Black Holes and String Theory:- Selected Topics

by

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Declaration

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Summary

This thesis is divided into three parts. In Part I the concept of holography in the context of the Maldacena conjecture and the three-dimensional black hole of Bañados, Teitelboim and Zanelli (BTZ) is studied. In particular, it is shown how a theorem of Sullivan provides a precise mathematical statement of a three-dimensional kinematic holographic principle, that is, the hyperbolic structure of a geometrically finite Kleinian manifold is completely determined in terms of the corresponding Teichmüller space of the boundary. It is shown that the Euclidean BTZ black hole is a geometrically finite Kleinian manifold and some consequences of the theorem for this space-time are explored.

In Part II string theory and D-branes in the context of Ashoke Sen's work on tachyon condensation in type II brane-antibrane systems are studied. A calculation of various tree-level scattering amplitudes on the upper half-plane describing the interaction of closed string R-R fields and open string tachyons is performed. The calculations lead to a proposal for the generalisation of the Wess-Zumino term to brane-antibrane systems. The proposal is written cleanly in terms of Quillen's superconnection.

The final part of the thesis is concerned with generalised Kaluza-Klein theory in the form of the Randall-Sundrum models and cosmological extensions such as the Binétruy, Deffayet and Langlois model. A single-brane cosmological scenario in which five-dimensional gravity is coupled to a scalar field sigma-model with indefinite metric and matter on the brane is induced by a bulk *a priori* anisotropic fluid is also considered. A range of solutions for which the induced metric on the brane is of Friedmann-Robertson-Walker (FRW) type and for which the fifth radius is non-static is found provided that the fluid is isotropic and stiff. Other FRW-type solutions with a static fifth radius are also found. In all cases it is found possible to achieve standard cosmology, that is, the Hubble parameter on the brane is proportional to the square root of the density of the fluid on the brane. Einstein's equations are linearised for the isotropic, stiff fluid case and a stability analysis of the transverse, traceless Kaluza-Klein modes is carried out. The analysis suggests that those solutions where the scale factor on the brane goes like $\sim t^{(2-q)/6}$, $-1 < q < 2$ and that of the fifth dimension goes like $\sim t^{q/2}$ (where t is the comoving time in the Jordan frame) are stable since the normalisable perturbations die away to zero as $t \to \infty$. It is also found that these solutions have finite four-dimensional Planck mass and do not violate the null energy condition on the brane in the case that the brane has positive tension, despite the assumption of an indefinite sigma-model metric. It is suggested that these solutions might be capable of localising gravity to four dimensions.

Contents

General Introduction

This thesis is based on four papers done in collaboration with other researchers and is divided into three parts, each dealing with one of the major current issues in modern theoretical particle physics. Each part begins with a discussion of the relevant background material and concludes with a presentation of the results contained in the paper(s) pertaining to that part.

In Part I we consider the concept of holography as applied to the three-dimensional black hole discovered by Bañados, Teitelboim and Zanelli (BTZ). We start in Chapter 1 with a review of the BTZ black hole, discussing both the Lorentzian and the Euclidean cases. We focus in particular on the global geometry and how the black hole can be obtained via a quotienting construction from three-dimensional anti-de Sitter space.

The *holographic principle* and the Maldacena conjecture are the subject of Chapter 2. We are all familiar with the notion of a hologram in which three-dimensional information is stored on a two-dimensional surface. Likewise, there is such a concept within string theory. It has its roots in the famous Bekenstein-Hawking (BH) entropy-area law which states that the entropy of a $(3+1)$-dimensional black hole, an inherently three-dimensional quantity, is proportional to the surface area of the event horizon. This law is in fact the limiting case of the more general statement that the entropy within a three-dimensional closed region containing gravity cannot exceed the surface area of the boundary of that region. This bound is known as the *Bekenstein bound* and is discussed in section 2.1. 't Hooft and Susskind interpreted this as meaning that it must be possible to describe all phenomena within the region by a set of degrees of freedom living on the boundary. This is essentially the content of the Maldacena conjecture which we outline in section 2.2. The conjecture states that type IIB string theory on a certain space-time is mathematically equivalent to a supersymmetric Yang-Mills theory living on the boundary, given certain conditions[1]. Central to the conjecture is the fact that the isometry group of the space-time acts as the conformal group on the boundary. A concrete realisation of these ideas is presented in section 2.3. This is Strominger's derivation of the BTZ entropy using the Cardy formula. The Cardy formula relates the entropy, as measured by the logarithm of the number of states, to the central charge of a conformal field theory. Conformal field theory arises in the context of the BTZ black hole because the asymptotic symmetry group of the BTZ metric is just the conformal group in $1+1$ dimensions. Two copies of the Virasoro algebra appear and the central charge can be calculated. This allows a value to be placed on the entropy which can be compared with the value obtained using the BH formula — the two are found to agree.

In Chapter 3, we discuss how an analogous concept of a kinematic holographic principle as embodied in a theorem of Sullivan exists within three-dimensional hyperbolic space. The theorem states that the hyperbolic structure of a certain class of three-dimensional manifolds is completely determined in terms of the corresponding Teichmüller space of the boundary. This result is a deep theorem and necessitates that we develop various aspects of hyperbolic geometry in some detail. This is most elegantly achieved through the use of Hamilton's quaternions and involves the quotienting construction introduced in the first chapter. We also explain the notion of *geometrical finiteness*, which is a necessary condition for Sullivan's theorem to apply. As Chapter 3 is rather abstract, we illustrate the ideas throughout using the Euclidean BTZ black hole. In particular, we prove that it is geometrically finite. This is the main result of Part I and allows us to deduce several consequences of the theorem for the BTZ black hole, one of which is to lend support to Strominger's derivation of the entropy.

Part II of the thesis is concerned with various topics in string theory. For the

[1]To properly understand the Maldacena conjecture a knowledge of string theory and D-branes is required. However, our aim in Chapter 2 is to stress the holographic principle behind the conjecture — we do not discuss string theory and D-branes in depth until Chapter 4.

sake of completeness, we begin in Chapter 4 with a review of the elements of string theory and D-branes that we require. We introduce the bosonic closed and open string in section 4.1.1. Bosonic string theory was born in the late 1960s out of an attempt to explain experimental observations in strong interaction physics through dual resonance models [1, 2], a role now filled by quantum chromodynamics. It was subsequently realised [3] that the spectrum of states of the closed string contains a massless spin-2 particle, which is just the right amount of degrees of freedom for describing the graviton, the particle conjectured to mediate gravitational interactions. This provided an alternative use for string theory as a theory of quantum gravity.

Unfortunately, the bosonic string cannot accommodate fermions and also contains a tachyon, a particle of negative mass-squared which makes the theory unstable. These problems are circumvented by supersymmetrising the string and by performing a projection (the GSO projection) on the resulting spectrum of physical states. The superstring and the GSO projection are the subject of section 4.1.3. Several consistent supersymmetric string theories exist and we focus mainly on two of these — type IIA and type IIB theory, both of which live in ten space-time dimensions[2]. The type II theories are closed string theories and the GSO projection is such that they consist of four sectors: NS-NS, R-R, R-NS and NS-R, where NS stands for "Neveu-Schwarz" and R for "Ramond." In particular, the R-R sector contains massless k-form field strengths $H_k = dC_{k-1}$ corresponding to potentials C_{k-1}, where k is even for type IIA and odd for type IIB. For both the bosonic string and the superstring, we concentrate on the conformal field theory of the worldsheet and in section 4.1.6 we describe how to calculate tree-level scattering amplitudes in the case of interacting strings. Section 4.1.7 then deals with the low-energy effective actions that can be deduced from knowledge of these amplitudes.

In section 4.2 we introduce Dp-branes in more depth, having first met them in the context of the Maldacena conjecture. A Dp-brane is a $(p+1)$-dimensional hyperplane in space-time on which open strings can end and which is positively charged under the R-R $(p+1)$-form potential C_{p+1}. Such objects play a crucial role in much

[2]Another consistent theory, which we do not actually discuss, is heterotic string theory with gauge group $E_8 \times E_8$. It also lives in ten space-time dimensions.

of modern-day string theory. We describe how the low-energy effective action on several coincident branes is composed of two parts — the Born-Infeld action and the Wess-Zumino (or Chern-Simons) term. The Wess-Zumino term is essential for the consistency of D-branes and contains a coupling of the R-R potentials to the ordinary non-abelian gauge field that arises from open strings stretched between the coincident branes. We conclude the chapter with a short discussion of the phenomenon of T-duality in section 4.3. T-duality is a symmetry of string theory which relates type IIA to type IIB causing Dp-branes in one theory to become D$(p \pm 1)$-branes in the other.

Chapter 5 deals with R-R couplings on brane-antibrane systems. An antibrane, or $\overline{\mathrm{D}p}$-brane, is simply a brane negatively charged under the potential C_{p+1}. When a brane and antibrane are coincident, the system as a whole preserves no supersymmetry and two open string tachyons exist in the spectrum. The system is therefore unstable and can decay to branes of lower dimension by tachyon condensation. Such systems were the subject of an important series of papers by Ashoke Sen in his attempts to provide evidence for various *duality conjectures* that relate one string theory to another. The power of T-duality proved to be a useful tool in his analyses. After first introducing brane-antibrane systems in section 5.1.1 and 5.1.2, we go on to describe Sen's work in section 5.1.3. It was subsequently realised by Witten that Sen's work could be reinterpreted in the language of K-theory and he was the first to mention Quillen's superconnection in the context of Sen's work. We discuss K-theory and the superconnection (which has its origin in the theory of superbundles) in section 5.1.4. The main result of Part II is contained in section 5.2. Here, we use the string theory developed in Chapter 4 to calculate some tree-level open-closed scattering amplitudes on the upper half-plane describing the interaction of tachyons with R-R fields on the worldvolume of the brane-antibrane system. From knowledge of these amplitudes we are able to infer the low-energy effective action for the dynamics. This allows us to conjecture a generalised form for the Wess-Zumino term to Dp-$\overline{\mathrm{D}p}$ systems. The proposed form is cleanly written in terms of the superconnection.

The final part of the thesis deals with what may be termed generalised Kaluza-

Klein theory. A brief review of standard Kaluza-Klein theory is contained in Chapter 6. The original theory envisaged by Kaluza [4] and developed by Klein [5] was an attempt to unify Einstein's gravitation with Maxwell's electromagnetism. The basic idea of the theory was that space-time was a five-dimensional manifold with the extra spacelike dimension forming a circle of very small radius r_0. The action of the theory was taken as the Einstein-Hilbert action in five dimensions. Both four-dimensional gravity and gauge theory were treated on a similar footing in that both were described as parts of the five-dimensional metric tensor. One then performed a dimensional reduction of the action down to four dimensions. In the process, one obtained the four dimesional action of gravity plus abelian gauge theory and a spectrum of massive modes which could however be neglected since the states had large masses $\sim 1/r_0$ and so were beyond experimental reach.

Unfortunately, this approach proved to be inconsistent in general, as we now explain. Klein set the component of the five-dimensional metric in the extra direction (g_{44}, say) to be a constant *before* performing the dimensional reduction. This was because if this component were non-constant, it would give rise to an additional field, a scalar field $\phi = \sqrt{g_{44}}$, to which no suitable interpretation could be attached. However, as was first pointed out by Jordan [6] and Thiry [7] by retaining the scalar field throughout the dimensional reduction process and setting it to a constant afterwards[3], $\phi = constant$ is only consistent provided the gauge field strength satisfies the condition $F_{ij}F^{ij} = 0$. (Here, i is a four-dimensional index.) For historical reasons, the scalar field ϕ is known as the *Brans-Dicke scalar* [8].

Despite the drawback, the Kaluza-Klein concept is one of the most beautiful in all of theoretical physics and still finds application today. Indeed, if superstring theory is to realistically describe our world one must reduce the ten-dimensional superstring action down to four dimensions by compactifying on T^6 or a Calabi-Yau manifold in much the same way as the original Kaluza-Klein theory was compactified on a circle. Consistency of the process results in the generation of many scalar fields (known as *dilatons*) which are analogous to the Brans-Dicke scalar. For further details of the

[3]This was some twenty years after Klein's paper!

application to string theory and supergravity the reader should consult the reviews by Duff [9] and by Duff *et al* [10].

We encounter the Randall-Sundrum (RS) models in Chapter 7. These are models in which we, and therefore the Standard Model, are conjectured to live on a four-dimensional hypersurface in a five-dimensional space-time. In brane language, we are conjectured to live on a three-brane. In addition, there exists in the five-dimensional space-time a second three-brane (the "hidden" brane) at either a finite or an infinite distance away from our "visible" brane. The models, while not directly related to string theory, are definitely string-inspired. Indeed, the RS set-up is very reminiscent of the Hořava-Witten solution [11] for the strongly-coupled $E_8 \times E_8$ heterotic string where each E_8 lives on a ten-dimensional boundary of an eleven-dimensional space-time — one E_8 can be considered to belong to the visible sector and to provide the Standard Model[4], whilst the other belongs to the hidden sector. The first RS model, discussed in sections 7.1 to 7.3, is an attempt to solve the age-old hierarchy problem that exists within particle physics — namely, how to explain the huge difference between the electroweak scale and the Planck scale. The second model, which we describe in sections 7.4 and 7.5, shows how one can localise a theory of gravity with five non-compact dimensions to four dimensions. Since the extra dimension is no longer very small the massive Kaluza-Klein modes may no longer be beyond experimental reach. An analysis of the Kaluza-Klein spectrum is therefore necessary to check that the massive modes do not significantly alter the inverse-square force law of four-dimensional gravitation.

The RS models are static — the induced metric on the visible three-brane is just that of Minkowski space-time. It is natural to generalise the models to allow Friedmann-Robertson-Walker (FRW) cosmologies on the brane. Such generalisations are the subject of our final chapter, Chapter 8. We begin with a discussion of the Binétruy, Deffayet and Langlois (BDL) model in section 8.1. Unfortunately, this model results in non-standard cosmology in that the Hubble parameter is found to be proportional to the density of matter on the brane (in the form of a perfect fluid)

[4] E_8 is connected through a series of subgroups to groups such as $SU(5)$ and $SO(10)$ that contain the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$ and appear in grand unification schemes.

rather than to its square root, as in conventional big bang cosmology.

In the BDL model, the matter is assumed to lie entirely within the brane, despite the absence of a well-defined non-gravitational mechanism to confine it. It might therefore be of interest to consider what happens in the case of a bulk fluid. We investigate this in sections 8.3 and 8.4, which form the main components of the chapter. Specifically, we consider a single-brane scenario in which five-dimensional gravity is coupled to a bulk scalar field sigma-model with indefinite metric and matter on the brane is induced via a bulk *a priori* anisotropic fluid. We find a range of FRW-type solutions with flat spatial sections for which the scale factor on the brane goes like $\sim e^{-\gamma t/6}$ or $\sim t^{(2-q)/6}$ and that of the fifth dimension goes like $\sim e^{\gamma t/2}$ and $\sim t^{q/2}$, respectively. Here, t is the comoving time in the Jordan frame and γ and q are arbitrary non-zero constants. We also obtain standard cosmology provided certain conditions are satisfied, one of which is that the fluid must be isotropic and *stiff*. A stiff fluid is one for which the pressure is equal to the density, implying that the speed of sound in the fluid is equal to the velocity of light. Other FRW-type solutions also obeying conventional cosmology are obtained when the radius of the fifth dimension is assumed to be static. In sections 8.3.3 and 8.3.4 we perform a perturbation analysis of the solutions in the case of an isotropic, stiff fluid and examine the stability of the transverse, traceless Kaluza-Klein modes. The analysis suggests that the power law solutions with $-4 < q \leq 2$ are stable since the normalisable perturbations die away to zero as $t \to \infty$. We also find that the effective four-dimensional Planck mass is finite for these solutions, given certain conditions. In section 8.4 we examine the null energy condition on the brane. Since a sigma-model with indefinite metric has "wrongly-signed" kinetic terms for some of the scalars, the scenario might be expected to violate various energy conditions, the weakest of which is the null condition. These energy conditions are required to be conserved if the scenario is not to have serious repercussions for physics. We find that the null condition on a brane of positive tension is not violated if $-1 < q < 2$. Interestingly, this is precisely the range of q that would have been obtained had we started with a standard positive-definite sigma-model metric. We conclude that for this range of q gravity might be localised in our scenario, thus providing a generalisation of the second RS model to a non-static metric.

