^{-x})$, with positive limiting constant greater than $\frac{3K}{4} > 0$: this is incongruous with the orders of the first two terms expressed in (2.46)-(2.48), giving us a contradiction. Therefore $u \equiv 0$. ∎

Theorem 2.2.4 *For $s > 3/2$, let $T > 0$ be the maximal existence time of the strong solution $u \in C([0,T), \mathbb{H}^s(\mathbb{R}))$ to equation (2.14) with initial data $u_0 = u(x,0)$. Suppose there is a $\delta \in (1/2, 1)$ such that*

$$|u_0(x)| \sim O(e^{-|x|}) \text{ and } |u_{0,x}(x)| \sim O(e^{-\delta|x|}) \text{ as } |x| \to \infty. \tag{2.53}$$

Then

$$|u(x,t)| \sim O(e^{-|x|}) \text{ as } |x| \to \infty, \tag{2.54}$$

uniformly in $[0, T-\epsilon]$ with $\epsilon \in (0,T)$.

Proof The proof follows in a similar fashion to Theorem 2.2.1. ∎

It was proven in [57] that the function m defined in (2.13) remains compactly supported if its initial data $m_0 = u_0 - u_{0,xx}$ has compact support. This parallels a similar result for solutions of the Camassa-Holm equation [19, 56]. In the case of the DP equation this result follows at once from the following relation, geometrically motivated for the Camassa-Holm equation [30], but which holds nonetheless for the DP equation [84, 57],

$$m(\varphi(x,t),t)\cdot\varphi_x^3(x,t)=m(x,0), \tag{2.55}$$

where $\varphi(x,t)$ is an increasing diffeomorphism of the line which solves the Cauchy problem

$$\begin{cases} \frac{\mathrm{d}\varphi(x,t)}{\mathrm{d}t}=u(\varphi(x,t),t), \\ \varphi(x,0)=x. \end{cases}$$

Let us now prove the following generalisation of this result.

Theorem 2.2.5 *Let u be a nontrivial solution of equation* (2.14) *with maximal time of existence $T>0$ and $u\in C([0,T),\mathbb{H}^s(\mathbb{R}))$ for $s>5/2$.*

(a) If the initial data $u_0(x)=u(x,0)$ is initially compactly supported on $[\alpha,\beta]$ then for $t\in[0,T)$ we have

$$u(x,t)=\begin{cases} 1/2E_+(t)e^{-x} & \text{for } x>\varphi(\beta,t), \\ 1/2E_-(t)e^{x} & \text{for } x<\varphi(\alpha,t), \end{cases} \tag{2.56}$$

where $E_+>0$ and $E_-<0$ are continuous nonvanishing functions with E_+ strictly increasing and E_- strictly decreasing for $t\in[0,T)$.

(b) Suppose for some constant $\mu>0$ we have

$$u_0,u_{0,x},u_{0,xx}\sim O(e^{-(1+\mu)|x|}) \text{ as } |x|\to\infty, \tag{2.57}$$

then for $t\in[0,T)$ we have

$$m(x,t)\sim O(e^{-(1+\mu)|x|}) \text{ as } |x|\to\infty, \tag{2.58}$$

and

$$\lim_{x\to\pm\infty} e^{\pm x}u(x,t)=1/2E_\pm(t). \tag{2.59}$$

Proof Since $u = p * m$ let us re-express u in the form

$$u(x,t) = \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y}m(y,t)\mathrm{d}y + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-y}m(y,t)\mathrm{d}y. \tag{2.60}$$

Notice equation (2.55) tells us that if u_0 is initially supported on the compact interval $[\alpha, \beta]$ then, for any $t \in [0, T)$, our function $m(\cdot, t)$ is supported on the compact interval $[\varphi(\alpha, t), \varphi(\beta, t)]$. To prove (a) we define

$$E_{+}(t) = \int_{\varphi(\alpha,t)}^{\varphi(\beta,t)} e^{y}m(y,t)\mathrm{d}y \text{ and } E_{-}(t) = \int_{\varphi(\alpha,t)}^{\varphi(\beta,t)} e^{-y}m(y,t)\mathrm{d}y, \tag{2.61}$$

with

$$\begin{aligned} u(x,t) &= p(x) * m(x,t) = \frac{1}{2}e^{-x}E_{+}(t), \quad x > \varphi(\beta,t), \\ u(x,t) &= p(x) * m(x,t) = \frac{1}{2}e^{x}E_{-}(t), \quad x < \varphi(\alpha,t). \end{aligned} \tag{2.62}$$

It follows from the relations in (2.62) that

$$\begin{aligned} u(x,t) &= -u_x(x,t) = u_{xx}(x,t) = \frac{1}{2}e^{-x}E_{+}(t), \quad x > \varphi(\beta,t), \\ u(x,t) &= u_x(x,t) = u_{xx}(x,t) = \frac{1}{2}e^{x}E_{-}(t), \quad x < \varphi(\alpha,t). \end{aligned}$$

Now m_0 is compactly supported and therefore [57, Lemma 2.3] we have $E_{+}(0) = 0$. Since $m(\varphi(\alpha, t)) = m(\varphi(\beta, t)) = 0$, for fixed t we have

$$\frac{\mathrm{d}E_{+}(t)}{\mathrm{d}t} = \int_{\varphi(\alpha,t)}^{\varphi(\beta,t)} e^{y}m_t(y,t)\mathrm{d}y = \int_{-\infty}^{\infty} e^{y}m_t(y,t)\mathrm{d}y. \tag{2.63}$$

Therefore from (2.13) and integration by parts, and using the fact that both u and consequently m have compact support, we get

$$\begin{aligned} \frac{\mathrm{d}E_{+}(t)}{\mathrm{d}t} &= -3\int_{-\infty}^{\infty} e^{y}u_x m\mathrm{d}y - \int_{-\infty}^{\infty} e^{y}u m_x\mathrm{d}y \\ = -2\int_{-\infty}^{\infty} e^{y}u_x m\mathrm{d}y &+ \int_{-\infty}^{\infty} e^{y}u m\mathrm{d}y = \frac{3}{2}\int_{-\infty}^{\infty} e^{y}u^2\mathrm{d}y > 0. \end{aligned} \tag{2.64}$$

Thus $E_{+}(t)$ is initially zero and strictly increasing for all $t \in [0, T)$. In a similar manner one can show that E_{-} is also initially zero but strictly decreasing for $t \in [0, T)$. This proves part (a) of the theorem.

To prove (b) we write equation (2.13) in the form

$$m_t(x,t) + u(x,t)m_x(x,t) = -3u_x(x,t)m(x,t), \quad x \in \mathbb{R}, t \in [0,T), \tag{2.65}$$

and proceeding as in Theorem 2.2.1 we find that

$$\sup_{t\in[0,T-\epsilon]} \|m(t)e^{(1+\mu)|x|}\|_\infty \leq c(\epsilon)\|m(0)e^{(1+\mu)|x|}\|_\infty, \ \epsilon \in (0,T), \tag{2.66}$$

where $c(\epsilon)$ is a constant depending on $\sup_{t\in[0,T-\epsilon]}(\|u_x(\tau)\|_\infty + \|u(\tau)\|_\infty)$. We note that for any $\theta \in (0,1)$

$$u(t), u_x(t), u_{xx}(t) \sim O(e^{-\theta|x|}) \text{ as } |x| \to \infty, \tag{2.67}$$

meaning that all of the integrals in the computations above are well-defined. The property (2.59) follows by an approach similar to the one performed before, taking into account (2.60) and defining $E_\pm$ as in (2.61) but with $\alpha = -\infty$, $\beta = \infty$. ∎

Chapter 3

Solutions for a class of edge-wave equations

3.1 The edge-wave problem

While classically considered to be a mere curiosity, edge waves have recently become a subject of great interest in the geophysical research literature [74] due to the fact that they play an essential role in the nearshore sediment transport. Edge waves are curious water waves: they propagate along the shoreline and have maximal amplitude near the shore with a rapid offshore decay [67]. This explains their role in sediment transport. Indeed, while the water waves coming in from the far sea lose most of their energy due to wave breaking that occurs mostly offshore, edge waves propagate along the shoreline and may reach, in certain circumstances, amplitudes that exceed 0.5 m; cf. the discussion in [18]. Mathematically, the edge waves are modelled by the Euler equations with appropriate kinematic and dynamic boundary conditions [67]. The resulting free-boundary problem is very complicated and apparently only one non-trivial explicit solution is known for the special case of a plane beach [18]. This edge-wave solution can be regarded as a natural correspondent of Gerstner's explicit solution for deep water waves [17, 51]. Due to the lack of large families of explicit solutions for the nonlinear governing equations, the linearisation procedure was pursued [67], mostly for seabeds with a special profile (e.g. planar beaches). For a general seabed, the equation

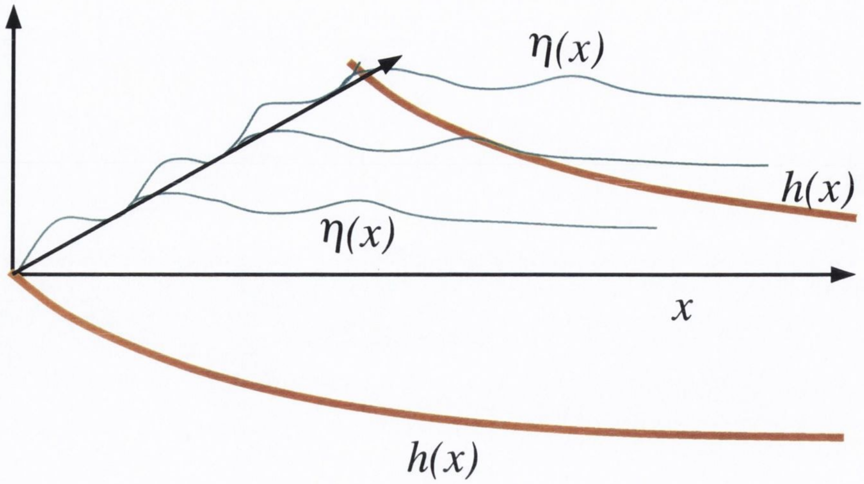
$$[h(x)\eta'(x)]' + \left[\frac{\sigma^2}{g} - k^2 h(x)\right]\eta(x) = 0, \qquad x \geq 0 \tag{3.1}$$

was recently derived [65]. Of interest is the existence of monotonous, convergent to zero (as x increases indefinitely) solutions $\eta(x)$ of this equation, classically describing the propagation of edge waves in the absence of a current [64], but also found recently to model the same problem in the presence of longshore currents and a variable seabed depth profile [65]. Following [65, eq. (11)], the seabed depth profile $h(x)$ and the mean longshore current $V(x)$ interplay via the formula

$$h(x) = \frac{h_0(x)}{[1 - (V(x)/c)]^2},$$

where

$$c = \frac{\sigma}{k}$$



Sketch of a typical edge wave with wave profile $\eta(x)$ and seabed $h(x)$. The wave is confined to the nearshore and moves in the longshore direction.

is the wave celerity, k and σ are the longshore radial wave number and wave frequency respectively. Various formulas have been given for the functional coefficient $h_0(x)$, e.g. βx (beach with plane sloping profile [46]) or $h_0\left(1-e^{-\alpha x}\right)$ (the exponential sloping profile case [5]). As for longshore currents, researchers have focused on currents constantly increasing in the offshore direction ($V(x) = \vartheta x$) that remain very small in comparison with the celerity of the edge wave [73]. The detailed account of the literature regarding edge waves in [65] shows that no mathematically rigorous investigation has been done up to this day about the existence of edge-wave solutions to (3.1). By an *edge-wave solution* of the equation (3.1) we understand a solution $\eta(x)$ that is defined for all positive values of $x \geq X$ — for some appropriately large $X > 0$ — which is monotonic sufficiently far out (it does not oscillate indefinitely) and which tends to zero as x increases indefinitely. This will allow, loosely speaking, for any entrapments (under appropriate hypotheses) in the interval $\hat{\mathcal{I}}$ that precedes the existence domain of the edge wave solution, the interval being $\hat{\mathcal{I}} = \{x : 0 \leq x < X\}$. The present section is devoted to establishing, in a rigorous way, the existence of edge wave solutions for the equation (3.1) when the seabed depth profile belongs to a large class of continuously differentiable functions. In this thesis we do not present an analysis of the behaviour of the edge-wave solutions on the interval $\hat{\mathcal{I}}$.

3.2 Asymptotic behaviour of monotonic solutions

The equation (3.1), after the convenient rescaling $x \longmapsto k^{-1}x$, can be written as

$$[(\lambda + \alpha(x))\,\eta'(x)]' = \alpha(x)\eta(x), \qquad x \geq X > 0, \tag{3.2}$$

where

$$\lambda = \frac{c^2}{g}$$

and we have chosen X sufficiently large to ensure that α is a positive continuously differentiable function. Here, $h(x) = \lambda + \alpha(x)$ for all $x \geq 0$. In this way, the equation (3.2) belongs to the larger class of second order self-adjoint equations

$$[p(x)\eta'(x)]' = q(x)\eta(x), \qquad x \geq 0, \tag{3.3}$$

where $p(x)$, $q(x) > 0$, the function p is continuously differentiable and the function q is continuous on the nonnegative half axis. A complete classification of solutions to the equation (3.3) from the asymptotic-behaviour viewpoint has been performed in the papers by Cecchi, Marini and Villari [10] and Marini and Zezza [80]. The dual case of $p(x) > 0$, $q(x) < 0$ was investigated in full detail by Cecchi, Marini and Villari [11].

Following [80, Theorem 1], the equation (3.3) always has a solution $\eta(x)$ enjoying the so-called $B-$*class property* which reads as

$$\eta(x)\eta'(x) < 0$$

for all sufficiently large x. It is obvious that such a solution is bounded (which means that it exists at all future times) and is monotonic. Further, according to [10, Theorem 2(i)], every solution $\eta(x)$, where $x \geq X > 0$, with the $B-$class property tends to zero if and only if

$$I = \int_X^{+\infty} q(x) \int_X^x \frac{dr}{p(r)} dx = +\infty. \tag{3.4}$$

Under a complementary restriction, namely

$$\int_X^{+\infty} \frac{1}{|p(x)|} \int_X^x |q(r)|\, dr dx < +\infty,$$

it was established in [92] (see the pioneering paper by Weyl [91] for the case $p(x)$, $q(x) > 0$) that all nontrivial solutions $\eta(x)$ of the equation (3.3) tend to a nonzero limit as x tends to $+\infty$. More detailed asymptotic characterisations of the solutions in this case are given in [10, 80].