Part I

Geometrical Finiteness, Holography and the Bañados-Teitelboim-Zanelli Black Hole

Chapter 1

The BTZ Black Hole

In this chapter we review the three-dimensional "BTZ" black hole discovered by Bañados, Teitelboim and Zanelli [12]. We shall make extensive use of the work of Carlip [13, 14], Bañados *et al* [15], Carlip and Teitelboim [16] and Satoh [17].

1.1 The Lorentzian BTZ Black Hole

The rotating BTZ black hole in Schwarzchild coordinates is described in units $8G = 1$ by the stationary, axially-symmetric metric

$$ds^2 = -f(r)^2 dt^2 + \frac{dr^2}{f(r)^2} + r^2(d\phi + N^\phi(r)dt)^2 \ , \tag{1.1}$$

where the lapse and shift functions are given by

$$f(r) = \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)^{1/2} \ , \ N^\phi(r) = -\frac{J}{2r^2} \quad (|J| \leq Ml, M > 0) \ , \tag{1.2}$$

and $-\infty < t < \infty$ is the time coordinate, $0 < r < \infty$ is a radial distance and ϕ an angle of period 2π. The parameters M and J are identified with the standard ADM mass and angular momentum associated with the Killing vectors ∂_t (which is asymptotically timelike) and ∂_ϕ. It is easily verified that this metric is a solution of Einstein's equations with negative cosmological constant:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \ , \tag{1.3}$$

where $\Lambda = -1/l^2$.

The locations of the horizons of the black hole are given by the zeros of the lapse function; there is both an inner and an outer horizon:

$$r_\pm^2 = \frac{Ml^2}{2}\left[1 \pm \left(1 - \frac{J^2}{M^2l^2}\right)^{1/2}\right] , \tag{1.4}$$

from which we determine

$$M = \frac{r_+^2 + r_-^2}{l^2} \; , \; |J| = \frac{2r_+r_-}{l} \; . \tag{1.5}$$

Note that the areas of the horizons are given by

$$A_\pm = \int_{r_\pm} \sqrt{g_{\phi\phi}}\, d\phi = 2\pi r_\pm \; . \tag{1.6}$$

Like the Kerr solution in $(3+1)$-dimensions, the BTZ black hole possesses an *ergosphere* — this is the boundary of the region in which ∂_t is spacelike and its location is given by the zero of the g_{tt} component of the metric. If a particle decays in this region it is possible to extract energy from the hole via the Penrose process [18]. We find

$$r_{erg} = M^{1/2} l \; . \tag{1.7}$$

Therefore, the ergosphere lies beyond the outer horizon r_+.

If we change to ingoing Eddington-Finkelstein-type coordinates via

$$dv = dt + \frac{dr}{f(r)^2} \; , \; d\tilde{\phi} = d\phi - \frac{N^\phi(r)}{f(r)^2} dr \; , \tag{1.8}$$

the metric (1.1) is transformed to

$$ds^2 = -f(r)^2 dv^2 + 2dvdr + r^2(d\tilde{\phi} + N^\phi(r)dv)^2 \; . \tag{1.9}$$

From this form of the metric it is easy to see that the normal to the hypersurface $S(x) \equiv r = constant$ is given by $n_\mu = \partial_\mu S(x) = \delta^r_\mu$ or, equivalently,

$$\mathbf{n}(r) = g^{\mu\nu} n_\nu \partial_\mu = \partial_v + f(r)^2 \partial_r - N^\phi(r) \partial_{\tilde{\phi}} \; . \tag{1.10}$$

Hence, the horizons are null since $\mathbf{n}(r)^2|_{r_\pm} = g^{rr}|_{r_\pm} = f(r_\pm)^2 = 0$. Furthermore, $\mathbf{n}(r_\pm) = \partial_v - N^\phi(r_\pm)\partial_\phi$ is a Killing vector on $r = r_\pm$, so the horizons are also Killing horizons with surface gravities determined by

$$\kappa_\pm^2 = -\frac{1}{2} g_{\nu\rho} \nabla^\mu n^\nu \nabla_\mu n^\rho \,|_{r_\pm} \; , \tag{1.11}$$

where it is understood that $\mathbf{n}(r)$ is evaluated on $r = r_\pm$ *before* differentiating. Therefore, we obtain

$$\kappa_\pm = \frac{r_\pm^2 - r_\mp^2}{l^2 r_\pm} \ . \qquad (1.12)$$

To demonstrate that the BTZ metric describes a genuine black hole we show that the outer horizon r_+ is an *apparent horizon*. The interior of a black hole generally contains trapped surfaces. These are surfaces for which both sets of ingoing and outgoing null geodesics are everywhere converging. In the absence of naked singularities, their presence is a sufficient (but not a necessary) condition for the existence of a black hole [19]. The apparent horizon is the outer boundary of a region of trapped surfaces for which the null geodesics have zero convergence [20]. Under certain conditions, the apparent horizon coincides with the event horizon for a stationary black hole [19]. The convergence of the geodesics is characterised by the optical expansion scalar θ defined by

$$\theta = \frac{1}{2}\nabla_\mu n^\mu \ , \qquad (1.13)$$

which we find vanishes on r_+.

The Penrose Diagram

In the non-extremal case $|J| \neq Ml$ (that is, $r_+ \neq r_-$) with both J and M non-zero, a Kruskal-like coordinate patch can be defined around each of the horizons. To do so, define the transformation

$$U = \rho(r)e^{-at} \ , \ V = \rho(r)e^{at} \ , \qquad (1.14)$$

where

$$\frac{d\rho}{dr} = \frac{a\rho}{f(r)^2} \ , \qquad (1.15)$$

and where the constant a can be different in each patch. Integrating (1.15) we obtain

$$\rho(r) = \left[\left|\frac{r - r_+}{r + r_+}\right|^{r_+ b} \left|\frac{r + r_-}{r - r_-}\right|^{r_- b}\right]^{1/2} \ , \qquad (1.16)$$

where $b = al^2/(r_+^2 - r_-^2)$.

We find that

$$\Omega^2 dUdV = -\Omega^2 a^2 \rho^2 dt^2 + \Omega^2 a^2 \rho^2 \frac{dr^2}{f^4} \ , \tag{1.17}$$

and therefore choosing $\Omega^2 = \frac{f^2}{a^2\rho^2}$ we can rewrite (1.1) in the form

$$ds^2 = \Omega^2 dUdV + r^2(d\phi^2 + N^\phi(r)dt)^2 \ . \tag{1.18}$$

where r and t are functions of U and V.

In the patch K_+ around r_+ there are two regions: $r_- < r \leq r_+$ and $r_+ \leq r < \infty$. We choose $a_+ = \kappa_+$ and, therefore,

$$\Omega_+^2(r) = \frac{(r^2 - r_-^2)(r + r_+)^2}{\kappa_+^2 r^2 l^2} \left(\frac{r - r_-}{r + r_-} \right)^{r_-/r_+} \ . \tag{1.19}$$

In the patch K_- around r_- there are two regions: $0 < r \leq r_-$ and $r_- \leq r < r_+$. We choose $a_- = \kappa_-$ and, therefore, obtain

$$\Omega_-^2(r) = \frac{(r_+^2 - r^2)(r + r_-)^2}{\kappa_-^2 r^2 l^2} \left(\frac{r_+ - r}{r_+ + r} \right)^{r_+/r_-} \ . \tag{1.20}$$

From the above forms of $\Omega^2(r)$ it is clear that the singularities at $r = r_\pm$ in the original form of the metric (1.1) are merely coordinate singularities. The real singularity is at $r = 0$, as we will show in the next section. However, unlike the Schwarzchild or Kerr solutions in four dimensions, the singularity is *not* a curvature one — one can easily check that the curvature tensor of the metric (1.1) is everywhere regular and given by

$$R_{\mu\nu\rho\sigma} = -l^{-2}(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}) \ , \tag{1.21}$$

and hence the blowing up of the curvature scalars R, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ at $r = 0$ which one usually associates with black hole singularities does not occur. Therefore, the BTZ space-time is a space of constant negative curvature and as such is locally isomorphic to the universal covering space $\widetilde{AdS}_3$ of three-dimensional anti-de Sitter space. We will return to this point in the next section.

An infinite number of Kruskal patches may be joined together to form a maximal solution. The Penrose diagram is obtained from the coordinate transformation

$$U = \tan\left(\frac{p+q}{2}\right) \;,\; V = \tan\left(\frac{p-q}{2}\right) \;, \tag{1.22}$$

where $-\frac{\pi}{2} < p, q < \frac{\pi}{2}$. Using (1.16), it is clear that in patch K_+ the horizon r_+ maps to $p = \pm q$ while $r = \infty$ maps to $p = \pm\frac{\pi}{2}$. Similarly, in patch K_-, r_- maps to $p = \pm q$ while $r = 0$ maps to $p = \pm\frac{\pi}{2}$. The diagram is given in figure 1.1 below.

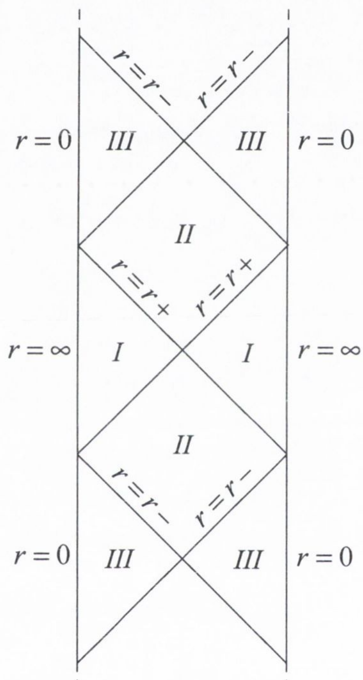

Figure 1.1: The Penrose diagram for $|J| \neq Ml$ with both J and M non-zero. Region I is the region $r_+ \leq r < \infty$, region II corresponds to $r_- \leq r \leq r_+$ and region III is the region $0 < r \leq r_-$. (Courtesy of Y. Satoh, taken from [17].)

Similar Penrose diagrams can be drawn for the cases: a) $J = 0$ and $M \neq 0$; b) $J = M = 0$; and c) $|J| = Ml \neq 0$. The reader is referred to [15] for details.

1.2 Global Geometry

In the last section we mentioned that the BTZ space-time was locally isomorphic to $\widetilde{AdS}_3$. Indeed, in this section we show how the BTZ black hole can be represented as a quotient space of $\widetilde{AdS}_3$. We shall restrict our attention to the case $|J| \neq Ml$ with both J and M non-zero. A discussion of the other cases may be found in [15].

Lorentzian AdS_3 is the three-dimensional hyperboloid

$$x_0^2 + x_1^2 - x_2^2 - x_3^2 = -l^2 \ , \tag{1.23}$$

embedded in flat $\mathbb{R}^{2,2}$ with the metric

$$ds^2 = dx_0^2 + dx_1^2 - dx_2^2 - dx_3^2 \ , \tag{1.24}$$

It can be shown shown that AdS_3 contains closed timelike curves[1] and so it is common to go to the universal covering space $\widetilde{AdS}_3$ by decompactifying the timelike direction.

Evidently, the isometry group of (1.23) is $SO(2,2)$. Alternatively, we can combine (x_0, x_1, x_2, x_3) into the $SL(2,\mathbb{R})$ matrix

$$\mathbf{X} = \frac{1}{l}\begin{pmatrix} x_3 + x_0 & x_1 + x_2 \\ x_1 - x_2 & x_3 - x_0 \end{pmatrix} \ , \ \det \mathbf{X} = 1 \ . \tag{1.25}$$

The isometries are then represented by the group $(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))/\mathbb{Z}_2 \approx SO(2,2)$ — the two copies of $SL(2,\mathbb{R})$ act on $\mathbf{X}$ via $\mathbf{X} \to \rho_L \mathbf{X} \rho_R$ with the $\mathbb{Z}_2$ symmetry $(\rho_L, \rho_R) \sim (-\rho_L, -\rho_R)$.

Let us consider the three regions of the above Penrose diagram along with the parametrisations

I. $(r_+ \leq r < \infty)$

$$x_0 = l\sqrt{\alpha}\sinh\left(\frac{r_+}{l}\phi - \frac{r_-}{l^2}t\right) \ , \ x_1 = l\sqrt{\alpha - 1}\cosh\left(\frac{r_+}{l^2}t - \frac{r_-}{l}\phi\right)$$
$$x_3 = l\sqrt{\alpha}\cosh\left(\frac{r_+}{l}\phi - \frac{r_-}{l^2}t\right) \ , \ x_2 = l\sqrt{\alpha - 1}\sinh\left(\frac{r_+}{l^2}t - \frac{r_-}{l}\phi\right)$$

II. $(r_- \leq r \leq r_+)$

$$x_0 = l\sqrt{\alpha}\sinh\left(\frac{r_+}{l}\phi - \frac{r_-}{l^2}t\right) \ , \ x_1 = -l\sqrt{1 - \alpha}\sinh\left(\frac{r_+}{l^2}t - \frac{r_-}{l}\phi\right)$$
$$x_3 = l\sqrt{\alpha}\cosh\left(\frac{r_+}{l}\phi - \frac{r_-}{l^2}t\right) \ , \ x_2 = -l\sqrt{1 - \alpha}\cosh\left(\frac{r_+}{l^2}t - \frac{r_-}{l}\phi\right)$$

[1]See section 2.3 in this regard.

III. $(0 < r \leq r_-)$

$$x_0 = l\sqrt{-\alpha}\cosh\left(\frac{r_+}{l}\phi - \frac{r_-}{l^2}t\right) \quad , \quad x_1 = -l\sqrt{1-\alpha}\sinh\left(\frac{r_+}{l^2}t - \frac{r_-}{l}\phi\right)$$
$$x_3 = l\sqrt{-\alpha}\sinh\left(\frac{r_+}{l}\phi - \frac{r_-}{l^2}t\right) \quad , \quad x_2 = -l\sqrt{1-\alpha}\cosh\left(\frac{r_+}{l^2}t - \frac{r_-}{l}\phi\right) \tag{1.26}$$

where

$$\alpha(r) = \left(\frac{r^2 - r_-^2}{r_+^2 - r_-^2}\right), \quad \phi \in (-\infty, \infty), \ t \in (-\infty, \infty) \ . \tag{1.27}$$

Then, in each region we find that the metric (1.24) transforms into the BTZ metric (1.1,1.2), albeit with the angle ϕ having infinite range. To make it into a true angular variable we must identify ϕ with $\phi + 2n\pi, n \in \mathbb{Z}$. These identifications are actually isometries of $\widetilde{AdS}_3$ (they are boosts in the $x_0 - x_3$ and $x_1 - x_2$ planes) and correspond to the $(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))/\mathbb{Z}_2$ elements $\{(\rho_L^{(n)}, \rho_R^{(n)}) | n \in \mathbb{Z}\}$ with

$$\rho_L^{(n)} = \begin{pmatrix} e^{n\pi(r_+ - r_-)/l} & 0 \\ 0 & e^{-n\pi(r_+ - r_-)/l} \end{pmatrix} \ , \ \rho_R^{(n)} = \begin{pmatrix} e^{n\pi(r_+ + r_-)/l} & 0 \\ 0 & e^{-n\pi(r_+ + r_-)/l} \end{pmatrix} \ . \tag{1.28}$$

Thus, the BTZ black hole may be viewed as the quotient $\widetilde{AdS}_3/\langle(\rho_L^{(1)}, \rho_R^{(1)})\rangle$, where $\langle(\rho_L^{(1)}, \rho_R^{(1)})\rangle$ is the group generated by $(\rho_L^{(1)}, \rho_R^{(1)})$.

However, as shown in [15], there is a subtlety involved which determines the location of the singularity in the coordinates (t, r, ϕ). It would seem from the parametrisations (1.26,1.27) that the metric could be extended in region III to negative values of r^2. This is indeed possible. However, the identification process makes the curves joining two distinct points of $\widetilde{AdS}_3$ that are on the same orbit closed in the quotient space. In order that the quotient space have a proper causal structure we must require that these closed curves not be timelike or null. A necessary condition for the absence of closed timelike curves is that the Killing vector ξ associated with the isometry (1.28) be spacelike, that is, we must excise the regions where $\xi \cdot \xi \leq 0$. The singularity is located at the boundary, $\xi \cdot \xi = 0$, and is therefore *causal* in nature[2].

[2] We stress that the singularity is not a conical curvature singularity of the type discussed in [21]. As mentioned in the text surrounding (1.21), the BTZ space-time is smooth (that is, Hausdorff) provided $J \neq 0$. However, in [15] it was shown that the geometry becomes non-Hausdorff at $\xi \cdot \xi = 0$ when $J = 0$; the singularity then resembles that of a Taub-NUT space.

According to [15], the Killing vector is

$$\xi = \frac{r_+}{l}(x_0\partial_{x_3} + x_3\partial_{x_0}) - \frac{r_-}{l}(x_1\partial_{x_2} + x_2\partial_{x_1}) = \partial_\phi \ . \tag{1.29}$$

and the norm is (using (1.24))

$$\xi \cdot \xi = \frac{r_+^2}{l^2}(x_3^2 - x_0^2) + \frac{r_-^2}{l^2}(x_2^2 - x_1^2) \ , \tag{1.30}$$

or, using (1.23), (1.26) and (1.27),

$$\xi \cdot \xi = r^2 \ , \tag{1.31}$$

in all three regions, showing that the singularity is indeed located at $r = 0$.

Remark. It is interesting to clarify the physical significance of the quotient space representation. To do this, we appeal to the Chern-Simons formulation of $(2+1)$-dimensional gravity with a negative cosmological constant [22, 23]. In this formulation, the Einstein-Hilbert action may be re-expressed as

$$S_{EH} = S_{CS}[A^{(+)}] - S_{CS}[A^{(-)}] \ , \tag{1.32}$$

where S_{CS} is the Chern-Simons action

$$S_{CS}[A] = \frac{k}{4\pi}\int_{\Sigma_3} \mathrm{Tr}\,(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) \ , \tag{1.33}$$

where k is a constant depending on normalisation conventions and $A^{(\pm)}$ are $SL(2,\mathbb{R})$ gauge fields given by

$$A^{(\pm)} = (\omega^a \pm \frac{1}{l}e^a)T_a \ . \tag{1.34}$$

In (1.34), the matrices T_a satisfy the $SL(2,\mathbb{R})$ Lie algebra

$$[T_a, T_b] = \epsilon_{ab}{}^c T_c \ , \ \mathrm{Tr}\,(T_a T_b) = \frac{1}{2}\eta_{ab} \ , \tag{1.35}$$

where $\eta_{ab} = \mathrm{diag}\,(-1,1,1)$ and $\epsilon^{012} = 1$. Additionally, we identify $e^a = e^a_\mu dx^u$ and $\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{\mu bc}dx^\mu$ as the one-forms associated with the dreibein ($g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$) and the spin-connection (obtained from Cartan's structure equations $de^a + \epsilon^a{}_{bc}\omega^b \wedge e^c = 0$).

With these conventions, the constant k is equal to $l/4G$. Note that the equations of motion resulting from the Chern-Simons action are just

$$F \equiv dA + A \wedge A = 0 \ , \tag{1.36}$$

(so the gauge fields are flat) and these imply the structure equations for the spin connection.

We begin to see the significance of the identifications (1.28) when we examine the holonomies in the Chern-Simons formulation. Any connection is completely determined by its holonomies, that is, by the Wilson loops

$$H[\gamma] = \mathcal{P} \exp \left\{ \int_\gamma A \right\} \ , \tag{1.37}$$

around closed curves γ, where $\mathcal{P}$ denotes path ordering. The Wilson loops are not gauge-invariant; however, it can be shown that a gauge transformation

$$A \mapsto A^\Omega = \Omega^{-1} A \, \Omega + \Omega^{-1} d\Omega \ , \tag{1.38}$$

is equivalent to conjugation of H by the associated $SL(2,\mathbb{R})$-valued gauge parameter:

$$H \mapsto H^\Omega = \Omega^{-1} A \, \Omega \ . \tag{1.39}$$

The trace of H is therefore the gauge-invariant physical observable characterising the geometry. Now, up to a gauge transformation, it can be shown that the gauge fields corresponding to the BTZ geometry are

$$A^{(\pm)} = \frac{r_+ \mp r_-}{2l} \left(\frac{dt}{l} \pm d\phi \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{1.40}$$

It is also known that for a flat connection the holonomies depend only on the homotopy class of γ. Therefore, given the homotopically non-trivial closed curves

$$\gamma_n : \phi(s) = 2\pi n s \ , \quad s \in [0,1] \ , \tag{1.41}$$

where $n \in \mathbb{Z}$, it is straightforward to read off that the holonomies up to conjugation are given by

$$H^{(+)}[\gamma_n] = \rho_L^{(n)} \ , \quad H^{(-)}[\gamma_n] = \rho_R^{(-n)} \ . \tag{1.42}$$

Thus, the same $SL(2,\mathbb{R})$ group elements that determine the Chern-Simons connection also give the identifications that fix the BTZ geometry.

1.3 The Euclidean BTZ Black Hole

In this section we discuss the Euclidean BTZ black hole which will be of more relevance in the sequel than its Lorentzian counterpart.

The Euclidean black hole is obtained from (1.1) by the Riemannian continuation

$$t = i\tau_E \ , \ J = -iJ_E \ , \tag{1.43}$$

in region I of the Lorentzian space-time (that is, the region $r_+ \leq r < \infty$), yielding:

$$ds^2 = f_E(r)^2 d\tau_E^2 + \frac{dr^2}{f_E(r)^2} + r^2(d\phi^2 + N_E^\phi(r) d\tau_E)^2 \ , \tag{1.44}$$

where the lapse and shift functions are now given by

$$f_E(r) = \left(-M + \frac{r^2}{l^2} - \frac{J_E^2}{4r^2}\right)^{1/2} \ , \ N_E^\phi(r) = iN^\phi(r) = -\frac{J_E}{2r^2} \ , \tag{1.45}$$

and the restriction $|J_E| \leq Ml$ is no longer necessary. The coordinate ϕ is again an angle of period 2π. The lapse function now has roots at

$$r_\pm^2 = \frac{Ml^2}{2}\left[1 \pm \left(1 + \frac{J_E^2}{M^2 l^2}\right)^{1/2}\right] \ , \tag{1.46}$$

with $|r_-| = ir_-$. Furthermore, $g_{\tau\tau}$ vanishes at $r_{erg}^2 = Ml^2 \leq r_+^2$. Hence, the restriction of the continuation to region I of the Lorentzian space-time is necessary to ensure that the metric (1.44) is of Euclidean signature and everywhere regular except at $r = r_+$ where it becomes singular.

Like its Lorentzian version, the metric (1.44) is of constant negative curvature and as such is locally isomorphic to hyperbolic three-space $\mathbb{H}^3$, the Euclidean analogue of $\widetilde{AdS_3}$ [3]. It is known that *any* geodesically complete three-dimensional space of Euclidean signature and constant negative curvature can be expressed as a quotient of $\mathbb{H}^3$ by a discrete subgroup of its isometry group [24,25]. Therefore, we should expect the Euclidean black hole to be obtainable from $\mathbb{H}^3$ by a quotienting construction in much the same way as the Lorentzian version is obtainable from $\widetilde{AdS_3}$ [4]. To see how

[3] We shall consider the geometry of $\mathbb{H}^3$ in more detail in Chapter 3.

[4] Note however that it is perhaps surprising that such a quotienting procedure works in the Lorentzian case because, unlike the Euclidean case, there is no clear-cut theorem which guarantees the success of the construction. That said, for Lorentzian signature Mess [26] has proven the existence of the construction in certain cases.

the quotienting construction works, perform the coordinate transformation

$$\begin{aligned}
x &= \left(\frac{r^2 - r_+^2}{r^2 - r_-^2}\right)^{1/2} \cos\left(\frac{r_+}{l^2}\tau_E + \frac{|r_-|}{l}\phi\right) \exp\left\{\frac{r_+}{l}\phi - \frac{|r_-|}{l^2}\tau_E\right\} , \\
y &= \left(\frac{r^2 - r_+^2}{r^2 - r_-^2}\right)^{1/2} \sin\left(\frac{r_+}{l^2}\tau_E + \frac{|r_-|}{l}\phi\right) \exp\left\{\frac{r_+}{l}\phi - \frac{|r_-|}{l^2}\tau_E\right\} , \\
z &= \left(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\right)^{1/2} \exp\left\{\frac{r_+}{l}\phi - \frac{|r_-|}{l^2}\tau_E\right\} , \qquad (1.47)
\end{aligned}$$

to bring (1.44) to the form

$$ds^2 = \frac{l^2}{z^2}(dx^2 + dy^2 + dz^2) , \qquad (1.48)$$

with $z > 0$. The metric (1.48) is the so-called *upper half-space* representation for the metric on $\mathbb{H}^3$. Note that under the transformation (1.47) the horizon $r = r_+$ maps to the positive z-axis and $r = \infty$ maps to $z = 0$.

It is straightforward to verify that the periodicity in the Schwarzchild coordinate $\phi \sim \phi + 2n\pi$, $n \in \mathbb{Z}$ is implemented via the identifications

$$(x, y, z) \sim e^{2n\Sigma} (x \cos 2n\Theta - y \sin 2n\Theta, x \sin 2n\Theta + y \cos 2n\Theta, z) , \qquad (1.49)$$

where

$$\Sigma = \frac{\pi r_+}{l} , \quad \Theta = \frac{\pi |r_-|}{l} . \qquad (1.50)$$

Therefore, the orbits have a helical-like ("whirlwind") structure in the upper half-space. Equation (1.49) is the Euclidean analogue of the identifications $\langle(\rho_L, \rho_R)\rangle$ of section 1.2.

It is instructive to change to spherical polar coordinates in the upper half-space via the transformation

$$(x, y, z) = (R \cos\theta \cos\chi, R \sin\theta \cos\chi, R \sin\chi) , \qquad (1.51)$$

where $\chi \in [0, \pi/2]$. The line element (1.48) is then written as

$$ds^2 = \frac{l^2}{\sin^2\chi} \left(\frac{dR^2}{R^2} + d\chi^2 + \cos^2\chi d\theta^2\right), \qquad (1.52)$$

and the identifications (1.49) become

$$(R, \theta, \chi) \sim (e^{2n\Sigma} R, \theta + 2n\Theta, \chi) . \qquad (1.53)$$

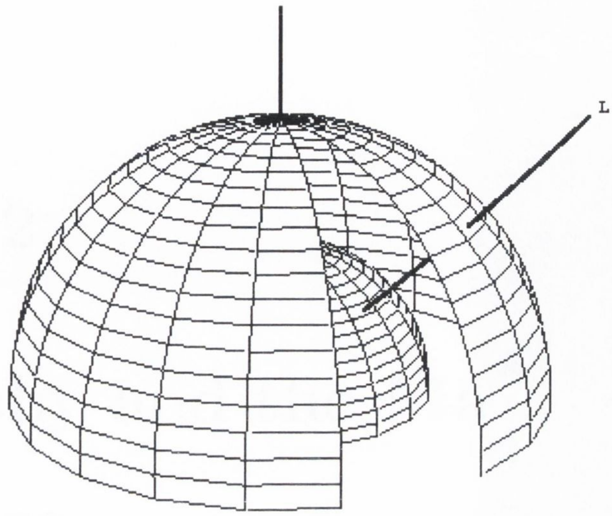

Figure 1.2: A sketch of a fundamental region for the Euclidean BTZ black hole in the coordinates (R, θ, χ). Points on the inner and outer hemispheres are identified along lines such as L. (Courtesy of S. Carlip, taken from [13].)

A fundamental region is the space between the hemispheres $R = 1$ and $R = e^{2\Sigma}$ with the inner and outer boundaries identified along a radial line, followed by a 2Θ rotation about the z-axis (see figure 1.2 above). Topologically, the resulting manifold is a solid torus [14, 16]. For $\chi \neq \pi/2$, each slice of fixed χ is a 2-torus, with circumferences parametrised by the periodic coordinates $\ln R$ and θ, while the degenerate surface $\chi = \pi/2$ (which corresponds to the horizon $r = r_+$) is a circle at the core of the torus[5].

This concludes our review of the BTZ black hole. We shall however return to the quotienting construction of the Euclidean version in Chapter 3.

[5]We remind the reader that "ln" denotes the natural logarithm (to base e).

Chapter 2

Holography and the Maldacena Conjecture

In this chapter we review the concept of holography in quantum field theory. This concept will play a fundamental role in the application of a theorem of Sullivan to the Euclidean BTZ black hole in the next chapter. We first discuss the notion of holography in a general sense and then go on to examine a realisation of it in the form of the Maldacena conjecture [27]. Our treatment of the conjecture will be brief as it is not directly relevant to the sequel. For a more thorough account, the reader is referred to the vast amount of literature on the conjecture[1] and to the numerous review articles [28–35] from which we shall borrow heavily. Finally, we review an old result concerning the asymptotic symmetry group at spatial infinity of $(2+1)$-dimensional gravity coupled to a negative cosmological constant and show how it naturally fits in with the notion of holography and the Maldacena conjecture.

2.1 The Holographic Principle

The *holographic principle*, as originally formulated by 't Hooft [36] and Susskind [37] and relying on the work of Bekenstein [38], is about the counting of quantum states

[1]At the time of writing, Maldacena's paper has received in excess of 1700 citations according to the Spires database at SLAC, making it one of the most highly-cited theoretical articles since records began.

of a system. It asserts that the number of possible states of a closed region of space containing gravity is the same as that of a system of binary degrees of freedom distributed on the boundary of the region. Furthermore, the number of such degrees of freedom is not indefinitely large but is bounded by the area of the region in Planck units. Since the entropy of a system measures the number of degrees of freedom, the principle effectively relates entropy, which is a bulk quantity, to area which is of one dimension less.

We shall present the Bekenstein/'t Hooft/Susskind argument momentarily but first we need to review the laws of black hole mechanics.

2.1.1 Black Hole Mechanics

Under certain conditions, black holes in 3+1 dimensions satisfy the following laws [18, 39, 40]:

0. The surface gravity κ is constant over the event horizon of the black hole;
1. In physical processes the changes of physical properties, in units $\hbar = c = 1$, obey the relation

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega_H \delta J + \Phi_H \delta Q \ , \tag{2.1}$$

   where G is Newton's constant, A is the area of the event horizon and J and Q are the angular momentum and charge of the black hole, respectively. Ω_H is the angular velocity and Φ_H the corotating electric potential of the event horizon;
2. The area of the event horizon never decreases, $\delta A \geq 0$;
3. It is impossible to achieve $\kappa = 0$ by any "physical process".

The analogy with the ordinary laws of thermodynamics should be apparent. The first law shows that there is a formal correspondence $M \leftrightarrow E$ (energy), $\alpha\kappa \leftrightarrow T$ (temperature) and $A/(8\pi G\alpha) \leftrightarrow S$ (entropy), where α is some (as yet) undetermined constant. It is thus conservation of energy, with Ω_H and J playing the roles of pressure and volume and Φ_H and Q the roles of chemical potential and particle number, respectively. The zeroth law then resembles the zeroth law of thermodynamics which states that temperature is constant in thermodynamic equilibrium. Similarly, the second law is analogous to the second law of thermodynamics, that is, $\delta S \geq 0$ in any process. The third law is less well understood[2]. It should be noted however that it is not analogous to the Planck-Nernst form of the third law of thermodynamics which states that $S \to 0$ (or a constant) as $T \to 0$ since there exist "extremal" black holes with $\kappa = 0$ but with non-vanishing A (the BTZ black hole with $|J| = Ml \neq 0$, that is, $r_+ = r_- \neq 0$ being the prime example of current interest in three dimensions (cf. equations (1.6) and (1.12)). Further discussion of the third law may be found in [18, 41].

Classically, the analogy is only formal since the temperature of a black hole is absolute zero since it is a perfect absorber and does not emit anything. However, in his famous paper on quantum particle creation effects in the vicinity of a black hole [42], Hawking showed that a black hole emits blackbody radiation at a temperature $T = \kappa/2\pi$. This fixes the constant $\alpha = 1/2\pi$ and, in particular, gives the celebrated Bekenstein-Hawking (BH) entropy-area law:

$$S = \frac{A}{4G} \ . \tag{2.2}$$

2.1.2 The Bekenstein/'t Hooft/Susskind Argument

The following argument is taken almost verbatim from [34] and [37].