3.3 Edge-wave solutions

From the preceding section we know that a necessary and sufficient condition for the existence of edge-wave solutions of the equation (3.2) is

$$\int_X^{+\infty} \alpha(x) \int_X^x \frac{dr}{\lambda + \alpha(r)} dx = +\infty. \tag{3.5}$$

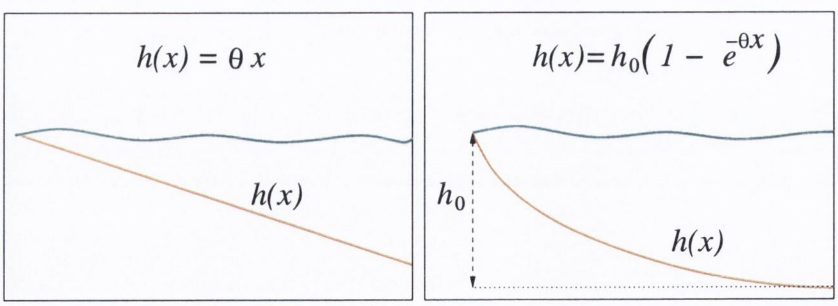
We are interested here in some simple conditions to be satisfied by the function α that will lead to (3.5). The function α being positive, it is obvious that the inner integral in (3.5) is bounded from below by a positive constant for all large x, say $x \geq 2X$. This means that (3.5) is implied by the simpler but more restrictive condition

$$\int_X^{+\infty} \alpha(x) dx = +\infty. \tag{3.6}$$

A pre-eminent example of such a function is given by α with

$$\liminf_{x \to +\infty} \alpha(x) > 0. \tag{3.7}$$

This includes the depth profiles investigated in [46, 5], namely:



Sketch of the two depth profiles classically featured in the literature.

In the dual situation when

$$\int_X^{+\infty} \alpha(x)dx < +\infty \tag{3.8}$$

we proceed as follows. For $x \geq X$, using the Cauchy-Schwarz inequality we get

$$(x-X)^2 = \left(\int_X^x \frac{(\lambda+\alpha(r))^{1/2}}{(\lambda+\alpha(r))^{1/2}}dr\right)^2 \leq \left(\int_X^x \frac{dr}{\lambda+\alpha(r)}\right)\left(\int_X^x (\lambda+\alpha(r))\,dr\right). \tag{3.9}$$

Furthermore,

$$\frac{1}{x}\int_X^x (\lambda+\alpha(r))\,dr = \lambda(1-\frac{X}{x}) + \int_X^x \frac{\alpha(r)}{x}dr \leq \lambda + \int_X^x \frac{\alpha(r)}{r}dr,$$

and using this relation together with (3.9) it follows that

$$\frac{1-\frac{X}{x}}{\lambda+\int_X^x \frac{\alpha(r)}{r}dr}(x-X) \leq \int_X^x \frac{dr}{\lambda+\alpha(r)}. \tag{3.10}$$

Since α is nonnegative we have

$$\int_X^x \frac{dr}{\lambda+\alpha(r)} \leq \frac{x-X}{\lambda}, \tag{3.11}$$

and putting relations (3.10) and (3.11) together we get

$$\frac{1-\frac{X}{x}}{\lambda+\int_X^x \frac{\alpha(r)}{r}dr}(x-X) \leq \int_X^x \frac{dr}{\lambda+\alpha(r)} \leq \frac{x-X}{\lambda}, \qquad x \geq X.$$

This means that (3.5) is equivalent to

$$\int_X^{+\infty} x\alpha(x)dx = +\infty. \tag{3.12}$$

A summary of these considerations is given below.

Theorem 3.3.1 *Let α be a positive, continuously differentiable function defined on the nonnegative half axis. Then equation* (3.2) *has an edge-wave solution if and only if one of the following sets of conditions holds:* (i) (3.6)*;* (ii) (3.8) *and* (3.12).

3.4 Edge-wave representation

An interesting mathematical problem can be introduced which is closely related to the Cecchi-Marini-Villari hypothesis (3.5). Precisely, we ask which are the positive, continuously differentiable functions w, defined on the nonnegative half axis, that satisfy the integral restriction

$$\int_0^{+\infty} w(x)dx = +\infty \tag{3.13}$$

and for which there exists an $X > 0$ such that the integral equation below

$$\alpha(x)\int_0^x \frac{dr}{\lambda + \alpha(r)} = w(x), \qquad x \geq X, \tag{3.14}$$

has at least one solution? Such a function α will be called an *edge-wave representation* of w in the sequel. Although in its entire generality the problem remains unsettled, certain particular cases lead to nontrivial results. For example, in many common situations it is difficult to compute the edge wave representation. The next example illustrates this fact. Consider $w(x) = x$ for $x \geq X$. By denoting the integral factor in (3.14) as $y(x)$, we obtain the first order differential equation

$$y'(x) = \frac{y(x)}{w(x) + \lambda y(x)}, \qquad x \geq X. \tag{3.15}$$

The equation (3.15) is homogeneous and has the solution in an implicit form

$$Cy = e^{\frac{x}{\lambda y}}, \qquad x \geq X,$$

where $C = \frac{1}{y_0}e^{\frac{X}{\lambda y_0}}$. Here, $\alpha(x) = \frac{x}{y(x)}$. A particular class of functions α that satisfy (3.6), namely the α for which

$$\int_0^{+\infty} \frac{dx}{\lambda + \alpha(x)} < +\infty, \tag{3.16}$$

have a large family of edge wave representations, as we show in Section 3.4.1.

3.4.1 Existence of edge-wave solutions

We shall establish here that every function α in the class (3.6), with (3.16), has a large family of edge wave representations. To this end, assume that the positive, continuously differentiable function w defined on the nonnegative half axis satisfies (3.13) and that

$$\int_0^{+\infty} \frac{dx}{\lambda + w(x)} < +\infty. \tag{3.17}$$

Let $a > 0$. Introduce also $X > 0$ such that

$$4\int_X^{+\infty} \frac{dx}{\frac{\lambda}{a} + w(x)} \leq k < 1 \tag{3.18}$$

for a fixed k. The set

$$D = \{z \in C([X, +\infty), \mathbb{R}) : a \leq z(x) \leq 2a \text{ for all } x \geq X\},$$

endowed with the distance below

$$d(z_1, z_2) = \sup_{x \geq X} |z_1(x) - z_2(x)|, \qquad z_{1,2} \in D,$$

is a complete metric space. Introduce further the operator $T : D \to D$ by

$$T(z)(x) = a + \int_x^{+\infty} \frac{z^2(s)}{\lambda + w(s)z(s)} ds$$

for all $x \geq X$ and $z \in D$. Since, for any $z_{1,2} \in D$, we have

$$\frac{\lambda[z_1(x) + z_2(x)] + w(x)z_1(x)z_2(x)}{\lambda^2 + w(x)\{\lambda[z_1(x) + z_2(x)] + w(x)z_1(x)z_2(x)\}} \leq \frac{4}{\frac{\lambda}{a} + w(x)},$$

it follows directly that

$$\begin{aligned} |T(z_1)(x) - T(z_2)(x)| &\leq \int_x^{+\infty} \frac{4}{\frac{\lambda}{a} + w(s)} |z_1(s) - z_2(s)| \, ds \\ &\leq k \cdot d(z_1, z_2) \end{aligned}$$

and, correspondingly

$$d(T(z_1), T(z_2)) \leq k \cdot d(z_1, z_2), \qquad z_{1,2} \in D.$$

The operator T being a contraction, the Banach contraction principle implies the existence of a unique fixed point z_a of T in D that can be computed through successive approximations; see [45]. We now have the following result.

Theorem 3.4.1 *Let w satisfy* (3.13), (3.17) *and $a > 0$ be given. Then there exists a positive, continuously differentiable function α defined on the non-negative half-axis such that* (3.14) *holds for all $x \geq X$ and*

$$\lim_{x \to +\infty} \frac{\alpha(x)}{w(x)} = a.$$

Proof Given the fixed point z_a of operator T in D, we introduce $\alpha(x) = w(x)z_a(x)$ for all $x \geq X$, where X satisfies (3.18). Then, α is a solution of the differential equation

$$\left(\frac{w(x)}{\alpha(x)}\right)' = \frac{1}{\lambda + \alpha(x)}, \qquad x \geq X.$$

The proof is completed by noticing that $z_a(x)$ tends to a as x tends to $+\infty$. ∎

For any function α in the class (3.6), with (3.16), the function w given by (3.14) belongs to the same class. We can backward-continue w appropriately towards 0. Then Theorem 3.4.1 shows that the edge-wave representations $\{w \mapsto \alpha_a : a > 0\}$ act as bijections on this class.

Chapter 4

The motion of fluid particles in water waves

4.1 Introduction to the water-wave problem

The goal of this chapter is to perform an analysis of water particle trajectories in three different wave-motion settings. We will first deal in Section 4.2.1 with linear periodic capillary and capillary-gravity waves in water flowing over a flat bed, and following that in Section 4.2.2 we deal with these waves in the deep-water setting (where the fluid is of an infinite depth). In both cases we will construct linearised versions of Euler's equations of motion. The transition from the finite- to infinite-depth problem involves changes to the boundary setting of the problem which lead to some differences in the trajectories of the particles. Finally in Section 4.3 we perform an analysis of particle motion in deep-water Stokes waves. Analysis of the particle motion in Stokes waves presents inherent challenges due to the nonlinear nature of the governing equations, and some results from harmonic analysis such as maximum principles and properties of level sets will be employed to this end.

4.1.1 Governing equations for water wave motion

In keeping with general practice we adopt the **continuum hypothesis**, which presumes that water is a continuous medium. Throughout this chapter we assume **homogeneity** of the water, that is, the constant density ρ of the liquid. While it is true that the density of a liquid varies with the depth of the liquid, or with the presence of dissolved solids, it turns out that in the context which most concerns us— the propagation of waves on the surface of water— homogeneity is a reasonable simplification for us to make. Furthermore throughout the following we assume that water is **inviscid**, that is, there is no internal friction in the liquid. Discussions on the rationale behind such simplifications can be found in Johnson [67, 77].

In this chapter we will deal with waves that are two dimensional, that is, the motion is identical in any direction parallel to the crest line. Let consider a cross section of the flow in the direction perpendicular to the crest line with Cartesian coordinates (x, y), the $x-$axis being in the direction of wave propagation while the $y-$axis points vertically upwards. As such, let $\mathbf{U} = (u(t,x,y), v(t,x,y))$ be the velocity field of the flow. In the case of water of finite depth we set the flat bed to be $y = 0$ and let $y = h_0 + \eta(t,x)$ be the water's free surface, where $h_0 > 0$ is the mean water level. In the case of water of infinite depth we let $y = 0$ denote the mean water level and $y = \eta(t,x)$ is the water's free surface. The homogeneity assumption yields the **equation of mass conservation**

$$\nabla \cdot \mathbf{U} = u_x + v_y = 0, \tag{4.1}$$

also known as the **continuity equation**. The equations of motion are given by **Euler's equation**

$$\frac{D\mathbf{U}}{Dt} = -\frac{1}{\rho}\nabla P + \mathbf{F} \tag{4.2}$$

where P is the pressure function, $\mathbf{F}$ is the body force of the liquid and $\frac{D}{Dt}$ represents the total or material derivative with respect to $\mathbf{U}$. The total derivative of a scalar function H with respect to a velocity field $\mathbf{U}$ is defined to be the rate of change of the H when H is associated to a particular fluid particle, with the particle moving about according to $\mathbf{U}$: explcitly, in the case above, this is described by

$$\frac{DH}{DT} = H_t + \mathbf{U} \cdot \nabla H.$$

In order to decouple the motion of the air from that of the free surface particles [67] we introduce the **dynamic boundary condition**

$$P = P_0 - \frac{\Gamma}{R} \tag{4.3}$$

on the surface of the water, where P_0 is the constant atmospheric pressure, the parameter Γ is the coefficient of surface tension, possibly zero in the absence of surface tension, and $1/R$ is the curvature in the $x-$direction given by

$$\frac{1}{R} = \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}.$$

Furthermore, since the free-surface, given by $S(t,x,y) := y - \eta(t,x) = 0$, is always composed of the same particles, we have the **kinematic boundary conditions**

$$\frac{DS}{Dt} = 0$$

or

$$v = \eta_t + u\eta_x \tag{4.4}$$

on the water surface, while for water of finite depth we use

$$v = 0 \qquad \text{on } y = 0, \tag{4.5}$$

so we assume that the rigid bed is impenetrable; for water of infinite depth we have

$$(u, v) \to (0, 0) \text{ as } y \to -\infty, \text{ uniformly for } x \in \mathbb{R}, t \geq 0, \tag{4.6}$$

cf. [67], expressing the fact that at great depths there is no motion. In this chapter we make the further assumption that the water is **irrotational** giving us the equation

$$u_y = v_x, \tag{4.7}$$

a simplification which is justified in the problems that we are concerned with since Kelvin's Circulation theorem assures us that water initially in a state of rest will remain irrotational at all later times [67, 77].

The equations (4.1)-(4.7) encompass the two-dimensional equations of motion for an idealised (homogeneous and inviscid) fluid lacking vorticity. For a complete derivation of the equations of motion see Johnson [67].