In 3+1 dimensions let us consider a large, closed three-dimensional spacelike surface Γ of volume V, with boundary $\partial\Gamma$ of area A. For simplicity we shall take $\partial\Gamma$ to have S^2 topology. Let us further suppose that Γ is a discrete lattice of spin-$\frac{1}{2}$-like degrees of freedom with lattice-spacing the Planck length l_p. This reflects the belief that a small distance cut-off, such as the Planck length, is needed in order to make sense of quantum gravity. Now consider the space of states that describe arbitrary systems that can fit into Γ such that the region outside Γ is empty space. The number of distinct orthogonal quantum states in Γ is

$$N(V) = 2^{V/l_p^3} \ , \tag{2.3}$$

[2]The obvious problem is to clearly define what is meant by a "physical process".

where V/l_p^3 is the number of lattice sites in Γ. Ignoring gravity for the moment, the logarithm of $N(V)$ is the maximum possible entropy in Γ and satisfies

$$\ln N(V) = \frac{V}{l_p^3} \ln 2 \ , \tag{2.4}$$

and is therefore proportional to the volume of Γ.

Suppose now that we include gravity and that we have a thermodynamic system with entropy S that is entirely contained within Γ. For the states of the system to be observable to the outside world, the total mass of the system cannot exceed the mass of a Schwarzchild black hole which just fills the region (that is, the black hole event horizon area is A and the mass is $m_{BH} = (A/16\pi)^{1/2}$). Otherwise a black hole of size larger than the region Γ would form; the states would then lie inside the Schwarzchild radius and so be unobservable. Now let us throw just the right amount of extra massive matter into the region so that together with the original mass it forms a black hole which just fills the region. The black hole has entropy *equal* to $A/4G$. If S were larger than $A/4G$ then the second law of thermodynamics would be violated. Hence, we conclude that $S \leq A/4G$.

This bound on the entropy[3] is due to Bekenstein and is known as the "Bekenstein bound". It is clearly saturated by black holes. 't Hooft [36] and later Susskind [37], interpreted Bekenstein's result as meaning that it must be possible to describe all phenomena within Γ by a set of degrees of freedom which reside on the boundary $\partial\Gamma$. The number of degrees of freedom should be no larger than that of a two-dimensional lattice with approximately one binary degree of freedom per Planck area[4].

Although the laws of black hole mechanics and the Bekenstein bound were originally formulated in 3+1 dimensions, they are universal and, therefore, are also valid in an arbitrary number of space-time dimensions. Thus, they relate S, a d-dimensional quantity to A, a $(d-1)$-dimensional quantity, where d is the number of spatial dimensions of the space-time. Taking the 't Hooft and Susskind interpretation to its logical conclusion, it might be thought possible to represent a

[3]See [43] for a recent review.

[4]In $(d+1)$-dimensions, Newton's constant G is proportional to $l_p^{(d-1)}$, where l_p is the $(d+1)$-dimensional Planck length.

$(d+1)$-dimensional bulk gravitational theory on some manifold with boundary by a different, *dual* d-dimensional theory without gravity living on the boundary. This is precisely the content of the Maldacena conjecture, to which we now turn.

2.2 The Maldacena Conjecture

This section will be brief because it is the general spirit of the holographic principle that is relevant to the sequel and not the specific technical details of the conjecture itself.

The Maldacena conjecture [27], also known as the "AdS/CFT correspondence", is an explicit realisation[5] of the holographic principle. It suggests that the degrees of freedom of a quantum field theory with gravity in the universal cover[6] $\widetilde{AdS}_{d+1}$ of $(d+1)$-dimensional Lorentzian anti-de Sitter space can be completely identified with the degrees of freedom of a conformal quantum field theory without gravity in the universal cover of M_d, the conformal completion[7] of Minkowski space of one dimension less. Central to the conjecture are the facts: (a) that M_d is the boundary at spatial infinity of AdS_{d+1} and (b) that the isometry group of AdS_{d+1}, which is $SO(2,d)$, acts as the conformal group on M_d. The conjecture was first formulated for $d=4$ and soon after was placed on a firmer footing by Witten [46] and Gubser *et al* [47].

Maldacena arrived at his conjecture by considering the space-time geometry in the vicinity of a number N of coincident, extremal, electrically-charged D3-branes lying along a $(3+1)$-dimensional plane in the ten-dimensional space-time of type IIB superstring theory[8]. The bulk metric for the configuration can be obtained by solving the equations of motion which result from the bosonic part of the type IIB supergravity action (which is the low-energy effective action for the theory). We shall

[5]The conjecture has not yet been proven, but see [44, 45] in this regard.

[6]One goes to the universal cover to avoid the existence of closed timelike curves.

[7]The conformal completion of d-dimensional Minkowski space is Minkowski space with some "points at infinity" added. Its universal cover is topologically the cylinder $\mathbb{R} \times S^{d-1}$, where $\mathbb{R}$ is the timelike direction. The reader should consult Witten [46] on these points.

[8]We discuss string theory and D-branes in more depth in Chapter 4.

simply quote the result and refer the reader to the excellent reviews [30, 48–52] for an explicit derivation (which is straightforward but lengthy). The metric is

$$\begin{aligned} ds^2 &= f^{-1/2}dx_{\parallel}^2 + f^{1/2}(dr^2 + r^2 d\Omega_5^2) \ , \\ f &= 1 + \frac{\rho^4}{r^4} \ , \end{aligned} \tag{2.5}$$

where $x_{\parallel}$ denotes the four coordinates along the worldvolume of the three-brane, r is the radial coordinate in the transverse direction and $d\Omega_5^2$ is the metric on the unit five-sphere. The characteristic length ρ is given by $\rho^4 = 4\pi g_s N l_s^4$, where g_s is the closed string coupling and l_s the string length[9]. The space-time geometry described by the metric (2.5) has a horizon at the location of the branes at $r = 0$ and is completely non-singular [54]. The background also has a constant dilaton and a self-dual 5-form field strength with N units of flux on the sphere. The low-energy approximation is a good description provided that $\rho \gg l_s$, which requires

$$g_s N \gg 1 \ , \tag{2.6}$$

or, using *S-duality*[10], $N/g_s \gg 1$. In other words, we need large (but fixed) N.

To study the near-horizon geometry, Maldacena took the limit

$$\alpha' \equiv l_s^2 \to 0 \ , \ g_s = \text{fixed} \ , \ U \equiv \frac{r}{l_s^2} = \text{fixed} \ . \tag{2.7}$$

In this limit the 1 can be neglected in the harmonic function f of (2.5) and the metric becomes

$$ds^2 = \alpha' \left[\frac{U^2}{l^2} dx_{\parallel}^2 + l^2 \frac{dU^2}{U^2} + l^2 d\Omega_5^2 \right] \ , \tag{2.8}$$

where $l^2 = \sqrt{4\pi g_s N}$. The change of variables $u = l^2/U$ transforms the metric into

$$ds^2 = \alpha' l^2 \left[\frac{\eta_{ij} dx^i dx^j + du^2}{u^2} + d\Omega_5^2 \right] \ , \tag{2.9}$$

[9]Throughout this section we use the normalisation of Polchinski [53].

[10]S-duality is an $SL(2,\mathbb{Z})$ symmetry of type IIB supergravity. In particular it implies that the weakly-coupled theory with $g_s < 1$ is equivalent to the same theory in the strongly-coupled regime with $g_s > 1$. See, for example, Sen [55] in this regard. In writing equation (2.6) we have implicitly assumed $g_s < 1$.

which is the metric of the product space $AdS_5 \times S^5$, where both factors have radius l in units of the string length. As $\alpha' \to 0$ the metric naively tends to zero also. However, the type IIB action in the Einstein frame is $S = \frac{1}{16\pi G_{10}} \int d^{10}x\sqrt{-G}(R + \cdots)$ (where $G_{10} = 8\pi^6 g_s^2 \alpha'^4$ is the ten-dimensional Newton constant), so that we can cancel the factor of α' in the metric and the Newton constant, leaving a five-dimensional theory (after dimensional reduction over the sphere) with finite Planck length in the limit. The isometry group of $AdS_5 \times S^5$ is $SO(2,4) \times SO(6) \approx SU(2,2) \times SU(4)$; in the presence of supersymmetry this is extended to $SU(2,2|4)$.

Now, according to Polchinski [56], D3-branes can be equivalently thought of in terms of open strings with end points on the branes. Since we are dealing with IIB string theory, which is a theory of oriented strings, and there are N different branes on which the strings can end, these open strings are equipped with $U(N)$ Chan-Paton labels. The low-energy effective-action for the dynamics *on* the D-branes was first derived (in the case of a single, bosonic brane) by Leigh [57]. He showed that the equations of motion are those of the *Born-Infeld action*[11]. After allowing for more than one brane[12] and supersymmetrising [58], it is found that the low-energy limit $\alpha' \to 0$ of the Born-Infeld action is precisely the dimensional reduction to 4-dimensional Minkowski space of 10-dimensional $\mathcal{N} = 1$ super Yang-Mills theory with gauge group $U(N)$ and gauge coupling constant $g_{YM}^2 = 2\pi g_s$. The reduced theory is conformally invariant and possesses $\mathcal{N} = 4$ supersymmetry in four dimensions. Furthermore, the group $SU(2,2|4)$ acts as the superconformal group of the gauge theory.

The conjecture is, therefore, that type IIB superstring theory on (the universal cover of) AdS_5 times S^5 is mathematically equivalent, that is, *dual* to the large N limit of $\mathcal{N} = 4$, $d = 4$ super Yang-Mills theory with gauge group $U(N)$. This is the weakest form of the conjecture. A stronger form is that the two theories are exactly the same for all values of g_s and N. Note that the duality is of strong-weak type. We have stated that the supergravity description is valid provided $g_s N \sim \lambda \gg 1$, where

[11]More explicit details of the Born-Infeld action are given later in section 4.2.3.

[12]The non-abelian generalisation of the original $U(1)$ action of Leigh is not straightforward. See section 4.2.4 for more details.

$\lambda = g_{YM}^2 N$ is the 't Hooft coupling relevant to large N Yang-Mills theory. However, perturbation analysis in the gauge theory is valid provided $\lambda \ll 1$. Hence, the gauge theory is strongly coupled.

Maldacena did not give a precise dictionary for the correspondence between the two theories. This was furnished by Witten [46] and, independently, by Gubser *et al* [47] who gave an explicit prescription for relating correlation functions of the Euclideanised conformal field theory to the bulk theory path integral for specified boundary behaviour of the bulk fields. More precisely, the proposal is that

$$\begin{aligned} \langle e^{\int d^4x\, \phi_0(\vec{x})\mathcal{O}(\vec{x})} \rangle_{CFT} &\equiv \sum_q \frac{1}{q!} \int (\prod_{k=1}^{q} d^4x_k) \langle \mathcal{O}(\vec{x}_1) \cdots \mathcal{O}(\vec{x}_q) \rangle \phi_0(\vec{x}_1) \cdots \phi_0(\vec{x}_q) \\ &= Z_{str}\left[\phi = \phi_0\right] \ , \end{aligned} \tag{2.10}$$

where the first line is the generator of connected Green's functions in the conformal field theory and the second line is the full string theory partition function with the boundary condition that at the boundary of the AdS space the bulk field ϕ approaches the given function ϕ_0. When stringy α' corrections and loop corrections (which are governed by the gravitational coupling $\kappa \sim g_s \alpha'^2$) can be ignored Z_{str} can be replaced by $e^{-I_S(\phi)}$, where I_S is the classical supergravity action. The correspondence is valid in general, for any field ϕ.

To illustrate how formula (2.10) supports the conjecture, we quote the result of Witten and Gubser *et al* for the two point function of a massless scalar field in the (Euclidean) AdS_{d+1} theory[13]. Similar calculations have been performed in the massive case and in the case of a massless abelian gauge field. The reader is referred to the original papers [46] and [47] and to the review by Petersen [30] for explicit details.

The action for the scalar field is

$$I(\phi) = \frac{1}{2} \int_{AdS_{d+1}} d^{d+1}x \sqrt{g}\, \partial_\mu \phi \partial^\mu \phi \ . \tag{2.11}$$

[13]As mentioned earlier in section 1.3, we shall consider some geometrical aspects of Euclidean AdS_{d+1} (for the case $d = 2$) in the next chapter. For now, simply note that the boundary of this space, in the upper-half space representation akin to (1.48), is $\mathbb{R}^d$ plus a point at infinity and is thus topologically S^d.

By calculating the Green's function of Laplace's equation with appropriate boundary conditions they show that the action can be reexpressed as

$$I(\phi) = \frac{cd}{2} \int d^d x \, d^d x' \, \frac{\phi_0(\vec{x})\phi_0(\vec{x}')}{|\vec{x} - \vec{x}'|^{2d}} \,, \tag{2.12}$$

where $\vec{x}, \vec{x}' \in \mathbb{R}^d$ and c is a normalisation constant for the Green's function. Therefore, upon using (2.10) in the classical (super)gravity limit, we obtain

$$\langle \mathcal{O}(\vec{x}) \mathcal{O}(\vec{x}') \rangle \sim \frac{1}{|\vec{x} - \vec{x}'|^{2d}} \,, \tag{2.13}$$

which is the expected result for a field $\mathcal{O}$ of conformal dimension d.

There is by now overwhelming support for the conjecture (for example, a description of the matching of the spectra of both sides of the correspondence can be found in the review by Aharony *et al* [31]) and, by extension, for the holographic principle as envisaged by 't Hooft and Susskind. In the next section we specialise to $d = 2$ and present another argument in favour of the AdS/CFT correspondence.

2.3 AdS_3/CFT_2

In this section we briefly review an old result of Brown and Henneaux [59] concerning the asymptotic symmetry group at spatial infinity of $(2+1)$-dimensional gravity coupled to a negative cosmological constant. We shall show how it naturally fits into our discussion of the holographic principle and the Maldacena conjecture when applied to the determination of the BTZ black hole entropy.

Firstly, recall from chapter 1 that the BTZ metric (1.1) is a solution to the $(2+1)$-dimensional Einstein equations with negative cosmological constant. If we formally set $J = 0$ and $M = -1$ (for which the horizons and singularity disappear) the metric becomes

$$ds^2 = -\left(\frac{r^2}{l^2} + 1\right) dt^2 + \left(\frac{r^2}{l^2} + 1\right)^{-1} dr^2 + r^2 d\phi^2 \,, \tag{2.14}$$

which is another representation of the metric on Lorentzian $\widetilde{AdS}_3$. To see this, perform the transformation

$$x_0 = r\cos\phi \,, \; x_1 = r\sin\phi \,, \; x_2 = \sqrt{r^2 + l^2}\cos\frac{t}{l} \,, \; x_3 = \sqrt{r^2 + l^2}\sin\frac{t}{l} \,, \tag{2.15}$$

to bring the metric (1.24) to the above form[14]. The BTZ metric for general J and $M > 0$ is "asymptotically anti-de Sitter" in the sense that the metric components of both (1.1) and (2.14) satisfy the fall-off conditions

$$g_{tt} = -\frac{r^2}{l^2} + \mathcal{O}(1) \ , \tag{2.16}$$

$$g_{t\phi} = \mathcal{O}(1) \ , \tag{2.17}$$

$$g_{tr} = \mathcal{O}(\frac{1}{r^3}) \ , \tag{2.18}$$

$$g_{rr} = \frac{l^2}{r^2} + \mathcal{O}(\frac{1}{r^4}) \ , \tag{2.19}$$

$$g_{r\phi} = \mathcal{O}(\frac{1}{r^3}) \ , \tag{2.20}$$

$$g_{\phi\phi} = r^2 + \mathcal{O}(1) \ , \tag{2.21}$$

as $r \to \infty$. This ensures that the BTZ metric has at least $SO(2,2)$ as an asymptotic symmetry group since this group is the exact symmetry group of $\widetilde{AdS}_3$.

Secondly, using (1.6), we find that the Bekenstein-Hawking entropy (2.2) of the BTZ black hole is

$$S = 4\pi r_+ \ , \tag{2.22}$$

in units with $8G = 1$.

The asymptotic symmetries of asymptotically anti-de Sitter metrics are described by vector fields which leave invariant the boundary conditions (2.16)-(2.21). By analysing the Lie transformation equations of these boundary conditions, Brown and Henneaux showed that the asymptotic symmetry group is actually the conformal group in 1+1 dimensions. The Lie algebra of the conformal group consists, after canonical quantisation, of two copies of the Virasoro algebra

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n} \ , \\ [\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n} \ , \\ [L_m, \bar{L}_n] &= 0 \ , \end{aligned} \tag{2.23}$$

[14]If t is identified with period $2\pi l$ then the space-time is AdS_3 and closed timelike curves exist; if t is not identified then the space-time is the universal cover $\widetilde{AdS}_3$.

where $m, n \in \mathbb{Z}$. The $SO(2,2)$ isometries correspond to $m, n = 0, \pm 1$. Brown and Henneaux further showed that the central charge is given by

$$c = 12l = \frac{3l}{2G} , \tag{2.24}$$

and that the conserved charges associated with ∂_t and ∂_ϕ are given by

$$M = (L_0 + \bar{L}_0 - \frac{c}{12})/l \ , \ \ J = L_0 - \bar{L}_0 \ . \tag{2.25}$$

The constants in (2.25) are somewhat arbitrary and are chosen so that the "zero-mass black hole" — the space-time given by the $M \to 0$ limit of the BTZ space-time — has $L_0 = \bar{L}_0 = \frac{c}{24}$, whilst Lorentzian $\widetilde{AdS}_3$ (with $M = -1$ and $J = 0$) has $L_0 = \bar{L}_0 = 0$. This is consistent with the supersymmetry argument of Coussaert and Henneaux [60] who showed by an analysis of the Killing spinor equations associated with the metric (1.1, 1.2) that the zero-mass black hole may be viewed as ground state of the Ramond sector of $(1,1)$-AdS supergravity. Similarly, Lorentzian $\widetilde{AdS}_3$ may be viewed as the ground state of the Neveu-Schwarz sector.

Very shortly after Maldacena's conjecture first appeared, Strominger [61], inspired by the work of Brown and Henneaux and so in a different setting to Maldacena's work, calculated the entropy of the BTZ black hole using conformal field theory techniques[15]. To do so he used the Cardy formula [63, 64] for the asymptotic growth of the number of states of a conformal field theory with central charge c:

$$S \sim 2\pi\sqrt{\frac{c(n - c/24)}{6}} + 2\pi\sqrt{\frac{c(\bar{n} - c/24)}{6}} , \tag{2.26}$$

where $n(\bar{n})$ is the eigenvalue of $L_0(\bar{L}_0)$. Using (2.25), we find

$$\begin{aligned} S &\sim 2\pi\sqrt{l(Ml + J)} + 2\pi\sqrt{l(Ml - J)} \\ &\sim 4\pi r_+ \ , \end{aligned} \tag{2.27}$$

where we have used (1.5). This is in exact agreement with the BH result (2.22). Note that use of the Cardy formula implies that the black hole is large.

We are thus lead to the conjecture that quantum gravity on an asymptotically $\widetilde{AdS}_3$ space-time is equivalent to a conformal field theory of central charge $c = \frac{3l}{2G}$

[15] See also [62].

living on the (t, ϕ) cylinder on the boundary at spatial infinity. Strominger's derivation of the BH entropy is independent support for the 't Hooft/Susskind holographic principle and for the (generalised) Maldacena conjecture. Indeed, the support is even stronger if the BTZ black hole is embedded within type IIB string theory. To do this one starts with a system of Q_5 D5-branes wrapped on M^4, where M^4 is either T^4 or $K3$, giving a string in six dimensions and Q_1 D1-branes parallel to the string [65]. All the branes are coincident in the transverse dimensions. The configuration is a black string whose near-horizon limit, as $\alpha' \to 0$, in the string frame is locally $\widetilde{AdS}_3 \times S^3 \times M^4$ [66, 67]. In units of the string length, both the anti-de Sitter factor and the sphere factor have radius $l = 2\pi l_s \sqrt{g_s}(Q_1 Q_5)^{1/4}/V^{1/4}$ (where V is the volume of M^4 in the near-horizon geometry) and the three-dimensional Newton constant is $G^{-1} = 2V^{1/4}(Q_1 Q_5)^{3/4}/(\pi l_s \sqrt{g_s})$. The near-horizon limit is taken with the ratio $l_s/V^{1/4}$ fixed. If the string direction is periodically identified, with period $2\pi l$, the $\widetilde{AdS}_3$ part of the metric is precisely the BTZ black hole[16].

Strominger's argument raises the question of where the relevant degrees of freedom that contribute to the entropy are located. This question becomes even more significant if we also consider Carlip's derivation of the BTZ entropy [71]. Carlip used the Chern-Simons formulation of $(2+1)$-dimensional gravity with a negative cosmological constant (see the remark on page 9) and the fact that a Chern-Simons theory on a manifold with boundary induces a dynamical Wess-Zumino-Witten (WZW) theory on the boundary [72, 73]. Now, the event horizon of a black hole is not a true boundary but it can often be treated as such. Taking the latter approach, Carlip found that the theory at the horizon is described by a $SO(2,1) \times SO(2,1)$ WZW model whose associated Virasoro algebra, for large black holes, has central charge $c \sim 6$ and $n \sim (\frac{r_+}{4G})^2$. The conformal field theory is not the same as the one at

[16] In his original paper [27], Maldacena also considered the D1-D5 system (and several other D-brane and M-brane systems) and put forward the supersymmetrised version of the above conjecture, which relates an $\mathcal{N} = (4,4)$ $(1+1)$-dimensional SCFT with central charge $c = 6(Q_1 Q_5 + 1) \approx 6Q_1 Q_5 = \frac{3l}{2G}$ (see, for example, [31, 68] for explicit details of this theory) to type IIB string theory on $\widetilde{AdS}_3 \times S^3 \times M^4$ (see [69, 70]) in the large (but fixed) Q_i limit. However, in the last section we chose to stick to the D3-brane case for simplicity.

spatial infinity since the central charge is different[17]. Calculation of the entropy via the Cardy formula again results in agreement with the BH entropy.

In the next chapter, we present a kinematical view of holography (in contrast to the dynamical one of Maldacena) for AdS_3/CFT_2, in the form of Sullivan's theorem, which suggests that the entropy is determined in terms of the boundary data at spatial infinity as advocated by Strominger.

[17]The connection between the two conformal field theories has been studied in [74].

Chapter 3

Geometrical Finiteness, Sullivan's Theorem and the BTZ Black Hole

In the last chapter we saw an explicit realisation of the holographic principle in the form of the Maldacena conjecture which posits a dynamical correspondence between string theory on anti-de Sitter backgrounds and conformal field theory on the boundary of anti-de Sitter space. In particular, we saw how this correspondence specialised to Lorentzian $\widetilde{AdS_3}$ was used to calculate, à la Strominger, the entropy of the BTZ black hole. In this chapter, we show that for hyperbolic space $\mathbb{H}^3$, the Euclidean analogue of AdS_3, there is a precise notion of holography in the kinematical sense[1]. This depends on a theorem of Sullivan [75], see also [76], which states that the inequivalent hyperbolic structures of a three-dimensional geometrically finite Kleinian manifold are parametrised by the Teichmüller space of the boundary. In order to understand this last sentence we must first study in more detail the geometry of $\mathbb{H}^3$. We then go on to explore the consequences of the theorem in the context of the Euclidean BTZ black hole.

[1]It should be obvious that since there are no closed timelike curves in a Euclidean signature space-time, the Euclidean analogues of both Lorentzian AdS_3 and its universal cover are one and the same.

3.1 Three Models of $\mathbb{H}^3$

Following [25,46], we briefly describe three equivalent models of hyperbolic space $\mathbb{H}^3$ with radius l. Depending on the context, one model may be more suitable than the others for calculations.

3.1.1 The Ball Model

Consider Euclidean space $\mathbb{R}^3$ with coordinates y_0, y_1, y_2 and let $\mathbb{B}^3$ be the open ball, $|y|^2 \equiv \sum_{i=0}^{2} y_i^2 < l^2$. The space $\mathbb{B}^3$ together with the Riemannian line element

$$ds^2 = 4l^4 \frac{\sum_{i=0}^{2} dy_i^2}{(l^2 - |y|^2)^2} , \tag{3.1}$$

is the ball model of $\mathbb{H}^3$.

We can compactify $\mathbb{B}^3$ to get the closed ball $\overline{\mathbb{B}^3}$, defined by $|y|^2 \leq l^2$. Its boundary is the sphere S^2, defined by $|y|^2 = l^2$, and is the Euclidean version of the conformal compactification of Minkowski space. The metric (3.1) is not defined on the boundary because it is singular there. To get a metric which extends over the full $\overline{\mathbb{B}^3}$, one picks a function f with a first order zero on the boundary (for example, $f = l^2 - |y|^2$) and replaces ds^2 by $d\tilde{s}^2 = f^2 ds^2$. Since there is no natural choice of f, the metric $d\tilde{s}^2$ is only well-defined up to the conformal transformation $f \to e^w \ , \ d\tilde{s}^2 \to e^{2w} d\tilde{s}^2$, where w is any real function on $\overline{\mathbb{B}^3}$. Therefore, S^2 has only a conformal structure.

3.1.2 The (Upper) Hyperboloid Model

We have already encountered the hyperboloid model of Lorentzian AdS_3 in section 1.2. The Euclidean analogue is defined by the hypersurface

$$\sum_{i=0}^{2} x_i^2 - x_3^2 = -l^2 \quad (x_3 \geq l > 0) , \tag{3.2}$$

embedded in $\mathbb{R}^{1,3}$ with coordinates x_0, x_1, x_2, x_3 and metric

$$ds^2 = \sum_{i=0}^{2} dx_i^2 - dx_3^2 . \tag{3.3}$$

This can, in fact, be seen from the Riemannian continuation of region I of the Penrose diagram, as described in sections 1.2 and 1.3. Using (1.26), it is easy to see that the effect of the continuation (1.43) is to take $x_1^2 - x_2^2$ to $x_1^2 + x_2^2$ and thus (1.23) and (1.24) to (3.2) and (3.3), respectively.

By means of the parametrisation

$$x_0 = l \sinh\chi \cos\theta\,,\; x_1 = l \sinh\chi \sin\theta \cos\phi\,,\; x_2 = l \sinh\chi \sin\theta \sin\phi\ ,$$

$$x_3 = l \cosh\chi\ , \tag{3.4}$$

with $0 \leq \chi < \infty$, the metric (3.3) can be rewritten in the more usual form

$$ds^2 = l^2(d\chi^2 + \sinh^2\chi d\Omega^2)\ , \tag{3.5}$$

where $d\Omega^2$ is the metric on the unit sphere. In this representation, the boundary is the sphere at $\chi = \infty$.

The hyperboloid projects stereographically from the point $(0,0,0,-l)$ onto the open ball, as can be seen from the transformation

$$x_i = 2l^2 \frac{y_i}{l^2 - |y|^2}\ ,\quad x_3 = l\frac{l^2 + |y|^2}{l^2 - |y|^2}\ . \tag{3.6}$$

The transformation of the boundary is obtained via the limit of this mapping, from which it is easily seen that $\chi = \infty$ maps to $|y|^2 = l^2$.

3.1.3 The Upper Half-Space Model

The upper half-space model of $\mathbb{H}^3$ was introduced in section 1.3. It is the space $\mathcal{H} := \{(x,y,z)|x,y \in \mathbb{R}, z > 0\}$ equipped with the Riemannian metric

$$ds^2 = \frac{l^2}{z^2}(dx^2 + dy^2 + dz^2)\ . \tag{3.7}$$

We shall take the coordinates x, y, z to be dimensionless. In this representation, the boundary consists of the $x-y$ plane at $z = 0$ together with a single point[2] P at $z = \infty$

[2] As noted in [46], there is only a single point at infinity because the metric (3.7) vanishes in the x, y directions as $z \to \infty$.

and is thus topologically S^2. It is equivalent to the Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. The closure of $\mathcal{H}$ is then $\overline{\mathcal{H}} = \mathcal{H} \cup \mathbb{C}_\infty$.

The transformation between this representation and the ball model is most succinctly written using the language of quaternions [77,78]. A quaternion is an element of a four-dimensional real vector space and can be expressed as

$$q = x + yi + zj + ak \ , \tag{3.8}$$

where $x, y, z, a \in \mathbb{R}$. Quaternions may be added in an abelian manner similarly to complex numbers but multiplication is non-commutative (but distributive) and subject to the rules

$$i^2 = j^2 = k^2 = ijk = -1 \ . \tag{3.9}$$

Using these rules, (3.8) can be reexpressed as

$$q = w + uj \tag{3.10}$$

where $w \equiv x + iy$ and $u \equiv z + ia$ are both in $\mathbb{C}$. Note that $ju = \bar{u}j$ for arbitrary complex u and hence

$$\begin{aligned} (w + uj)(\bar{w} - uj) &= |w|^2 + |u|^2 \\ \Rightarrow \quad (w + uj)^{-1} &= (\bar{w} - uj)/(|w|^2 + |u|^2) \ . \end{aligned} \tag{3.11}$$

We can now write the point (x, y, z) of $\mathcal{H}$ as the quaternion

$$q = w + zj \ , \tag{3.12}$$

where $w \equiv x + iy \in \mathbb{C}$ and $z \in \mathbb{R}_+$. It is easily verified that the mapping to the ball model is given by

$$q \mapsto q' = \Phi(q) = (q - j)(-jq + 1)^{-1} \ , \tag{3.13}$$

where $q' = \frac{1}{l}(y_0 + y_1 i + y_2 j)$. Equivalently,

$$\begin{aligned} y_0 &= l\tfrac{2x}{x^2+y^2+(z+1)^2} \ , \\ y_1 &= l\tfrac{2y}{x^2+y^2+(z+1)^2} \ , \\ y_2 &= l\tfrac{x^2+y^2+z^2-1}{x^2+y^2+(z+1)^2} \ . \end{aligned} \tag{3.14}$$

The transformation of the boundary is again obtained in the limit. For example, the point P at $z = \infty$, which we define to be equivalent to $q = \infty$, maps to the North Pole of $\overline{\mathbb{B}^3}$, whilst the origin maps to the South Pole. The rest of the positive z-axis maps to the diameter joining the two poles and the point $q = j$ maps to the centre of the ball.

3.2 The Isometries of $\mathbb{H}^3$

The 2×2 complex matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \tag{3.15}$$

with determinant $\det M = ad - bc \neq 0$ induces the orientation-preserving conformal homeomorphism

$$w \mapsto w^* = M(w) = \frac{aw + b}{cw + d} , \tag{3.16}$$

of $\mathbb{C}_\infty$ onto itself[3]. Such maps are called Möbius transformations. Note that the matrices M, $M/\sqrt{\det M}$ and $-M$ all induce the same transformation $M(w)$. Hence, the Möbius group **Möb** is isomorphic to $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\mathbb{Z}_2$. It can be shown that these transformations are the *unique* orientation-preserving bijections $\mathbb{C}_\infty \to \mathbb{C}_\infty$.

The action of **Möb** can be extended from $\mathbb{C}_\infty$ to $\overline{\mathcal{H}}$. The extension is given by

$$q \mapsto q^* = M(q) = (aq + b)(cq + d)^{-1} \quad (ad - bc = 1) , \tag{3.17}$$

where q is the quaternion (3.12). More explicitly, we find

$$w^* = \frac{(aw + b)(\bar{c}\bar{w} + \bar{d}) + a\bar{c}z^2}{|cw + d|^2 + |c|^2 z^2} , \tag{3.18}$$

$$z^* = \frac{z}{|cw + d|^2 + |c|^2 z^2} . \tag{3.19}$$

[3]If M is singular, it is easily seen that $M(w)$ is not invertible. If M is non-singular, the map $M(w)$ is a bijection provided we define $M(-d/c) = \infty$ and $M(\infty) = a/c$ for c non-zero or $M(\infty) = \infty$ for $c = 0$.

We recover (3.16) in the limits $z \to 0$ and z (or q) $\to \infty$. Quaternions q defined in $\mathcal{H}$ are termed *proper*, while those defined on the boundary $\mathbb{C}_\infty$ are termed *improper*.

It is straightforward to show that that the metric (3.7) is $PSL(2,\mathbb{C})$-invariant. Furthermore, it is shown in [78] how a notion of orientability can be defined in $\mathcal{H}$ and this is used to show that $PSL(2,\mathbb{C})$ is the group of orientation-preserving motions $ISO^+(\mathcal{H})$ of the upper half-space. The proof is not relevant for the sequel. The map (3.13) shows that $PSL(2,\mathbb{C})$ is also the group $ISO^+(\mathbb{B}^3)$. Indeed, M is an isometry of $\mathcal{H}$ if and only if $\Phi M \Phi^{-1}$ is an isometry of $\mathbb{B}^3$. On the other hand, from equations (3.2) and (3.3), it is clear that the group of orientation-preserving motions of the hyperboloid model is $SO^+(1,3)$, the connected component of the identity in $SO(1,3)$. The equivalence of all three models follows from the exceptional isomorphism $PSL(2,\mathbb{C}) \approx SO^+(1,3)$. The reader is referred to [25] for a proof of this isomorphism.