4.2 Particle trajectories for linear periodic capillary and capillary-gravity water waves

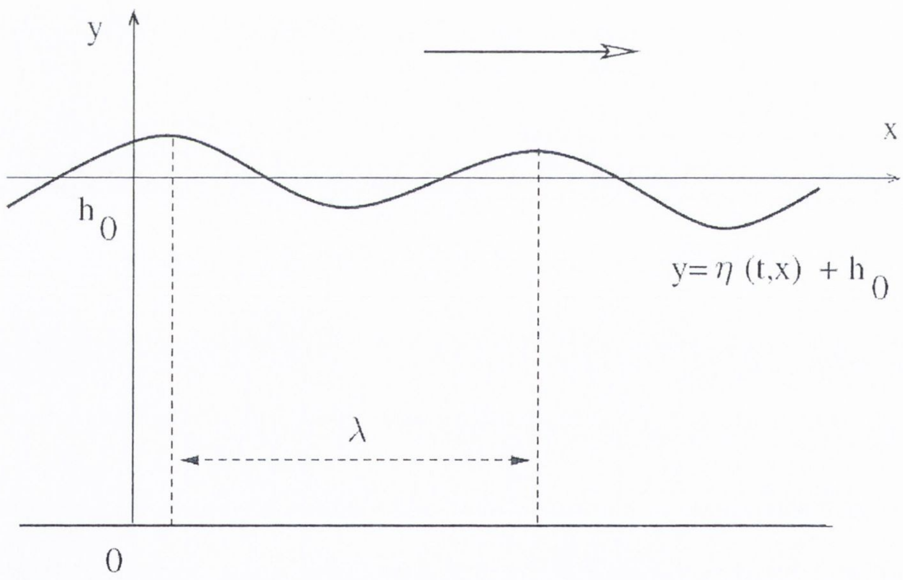
Wind forces are the single largest factor in the generation of water waves. Ripples (waves of small amplitude and wavelength) are formed when the friction between the air and the water deforms the originally flat water surface resulting in what is known as capillary waves. Surface tension is the dominant restoring force for these capillary waves. The presence of ripples enables the wind to 'grip' the agitated surface, thus increasing the amplitude and wavelength of the waves, reaching a stage where gravity plays a significant role as a restoring force in addition to the surface tension. The resulting waves are known as capillary-gravity waves. With ever-increasing amplitude, the influence of surface tension on the motion of the wave diminishes to the point where it is negligible and the resulting waves are known as gravity water waves. The three main types of water waves (capillary, capillary-gravity, and gravity waves) have generally different properties; in Sections 4.2.1 and 4.2.2 we are interested in the particle trajectories of linear periodic capillary and capillary-gravity waves. In Section 4.2.1 we focus on the propagation of waves over water of finite depth, while in Section 4.2.2 we deal with deep-water waves (propagating over water of infinite depth). In both cases we work within the framework of linear theory. Since the nonlinear governing equations are highly intractable (while there exist some results for the nonlinear governing equations — see [25, 26, 35, 36, 33, 38, 47, 53, 70, 88, 89, 90] — the information available is not sufficiently detailed as to enable a study of the particle paths in the fluid), it appears that the linear framework is appropriate for a first study of the particle trajectories in capillary and capillary-gravity water waves. Furthermore, imposing periodicity is not too much of a restriction in this context since, when we observe waves that are not near breaking, they appear to be two-dimensional periodic wave trains.

Previously, formal considerations have suggested that particle trajectories in the fluid are closed (see [8, 41, 77, 86]) and, while there are special solutions to the nonlinear governing equations with all particle trajectories closed (see [17, 18]), what we will show is that, over water of finite and of infinite depth, within linear capillary and capillary-gravity theory for steady waves, this is not the case: if the surface is not flat there are no closed orbits in the fluid — see Theorems 4.2.1 and 4.2.3. It is anticipated that further studies will permit the extension of these results to the nonlinear governing equations (for gravity waves the features observed within the linear theory in [37] were recently proven [20, 60] to hold true for the nonlinear governing equations; see Section 4.3).

4.2.1 Particle trajectories in water of finite depth

Governing equations

We use Cartesian coordinates (x, y) with the $x-$axis in the direction of wave propagation and the $y-$axis pointing vertically upwards. We let $y = 0$ represent the flat bed and let $y = h_0 + \eta(t, x)$ be the water's free surface, where $h_0 > 0$ is the mean water level. As



Sketch of a periodic water wave propagating over a flat bed.

detailed in Section 4.1.1, the following equations govern the motion of the water. We have the equation of mass conservation

$$u_x + v_y = 0, \tag{4.8}$$

together with Euler's equation (written in component form)

$$\begin{cases} u_t + uu_x + vu_y = -P_x, \\ v_t + uv_x + vv_y = -P_y - g, \end{cases} \tag{4.9}$$

throughout the fluid, where $P(x, y, t)$ denotes the pressure and g is the gravitational constant of acceleration. Decoupling the motion of the air from that of the free surface particles we have the dynamic boundary condition

$$P = P_0 - \frac{\Gamma}{R} \qquad \text{on } y = h_0 + \eta(t, x), \tag{4.10}$$

where P_0 is the constant atmospheric pressure, and here the coefficient of surface tension Γ will be strictly positive, with $1/R$ the curvature of the surface

$$\frac{1}{R} = \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}}. \tag{4.11}$$

We have the kinematic boundary condition at the free surface

$$v = \eta_t + u\eta_x \qquad \text{on } y = h_0 + \eta(t, x), \tag{4.12}$$

and the kinematic boundary condition on the flat bed

$$v = 0 \qquad \text{on } y = 0, \tag{4.13}$$

which tells us that the rigid bed is impenetrable. Thus the nonlinear free boundary problem (4.8)–(4.13) governs the capillary and capillary-gravity water wave problem in water of finite depth, cf. [67] and Section 4.1.1. In addition we shall impose the condition of irrotational flow

$$u_y = v_x. \tag{4.14}$$

Linearising the gravity-capillary water-wave problem

We now begin by nondimensionalising the problem (4.8)–(4.14) using a typical wavelength λ and a typical amplitude a of the wave. Define the set of nondimensional variables

$$x \mapsto \lambda x, \ y \mapsto h_0 y, \ t \mapsto \frac{\lambda}{\sqrt{gh_0}} t, \ u \mapsto u\sqrt{gh_0}, \ v \mapsto v\frac{h_0\sqrt{gh_0}}{\lambda}, \ \eta \mapsto a\eta,$$

where, for example, we replace x by λx, with the variable x now being nondimensional, thus avoiding new notation. Setting the constant water density $\rho = 1$, for convenience, the pressure in the new nondimensional variables is given by

$$P = P_0 + gh_0(1 - y) + gh_0 p,$$

where the nondimensional pressure variable p measures the deviation from the hydrostatic pressure distribution. This gives us the following boundary-value problem in nondimensional variables

$$\begin{cases} u_t + uu_x + vu_y = -p_x, \\ \delta^2(v_t + uv_x + vv_y) = -p_y, \\ u_x + v_y = 0, \ u_y = \delta^2 v_x, \\ v = \epsilon(\eta_t + u\eta_x) \quad \text{on } y = 1 + \epsilon\eta, \\ v = 0 \quad \text{on } y = 0, \\ p - \epsilon\eta = -\epsilon\left(\frac{\Gamma}{g\lambda^2}\left\{\frac{\eta_{xx}}{(1+\epsilon^2\delta^2\eta_x^2)^{3/2}}\right\}\right) \quad \text{on } y = 1 + \epsilon\eta, \end{cases} \tag{4.15}$$

where $\epsilon = a/h_0$ is the amplitude parameter and $\delta = h_0/\lambda$ is the shallowness parameter. It is conventional to write $\Gamma/(\rho g\lambda^2) = \delta^2 W_e$, with $W_e = \Gamma/(\rho g h_0^2)$ being the Weber number. This nondimensional parameter measures the size of the surface tension contribution. From the fourth and sixth equations in (4.15) it is obvious that both v and p, if evaluated on $y = 1 + \epsilon\eta$, are essentially proportional to ϵ. Indeed, physically as $\epsilon \to 0$ we must have $v \to 0$ and $p \to 0$. This leads us to the scaling of the nondimensional variables

$$p \mapsto \epsilon p, \ (u, v) \mapsto \epsilon(u, v),$$

avoiding again the introduction of new variables. Now problem (4.15) becomes

$$\begin{cases} u_t + \epsilon(uu_x + vu_y) = -p_x, \\ \delta^2\{v_t + \epsilon(uv_x + vv_y)\} = -p_y, \\ u_x + v_y = 0, \ u_y = \delta^2 v_x, \\ p = \eta - \delta^2 W_e \frac{\eta_{xx}}{(1+\epsilon^2\delta^2\eta_x^2)^{3/2}} \quad \text{and} \quad v = \eta_t + \epsilon u\eta_x \quad \text{on } y = 1 + \epsilon\eta, \\ v = 0 \quad \text{on } y = 0. \end{cases} \tag{4.16}$$

The linearised problem is obtained by letting $\epsilon \longrightarrow 0$ in (4.16). The resulting equations are

$$\begin{cases} u_t = -p_x, \quad \delta^2 v_t = -p_y, \\ u_x + v_y = 0, \; u_y = \delta^2 v_x, \\ v = \eta_t \quad \text{and} \quad p = \eta - \delta^2 W_e \eta_{xx} \quad \text{on } y = 1, \\ v = 0 \quad \text{on } y = 0. \end{cases} \tag{4.17}$$

We will seek travelling-wave solutions of (4.17), that is waves for which the $(t, x)-$dependence of u, v, p, η is in the form of a periodic dependence in $x - c_0 t$, where $c_0 > 0$ represents the nondimensionalised speed of the wave. With the ansatz Fourier mode

$$\eta(t, x) = \cos[2\pi(x - c_0 t)],$$

manipulating the various equations in (4.17) we see that

$$v_{yyt} = -u_{xyt} = p_{yxx} = -\delta^2 v_{xxt},$$

and so

$$v_{yy} + \delta^2 v_{xx} = f(x, y),$$

for some function $f(x, y)$ which we choose for simplicity to be equal to 0. For a fixed t, and under the change of variables $\bar{x} = \frac{x - c_0 t}{\delta}$, we get

$$v_{yy} + v_{\bar{x}\bar{x}} = 0. \tag{4.18}$$

We solve this using the method of separation of variables. Assuming

$$v(t, x, y) = X(\bar{x}) Y(y), \tag{4.19}$$

and upon substituting this form of v into (4.18) we obtain, whenever $X \neq 0 \neq Y$,

$$-\frac{X''(\bar{x})}{X(\bar{x})} = \frac{Y''(y)}{Y(y)},$$

where the prime denotes differentiation. As the left hand side is independent of the y variable, and the right hand side is similarly independent of the $\bar{x}$ variable, it immediately follows that

$$-\frac{X''(\bar{x})}{X(\bar{x})} = K = \frac{Y''(y)}{Y(y)},$$

for some real constant K. Due to the required periodicity in the $\bar{x}$-variable, we must have $K = (2\pi\alpha)^2$ for some $\alpha \in \mathbb{R}$. Thus

$$\begin{cases} X''(\bar{x}) = -(2\pi\alpha)^2 X(\bar{x}) \\ Y''(y) = (2\pi\alpha)^2 Y(y), \end{cases} \tag{4.20}$$

the solutions of which are

$$\begin{cases} X(\bar{x}) = Ae^{-i2\pi\alpha\bar{x}} + Be^{i2\pi\alpha\bar{x}} \\ Y(y) = Ce^{-2\pi\alpha y} + De^{2\pi\alpha y}, \end{cases} \tag{4.21}$$

where A, B, C, D are constants to be determined using the initial conditions. On $y = 0$ we have $v = 0$, and so it follows that $C = -D$ (as we cannot have $A = B = 0$ unless $v \equiv 0$ — and $v \equiv 0$ in (4.17) is not admissible since it forces p to be constant which is incompatible with our choice of η). Equation (4.19) now becomes

$$v(\bar{x}, y) = 2D\sinh(2\pi\alpha y)\left\{Ae^{-i2\pi\alpha\bar{x}} + Be^{i2\pi\alpha\bar{x}}\right\}. \tag{4.22}$$

For $y = 1$, $v = \eta_t$ becomes

$$2D\sinh(2\pi\alpha)\left\{Ae^{-i2\pi\alpha\bar{x}} + Be^{i2\pi\alpha\bar{x}}\right\} = 2\pi c_0\sin(2\pi\delta\bar{x}).$$

The functional part of the left hand side must be equal to the functional part of the right hand side, and this can only happen if $A = -B$ and $\alpha = \delta$. Equating the constants on each side, it follows that $4DB\sinh(2\pi\delta) = 2\pi c_0$. Thus

$$v(t, x, y) = 2\pi c_0\frac{\sinh(2\pi\delta y)}{\sinh(2\pi\delta)}\sin[2\pi(x - c_0t)].$$

It is a straightforward exercise in integration of the equations in (4.17) to show that, consistent with this $v(t, x, y)$,

$$u(t, x, y) = 2\pi\delta c_0\frac{\cosh(2\pi\delta y)}{\sinh(2\pi\delta)}\cos[2\pi(x - c_0t)],$$

$$p(t, x, y) = (1 + (2\pi\delta)^2 W_e)\frac{\cosh(2\pi\delta y)}{\cosh(2\pi\delta)}\cos[2\pi(x - c_0t)],$$

provided

$$c_0^2 = \frac{\tanh(2\pi\delta)}{2\pi\delta}(1 + (2\pi\delta)^2 W_e).$$

We return to the original physical variables, using the change of variables

$$x \mapsto \frac{x}{\lambda}, \; y \mapsto \frac{y}{h_0}, \; t \mapsto t\frac{\sqrt{gh_0}}{\lambda}, \; u \mapsto \frac{u}{\sqrt{gh_0}}, \; v \mapsto v\frac{\lambda}{h_0\sqrt{gh_0}}, \; \eta \mapsto \frac{\eta}{a}.$$

A linear wave solution, written in the physical variables, is

$$\begin{cases} \eta(t,x) = \epsilon h_0 \cos(kx - \omega t), \\ u(t,x,y) = \epsilon \omega h_0 \frac{\cosh(ky)}{\sinh(kh_0)} \cos(kx - \omega t), \\ v(t,x,y) = \epsilon \omega h_0 \frac{\sinh(ky)}{\sinh(kh_0)} \sin(kx - \omega t), \\ P(t,x,y) = P_0 + g(h_0 - y) + \epsilon g h_0 \Big(1 + (2\pi\delta)^2 W_e\Big) \frac{\cosh(ky)}{\cosh(kh_0)} \cos(kx - \omega t), \end{cases} \tag{4.23}$$

of amplitude $\epsilon h_0 > 0$ and wavelength $\lambda > 0$, propagating over the flat bed $y = 0$ and with mean water level $h_0 > 0$. Here

$$k = \frac{2\pi}{\lambda}, \qquad \omega = \sqrt{gk\Big(1 + (2\pi\delta)^2 W_e\Big) \tanh{(kh_0)}},$$

are the wavenumber and the frequency respectively, and the dispersion relation is

$$c = \frac{\omega}{k} = \sqrt{g\Big(1 + (2\pi\delta)^2 W_e\Big)\frac{\tanh(kh_0)}{k}} \tag{4.24}$$

which determines the speed c of the linear wave for right-travelling waves ($c > 0$). The period of this wave is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{gk\Big(1 + (2\pi\delta)^2 W_e\Big) \tanh(kh_0)}}.$$