3.2.1 Classification of Isometries

We are interested in the the conjugacy classes of **Möb**. In $PSL(2,\mathbb{C})$ there is essentially only one conjugate-invariant function, the square of the trace, that is, $tr^2 M = (a+d)^2$ for a matrix M of the form (3.15). We shall denote the identity of $PSL(2,\mathbb{C})$ by I. We have the following classification (see, for example, [25, 79, 80]):

Definition. Let $M\,(\neq I)$ be any Möbius transformation. Then

(i) M is *parabolic* if and only if it has it has a unique fixed point in $\mathbb{C}_\infty$;

(ii) M is *elliptic* if and only if it has two fixed points in $\mathbb{C}_\infty$ and if the points on the geodesic in $\mathcal{H}$ joining these two points are also left fixed;

(iii) M is *hyperbolic* if and only if it has two fixed points in $\mathbb{C}_\infty$ and if any circle in $\mathbb{C}_\infty$ together with its interior is left invariant. The line in $\mathcal{H}$ joining these two points is then left invariant, but M has no fixed points in $\mathcal{H}$;

(iv) M is *loxodromic* in all other cases. M then has two fixed points in $\mathbb{C}_\infty$ and no fixed points in $\mathcal{H}$. The geodesic joining the two fixed points is the only geodesic in $\mathcal{H}$ which is left invariant. M may leave the circles joining the two fixed points invariant, but it then interchanges interior and exterior.

Let us introduce the following *standard forms* for a Möbius transformation (3.17) acting on $\overline{\mathcal{H}}$:

$$M_1(q) = q + 1 , \quad (3.20)$$

$$M_k(q) = kw + |k|zj \quad (k \neq 1) , \quad (3.21)$$

for non-zero $k \in \mathbb{C}$. Then, we find $tr^2 M_k = k + 1/k + 2$ for *all* k. Note that M_1 fixes the point P at $q = \infty$ and M_k fixes the origin and P if $|k| \neq 1$ or the whole of the z-axis (including the origin and P) if $|k| = 1, k \neq 1$.

We say that two elements $M, N \in$ **Möb** are *conjugate* ($M \sim N$) if $N = LML^{-1}$ for some other transformation $L \in$ **Möb**. Note that $tr^2 M = tr^2 N$ for conjugate M and N. Then, we have the following theorem:

Theorem. *Let M ($\neq I$) be any Möbius transformation. Then*

(i) *M is parabolic if and only if $M \sim M_1$ (equivalently, $tr^2 M = 4$);*

(ii) *M is elliptic if and only if $M \sim M_k$, with $|k| = 1, k \neq 1$ (equivalently, $tr^2 M \in [0, 4)$);*

(iii) *M is hyperbolic if and only if $M \sim M_k$, with $k > 0, k \neq 1$ (equivalently, $tr^2 M \in (4, +\infty)$);*

(iv) *M is loxodromic if and only if $M \sim M_k$, with $|k| \neq 1$ and $tr^2 M \notin [0, +\infty)$.*

Proof. The reader is referred to Beardon [79] or Maskit [80]. ▮

Example. Recall from (1.49) that for the Euclidean BTZ black hole periodicity in the Schwarzchild angle $\phi \sim \phi + 2n\pi$, $n \in \mathbb{Z}$ requires the following identifications in $\mathcal{H}$:

$$(x, y, z) \sim e^{2n\Sigma} (x \cos 2n\Theta - y \sin 2n\Theta, x \sin 2n\Theta + y \cos 2n\Theta, z) ,$$

where

$$\Sigma = \frac{\pi r_+}{l} \ , \ \Theta = \frac{\pi |r_-|}{l} \ .$$

These identifications can be written alternatively as

$$w \sim e^{2n(\Sigma + i\Theta)} w \ , \ z \sim e^{2n\Sigma} z \ . \tag{3.22}$$

Using (3.17), it is easy to verify that these identifications are associated with the $PSL(2,\mathbb{C})$ matrices

$$\begin{pmatrix} e^{n(\Sigma+i\Theta)} & 0 \\ 0 & e^{-n(\Sigma+i\Theta)} \end{pmatrix} , \tag{3.23}$$

corresponding to the standard form (3.21) with $k_n = e^{2n(\Sigma+i\Theta)}$. It follows from the theorem that in the generic case[4] $|r_-| \neq 0$ (which corresponds to the spinning black hole, according to (1.46)) M_{k_n} is loxodromic with fixed points at the origin and P for all $n \neq 0$. In the special (spinless) case $|r_-| = 0$, M_{k_n} is hyperbolic, again with fixed points at the origin and P for all $n \neq 0$. Furthermore, it is easily seen that the collection of all such M_{k_n} (including the identity with $n = 0$) forms an infinite abelian cyclic group generated by the single element M_{k_1}, with $|k_1| = e^{2\Sigma} > 1$. This group, which shall henceforth be denoted by Γ_{BTZ}, is a subgroup of $PSL(2,\mathbb{C})$ and will play a pivotal role in what follows. Clearly, it is isomorphic to $\mathbb{Z}$.

3.3 Geometrical Finiteness and Sullivan's Theorem

This section consists mainly of a sequence of definitions and theorems from hyperbolic geometry culminating in the definition of geometrical finiteness and a statement of Sullivan's Theorem. As such, it is rather abstract in nature and so we have illustrated the concepts in the case of the BTZ black hole. We follow [24, 80–82].

[4]Recall that the black hole exists for $M > 0$, which implies $r_+ > 0$ by (1.46).

Definition. Let X be any topological space and $\mathcal{M}$ a group of homeomorphisms of X onto itself. We say that $\mathcal{M}$ acts *(properly) discontinuously* on X if and only if for every compact subset K of X,

$$M(K) \cap K = \emptyset \ , \tag{3.24}$$

except for a finite number of distinct M in $\mathcal{M}$.

Example. Take $X = \mathcal{H}$ with the usual topology and $\mathcal{M} \subset PSL(2, \mathbb{C})$. Then $\mathcal{M}$ acts discontinuously on $\mathcal{H}$ if and only if for arbitrary $q \in \mathcal{H}$, $M(q) \cap q$ is nonempty only for finitely many distinct $M \in \mathcal{M}$. In particular, take $\mathcal{M} = \Gamma_{BTZ}$. Since the non-trivial elements of Γ_{BTZ} are either all loxodromic or all hyperbolic and therefore have no fixed points in $\mathcal{H}$ then $M(q) \cap q \neq \emptyset$ only for $M = I$. So Γ_{BTZ} acts discontinuously on $\mathcal{H}$.

Definition. A subgroup of $PSL(2, \mathbb{C})$ acting discontinuously on $\mathcal{H}$ is called *discrete*. We denote such a group by Γ in what follows. We shall denote the identity of the group by I.

Definition. The Γ-*orbit* of any quaternion $q \in \mathcal{H}$ under the action of Γ is the set

$$\Gamma_q := \{M(q) \in \mathcal{H} | M \in \Gamma\} \ . \tag{3.25}$$

A point $p \in \mathbb{C}_\infty$ is an *accumulation point* of Γ_q if there exists a sequence $\{M_i\}$ of distinct elements of Γ such that $\lim M_i(q) \to p$. The set Γ_q together with its accumulation points is the closure of Γ_q, denoted $\overline{\Gamma}_q$.

We define the *limit set* L_Γ of Γ as the set

$$L_\Gamma := \overline{\Gamma}_q \cap \mathbb{C}_\infty \ , \tag{3.26}$$

that is, the set of accumulation points of some Γ-orbit in $\mathcal{H}$. One can show that the limit set is independent of the choice of q. Furthermore, it can be shown that the limit set coincides with the closure of the set $Fix(\Gamma)$ of fixed points of all but the elliptic elements of Γ.

Example. Consider $\Gamma = \Gamma_{BTZ}$. It is straightforward to see that the origin and the point P at $q = \infty$ are accumulation points of Γ_q for any $q \in \mathcal{H}$. For example, taking the sequence $\{M_{k_n}^m | m \in \mathbb{Z}, n \in \mathbb{Z} - \{0\} \text{ fixed}\}$ we find

$$M_{k_n}^m(q) = M_{k_{nm}}(q) = M_{k_1}^{nm}(q) = k_1^{mn} w + |k_1|^{mn} zj \to \begin{cases} 0 & m \to -\infty, \\ \infty & m \to +\infty, \end{cases} \tag{3.27}$$

for *every* q and n. Indeed, these two points are the *only* accumulation points in $\mathbb{C}_\infty$ and so form the set $L_{\Gamma_{BTZ}}$. Furthermore, since the non-trivial elements of Γ_{BTZ} are either all loxodromic or all hyperbolic and of standard form, the limit set coincides with the closure of $Fix(\Gamma_{BTZ})$.

Definition. The *domain of discontinuity* $\Omega(\Gamma)$ of Γ is defined as the complement of the limit set L_Γ in $\mathbb{C}_\infty$. It is Γ-invariant. It is possible for it to be empty.

Example. For $\Gamma = \Gamma_{BTZ}$, it is clear that $\Omega(\Gamma_{BTZ})$ is the punctured plane $\mathbb{C} - \{0\}$.

Remark. Even if Γ acts discontinuously on $\mathcal{H}$, it is clear that Γ does not act discontinuously on L_Γ. For example, since Γ_{BTZ} is infinite there are an infinite number of transformations $M \in \Gamma_{BTZ}$ such that $M(q) \cap q \neq \emptyset$ for q the origin or the point P. On the other hand, it can be shown that Γ does act discontinuously on $\Omega(\Gamma)$ (hence its name). This is enough to ensure that the quotient space $\Omega(\Gamma)/\Gamma$ is Hausdorff (whereas $\mathbb{C}_\infty/\Gamma$ is non-Hausdorff). Indeed, it is a well-known theorem that $\Omega(\Gamma)/\Gamma$ is a (not necessarily connected) Riemann surface. For the BTZ black hole, we find $\Omega(\Gamma_{BTZ})/\Gamma_{BTZ} \approx (\mathbb{C} - \{0\})/\mathbb{Z}$. The obvious fundamental domain is the annulus $\{w \in \mathbb{C} | 1 < |w| < |k_1| = e^{2\pi r_+/l}\}$, with points on the two sides identified with a twist of $\arg k_1 = 2\pi |r_-|/l$ about the origin. Thus, it is topologically a torus. This is in agreement with our discussion in section 1.3.

Definition. We can now form the *Kleinian manifold* $N_C(\Gamma) = (\mathcal{H} \cup \Omega(\Gamma))/\Gamma$ which has interior $N(\Gamma) = \mathcal{H}/\Gamma$ and boundary $\partial N(\Gamma) = \Omega(\Gamma)/\Gamma$. Note that $N_C(\Gamma)$ is actually an orbifold unless Γ acts freely.

Example. It should now be obvious that for $\Gamma = \Gamma_{BTZ}$, $N_C(\Gamma_{BTZ})$ is just the Euclidean BTZ black hole.

Definition. Suppose we are given two points q_α and q_β in $\mathcal{H}$. Let $q(\tau) = w(\tau) + z(\tau)j$, $\alpha \leq \tau \leq \beta$ be a continuous and piecewise continuously differentiable curve σ in $\mathcal{H}$ parametrised by $\tau \in \mathbb{R}$ and with endpoints $q(\alpha) = q_\alpha$ and $q(\beta) = q_\beta$. The *hyperbolic length* between q_α and q_β along σ is defined as

$$d_\sigma(q_\alpha, q_\beta) = \int_\sigma \frac{ds}{d\tau} d\tau \ , \tag{3.28}$$

where ds is the (square root of the) Riemannian line element (3.7).

We define the *hyperbolic distance* between q_α and q_β as

$$d(q_\alpha, q_\beta) = \inf_\sigma d_\sigma(q_\alpha, q_\beta) \ , \tag{3.29}$$

the infimum being taken over all continuous and piecewise continuously differentiable curves joining q_α and q_β. The invariance of ds under $PSL(2, \mathbb{C})$ transformations implies that the metric d is *point-pair invariant*:

$$d(q_\alpha, q_\beta) = d(M(q_\alpha), M(q_\beta)) \ , \tag{3.30}$$

for all $q_\alpha, q_\beta \in \mathcal{H}, M \in PSL(2, \mathbb{C})$.

Proposition. *In $\mathcal{H}$, the geodesic (that is, the curve of shortest hyperbolic length) joining the two points aj and bj ($b \geq a > 0$) is simply the segment of the z-axis joining them.*

Proof. The proof is straightforward. Let $q(\tau) = w(\tau) + z(\tau)j$, $\alpha \leq \tau \leq \beta$ be any continuous and piecewise continuously differentiable curve σ in $\mathcal{H}$ parametrised by $\tau \in \mathbb{R}$ and with endpoints $q(\alpha) = aj$ and $q(\beta) = bj$. Then, we have

$$d_\sigma(aj, bj) = \int_\sigma \frac{l d\tau}{z} \left(\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2 \right)^{1/2} \geq \int_a^b \frac{l dz}{z} = l \ln \frac{b}{a} . \tag{3.31}$$

The second integral in this expression is the hyperbolic length of the segment of the z-axis joining aj to bj. Equality in the above holds only if $dx/d\tau = 0 = dy/d\tau$, that is, only if σ is this segment. We conclude that for general $b \geq a > 0$, there is a unique geodesic in $\mathcal{H}$ between aj and bj; it is simply the segment of the z-axis joining them. ∎

Example. Geodesics in $\mathcal{H}$ can join two proper points, two improper points or one proper and one improper point. By taking the limit $a \to 0$, $b \to \infty$, the proposition shows that the positive z-axis is the unique geodesic in $\mathcal{H}$ joining the origin and the point P at infinity.

Proposition. *The group $PSL(2, \mathbb{C})$ acts in the following sense doubly transitively on $\mathcal{H}$: For all $p, p^*, q, q^* \in \mathcal{H}$ such that $d(p, p^*) = d(q, q^*)$ there exists an $M \in PSL(2, \mathbb{C})$ such that $M(p) = p^*$ and $M(q) = q^*$.*

Proof. See [25]. ∎

Proposition. *The hyperbolic distance between any two points $q = w + zj$ and $q^* = w^* + z^*j$ in $\mathcal{H}$ is given by*

$$\cosh\left(\frac{d(q, q^*)}{l}\right) = \delta(q, q^*) \ , \tag{3.32}$$

where δ is defined by

$$\delta(q, q^*) := \frac{|w - w^*|^2 + z^2 + z^{*2}}{2zz^*} \ . \tag{3.33}$$

Proof. Firstly, by using the second integral in (3.31) with $a = 1$, it is clear that the result (3.32) is true in the special case $q = j, q^* = bj$, $(b \geq 1)$. Secondly, by the previous proposition, there exists for all $q, q^* \in \mathcal{H}$ an $M \in PSL(2, \mathbb{C})$ such that $M(q) = j, M(q^*) = bj, b = e^{d(q,q^*)/l}$. The result follows when one notes that δ is actually point-pair invariant:

$$\cosh\left(\frac{d(q, q^*)}{l}\right) = \cosh\left(\frac{d(j, bj)}{l}\right) = \delta(t, bj) = \delta(M(q), M(q^*)) = \delta(q, q^*) \ . \quad ∎$$

Definition. A non-empty subset $\overline{X}$ of $\overline{\mathcal{H}}$ is *convex* if for any two (proper or improper) points of $\overline{X}$ the geodesic in $\mathcal{H}$ joining these points[5] is contained in $\overline{X}$. Clearly, the intersection of convex sets is again convex.

[5] This assumes that there is a unique geodesic in $\mathcal{H}$ joining any two points in $\overline{\mathcal{H}}$. This is in fact true and was used earlier in the classification of isometries in section 3.2.1. We have shown this explicitly for the origin and the point P at infinity and this is all that will be needed in what follows.