The two main classes of waves of interest in linear wave theory are shallow water waves and deep water waves [77]. The case $\delta = h_0/\lambda \to 0$ corresponds to long waves, or shallow water waves, whereas the deep water (or short wave) limit is given by $\delta \to \infty$. In the case of long waves (where $kh_0 \to 0$: see area **B** of Figure 4-1) and the dispersion relation for capillary-gravity waves yields

$$c \approx \sqrt{gh_0},$$

which is independent of both the wavelength λ (so in the long wave limit the waves are non-dispersive) and of the coefficient of surface tension Γ (capillary-gravity waves become

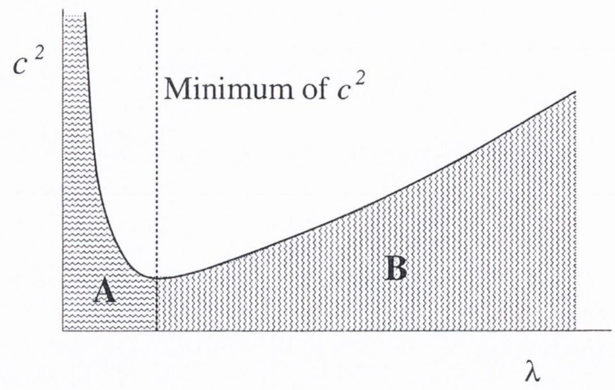


Figure 4-1: A sketch of the relationship described by equation (4.24) (plotted here for a fixed value of W_e) between the speed, represented by c^2, and the wavelength λ.

pure gravity waves in the long wave limit). In the short-wave limit (where $kh_0 \to \infty$: see area **A** of Figure 4-1) we obtain

$$c \approx \sqrt{k\Gamma} = \sqrt{\frac{2\pi\Gamma}{\lambda}}.$$

This displays, for capillary-gravity waves, a strong relationship between the wavelength λ, the coefficient of surface tension Γ, and the wavespeed c. Also, as we decrease the wavelength for capillary-gravity waves the wavespeed becomes independent of the coefficient of gravity g, leading us to pure-capillary waves where surface-tension is the sole restoring-force. In deep water, the shorter the capillary-gravity wave the faster it travels, and so the wave speed is proportional to the square root of the coeeficient of surface tension. It is interesting to note the contrast in this aspect of the capillary-gravity waves with that of pure gravity waves: in deep water the longer gravity waves propagate faster than shorter ones (see the discussion in [37, 67]).

A summary of some properties of the dispersion relation is shown in Figure 4-1 above. Interestingly, at any given speed above the minimum there can coexist a capillary and a gravity wave travelling with the same speed — the capillary wave is "generated" in **A** while the gravity wave is generated in **B**. This is often seen in reality, when an observer may notice many smaller capillary waves 'riding' on a larger gravity wave as they travel at the same speed.

The evolution of water particles

If $(x(t), y(t))$ is the path of the particle below the linear wave (4.23), then

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v,$$

so that the motion of the particle is described by the system

$$\begin{cases} \frac{dx}{dt} = M\cosh(ky)\cos k(x-ct) \\ \frac{dy}{dt} = M\sinh(ky)\sin k(x-ct), \end{cases} \tag{4.25}$$

with initial data (x_0, y_0), where we denoted

$$M = \frac{\epsilon\omega h_0}{\sinh(kh_0)}. \tag{4.26}$$

The right-hand side of the differential system (4.25) is smooth so that the existence of a unique local smooth solution is ensured [55]. Also, since y is bounded, the right-hand side of (4.25) is bounded and therefore this unique solution is defined globally [55].

If we now consider the case of pure capillary waves by letting $g \to 0$, the equations in (4.23) remain unchanged except for the last equation describing the pressure. This becomes

$$P(t,x,y) = P_0 + \epsilon h_0 k^2 \Gamma \frac{\cosh(ky)}{\cosh(kh_0)} \cos(kx - \omega t), \tag{4.27}$$

and we have

$$\omega = \sqrt{k^3\Gamma\tanh(kh_0)}; \quad c = \sqrt{k\Gamma\tanh(kh_0)}; \quad T = \frac{2\pi}{\sqrt{k^3\Gamma\tanh(kh_0)}}.$$

Therefore, the system (4.25) describes perfectly-well the motion of particles in pure capillary waves with no gravity foces, but with different c and M due to the different dependence of c and ω on Γ. If we let Γ tends to zero then the hydrostatic pressure becomes constant and the wavespeed vanishes, which makes perfect physical sense since is this case there are no outside forces disturbing the system from a position of inertia. An examination of the pure capillary wavespeed c, in the case $\Gamma > 0$, shows us that, in the deep water limit, we obtain the same wavespeed as the capillary-gravity waves. This fact is consistent with our previous observation that in waves with relatively small wavelength the surface-tension effects dominate over gravity forces. Also in the long wave, or shallow water, limit the wave speed becomes zero: gravity plays an important role in the motion of waves with long wavelength.

Qualitative analysis of solutions

We now proceed to an analysis of the solutions of (4.25). Since the right hand side of (4.25) is nonlinear, we will not try to solve this system of equations explicitly. Instead, we use phase plane analysis to examine the qualitative features of the solutions. Our aim is to show that there are no water particles travelling in closed orbits. In fact, we will see that every water particle experiences a forward drift as the wave progresses. We use the following transformation,

$$X(t) = kx(t) - \omega t, \qquad Y(t) = ky(t), \tag{4.28}$$

to give us the new system

$$\begin{cases} \frac{dX}{dt} = kM\cosh(Y)\cos(X) - kc, \\ \frac{dY}{dt} = kM\sinh(Y)\sin(X), \\ (X(0), Y(0)) = (x_0, y_0). \end{cases} \tag{4.29}$$

Since (4.29) is periodic in X, we need only consider the strip

$$\{(X, Y) \in \mathbb{R}^2 : \ -\pi \leq X \leq \pi\}.$$

Furthermore, as (4.29) is a description of our physical model we can restrict our attention to the values $Y > 0$.

The 0-isocline is defined to be the set where $dY/dt = 0$, and the ∞-isocline is the set where $dX/dt = 0$. Therefore the 0-isocline is given by

$$\{(X, Y) \in \mathbb{R}^2 : \ X \in \{0, \pm\pi\}\},$$

and the ∞-isocline is given by the curve $(X, \alpha(X))$, for $X \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\alpha(X) \in [Y^*, \infty)$, where $Y^* = \cosh^{-1}(\frac{c}{M})$ and α is defined as follows: on $[0, \frac{\pi}{2})$ we set α to be the inverse of the function $Y \mapsto \arccos\left(\frac{c}{M cosh(Y)}\right)$ defined on $[Y^*, \infty)$, and extend it by mirror symmetry to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Now, by (4.26) we have

$$kM = k\frac{\epsilon\omega h_0}{\sinh(kh_0)} = \epsilon\omega\frac{kh_0}{\sinh(kh_0)} < kc = \omega, \tag{4.30}$$

since $s < \sinh(s)$ for $s > 0$ and we assume that $\epsilon < 1$ within the confines of linear theory. It follows for $Y \geq Y^*$ that $\frac{c}{M\cosh(Y)} \leq 1$, and so α is well-defined. Furthermore the even function α is smooth, it takes on its infimum Y^* at $X = 0$, and satisfies

$$\lim_{X\to\pm\infty} \alpha(X) = \infty.$$

Now, for $X \in (\frac{\pi}{2}, \pi)$ we have $dX/dt < 0$, $dY/dt > 0$. If $X \in (0, \frac{\pi}{2})$ then $dX/dt < 0$ below the curve of $\alpha(X)$ and is positive above it, while dY/dt remains positive in this region. We obtain the corresponding signs for $X \in (-\pi, 0)$ by using the symmetric definition with respect to the Y-axis.

The only singular point of the system (4.29) in our region is $P = (0, Y^*)$. In order to

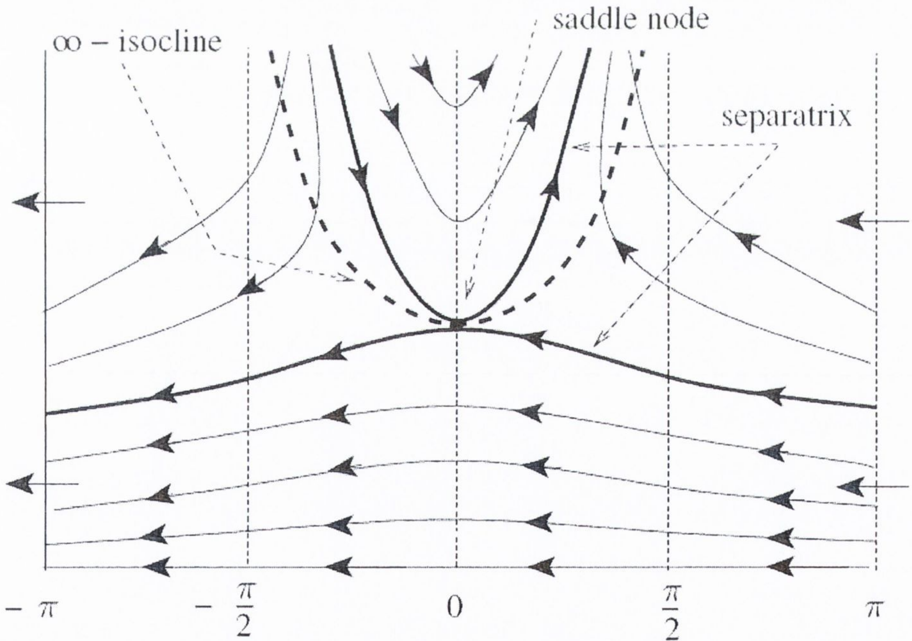


Figure 4-2: Phase portrait for system (4.29).

show that P is a saddle point we rewrite (4.29) as a Hamiltonian system

$$\begin{cases} \frac{dX}{dt} = H_Y, \\ \frac{dY}{dt} = -H_X, \end{cases} \tag{4.31}$$

with the Hamiltonian function $H(X,Y) \equiv kM \sinh(Y)\cos(X) - kcY$. Now P is a critical point of H, and as the Hessian of H at P is

$$\begin{pmatrix} -kM\sinh(Y^*) & 0 \\ 0 & kM\sinh(Y^*) \end{pmatrix},$$

it follows that P is a nondegenerate singular point. By Morse's lemma [82] in a neighbourhood of P there exists a diffeomorphic change of coordinates which sends the level lines of H to hyperbolae. Thus P is a saddle point for H. If (X,Y) is a solution of (4.29) then

$$\frac{d}{dt}H(X,Y) = H_X\frac{dX}{dt} + H_Y\frac{dY}{dt} = 0,$$

and so H is constant along the phase curves. Away from the critical point P the separatrix $H^{-1}\{H(P)\} = \{(X,Y) : H(X,Y) = H(P)\}$ is a smooth curve, since we can apply the implicit function theorem [55]. It intersects the vertical line $X = \pi$ at the point (π, β) if

$$-kM\sinh(\beta) - kc\beta = H(\pi,\beta) = H(P).$$

Suppose we have another point $Q = (\pi, Y)$ on this line. If $Y > \beta$, then the positive trajectory $\gamma^+(Q)$ of the phase curve is unbounded, whereas $\gamma^+(Q)$ will intersect the line $X = -\pi$ at $(-\pi, Y)$ if $Y \in (0,\beta)$.

Once we have plotted the phase diagram Figure 4-2 for the system (4.29) we obtain the particle trajectories for the linear wave (4.23) by applying the transformations

$$x(t) = \frac{X(t)}{k} + ct, \qquad y(t) = \frac{Y(t)}{k}. \tag{4.32}$$

At this point we should take note of the restrictions necessary to ensure that solutions are compatible with our physical model. Namely, from the above discussion it is clear that we require

$$h_0(1+\epsilon) \leq Y^* = \cosh^{-1}\left(\frac{\sinh(kh_0)}{\epsilon h_0 k}\right). \tag{4.33}$$

This condition is ensured if $\epsilon \cosh(h_0(1+\epsilon)) < 1$ (see [37]), a relation which gives a quantitative meaning to the notion that "$\epsilon < 1$ is small". Let $(X(t), Y(t))$ be a solution of (4.29) with $(X(0), Y(0)) = (\pi, Y_0)$, $Y_0 \in [0, \beta)$. We denote by $t_{-\pi}(Y_0)$ the time it takes for the phase curve $(X(t), Y(t))$ to intersect the line $X = -\pi$. Considerations similar to

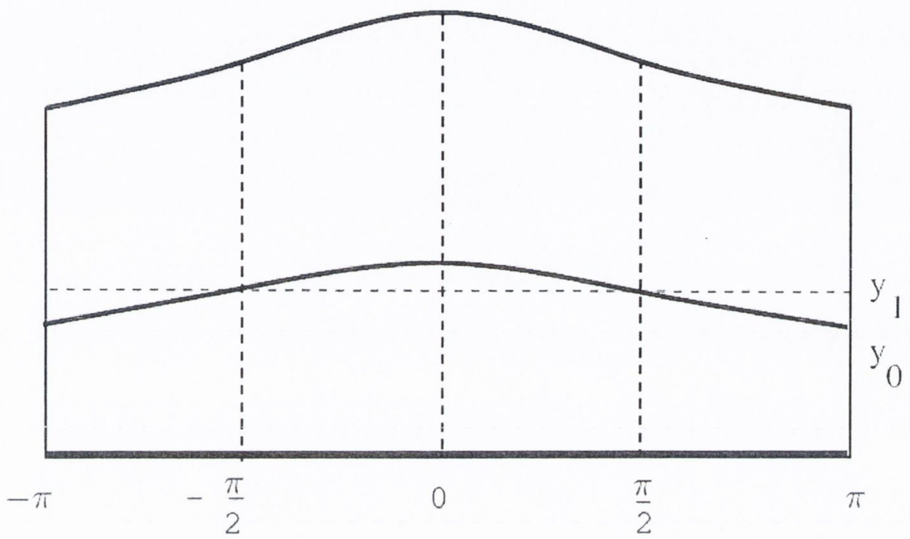


Diagram of the paths of particles with respect to the moving frame.

those made in the case of pure gravity water waves [37] show that the only possible period τ for a periodic particle path $(x(t), y(t))$ is $\tau = \frac{2\pi}{\omega}$ and, conversely, if $t_{-\pi}(Y_0) = \frac{2\pi}{\omega}$ then the corresponding particle path is periodic. We now prove our main result.

Theorem 4.2.1 *The system* (4.25) *has no periodic solutions.*

Proof As a consequence of the previous considerations, it suffices for us to show that $t_{-\pi}(Y_0) > \frac{2\pi}{ck}$ for $Y_0 \in [0, \beta)$ in order to prove the theorem.

We start with the case $Y_0 = 0$. From (4.29), it follows that the phase curve of $(X(t), Y(t))$ with $(X(0), Y(0)) = (\pi, 0)$ remains on the line $Y = 0$, and it can be obtained explicitly by solving the differential equation

$$\frac{dX}{dt} = kM\cos(X) - kc, \quad X(0) = \pi. \tag{4.34}$$

Keeping in mind $M < c$, we integrate to get

$$\int_0^z \frac{ds}{kc - kM\cos(s)} = \frac{2}{k}\sqrt{\frac{1}{c^2 - M^2}} \arctan\left(\sqrt{\frac{c+M}{c-M}} \tan\left(\frac{z}{2}\right)\right), \qquad z > 0.$$

Therefore

$$t_{-\pi}(0) = \int_{-\pi}^{\pi} \frac{ds}{kc - kM\cos(s)} = \frac{2\pi}{k}\sqrt{\frac{1}{c^2 - M^2}} > \frac{2\pi}{ck},$$

proving the theorem in the case $Y_0 = 0$.

For the case $Y_0 \in (0, \beta)$, we work as follows. Since $dY/dt > 0$ in the region $X \in (0, \pi)$, and $dY/dt < 0$ when $X \in (-\pi, 0)$, then if $Y_1 \in (Y_0, Y^*)$ is the value where the phase curve $(X(t), Y(t))$ intersects the line $X = \frac{\pi}{2}$, we have the phase curve lying below the line $Y = Y_1$ if $X(t) \in [-\pi, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ and lying above it if $X(t) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Thus,

$$\frac{dX}{dt} = kM\cosh(Y)\cos(X) - kc \geq kM\cosh(Y_1)\cos(X) - kc, \quad t \geq 0. \tag{4.35}$$

Let us introduce the differential equation

$$\frac{d\bar{X}}{dt} = kM\cosh(Y_1)\cos(\bar{X}) - kc, \quad \bar{X}(0) = \pi.$$

It follows immediately from (4.35), and the fact that $X(0) = \bar{X}(0) = \pi$, that $X(t) \geq \bar{X}(t)$ for $t \geq 0$, and thus $t_{-\pi}(Y_0) > t^*$ where t^* is the time when $\bar{X}(t^*) = -\pi$. However, we can now compute t^* explicitly as being

$$t^* = \frac{2\pi}{k}\sqrt{\frac{1}{c^2 - M^2\cosh^2(Y_1)}} > \frac{2\pi}{ck}, \tag{4.36}$$

in a manner similar to that of the solution of (4.34). Thus $t_{-\pi}(Y_0) > \frac{2\pi}{ck}$, completing the proof. ∎

The qualitative analysis performed above for the system (4.29) lets us describe the particle trajectories in linear capillary and capillary-gravity waves. We have, in view of (4.25) and (4.32),

$$\begin{aligned}
&\tfrac{dx}{dt} < 0,\ \tfrac{dy}{dt} < 0 \text{ for } && X(t) \in (-\pi, -\pi/2),\\
&\tfrac{dx}{dt} > 0,\ \tfrac{dy}{dt} < 0 \text{ for } && X(t) \in (-\pi/2, 0),\\
&\tfrac{dx}{dt} > 0,\ \tfrac{dy}{dt} > 0 \text{ for } && X(t) \in (0, \pi/2),\\
&\tfrac{dx}{dt} < 0,\ \tfrac{dy}{dt} > 0 \text{ for } && X(t) \in (\pi/2, \pi).
\end{aligned}$$

So, if we assume that at $t = 0$ and $X(0) = \pi$ a particle is at its greatest possible depth with $y(0) = y_0$, then (see Figure 4-3) the particle moves backward and up, then forward and up, then forward and down, and finally backward and down reaching the level $y = y_0$ in the time $t_{-\pi}(Y_0) > \frac{2\pi}{\omega}$ with

$$x(t_{-\pi}(Y_0)) - x(0) = \frac{t_{-\pi}(Y_0)\omega - 2\pi}{k} > 0.$$

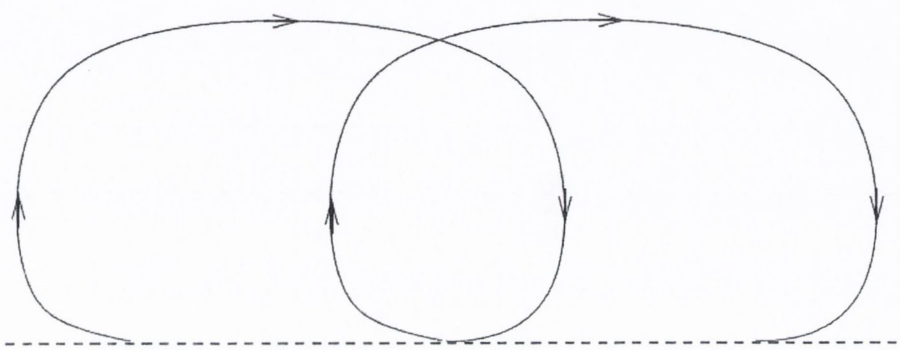


Figure 4-3: Trajectories of particles in linear capillary and capillary-gravity waves propagating above the flat bed.

4.2.2 Particle trajectories in water of infinite depth

Governing equations

The governing equations for two-dimensional capillary-gravity and pure capillary waves in water of infinite depth are similar to those of the finite depth case in Section 4.2.1 with, however, some modifications which have a significant bearing on the particle motion in the fluid.The governing equations for the capillary-gravity deep-water wave problem are encompassed by equations (4.8)–(4.12) together with the kinematic boundary condition expressing the fact that at great depths there is practically no motion

$$(u, v) \to (0, 0) \text{ as } y \to -\infty, \text{ uniformly for } x \in \mathbb{R}, t \geq 0, \tag{4.37}$$

cf. [67]. As in the previous section the equations for pure capillary water waves follow upon setting $g = 0$. We again assume that the flow is irrotational and so (4.14) holds. Following the method used in the case of water of finite depth, except this time we use

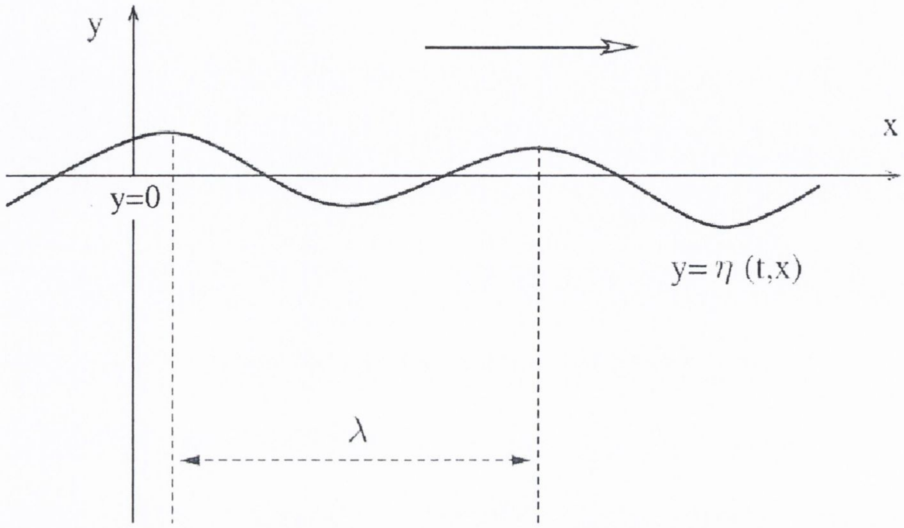


Figure 4-4: A periodic wave propagating in water of infinite depth.

a typical wavelength λ and a typical amplitude ϵ of the wave, we nondimensionalise the variables using

$$x \mapsto \lambda x, \; y \mapsto y, \; t \mapsto \frac{\lambda}{\sqrt{g}} t, \; u \mapsto u\sqrt{g}, \; v \mapsto v\frac{\sqrt{g}}{\lambda}, \; \eta \mapsto \epsilon\eta.$$

Following the scaling $p \mapsto \epsilon p$, $(u, v) \mapsto \epsilon(u, v)$ we obtain the boundary-value problem in nondimensional variables

$$\begin{cases} u_t + \epsilon(uu_x + vu_y) = -p_x, \\ \frac{1}{\lambda^2}\{v_t + \epsilon(uv_x + vv_y)\} = -p_y, \\ u_x + v_y = 0, \\ p = \eta - \frac{\Gamma}{g\lambda^2}\frac{\eta_{xx}}{(1+\frac{\epsilon^2}{\lambda^2}\eta_x^2)^{3/2}} \quad \text{and} \quad v = \eta_t + \epsilon u\eta_x \quad \text{on } y = \epsilon\eta, \\ (u, v) \to (0, 0) \quad \text{as } y \to -\infty, \text{ uniformly for } x \in \mathbb{R}, t \geq 0. \end{cases} \tag{4.38}$$

By letting $\epsilon \to 0$ in (4.38) we get the linearised problem

$$\begin{cases} u_t = -p_x, \quad \frac{1}{\lambda^2} v_t = -p_y, \\ u_x + v_y = 0, \\ v = \eta_t \quad \text{and} \quad p = \eta - \frac{\Gamma}{g\lambda^2}\eta_{xx} \quad \text{on } y = 0, \\ (u, v) \to (0,0) \quad \text{as } y \to -\infty, \text{ uniformly for } x \in \mathbb{R}, t \geq 0. \end{cases} \tag{4.39}$$

As before we look for travelling wave solutions for which the (t, x)−dependence of u, v, p, η is in the form of a periodic dependence in $x - c_0 t$, where $c_0 > 0$ represents the nondimensionalised speed of the wave. Choosing the Fourier mode

$$\eta(t, x) = \cos\big(2\pi(x - c_0 t)\big),$$

we obtain the solution

$$\begin{cases} \eta(t, x) = \cos\big(2\pi(x - c_0 t)\big), \\ u(t, x, y) = \frac{F(y)}{\lambda} \cos\big(2\pi(x - c_0 t)\big), \\ v(t, x, y) = F(y) \sin\big(2\pi(x - c_0 t)\big), \\ p(t, x, y) = \frac{c_0 F(y)}{\lambda} \cos\big(2\pi(x - c_0 t)\big), \end{cases} \tag{4.40}$$

where

$$c_0^2 = \frac{\lambda}{2\pi} + \frac{2\pi\Gamma}{g\lambda}, \qquad F(y) = 2\pi c_0 \exp\left(\frac{2\pi}{\lambda} y\right).$$

Returning to the original physical variables by means of the change of variables

$$x \mapsto \frac{x}{\lambda}, \; y \mapsto y, \; t \mapsto t\frac{\sqrt{g}}{\lambda}, \; u \mapsto \frac{u}{\epsilon\sqrt{g}}, \; v \mapsto v\frac{\lambda}{\epsilon\sqrt{g}}, \; \eta \mapsto \frac{\eta}{\epsilon}, \; p \mapsto \frac{p}{\epsilon},$$

if we define the wavenumber k and the frequency ω by

$$k = \frac{2\pi}{\lambda}, \qquad \omega = \sqrt{gk + k^3\Gamma}, \tag{4.41}$$

then the linear wave solution in the physical variables is

$$\begin{cases} \eta(t, x) = \epsilon \cos(kx - \omega t), \\ u(t, x, y) = \epsilon\omega \exp(ky) \cos(kx - \omega t), \\ v(t, x, y) = \epsilon\omega \exp(ky) \sin(kx - \omega t), \\ p(t, x, y) = P_0 - gy + \epsilon\,(g + k^2\Gamma) \exp(ky) \cos(kx - \omega t). \end{cases} \tag{4.42}$$

Notice that in the physical variables the wavespeed in (4.23), for right-travelling waves, is given by the dispersion relation

$$c = \frac{\omega}{k} = \sqrt{\frac{g\lambda}{2\pi} + \frac{2\pi\Gamma}{\lambda}}. \tag{4.43}$$

If the coefficient of surface-tension Γ is 0, then we retrieve the usual result for gravity waves in deep water, which is that longer waves propagate faster than shorter waves. If now we ignore the effects of gravity altogether, by setting $g = 0$, and focus solely on pure capillary waves, that is, waves acted upon by surface tension forces alone, then we find that in fact waves of shorter wavelength will move faster than waves of longer wavelength.

Description of particle trajectories

If $(x(t), y(t))$ is the path of the particle below the linear wave (4.42), then

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v,$$

so that the motion of the particle is described by the system

$$\begin{cases} \frac{dx}{dt} = M \exp(ky) \cos(kx - \omega t) \\ \frac{dy}{dt} = M \exp(ky) \sin(kx - \omega t), \end{cases} \tag{4.44}$$

with initial position (x_0, y_0), where

$$M = \epsilon\omega. \tag{4.45}$$

The right-hand side of the differential system (4.44) is smooth so that the existence of a unique local smooth solution is ensured [55]. Also, since y is bounded from above, the right-hand side of (4.44) is bounded and therefore this unique solution is defined globally [55].

In the case of pure capillary waves, that is $g = 0$, the equations in (4.42) remain unchanged except for the last equation describing the pressure. This becomes

$$P(t, x, y) = P_0 + \epsilon k^2 \Gamma \exp(ky) \cos(kx - \omega t), \tag{4.46}$$

and we have

$$\omega = \sqrt{k^3\Gamma}; \quad c = \sqrt{k\Gamma}.$$

Therefore, the system (4.44) provides a suitable description for the motion of particles in pure capillary deep-water waves, but now with a different c and M due to the different dependence of c and ω on Γ. If we also have Γ tending to zero, the hydrostatic pressure becomes constant and the wavespeed vanishes — a mathematical result which is in accordance with the physical situation.

Qualitative analysis of solutions

In performing an analysis of the solutions of (4.44) we will not try to solve the system of equations explicitly, since the right hand side of (4.44) is nonlinear. Rather, using phase plane analysis to examine the qualitative features of the solutions c.f. [37], it will be shown that there are no water particles travelling in closed orbits. In fact, we will see that every water particle experiences a forward drift as the wave progresses. We use the following transformation,

$$X(t) = kx(t) - \omega t, \qquad Y(t) = ky(t), \tag{4.47}$$

to give us the new system

$$\begin{cases} \frac{dX}{dt} = kM\exp(Y)\cos(X) - kc, \\ \frac{dY}{dt} = kM\exp(Y)\sin(X), \\ (X(0), Y(0)) = (x_0, y_0). \end{cases} \tag{4.48}$$

Since (4.48) is periodic in X, we need only consider the strip

$$\{(X, Y) \in \mathbb{R}^2 : \ -\pi \leq X \leq \pi\}.$$

This system corresponds directly to the system considered in [21] for pure-gravity deep-water waves, therefore we now present the following results obtained therein. Firstly, we present a necessary condition for the wave particles to be periodic.