Definition. The *convex hull* $H(L_\Gamma)$ of the limit set L_Γ of Γ is defined to be the intersection of all convex sets X in $\mathcal{H}$ whose closures $\overline{X}$ in $\overline{\mathcal{H}}$ contain L_Γ. It is thus the smallest convex set in $\mathcal{H}$ whose closure in $\overline{\mathcal{H}}$ contains L_Γ. The *convex core* $C(\Gamma)$ is defined to be the quotient $H(L_\Gamma)/\Gamma$.

Example. Consider $\Gamma = \Gamma_{BTZ}$. We have seen that $L_{\Gamma_{BTZ}}$ consists of precisely two points — the origin and the point P at infinity — and that the positive z-axis is the unique geodesic in $\mathcal{H}$ joining these two points. By the very definition of convexity, a geodesic is itself convex. Hence, the convex hull $H(L_{\Gamma_{BTZ}})$ is just the positive z-axis. The convex core $C(\Gamma_{BTZ})$ is then the quotient of the positive z-axis by $\Gamma_{BTZ} \approx \mathbb{Z}$. Thus, it is $\mathbb{R}_+/\mathbb{Z}$ which is a circle. The fundamental domain of the core can be taken to be $q(a) = aj$, $1 < a < |k_1| = e^{2\pi r_+/l}$. The hyperbolic length of this segment is

$$d(j, |k_1|j) = l \int_1^{|k_1|} \frac{dz}{z} = 2\pi r_+ \ . \tag{3.34}$$

This is in accord with section 1.3, where we noted that the horizon of the Euclidean BTZ black hole at $r = r_+$ maps to the z-axis under the transformation (1.47).

Theorem (Margulis Lemma). *Given $q \in \mathcal{H}$ and $\varepsilon > 0$, denote by $\Gamma_\varepsilon(q)$ the subgroup of Γ generated by all elements $M \in \Gamma$ such that $d(q, M(q)) \leq l\varepsilon$. Then there exists a universal constant $\varepsilon(3) > 0$, (called the Margulis constant), such that $\Gamma_{\varepsilon(3)}(q)$ has an abelian subgroup of finite index.*

Proof. See [83]. ∎

Definition. Given $\varepsilon > 0$, define the set

$$T_\varepsilon(\Gamma) := \{q \in \mathcal{H} | d(q, M(q)) \leq l\varepsilon \text{ for some } M \in Free(\Gamma)\} \ , \tag{3.35}$$

where $Free(\Gamma) := \{M \in \Gamma | M^n \neq I \text{ for any } n \neq 0\}$ is the set of infinite order elements of Γ. It is possible for $T_\varepsilon(\Gamma)$ to be empty.

We now define the *thin part* of $N(\Gamma)$ as $\text{thin}_\varepsilon(N(\Gamma)) = T_\varepsilon(\Gamma)/\Gamma \subseteq N(\Gamma)$. The *thick part* of $N(\Gamma)$, denoted $\text{thick}_\varepsilon(N(\Gamma))$, is defined as the closure of the complement of $\text{thin}_\varepsilon(N(\Gamma))$ in the Kleinian manifold $N_C(\Gamma)$. This is the "alternative" definition of Thurston's thick-thin decomposition, as described in [81, 84]. (The more usual definition of $T_\varepsilon(\Gamma)$ is as the set $T'_\varepsilon(\Gamma) := \{q \in \mathcal{H} | \Gamma_\varepsilon(q) \text{ is infinite}\}$, where $\Gamma_\varepsilon(q)$ is as defined in the Margulis Lemma. In general, we have $T_\varepsilon(\Gamma) \subseteq T'_\varepsilon(\Gamma)$. However, as we shall see momentarily, both definitions turn out to be equivalent for $\Gamma = \Gamma_{BTZ}$.)

Definition (Geometrical Finiteness). As shown in [81], there are five different definitions of geometrical finiteness. They are all equivalent when applied to $\mathbb{H}^3$ but some have a more natural generalisation than others when one goes to more than three dimensions. For our purposes, we shall use definition $GF4$ in the nomenclature of [81]:

A discrete subgroup Γ of $PSL(2, \mathbb{C})$ is *geometrically finite* if for some positive $\varepsilon < \varepsilon(3)$

$$C(\Gamma) \cap \text{thick}_\varepsilon(N(\Gamma))$$

is compact. The Kleinian manifold $N_C(\Gamma)$ is called geometrically finite if the quotient group Γ generating it is geometrically finite.

Theorem. Γ_{BTZ} *is geometrically finite.*

Proof. Firstly, we show that the set $T_\varepsilon(\Gamma_{BTZ})$ defined by (3.35) is empty for suitably chosen $\varepsilon > 0$. Since $\Gamma_{BTZ} = \{M^n_{k_1} | n \in \mathbb{Z}\}$, it is clear that $Free(\Gamma_{BTZ}) = \Gamma_{BTZ} \setminus \{I\}$. We now use (3.27), (3.32) and (3.33) to find

$$\cosh\left(\frac{d(q, M^n_{k_1}(q))}{l}\right) = \cosh 2n\Sigma + \frac{|w|^2}{z^2}\left(\cosh 2n\Sigma - \cos 2n\Theta\right) \ , \tag{3.36}$$

for all $q = w + zj \in \mathcal{H}, n \in \mathbb{Z}$. Thus, for the point q to be in the set $T_\varepsilon(\Gamma_{BTZ})$ we require

$$\cosh 2n\Sigma + \frac{|w|^2}{z^2}\left(\cosh 2n\Sigma - \cos 2n\Theta\right) \leq \cosh\varepsilon \ , \tag{3.37}$$

for some $n \neq 0$. We can rewrite (3.37) alternatively as

$$\frac{|w|^2}{z^2} \leq \frac{\cosh\varepsilon - \cosh 2n\Sigma}{\cosh 2n\Sigma - \cos 2n\Theta} \ , \tag{3.38}$$

where the denominator on the right side of this expression is always positive. If we choose $\varepsilon < \min(\varepsilon(3), 2\Sigma)$ then no such q exists and hence $T_\varepsilon(\Gamma_{BTZ})$ is empty. (Furthermore, for ε in this range, we have $\Gamma_\varepsilon(q) = \{I\}$ for all $q \in \mathcal{H}$ and hence $T'_\varepsilon(\Gamma_{BTZ})$ is also empty.) Moreover, since $T_\varepsilon(\Gamma_{BTZ})$ is empty, $\text{thin}_\varepsilon(N(\Gamma_{BTZ}))$ is empty and so $\text{thick}_\varepsilon(N(\Gamma_{BTZ})) = N_C(\Gamma_{BTZ})$. By the definition of geometrical finiteness, we see that Γ_{BTZ} is geometrically finite if the convex core $C(\Gamma_{BTZ})$ is compact. But in the last example we showed that $C(\Gamma_{BTZ})$ is a circle and so is compact. ∎

Theorem (Sullivan). *Let M denote a topological 3-manifold with boundary ∂M and let $GF(M)$ denote the space of geometrically finite Kleinian 3-manifolds $N_C(\Gamma)$ which are homeomorphic to M. Then, as long as M admits at least one hyperbolic realisation, there is a $1-1$ correspondence between hyperbolic structures on M and conformal structures on ∂M, that is,*

$$GF(M) \cong \text{Teich}(\partial M) \ , \tag{3.39}$$

where Teich(∂M) *is the Teichmüller space of ∂M.*

Proof. See [75] and also [76]. ∎

Remark. Let M be a solid torus. Then M has at least one hyperbolic realisation by a geometrically finite Kleinian manifold, namely $N_C(\Gamma_{BTZ})$ for given parameters r_+ and $|r_-|$. So Sullivan's theorem applies. We explore the consequences of this theorem for the BTZ black hole in the next section.

3.4 Implications of Sullivan's Theorem for the BTZ Black Hole

The following are consequences of Sullivan's Theorem for the BTZ black hole:

- The theorem allows us to declare that the Euclidean BTZ manifold is a holographic manifold such that the three-dimensional hyperbolic structures are in 1-1 correspondence with the Teichmüller parameters of the two-dimensional genus-one toroidal boundary.

- Since the Teichmüller space of the torus is parametrised by two real parameters, the theorem states that we have a 'No Hair' theorem; namely, the BTZ black hole can be parametrised by at most two parameters, thus excluding the construction of a charged, rotating generalisation *as a geometrically finite Kleinian manifold.* It should be noted, however, that such generalisations *do* exist (see [13] for a brief review).

- As the Bekenstein-Hawking entropy formula (2.2) is a geometrical quantity it is determined once the hyperbolic structure is fixed. Hence, the Bekenstein-Hawking entropy is determined by the Teichmüller space of the boundary. This is in agreement with Strominger's derivation in section 2.3 of the BTZ entropy based on the asymptotic symmetry algebra of anti-de Sitter space.

- In [67] (see also [85–87]), the actions for the BTZ black hole and thermal $\mathbb{H}^3$ were written in terms of the complex Teichmüller parameter $\tau = i\pi/(\Sigma + i\Theta)$ of the boundary 2-torus. Thermal $\mathbb{H}^3$ is simply Euclidean anti-de Sitter space with the angle ϕ and Euclidean time τ_E identified with periods

$$\Phi_E = \frac{2\pi|r_-|l}{r_+^2 + |r_-|^2} , \tag{3.40}$$

$$\beta_E = \frac{2\pi r_+ l^2}{r_+^2 + |r_-|^2} , \tag{3.41}$$

respectively. Formally, it is obtained by putting $M = -1, J_E = 0$ in the BTZ black hole metric (1.44,1.45) (or by the Riemannian continuation $t = i\tau_E$ of (2.14)) and then making the above identifications[6]. By transforming to the upper half-space model via

$$w = \left(\frac{r^2}{r^2 + l^2}\right)^{1/2} \exp\left\{\frac{\tau_E}{l} - i\phi\right\} , \tag{3.42}$$

$$z = \left(\frac{l^2}{r^2 + l^2}\right)^{1/2} \exp\left\{\frac{\tau_E}{l}\right\} , \tag{3.43}$$

we see that the identifications become

$$w \sim e^{2n\pi^2/(\Sigma + i\Theta)} w , \quad z \sim e^{2n\pi^2\Sigma/|\Sigma + i\Theta|^2} z , \tag{3.44}$$

[6]Here, the values of r_+ and $|r_-|$ are the same as those of the BTZ black hole. One does not substitute $M = -1, J_E = 0$ into (1.46) to obtain $r_+^2 = -l^2, |r_-| = 0$.

for $n \in \mathbb{Z}$. These identifications are similar to the BTZ identifications (3.22) except that $k_1 = e^{2(\Sigma+i\Theta)}$ has been replaced by $k_1' = e^{2\pi^2/(\Sigma+i\Theta)}$. In other words, $\Sigma + i\Theta$ has been replaced by $\pi^2/(\Sigma + i\Theta)$ or, equivalently, τ by $-1/\tau$. It was found that the resulting actions also transformed into each other under $S : \tau \mapsto -1/\tau$. This then suggested the existence of a $PSL(2, \mathbb{Z})$ family of solutions whose boundary data τ is related by the associated modular transformations. We see that the above theorem does indeed establish the existence of this class of hyperbolic geometries. Two such geometries whose Teichmüller parameters are related by a modular transformation are then equivalent as hyperbolic structures. Furthermore, it was noted in [67] that there is a correspondence between the $PSL(2, \mathbb{C})$ isometry used to construct the BTZ black hole and the $PSL(2, \mathbb{C})$ element used in the boundary conformal field theory. This correspondence finds a precise explanation in the theorem of Sullivan.

3.5 Conclusions and Further Work

In this chapter we have used machinery from three-dimensional hyperbolic geometry to show how Sullivan's theorem provides a precise notion of holography in the kinematical sense. We have used the theorem to obtain a clearer understanding of the underlying mathematical structure of the Euclidean BTZ black hole.

The observation that the Euclidean BTZ black hole is a geometrically finite Kleinian manifold with genus one boundary suggests that one might try to construct more general objects which have higher genus boundary. There is a well-known technique for doing this. One generalises the group Γ_{BTZ} to a classical Schottky group, Γ. This is a discrete, free, purely loxodromic[7] group on $g > 1$ generators [80]. The quotient $\Omega(\Gamma)/\Gamma$ is then a handlebody of genus g. Since the Kleinian manifold generated by Γ is geometrically finite [80, 88] the theorem of Sullivan again applies. Therefore, $N_C(\Gamma)$ is parametrised by $(3g - 3)$ complex Teichmüller parameters. Whether black holes are amongst these objects or not remains an open question. Work in this

[7] A purely loxodromic group is one which has only loxodromic or hyperbolic elements.

direction has recently appeared in [89].

Finally, one could study the AdS/CFT correspondence within the context of hyperbolic geometry. It is not clear how to quantise classical conformal fields on a non-Hausdorff space. This suggests that when one tries to associate a quantised "boundary theory" to a (semi)-classical bulk theory the n-dimensional Kleinian manifold $(\mathbb{H}^n \cup \Omega(\Gamma))/\Gamma$ is naturally involved[8]. Work along these lines has appeared in [90, 91].

[8]The 3-dimensional formalism presented in this chapter can be generalised relatively straightforwardly to $n > 3$ dimensions. See, for example, Maskit [80] for a general discussion and Bowditch [81] for geometrical finiteness.

Part II

Ramond-Ramond Couplings on Brane-Antibrane Systems

Chapter 4

String Theory and D-Branes

In this chapter we provide some background on strings and D-branes. This chapter is not intended to be comprehensive review; the reader is referred to the standard text of Polchinski [53] and to the older one of Green, Schwarz and Witten [92] for a more thorough treatment. Other useful texts can be found in [93, 94] and there are numerous excellent introductory lecture notes available [95]. Of these references the book by Lüst and Thiesen, Friedan's lectures and those of Ginsparg are particularly useful for understanding the conformal field theory contained in this chapter. Furthermore, this part of the thesis has significant overlap with that of Craps [96] and so much of the material in this chapter (and the next) can also be found in that reference.

4.1 String Theory

4.1.1 The Free Bosonic String

In this section we examine the first-quantised picture of a non-interacting string; string interactions will be considered in section 4.1.6.

The Polyakov action for a bosonic string with Lorentzian-signature worldsheet (Σ, g) moving in a D-dimensional Minkowski space-time $(\mathcal{M}, \eta)$ is

$$S = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma(-g(\sigma))^{1/2} g^{ab}(\sigma)\partial_a X^\mu(\sigma)\partial_b X^\nu(\sigma)\eta_{\mu\nu} \ , \qquad (4.1)$$

In this formula, the worldsheet has coordinates $\sigma^a, a = 1, 2$ and metric $g_{ab}(\sigma)$, whilst the ambient Minkowski space-time (the "target space") has coordinates $X^\mu, \mu = 0, 1, \ldots, D-1$ and metric $\eta_{\mu\nu}$. We use the "mostly-plus" convention for $\eta_{\mu\nu}$. The functions $X^\mu(\sigma)$ then provide the embedding of Σ into $\mathcal{M}$. The constant $l_s = \sqrt{\alpha'}$ is the string length.

The action (4.1) has the following symmetries:

1. D-dimensional Poincaré invariance:

$$\begin{aligned} X'^\mu(\sigma) &= \Lambda^\mu_\nu X^\nu(\sigma) + a^\mu \ , \\ g'_{ab}(\sigma) &= g_{ab}(\sigma) \ , \end{aligned} \tag{4.2}$$

   where $\Lambda^\mu_\nu \in SO(1, D-1)$ and a^μ is a constant D-dimensional vector.

2. Diffeomorphism invariance of the worldsheet:

$$\begin{aligned} X'^\mu(\sigma') &= X^\mu(\sigma) \ , \\ \frac{\partial\sigma'^c}{\partial\sigma^a}\frac{\partial\sigma'^d}{\partial\sigma^b} g'_{cd}(\sigma') &= g_{ab}(\sigma) \ , \end{aligned} \tag{4.3}$$

   for new coordinates $\sigma'^a \equiv \sigma'^a(\sigma)$.

3. Two-dimensional Weyl invariance:

$$\begin{aligned} X'^\mu(\sigma) &= X^\mu(\sigma) \ , \\ g'_{ab}(\sigma) &= e^{2\rho(\sigma)} g_{ab}(\sigma) \ . \end{aligned} \tag{4.4}$$

The equations of motion resulting from (4.1), subject to appropriate boundary conditions to be discussed momentarily, are

$$\begin{aligned} T_{ab} &\equiv -4\pi(-g)^{-1/2}\frac{\delta}{\delta g^{ab}} S \\ &\equiv \frac{1}{\alpha'}(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2} g_{ab}\partial_c X^\mu \partial^c X_\mu) \\ &= 0 \ , \end{aligned} \tag{4.5}$$

$$\Box X^\mu = 0 \ , \tag{4.6}$$

where T_{ab} is the stress-energy tensor of the string and $\Box$ is the D'Alembertian of the metric g_{ab}. Note that T_{ab} is traceless and conserved on shell.