Lemma 4.2.2 *[21] If the particle path $(x(t), y(t))$ is periodic, with period τ, then $\tau = \frac{2\pi}{\omega}$.*

Our main result now follows with a proof that is entirely along the lines of the proof of Theorem 4.2.1.

Theorem 4.2.3 *The system* (4.44) *has no periodic solutions.*

Orbital properties

We finally note two results from the paper [21], which tell us that as we reach greater depths the orbits become nearly closed circles.

Lemma 4.2.4 *[21] Let (X, Y) be a solution of* (4.48) *with* $(X(0), Y(0)) = (\pi, Y_\pi)$*. Then* $\theta \equiv \theta(Y_\pi)$ *defined by* $X(\theta) \equiv -\pi$ *is a strictly increasing function of* Y_π*, and* $\theta(Y) \to 2\pi/\omega$ *as* $Y \to -\infty$*.*

If (x, y) is the path of a particle we can consider the orbit traced by the particle as it goes from its point of greatest height until it reaches the same height again in the finite time θ, as defined in Lemma 4.2.4. Defining the *forward drift* of a fluid particle to be the horizontal distance $x(\theta) - x(0)$, we have the result

Corollary 4.2.5 *[21] The forward drift of a fluid particle is strictly decreasing with greater depth and vanishes as* $y \to -\infty$*.*

It is possible to present a qualitative description of the motion of water particles. Namely

$$\begin{aligned}
&\tfrac{dx}{dt} < 0,\ \tfrac{dy}{dt} < 0 \text{ for } && X(t) \in (-\pi, -\pi/2),\\
&\tfrac{dx}{dt} > 0,\ \tfrac{dy}{dt} < 0 \text{ for } && X(t) \in (-\pi/2, 0),\\
&\tfrac{dx}{dt} > 0,\ \tfrac{dy}{dt} > 0 \text{ for } && X(t) \in (0, \pi/2),\\
&\tfrac{dx}{dt} < 0,\ \tfrac{dy}{dt} > 0 \text{ for } && X(t) \in (\pi/2, \pi).
\end{aligned}$$

This tells us that the water particles move in an clockwise manner, and if starting at the point of greatest depth then (see Figure 4-5) they first move backward and up, then forward and up, then forward and down, then backward and down, until they again reach the point of greatest depth after the time of $\theta > \frac{2\pi}{\omega}$ with $x(\theta) - x(0) = \frac{\theta\omega - 2\pi}{k} > 0$.

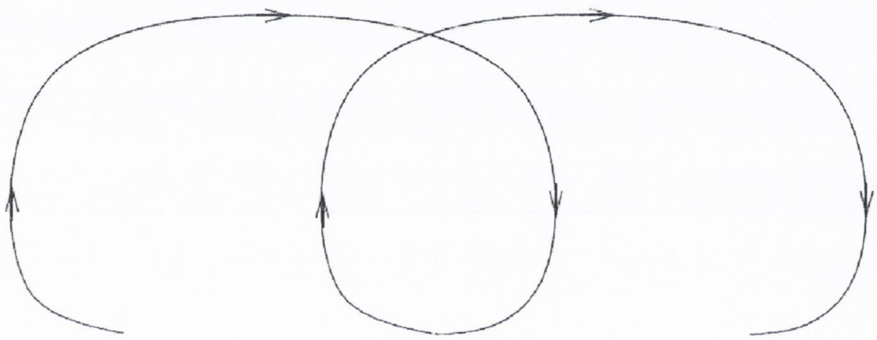


Figure 4-5: Particle trajectories in linear deep-water capillary and capillary-gravity waves.

4.3 The trajectories of particles in deep-water Stokes waves

4.3.1 The Stokes-wave problem

We have mentioned in previous discussions that within water wave theory it has long been conjectured that in shallow-water motion the trajectories of fluid particles take the form of closed ellipses, while in deep-water the paths more closely resemble circles (see [41, 67, 77, 86]). Recently it was proved that this conjecture is false for linear water waves propagating both over a flat bed [37, 62] and in water which is infinitely deep [21, 61] (see the previous Section 4.2 for some of these results). A proof of the fact that within the fully nonlinear framework of the governing equations for Stokes waves travelling over a flat bed there are no closed particle paths for nontrivial waves was recently achieved in [20]. In the course of this section we will prove the corresponding result for Stokes waves travelling over water of infinite depth. Our methods rely on results from the theory of harmonic functions, namely maximum principles and uniqueness properties of level sets.

4.3.2 Governing equations for Stokes waves

A deep-water Stokes wave is a two-dimensional periodic wave with a symmetric profile that rises and falls exactly once per wavelength and which is acted on by gravity, travelling at constant speed on the surface of irrotational water which is infinitely deep. To formulate the governing equations we will use Cartesian-coordinate axes with the $Y-$axis pointing vertically upwards and the $X-$axis perpendicular to the crestlines of the waves, the flow being in the positive $X-$direction. We let $Y = 0$ denote the mean water level and we assume the water domain is infinitely deep, that is $Y \to -\infty$. The velocity field of the water flow is given by $(u(t, X, Y), v(t, X, Y))$ with the water's free surface $Y = \eta(t, X)$; see Figure 4-6. Assuming that the water is both homogeneous (constant density) and inviscid (no internal friction forces) we obtain within the fluid domain the equation of

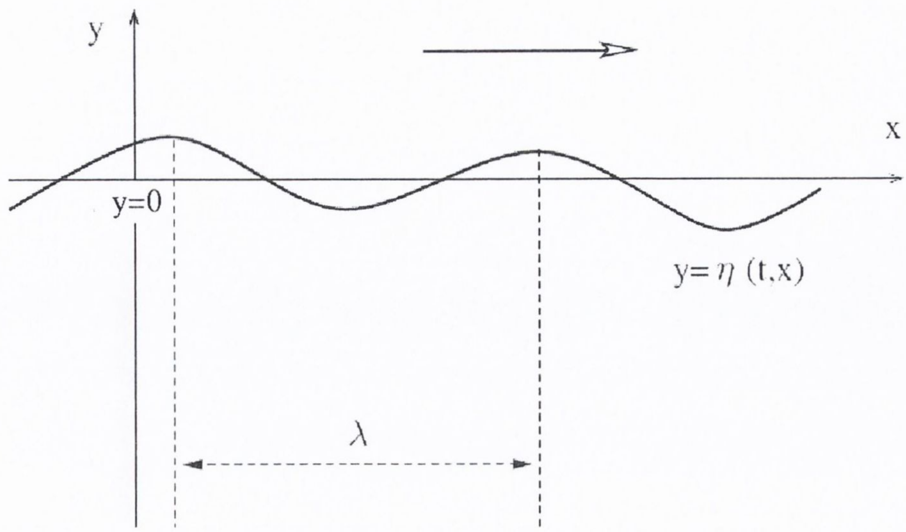


Figure 4-6: A deep-water Stokes wave.

mass conservation

$$u_X + v_Y = 0, \tag{4.49}$$

together with Euler's equation in component form

$$\begin{cases} u_t + uu_X + vu_Y = -P_X, \\ v_t + uv_X + vv_Y = -P_Y - g, \end{cases} \tag{4.50}$$

where $P(t, X, Y)$ denotes the pressure and g is the gravitational constant of acceleration. In order to decouple the motion of the air from that of the free surface particles [67], we introduce the dynamic boundary condition

$$P = P_0 \qquad \text{on } Y = \eta(t, X), \tag{4.51}$$

where P_0 is the constant atmospheric pressure and we have neglected the effects of surface tension. Since the free surface is always composed of the same particles we get the kinematic boundary condition

$$v = \eta_t + u\eta_X \qquad \text{on } Y = \eta(t, X). \tag{4.52}$$

The boundary condition for the bottom of the water domain which expresses the fact that at great depths there is practically no motion in the fluid is given, for all $t \geq 0$, by the limiting condition

$$(u, v) \to (0, 0) \qquad \text{as } Y \to -\infty \quad \text{uniformly for } X \in \mathbb{R}. \tag{4.53}$$

In a Stokes wave we assume the water to be irrotational: this is actually the case in the motion of water which starts from rest, since it is a consequence of Kelvin's Circulation theorem (that water which is irrotational at any instant remains so for all further times); therefore

$$u_Y = v_X. \tag{4.54}$$

The equations (4.49)–(4.54) are the governing equations for irrotational water waves [67] acting under the influence of gravity in water of infinite depth.

We look for steady periodic waves travelling at speed $c > 0$, i.e. for solutions which are periodic in X (of, say, period λ) and for which u, v, η, P depend on (X, t) only in the combination $(X - ct)$. We now pass to a reference frame moving with speed c in the positive X–direction, where the constant $c > 0$ is the speed of the wave, thus eliminating time from the problem. This is equivalent to the change of variables

$$(X - ct, Y) \mapsto (x, y). \tag{4.55}$$

In this new frame of reference the wavefront η is stationary and the flow (u, v) is steady. Experimental evidence suggests that for waves which are not near the breaking or spilling state, the speed of an individual particle is generally appreciably smaller than the wave propagation speed. In accordance with this we shall assume that $u < c$ throughout the fluid except possibly at the wave crest, where $u = c$ for the wave of greatest height— see [88]. Under the transformation to the moving reference frame, (4.50) and (4.52) become

$$\begin{cases} (u - c)u_x + vu_y = -P_x, \\ (u - c)v_x + vv_y = -P_y - g, \end{cases} \tag{4.56}$$

with

$$v = (u - c)\eta' \qquad \text{on } y = \eta(x). \tag{4.57}$$

Let $\overline{D_\eta} = \{(x,y) \in \mathbb{R}^2 : -\infty < y \leq \eta(x)\}$ represent the closure of the fluid domain. Thus a Stokes wave is a solution of (4.56)–(4.57) satisfying $u_x + v_y = 0$ throughout D_η with $P = P_0$ on $y = \eta(x)$ and with $(u,v) \to (0,0,)$ as $y \to -\infty$ uniformly for $x \in \mathbb{R}$, and for which $(\eta, u, v, P) \in C^3(\overline{D_\eta}) \times C^2(\overline{D_\eta}) \times C^2(\overline{D_\eta}) \times C^1(\overline{D_\eta})$ are all periodic in the $x-$variable. Furthermore, we require the functions η, u, P to be even and the function v to be odd in the $x-$variable. Upon integrating (4.56) we derive Bernoulli's law which states that the expression

$$E := \frac{(u-c)^2 - v^2}{2} + gy + P$$

is constant throughout D_η. We define the stream function $\psi(x,y) \in C^2(\overline{D_\eta})$ up to a constant by

$$\psi_y = u - c, \quad \psi_x = -v. \tag{4.58}$$

The irrotational flow assumption, $u_x + v_y = 0$ throughout D_η, tells us that ψ is harmonic, and the explicit integral formula

$$\psi(x,y) = \psi_0 - \int_0^x v(\xi, -d)d\xi + \int_{-d}^y [u(x,\xi) - c]d\xi, \qquad y \leq \eta(x),$$

where $\psi_0 \in \mathbb{R}$ is a constant and the line $y = -d$ lies beneath the wave trough-level shows that ψ is periodic in x. Furthermore, we have $u - c = \psi_y < 0$ throughout D_η. Writing (4.57) in terms of ψ we get $\psi_x + \psi_y \eta' = 0$ on $y = \eta(x)$, which tells us that the stream function is constant along the free surface $y = \eta(x)$ — it will be convenient to choose $\psi = 0$ as the constant value for the free surface. If we now express Bernoulli's law in terms of ψ we obtain

$$E := \frac{\psi_x^2 + \psi_y^2}{2} + gy + P,$$

and so the dynamic boundary condition (4.51) is equivalent to

$$|\nabla\psi|^2 + 2gy = E_0 \quad \text{on} \quad y = \eta(x), \tag{4.59}$$

where $E_0 = 2(E - P_0)$. Since $\eta > 0$ for at least some $x \in \mathbb{R}$ it follows from (4.51) that

$E_0 > 0$. Thus for Stokes waves the governing equations can be summarized as

$$\begin{cases} \Delta\psi = 0 & \text{in} \quad -\infty < y < \eta(x), \\ |\nabla\psi|^2 + 2gy = E_0 & \text{on} \quad y = \eta(x), \\ \psi = 0 & \text{on} \quad y = \eta(x), \\ \nabla\psi \to (0, -c) & \text{as} \quad y \to -\infty \text{ uniformly for } x \in \mathbb{R}, \end{cases} \tag{4.60}$$

where $\eta \in C^3(\mathbb{R})$ and $\psi \in C^2(\overline{D_\eta})$ are periodic in the $x-$variable, with wavelength λ as the period, and η rises and falls exactly once per period with $\eta'(x) \neq 0$ except at the maximum or minimum.

Now suppose y_0 and d are fixed depths below the wave trough, such that $y_0 < d < \eta$. The divergence theorem applied to the vector field (ψ_x, ψ_y) on the rectangular box enclosed by the lines $x = 0,\ x = \lambda,\ y = d,\ y = y_0$ gives

$$\begin{aligned} 0 &= \int_0^\lambda \int_{y_0}^d (\psi_{xx} + \psi_{yy}) dy dx \\ &= \int_0^\lambda \psi_y(x, y_0) dx + \int_{y_0}^d \psi_x(\lambda, y) dy + \int_\lambda^0 \psi_y(x, d) dx + \int_d^{y_0} \psi_x(0, y) dy \\ &= \int_0^\lambda \psi_y(x, y_0) dx - \int_0^\lambda \psi_y(x, d) dx, \end{aligned}$$

and so the mean horizontal velocity component per wavelength λ, at any fixed depth below the wave trough, is constant throughout D_η. A natural consequence of this is Stokes' definition of the wave speed as the mean horizontal velocity component in the moving frame of reference for which the wave is stationary, i.e.

$$c = -\frac{1}{\lambda}\int_0^\lambda \psi_y(x, y_0) dx > 0, \tag{4.61}$$

where y_0 is any fixed depth below the level of the wave trough.

4.3.3 Results for the velocity field

Without any loss of generality, we restrict our attention to Stokes waves of period 2π with the crest at $(0, \eta(0))$ and the trough at $(\pi, \eta(\pi))$.

Lemma 4.3.1 *The following strict inequalities hold*

$$\psi_x(x,y) < 0, \quad \frac{d}{dx}u(x,\eta(x)) < 0 \quad \text{for} \quad x \in (0,\pi), y \in (-\infty, \eta(x)]. \tag{4.62}$$

Proof Note that $\psi_x = 0$ on the half-lines $x = 0$ and $x = \pi$ since v is periodic and an odd function in the x variable. Also by (4.60) ψ_x is harmonic in D_η and $\psi_x \to 0$ as $y \to -\infty$. Since $\psi_x = -\psi_y\eta'$ on $y = \eta(x)$ and by assumption both $\eta'(x) < 0$ and $\psi_y(x,\cdot) < 0$ for $x \in (0,\pi)$, it follows that $\psi_x < 0$ along the upper boundary η of D_η when $x \in (0,\pi)$. Defining the bounded water domain

$$D_{\eta,k} = \{(x,y) \in \mathbb{R}^2 : x \in (0,\pi),\ -k < y < \eta(x)\},$$

for $k \in \mathbb{R}^+$ sufficiently large that $y = -k$ is below the trough level, we suppose that there is a point (x_0, y_0) in the interior of D_η for which $\psi_x(x_0,y_0) = \epsilon > 0$. Let $\hat{k} \in \mathbb{R}^+$ be large enough that $(x_0,y_0) \in D_{\eta,\hat{k}}$ and $\psi_x < \epsilon$ uniformly in x for $-\infty < y \leq -\hat{k}$, and also $-\hat{k} < \eta(x)$: it follows from the fourth equation in (4.60) that such a $\hat{k}$ exists. Then $\psi_x(x_0,y_0) = \epsilon$ contradicts the strong maximum principle for harmonic functions [50] applied to the domain $D_{\eta,\hat{k}}$. Therefore we must have $\psi_x \leq 0$ on D_η. Suppose that $\psi_x = 0$ at an interior point of D_η. Then we can choose k_0 so that the point is contained in D_{η,k_0}; we can apply the maximum principle on this bounded domain and get $\psi_x \equiv 0$ on $\overline{D_{\eta,k_0}}$, which contradicts $\psi_x < 0$ on the upper boundary of D_η. Therefore $\psi_x < 0$ in D_η.

To show the second inequaltity in (4.62) we note that, from (4.50) and (4.51), we have $P = P_0$ on $y = \eta(x)$ and $P_y \to -g$ as $y \to -\infty$. Differentiating (4.50) we see that P is superharmonic,

$$\Delta P = -2\psi_{xy}^2 - 2\psi_{xx}^2 \leq 0.$$

First suppose that the minimum of P is attained on the side boundary of the water domain D_η at the point $(0, y_0)$, say. Considering now the bounded water domain $D_{\epsilon,k} = \{(x,y) : x \in (-\epsilon,\pi),\ y \in (-k,\eta(x))\}$ for $-k < y_0$ and $\epsilon > 0$, it follows from the periodicity of the function that P attains its minimum in $D_{\epsilon,k}$ also at the interior point $(0,y_0)$. The maximum principle [50] then implies that P is constant on $D_{\epsilon,k}$ and since this holds for any $-k < y_0$ we must have P constant on D_η which is a contradiction. Since at great depths the pressure function P increases with increasing depth, which is a result of the limiting condition $P_y \to -g$ as $y \to -\infty$, we must have $P > P_0$ below the free surface η. Also, from (4.51) we get $P_x + P_y \eta' = 0$ on $y = \eta(x)$ for $x \in (0, \pi)$, and since $\eta'(x) < 0$ and $P_y(x, \eta(x)) < 0$ (by Hopf's lemma [50]) it follows that $P_x(x, \eta(x)) < 0$ for $x \in (0, \pi)$. But from (4.56) and (4.57) we have $P_x = (c - u)[u_x + \eta' u_y]$ on $y = \eta(x)$, and so the sum of the terms in the square brackets is strictly less than zero for $x \in (0, \pi)$, which proves the second inequality in (4.62). ∎

Lemma 4.3.2 *The function $y \mapsto u(0, y)$ is strictly decreasing as we move down the vertical half-line $[(0, \eta(0)), (0, -\infty)]$, whereas it is strictly increasing along $[(\pi, \eta(\pi)), (\pi, -\infty)]$ as we move downwards.*

Proof From Lemma 4.3.1 we know that $\psi_x < 0$ in $D_{\eta,k}$ for $k \in \mathbb{R}^+$, $-k < \eta$. Furthermore, since $\psi_x = -v = 0$ on $x = 0$ and $x = \pi$ we can apply Hopf's maximum principle [50] on $D_{\eta,k}$ to get

$$\psi_{xx}(0, y) < 0, \qquad y \in (-k, \eta(0)), \tag{4.63}$$

$$\psi_{xx}(\pi, y) > 0, \qquad y \in (-k, \eta(\pi)). \tag{4.64}$$

Since $u_y = \psi_{yy}$ and $\Delta\psi = 0$ the statement follows at once. ∎

Remark Prior to this we have assumed that $u < c$ in D_η. As a result of Lemma 4.3.1 and Lemma 4.3.2 we can now state that $u < c$ in the closure $\overline{D_\eta}$ of D_η, except in the case of a Stokes wave of greatest height, in which case, at the crest $(0, \eta(0))$, we have $u = c$ with $u < c$ at all other points of $\overline{D_\eta}$; cf. [88].

We now present some properties of the zero-level set $\{u = 0\}$ of a nontrivial harmonic function u. Let $u(x_0, y_0) = 0$, and let us define the analytic function

$$f(z) = u(z) + i\,(v(z) - v(z_0))\,, \qquad z = x + iy,\ z_0 = x_0 + iy_0,$$

where v is the harmonic conjugate of u. Since $f \not\equiv 0$ and $f(z_0) = 0$ there exists a unique positive integer $n \geq 1$ such that $f(z) = (z - z_0)^n f_1(z)$ in some neighbourhood of z_0 where f_1

is analytic and $f_1(z_0) = w \neq 0$. Choosing a single-valued branch of $z^{\frac{1}{n}}$ in a neighbourhood of w we can define a function f_2 which is analytic in a neighbourhood of z_0 by $f_2 = (f_1)^{\frac{1}{n}}$, that is

$$f(z) = [(z - z_0)f_2(z)]^n$$

in this neighbourhood. Now the analytic function $\varphi_1(z) = (z - z_0)f_2(z)$ is a local diffeomorphism in a neighbourhood $\mathcal{N}'(z_0)$ of z_0 as a result of the inverse function theorem. Let φ be the analytic inverse of φ_1 in a neighbourhood $\mathcal{N}(z_0)$ of z_0 suitably defined so that $\varphi\{\mathcal{N}(z_0)\} \subset \mathcal{N}'(z_0)$, with $\varphi_1(\varphi(z)) = z$ for $z \in \mathcal{N}(z_0)$. Then $f \circ \varphi(z) = [\varphi_1(\varphi(z))]^n = z^n$ and so $u \circ \varphi(z) = \Re(z^n)$. Let γ_j, for $j = 0, \ldots, n-1$, be a conformal homeomorphism of a line segment in the neighbourhood $\mathcal{N}(z_0)$ defined as follows: $\gamma_j(r) = \varphi\left(s_j(r)e^{i\left(\frac{\pi+2j\pi}{2n}\right)}\right)$ with each s_j a function with the property that $s_j(r)e^{i\left(\frac{\pi+2j\pi}{2n}\right)} \in \mathcal{N}(z_0)$ for $r \in (-1, 1)$, and $s_j(0) = 0$. Thus, for $j = 0, \ldots, n-1$, the γ_j map n line segments contained in $\mathcal{N}(z_0)$ passing through the point $(0, 0)$, with adjacent line segments separated by the angle $\frac{\pi}{n}$ at $(0, 0)$, conformally into n curves contained in $\mathcal{N}'(z_0)$ and intersecting only at the point (x_0, y_0) . Thus for a nontrivial harmonic function u we have a unique positive integer $n \geq 1$ such that

(i) $\{u = 0\} \cap \mathcal{N}(x_0, y_0) = \cup_{j=0}^{n-1}\gamma_j$ where $\gamma_j = \{\gamma_j(r) : r \in (-1, 1)\}$;

(ii) $\gamma_j(0) = (x_0, y_0)$ and the angle at (x_0, y_0) between γ_j and γ_{j+1} is $\frac{\pi}{n}$.

We are now in a position to prove the following.

Lemma 4.3.3 *The zero-level set $\{u = 0\}$ of the function u is a maximal non-self-intersecting infinite curve C originating at some point $(x_C, \eta(x_C))$ on the free surface for $x_C \in (0, \pi)$, and has the property that any streamline $\psi = \psi_0$ intersects C in exactly one point for $\psi_0 \in \{\psi(x, y) : (x, y) \in \overline{D_\eta}\} = \mathbb{R}^+$.*

Proof There cannot be a zero point of u located along $x = 0$ or $x = \pi$ as Lemma 4.3.2 tells us that u decreases strictly towards zero along $x = 0$ as y tends to minus infinity, whereas it increases strictly towards zero as $y \to -\infty$ along $x = \pi$. The monotonicity of u on η, ensured by Lemma 4.3.1 together with the fact that $u(0, \eta(0)) > 0$ and $u(\pi, \eta(\pi)) < 0$, imply the existence of a unique point $x_{\mathcal{C}} \in (0, \pi)$ such that $u(x_{\mathcal{C}}, \eta(x_{\mathcal{C}})) = 0$. Let $\mathcal{C}$ be a maximal extension of the local curves $\gamma_j(r)$ of the level set $\{u = 0\}$ which pass through $(x_{\mathcal{C}}, \eta(x_{\mathcal{C}}))$. We will show that $\mathcal{C}$ is a unique non-self-intersecting curve which descends towards the infinitely deep seabed in such a way that any streamline $\psi = \psi_0$ intersects $\mathcal{C}$ in exactly one point. Let $(x, \sigma(x))$ be the equation of a streamline $\psi = \psi_0$ for any $\psi_0 \in \{\psi(x, y) : (x, y) \in \overline{D_\eta}\} = \mathbb{R}^+$. Then $\psi_x + \psi_y \sigma'(x) = 0$ and so $0 > \sigma'(x) = -\frac{\psi_x}{\psi_y}$ for $x \in (0, \pi)$. Also $u(0, \sigma(0)) > 0 > u(\pi, \sigma(\pi))$ and so each streamline intersects $\mathcal{C}$ in at least one point. We need to show that $\mathcal{C}$ does not self-intersect and that each streamline intersects $\mathcal{C}$ at just one point.

First we show that at no point (x_0, y_0) does the curve $\mathcal{C}$ furcate into multiple branches. Supposing otherwise, let (x_0, y_0) be the first point of $\mathcal{C}$ where branching occurs (that is, y_0 is the largest possible value with this property) with $\mathcal{C}_1$ and $\mathcal{C}_2$ any two branches emanating from this point. Then we have two possible scenarios: (a) the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at some further point (x_1, y_1) with $y_1 \leq y_0$; (b) both curves head toward the infinitely deep seabed $y = -\infty$ without ever meeting again. In case (a) $\mathcal{C}_1$ and $\mathcal{C}_2$ form the boundary of a compact domain D^{com} on which u is harmonic and where $u = 0$ on the boundary $\mathcal{C}_1 \cup \mathcal{C}_2$. The strong maximum principle then dictates that $u \equiv 0$ on D^{com}. In case (b) the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ form the boundary of an infinite domain D^{inf} on which u is harmonic and having $u = 0$ on the infinite boundary $\mathcal{C}_1 \cup \mathcal{C}_2$, with $u \to 0$ as $y \to -\infty$ in D^{inf}. Therefore $u \equiv 0$ on D^{inf} as a consequence of the Phragmen-Lindelöf principle. In both cases (a) and (b), we have $u \equiv 0$ on some open set Ω of D_η. Therefore, for any point $\hat{z} = (\hat{x}, \hat{y})$ in Ω, there exists a neighbourhood $\mathcal{N}(\hat{z})$ of $\hat{z}$ contained in Ω, and for any integer $n \in \mathbb{N}$ we can construct n curves $\hat{\gamma}_i(t)$, for $i = 0, \ldots, n-1$ and $t \in (-1, 1)$, on which $u = 0$, $\hat{\gamma}(0) = (\hat{x}, \hat{y})$ and the angle at $\hat{z}$ between $\hat{\gamma}_i$ and $\hat{\gamma}_{i+1}$ is $\frac{\pi}{n}$. Since u is harmonic, this situation violates conditions (i) and (ii) unless u is the trivial solution $u \equiv 0$ on $\overline{D_\eta}$, which is a contradiction. This rules out the possibility of the curve $\mathcal{C}$ forking out at any point; a consequence of this is that the maximal curve $\mathcal{C}$ of $\{u = 0\}$ is unique. We now show that $\mathcal{C}$ intersects each streamline $\psi = \psi_0 \in [0, \infty)$ in exactly one point. This follows once we demonstrate that, in a certain sense, the curve $\mathcal{C}$ cannot bend or change direction too dramatically as it heads towards the bottom of the water domain. We can parameterise the curve $\mathcal{C}$ locally in either of two ways, by $(s, f(s))$ or by $(f(s), s)$, for s in some real interval $\mathcal{I}$. Suppose that $\mathcal{C}$ has a local maximum or minimum near a point Q which is parameterised by $Q = (s_0, f(s_0))$. Then as u is constant along $\mathcal{C}$ we get $u_x(s, f(s)) + u_y(s, f(s))f'(s) = 0$ for all $s \in \mathcal{I}$. Thus $u_x(s_0, f(s_0)) = 0$ since $f'(s_0) = 0$. Along streamlines we get $u_x(x, \sigma(x)) + u_y(x, \sigma(x))\sigma'(x) = 0$ and, since at least one streamline passes through $(s_0, f(s_0))$, we must also have $u_y = 0$ at $(s_0, f(s_0))$. It follows from Taylor's formula that at the point Q the function u has a zero of order at least two and accordingly we get at least two branches of $\mathcal{C} = \{u = 0\}$ intersecting all streamlines sufficiently close to Q, which contradicts the earlier result that $\mathcal{C}$ does not fork out into multiple branches. This completes the proof. ∎

4.3.4 Particle trajectories in deep-water Stokes waves

The unique solution $(X(t), Y(t))$ of the differential system

$$\begin{cases} \dot{X} &= u(X - ct, Y) \\ \dot{Y} &= v(X - ct, Y), \end{cases} \tag{4.65}$$

with initial position $(X(0), Y(0))$ describes the trajectory of a particle in the fluid [55]. Correspondingly, in the moving reference frame we have the Hamiltonian system

$$\begin{cases} \dot{x} &= u(x, y) - c \\ \dot{y} &= v(x, y), \end{cases} \tag{4.66}$$

with the Hamiltonian function the stream function $\psi(x, y)$. Solutions of (4.65) are mapped into solutions of (4.66) by the change of variables (4.55). Unless we are dealing with a wave of greatest height, then it follows from the remark in the previous section that $u - c \leq -\epsilon$ on D_η, for some $\epsilon > 0$. Therefore each solution of (4.66) originating in D_η will intersect the line $x = -\pi$ at a finite time in the future, having intersected the line $x = \pi$ a finite length of time in the past. This also holds for a Stokes wave of greatest height. In this case, since a solution of the Hamiltonian system (4.66) will remain on the same level set for all times — a property characteristic of Hamiltonian systems in general — a particle starting at some point in D_η will have $u - c \leq -\epsilon$ for some $\epsilon > 0$ (in view of the Remark in the previous section). For a solution of (4.66) starting on the boundary $\eta(x)$, $x \in (0, \pi)$, of D_η we know that, since $\eta'(x)$ is bounded away from zero in the case of a wave of greatest height, we get

$$(c - u(x, \eta(x)))^2 \leq E_0 - 2g\eta(x) = O(x) \quad \text{as } x \downarrow 0,$$

from (4.60) and the mean–value theorem. Therefore $c - u(x, \eta(x)) = O(\sqrt{x})$ as $x \downarrow 0$ and so

$$\int_0^{\frac{\pi}{2}} \frac{dx}{c - u(x, \eta(x))} < \infty.$$

For a solution of (4.66) with initial data $(0, \eta(0))$ we get $\dot{x}(0) = \dot{y}(0) = 0$, and so the solution is not unique. The constant solution is not physically reasonable as this would mean that other particles from the top boundary η would collide with it at the crest. Therefore we can say that, in the moving frame, any solution which starts on the boundary η reaches $(0, \eta(0))$ in finite time but does not stay there, rather it continues moving in the negative $x-$direction.

Lemma 4.3.4 *For $y_0 \in (-\infty, \eta(\pi)]$ let $\theta = \theta(y_0) > 0$ be the time necessary for the solution $(x(t), y(t))$ of (4.66), with initial data (π, y_0), to intersect the line $x = -\pi$. This solution represents a closed particle path if and only if $\theta = \frac{2\pi}{c}$.*

Proof The solution will intersect the line $x = -\pi$ at the point $(-\pi, y_0) = (x(\theta), y(\theta))$ by symmetry of the wave. If $\theta = \frac{2\pi}{c}$ then

$$X(\theta) - X(0) = [x(\theta) - c\theta] - x(0) = 0$$

with

$$(X(\theta) - c\theta) = -\pi = x(0) - 2\pi = X(0) - 2\pi$$

and so sufficiency follows from the periodicity of (4.65). Conversely if $(X(t), Y(t))$ is a closed path of (4.65) with period $\tau > 0$ then $y(0) = y(\tau)$ and so $\tau = n\theta$ for some $n \in \mathbb{Z}$.

But then $x(\tau) = x(0) - 2n\pi$ so that $X(0) = X(\tau)$ implies that

$$0 = X(\tau) - X(0) = [x(\tau) + c\tau] - x(0) = x(\tau) - x(0) + cn\theta = -2n\pi + cn\theta,$$

and so $\theta = \frac{2\pi}{c}$ is necessary. ∎

We can now prove the following result.

Proposition 4.3.5 *There are no closed particles in a deep-water Stokes wave.*

Proof We aim to show that $\theta(y_0) > \frac{2\pi}{c}$ for all $y_0 \in (-\infty, \eta(\pi)]$. Let $y = \sigma(x)$ denote the equation of the streamline $\psi = \psi(\pi, y_0)$ and let $k \in \mathbb{R}^+$ be such that $-k < \sigma(x) < \eta(x)$ for $x \in [0, \pi]$. If we apply the divergence theorem to the vector field (ψ_x, ψ_y) in the strip

$$\{(x, y) \in \mathbb{R}^2 : x \in (-\pi, \pi), \ -k < y < \sigma(x)\},$$

we find that

$$\int_{-\pi}^{\pi} (c - u(x, -k))\, dx = \int_{-\pi}^{\pi} \frac{[c - u(x, \sigma(x))]^2 + v^2(x, \sigma(x))}{c - u(x, \sigma(x))} dx,$$

and so from (4.61) we deduce that

$$\int_{-\pi}^{\pi} \frac{[c - u(x, \sigma(x))]^2 + v^2(x, \sigma(x))}{c - u(x, \sigma(x))} dx = 2\pi c.$$

It follows from Lemma 4.3.1 that

$$\int_{-\pi}^{\pi} [c - u(x, \sigma(x))] dx < 2\pi c. \tag{4.67}$$

Since $u - c < 0$ along $y = \sigma(x)$, except maybe at the point $(0, \sigma(0))$ in the case of a wave of greatest height where $y_0 = \eta(\pi)$, we have

$$\int_{-\pi}^{\pi} \frac{dx}{c - u(x, \sigma(x))} = \theta. \tag{4.68}$$

Using the Cauchy-Schwarz inequality together with (4.68) we get

$$\theta \int_{-\pi}^{\pi} [c - u(x, \sigma(x))] dx = \int_{-\pi}^{\pi} \frac{dx}{c - u(x, \sigma(x))} \int_{-\pi}^{\pi} [c - u(x, \sigma(x))] dx \geq 4\pi^2$$

and so from (4.67) we have

$$\theta \geq \frac{4\pi^2}{\int_{-\pi}^{\pi} [c - u(x, \sigma(x))] dx} > \frac{2\pi}{c},$$

and the proof is complete. ∎

We can summarise the motion of water particles in deep-water Stokes waves as follows. Along each streamline $y = \sigma(x)$ we have

$$u(-\pi, \sigma(-\pi)) = u(\pi, \sigma(\pi)) \; < \; 0 < u(0, \sigma(0)),$$
$$v(x, \sigma(x)) = -v(-x, \sigma(-x)) \; > \; 0, \quad x \in (0, \pi),$$

with the function $x \mapsto u(x, \sigma(x))$ changing sign when it reaches the infinite curve $\mathcal{C}$ located in both regions $x \in (-\pi, 0)$ and $x \in (0, \pi)$, the streamline $y = \sigma(x)$ intersecting each curve in a unique point. Let $X(t)$ be a solution of (4.65) with initial value $X(0) = \pi$ and let $\theta > 0$ be the first time when $Y(\theta) = Y(0)$. Then the wave period is $\frac{2\pi}{c}$ and we have shown in Proposition 4.3.5 that

$$\theta > \frac{2\pi}{c}, \quad X(\theta) = c\theta - \pi > \pi,$$

for any solution $X(t)$ of (4.65). Thus a water particle located initially at some point (x_0, y_0) with $x_0 \in (0, \pi]$ always stays above the line $y = y_0$. Its trajectory is composed (see Figure 4-7) of the following distinct motions: it first moves backwards and upwards; then while continuing upwards it moves forwards; then while moving forward it begins to go downwards; then at a certain point in the downward motion it begins to move backwards until it hits the line $y = y_0$ at a point (x_1, y_0) with $x_1 > x_0$ and the cycle repeats itself as before.

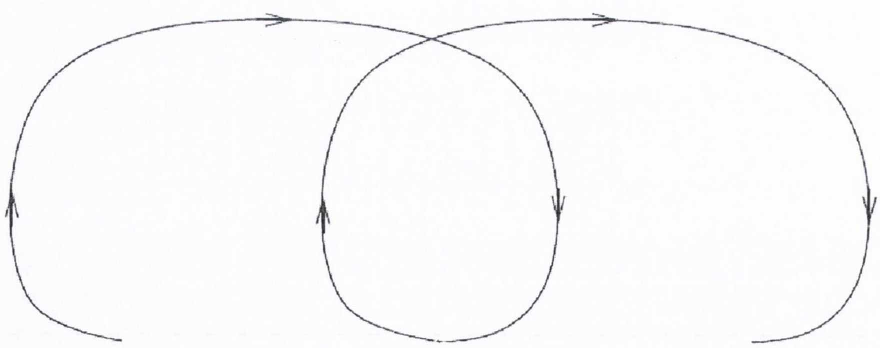


Figure 4-7: Particle trajectories in deep-water Stokes waves.

Chapter 5

Conclusions and further work

In Chapter 2 we study the propagation speed and persistence properties for solutions of the $b-$equations, expressed in the forms (2.1) or (2.2). The first section of this chapter is restricted to the solutions of these equations which are initially compactly supported. We addressed the question as to whether such solutions retain this property for any nontrivial length of time throughout its existence. The work contained in this section extends the results contained in Constantin [19] for the case of the Camassa-Holm (CH) equation (given by $b = 2$) and Mustafa [84] for the Degasperis-Procesi (DP) equation (given by $b = 3$). It is assumed in [19, 84] that the initial data is smooth, $u_0 \in C^\infty(\mathbb{R})$. As a result of Proposition 2.1.4 we lessen this restricion and merely require $u_0 \in \mathbb{H}^4(\mathbb{R})$. Furthermore, we expand on the cases $b = 2, 3$ (presented in [19, 84]) and prove that for all values $b \in \mathbb{R}$ the initially compactly supported solutions m to equation (2.2) remain compactly supported for all times in their existence, and that for b in the range $0 \leq b \leq 3$ all solutions u of equation (2.1) which are initially compactly supported instantly lose this property.

A future ambition would be to obtain results on solutions to equation (2.1) for values of b outside the range $0 \leq b \leq 3$ — however it seems such a hope may be beyond the capabilites of the approach presented in this thesis. Of course, the results we present in Chapter 2 do cover the two most important cases $b = 2, 3$ which represent the CH, DP equations, respectively.

The second section of Chapter 2 develops the results detailed above for the case of the DP equation. This extends and clarifies some results obtained in Himonas *et al.* [63] for the CH equation. The class of initial data for the DP equation is now expanded to include solutions with an asymptotic decay no slower than the inverse exponential function with constant weight less than or equal to one. For a variety of conditions on the asymptotic behaviour of the derivative of the initial data, we present results in Theorems 2.2.1 -2 -3 -4 relating to the persistence of these decay conditions throughout the evolution of the solutions over its interval of existence. A thorough description of solutions to the DP equation which are initially compactly supported is provided in Theorem 2.2.5.

We envisage that in the course of future work these results might be generalised to the entire family of $b-$equations, or at the very least to the equations in the range $0 \leq b \leq 3$.

In Chapter 3 we delve into the subject of edge waves. Edge waves are ocean waves which propagate in the longshore direction and are trapped in the nearshore — they betray little sign of activity far out from the shore. In 1846 Stokes found an edge-wave solution for the linear water-wave equations on a beach with a constant slope. Unfortunately for edge waves, they have long been regarded by many as a mere curiosity and consequently they have been relatively ignored in the literature. However, recent geophysical studies suggest that edge waves play a major role in sediment transport in the nearshore, and in the past half century or so a concerted effort has been made to place the study of the problem mathematically— the work of Ursell (1952), Stoker (1957), Whitham (1976,77) and Minzioni (1976,77) being among the more notable early contributions. In 2001, Constantin [18] presented an exact solution for the nonlinear edge-wave problem in the case of rotational water flowing over a constantly sloping beach. In 2005, Johnson [69] obtained a fully nonlinear edge-wave description for an arbitrary depth profile which has an exact solution which is irrotational. Results incorporating a longshore current into the depth profile producing an effective depth profile have also been obtained in [69].

In Chapter 3 we present a mathematically rigorous study of the linear edge-wave equation derived by Howd *et al.* [65] in 1992. This linear self-adjoint second order differential equation describes the propagation of edge waves over a bed of variable depth; a longshore current can also be incorporated into the effective depth profile. We define edge-wave solutions to be solutions which are monotonous and tend to zero far out, and in the course of our analysis we state integral conditions on the effective depth profile which are necessary and sufficient for the existence of edge waves, adapting results from Cecchi *et al.* [10]. We use fixed point theorems to show the existence of edge-wave solutions for a large class of suitable depth profiles which have an elementary description.

Anticipated future work in the area of edge waves would consist of the analysis of the nonlinear edge wave problem. Indeed, the work of Johnson [69] presents a number of nonlinear edge-wave model equations, which admit some exact solutions; these would be worthy candidates for future study. Analysis of the edge-wave problem in the nonlinear milieu is considerably more challenging than the linear case.

In Chapter 4 we study the trajectories of fluid particles in a number of water-wave settings. The behaviour of individual water particles during the passage of a water wave is a fundamental question, and one of no-little practical importance when it comes to the design of oil rigs, submarines and for providing a qualitative insight into the transport of sediment through the motion of waves. The question has long occupied the curiosity of researchers, and a consensus seems to have been reached in the literature, apparently backed up by photographic and empirical evidence, that particle trajectories, in general, describe closed orbits which decrease with the depth [8, 67, 78, 79]. However, recent results from the last couple of years by Constantin *et al.* [37, 21, 20] showed that in the linear setting for irrotational gravity waves, in water of both finite and infinite depth, and in the fully nonlinear setting for finite depth Stokes water waves, water particles do not in fact trace out a closed orbit but rather they experience a forward drift.

The results we present in Chapter 4 consists of two sections. In the first section we generalise results obtained in the literature above for the linear setting. We introduce the effect of surface tension on the development of the water waves and then prove that the particle paths are also not closed in linear periodic capillary and capillary-gravity waves — in the case of irrotational motion with both finite and infinite depth. Similar to the linear pure-gravity-wave-case, the particles experience a forward drift. In the second section we generalise the results of Constantin [20] for the finite-depth Stokes-wave to the deep water, i.e. the infinite-depth, Stokes-wave. Here the transition from the finite depth case to the deep water Stokes wave case is nontrivial — we must employ maximum principles and carefully use the intrinsic properties of level sets of harmonic functions in order to obtain results on the now semi-infinite domain. The end result shows, again, a forward drift for the water particles.

Future work concerning particle trajectories might consist of introducing the effects of surface tension on the classical Stokes wave. This would be a significant complication of the problem, introducing the curvature of the surface into the nonlinear boundary conditions.

Chapter 6

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