As remarked above, the validity of (4.6) depends on the vanishing of the surface terms upon variation of the action (4.1) with respect to X^μ. This variation gives

$$\delta_X S = -\frac{1}{2\pi\alpha'}\oint_{\partial\Sigma} d\tau(-g_{|\partial\Sigma})^{1/2}\delta X^\mu \partial_n X_\mu \ , \tag{4.7}$$

where τ parametrises the boundary and $\partial_n = n^a\partial_a$ is the normal derivative to the boundary. There are three simple ways of setting this to zero:

a. *Periodic* boundary conditions (see below).

b. *Neumann* boundary conditions: $\partial_n X^\mu|_{\partial\Sigma} = 0$.

c. *Dirichlet* boundary conditions: $\delta X^\mu|_{\partial\Sigma} = 0$.

Mode Expansion and Canonical Quantisation

Now, as is well-known, any two-dimensional surface Σ is conformally flat so that after a coordinate transformation $\sigma^a \to \sigma'^a(\sigma)$ the metric $g_{ab}(\sigma)$ can be written as $\frac{\partial\sigma'^c}{\partial\sigma^a}\frac{\partial\sigma'^d}{\partial\sigma^b}g'_{cd}(\sigma')$, where $g'_{cd}(\sigma') = e^{2\rho(\sigma')}\eta_{cd}$ and where $\eta_{cd} = \text{diag}(-1,1)$. The classical symmetries (4.3) and (4.4) then allow us to replace $g_{ab}(\sigma)$ by η_{ab} and the coordinates σ^a by σ'^a in the equations of motion (4.5,4.6). This is the so-called *conformal gauge.* Quantum mechanically, however, this procedure is valid only locally unless $D = 26$. For now we will ignore such technicalities. Equation (4.6) then becomes (we drop the primes on the coordinates σ'^a)

$$(-\partial_1^2 + \partial_2^2)X^\mu = 0 \ , \tag{4.8}$$

with the solution

$$X^\mu(\sigma) = \tilde{X}^\mu(\sigma^-) + X^\mu(\sigma^+) \ , \tag{4.9}$$

where $\sigma^\pm = \sigma^1 \pm \sigma^2$ are light-cone coordinates. Equations (4.5) in conformal gauge are then imposed as constraints on the space of physical states of the quantum theory (that is, they are zero when acting on physical states but are not identically zero). They take the form

$$T_{11} = T_{22} = \frac{1}{2\alpha'}(\partial_1 X^\mu \partial_1 X_\mu + \partial_2 X^\mu \partial_2 X_\mu) \ = \ 0 \ , \tag{4.10}$$

$$T_{12} = T_{21} = \frac{1}{\alpha'}(\partial_1 X^\mu \partial_2 X_\mu) \ = \ 0 \ . \tag{4.11}$$

These can be expressed alternatively as

$$T_{++} \equiv \frac{1}{2}(T_{11}+T_{12}) \equiv \frac{1}{\alpha'}\partial_+ X^\mu \partial_+ X_\mu = 0 \ , \tag{4.12}$$

$$T_{--} \equiv \frac{1}{2}(T_{11}-T_{12}) \equiv \frac{1}{\alpha'}\partial_- X^\mu \partial_- X_\mu = 0 \ , \tag{4.13}$$

where $\partial_\pm = \frac{1}{2}(\partial_1 \pm \partial_2)$.

We have not, as yet, specified the topology of the worldsheet. We take the timelike coordinate σ^1 to have range $-\infty < \sigma^1 < \infty$. Strings can be closed or open. For closed strings we compactify the spacelike coordinate σ^2 and identify it periodically with period 2π so that topologically the worldsheet is an infinite cylinder; for open strings we take $0 \le \sigma^2 \le \pi$ so that the worldsheet is an infinite strip.

Closed Strings

The solution (4.9) of (4.8) can now be mode-expanded as

$$\tilde{X}^\mu(\sigma^-) = x_R^\mu + \frac{\alpha'}{2}p_R^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-in\sigma^-} \ , \tag{4.14}$$

$$X^\mu(\sigma^+) = x_L^\mu + \frac{\alpha'}{2}p_L^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{1}{n}\alpha_n^\mu e^{-in\sigma^+} \ , \tag{4.15}$$

where

$$x_R^\mu = \frac{1}{2}x^\mu \ , \ p_R^\mu = p^\mu \ , \tag{4.16}$$

$$x_L^\mu = \frac{1}{2}x^\mu \ , \ p_L^\mu = p^\mu \ , \tag{4.17}$$

with x^μ the centre-of-mass position of the string and p^μ its centre-of-mass momentum at $\sigma^1 = 0$ and $\tilde{\alpha}_n^\mu$ and α_n^μ are its right-moving and left-moving non-zero oscillator modes, respectively. It proves convenient to define $\tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}p_R^\mu$, $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p_L^\mu$. These expansions imply that $X^\mu(\sigma)$ satisfies periodic boundary conditions:

$$X^\mu(\sigma^1,0) = X^\mu(\sigma^1,2\pi) \ , \ \partial_2 X^\mu(\sigma^1,0) = \partial_2 X^\mu(\sigma^1,2\pi) \ , \tag{4.18}$$

and hence (4.7) is zero[1]. The reality of X^μ implies

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^* \ , \tag{4.19}$$

[1]It is assumed that $\delta X^\mu = 0$ at $\sigma^1 = \pm\infty$.

and similarly for $\tilde{\alpha}_n^\mu$.

Additionally, we can use invariance of the action under (4.2) to derive, via the Noether theorem, the energy-momentum current associated with translations:

$$P_\mu^a \equiv \frac{\delta S}{\delta \partial_a X^\mu} \equiv -\frac{1}{2\pi\alpha'}(-g)^{1/2} g^{ab} \partial_b X_\mu = -\frac{1}{2\pi\alpha'} \partial^a X_\mu \ , \tag{4.20}$$

(in conformal gauge).

We can now canonically quantise in the standard fashion (Poisson brackets $\rightarrow$ commutators):

$$\begin{aligned} [X^\mu(\sigma^1,\sigma^2), X^\nu(\sigma^1,\sigma'^2)] &= [P^{1\mu}(\sigma^1,\sigma^2), P^{1\nu}(\sigma^1,\sigma'^2)] = 0 \ , \\ [X^\mu(\sigma^1,\sigma^2), P^{1\nu}(\sigma^1,\sigma'^2)] &= i\eta^{\mu\nu}\delta(\sigma^2 - \sigma'^2) \ , \end{aligned} \tag{4.21}$$

and thus we obtain the commutation relations

$$\begin{aligned} [x^\mu, p^\nu] &= i\eta^{\mu\nu} \ , \\ [\alpha_m^\mu, \alpha_n^\nu] &= [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu} \ , \\ [\alpha_m^\mu, \tilde{\alpha}_n^\nu] &= 0 \ . \end{aligned} \tag{4.22}$$

In addition, the conjugation relation (4.19) becomes hermitian conjugation rather than complex conjugation.

Note that due to the fact that they share common zero modes, $\tilde{X}^\mu(\sigma^-)$ and $X^\mu(\sigma^+)$ are not truly independent. For this reason, it is usual to replace them with independent fields of exactly the same form as (4.14) and (4.15) but with the modified commutation relations:

$$\begin{aligned} [x_R^\mu, p_R^\nu] &= [x_L^\mu, p_L^\nu] = i\eta^{\mu\nu} \ , \\ [x_R^\mu, p_L^\nu] &= [x_L^\mu, p_R^\nu] = 0 \ . \end{aligned} \tag{4.23}$$

This modification has the effect of making $\tilde{X}^\mu(\sigma^-)$ and $X^\mu(\sigma^+)$ truly independent while at the same time not affecting the commutation relations (4.21) of the sum $X^\mu(\sigma^1,\sigma^2) = \tilde{X}^\mu(\sigma^-) + X^\mu(\sigma^+)$.

Open Strings

Open strings present more varied possibilities for solutions of the equations of motion than closed ones. In particular, it is possible to impose either Neumann or Dirichlet boundary conditions independently for each μ and for each endpoint $\sigma^2 = 0, \pi$ of the string.

The mode expansion for strings with Neumann boundary conditions at both ends (N-N) in all D directions is

$$X^\mu_{NN}(\sigma) = x^\mu + 2\alpha' p^\mu \sigma^1 + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n}\alpha^\mu_n e^{-in\sigma^-} + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n}\alpha^\mu_n e^{-in\sigma^+} . \quad (4.24)$$

There is only one set of independent oscillators in this case, $\tilde{\alpha}^\mu_n$ getting linked to α^μ_n by the boundary conditions. Canonical quantisation leads to the commutation relations

$$[x^\mu, p^\nu] = i\eta^{\mu\nu} , \quad (4.25)$$

$$[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu} , \quad (4.26)$$

with $\alpha^\mu_{-n} = (\alpha^\mu_n)^\dagger$. It proves convenient to define $\alpha^\mu_0 = \sqrt{2\alpha'}p^\mu$.

We will discuss Dirichlet boundary conditions in more detail in section 4.2. Until then whenever we refer to open string theory in the following we mean the conventional theory with all directions N-N.

Fock Space and Virasoro Algebra

Let us consider the closed string. The relations (4.22) show that we can regard the negative modes $n < 0$ and the positive modes $n > 0$ as the creation and annihilation operators respectively in a Fock space. The Fock vacuum $|0, \tilde{0}; k\rangle$ is annihilated by all positive modes and is an eigenstate of p^μ with eigenvalue k^μ. A generic (non-normalised) state in the Fock space descended from this vacuum is then

$$|N, \tilde{N}; k\rangle = \prod_{\mu=0}^{D-1} \prod_{m=1}^{\infty} (\alpha^\mu_{-m})^{N_{\mu m}} (\tilde{\alpha}^\mu_{-m})^{\tilde{N}_{\mu m}} |0, \tilde{0}; k\rangle , \quad (4.27)$$

and is also an eigenstate of p^μ with eigenvalue k^μ. The full Hilbert space is the collection of all such states for all momenta k^μ.

Classically, the Virasoro generators are just the Fourier modes of the stress-energy tensor at $\sigma^1 = 0$ and so are are defined by

$$\tilde{L}^c_m \equiv \frac{1}{2\pi} \int_0^{2\pi} d\sigma^2 e^{im\sigma^-} T_{--} , \quad (4.28)$$

$$L^c_m \equiv \frac{1}{2\pi} \int_0^{2\pi} d\sigma^2 e^{im\sigma^+} T_{++} . \quad (4.29)$$

Alternatively,

$$T_{--} = \sum_m e^{-im\sigma^-} \tilde{L}^c_m \ , \ T_{++} = \sum_m e^{-im\sigma^+} L^c_m \ . \tag{4.30}$$

It is easy to show that they are given in terms of the oscillators by

$$\tilde{L}^c_m = \frac{1}{2} \sum_n \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n \ , \tag{4.31}$$

$$L^c_m = \frac{1}{2} \sum_n \alpha_{m-n} \cdot \alpha_n \ , \tag{4.32}$$

where the "$\cdot$" denotes contraction of the target space indices with the metric $\eta_{\mu\nu}$. The zero-mode oscillators $\alpha^\mu_0, \tilde{\alpha}^\mu_0$ are included in the summations. Classically, therefore, the constraints (4.12,4.13) become

$$L^c_m = \tilde{L}^c_m = 0 \text{ for all } m \in \mathbb{Z} \ , \tag{4.33}$$

and so impose relations among the oscillator modes.

Quantum mechanically, however, we face an operator ordering problem since the oscillator modes no longer commute. As is usual in field theory, we pick a normal ordering prescription for the operators. For the left-movers, it is given by

$$: \alpha^\mu_m \alpha^\nu_n := \begin{cases} \alpha^\mu_m \alpha^\nu_n & n \geq 0 \ , \\ \alpha^\nu_n \alpha^\mu_m & n < 0 \ , \end{cases} \tag{4.34}$$

and puts all positive frequency modes to the right of the negative frequency modes[2]. It is also conventional to include p^μ amongst the positive modes and x^μ amongst the negative modes. A similar prescription applies to the right-moving modes. For $m \neq 0$, the quantum Virasoro operators on the cylinder are then defined as the normal-ordered classical ones:

$$L_m \equiv : L^c_m : \ , \ \tilde{L}_m \equiv : \tilde{L}^c_m : \ . \tag{4.35}$$

In this case, it turns out that the normal ordering prescription has no effect so that $L_m = L^c_m$, etc. However, for the zero modes we find

$$L^c_0 \ = \ L_0 + b \ , \tag{4.36}$$

[2]In addition, for two anticommuting modes an extra minus sign appears when the modes are interchanged and for one commuting and one anticommuting mode no sign appears under interchange. These facts will be needed when we consider the superstring in section 4.1.3

where

$$L_0 \equiv \; :L_0^c: = \frac{1}{2}\alpha_0 \cdot \alpha_0 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \; , \tag{4.37}$$

$$b = \frac{D}{2}\sum_{n=1}^{\infty} n \; . \tag{4.38}$$

Note that the constant b is a formally infinite c-number. Using zeta-function regularisation it can be evaluated as $b_{reg} = -D/24$. Therefore, we define the quantum Virasoro operator on the cylinder as $L'_0 = L_0 + b_{reg}$. We shall see in the next section that L_0 is actually the zero-mode Virasoro operator *on the plane*[3]. We also find

$$L_{-m} = L_m^\dagger \; . \tag{4.39}$$

Similar considerations apply to the right-movers with the same constant b_{reg}. (For the open string the right-movers are, of course, absent and the Virasoro operators L_m are exactly as those above for the closed string, save for the modified definition of α_0^μ.)

The Virasoro algebra can now be calculated. It is conventional to formulate the quantum theory in terms of L_0 rather than L'_0. With great care (because of the normal ordering), one finds

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \; , \tag{4.40}$$

where $c = D$ is the central charge[4]. The right-movers satisfy the same algebra (with the same central charge[5]) and, of course, the left and right Virasoro operators commute.

As remarked earlier, the classical constraints (4.33) do not hold as operator identities at the quantum level but are satisfied when acting on *physical* states. Therefore,

[3]As we shall see in the next section, the non-zero mode Virasoro operators on the cylinder and on the plane turn out to be equal. Therefore, we do not distinguish between primed and unprimed operators in this case.

[4]If we had used L'_0 rather than L_0 the term linear in m in the central charge term would be absent.

[5]It can be shown that diffeomorphism invariance of the worldsheet implies that the central charge of the right-movers is the same as that of the left-movers. One could take the central charges to be different but this leads to an inconsistent theory containing a *diff* or *gravitational* anomaly.

we might naively assume that a state $|\text{phys}\rangle$ of the Hilbert space is physical if it satisfies

$$L_m|\text{phys}\rangle = 0 \quad (m \neq 0) , \tag{4.41}$$

$$(L_0 - a)|\text{phys}\rangle = 0 , \tag{4.42}$$

and similarly for the right-movers. The constant a is included because of the normal ordering ambiguity and is determined by requiring consistency of the quantum theory[6]. Unfortunately, this naive assumption fails. Equation (4.41) can be imposed for a physical state only for $m > 0$; it cannot be imposed for $m < 0$ as well. There is a standard argument for illustrating this. Suppose we could impose (4.41) for all $m \neq 0$. Then we would conclude $\langle\text{phys}|[L_m, L_{-m}]|\text{phys}\rangle = 0$ for $m > 0$. However, the Virasoro algebra (4.40), which is an operator identity, yields

$$\begin{aligned}\langle\text{phys}|[L_m, L_{-m}]|\text{phys}\rangle &= 2m\langle\text{phys}|L_0|\text{phys}\rangle + \tfrac{c}{12}m(m^2-1)\langle\text{phys}|\text{phys}\rangle \\ &= 2ma + \tfrac{c}{12}m(m^2-1) \\ &\neq 0 \quad (\text{in general}) ,\end{aligned} \tag{4.43}$$

where we have assumed that $|\text{phys}\rangle$ is normalised to unity. Hence, there is a contradiction. Therefore, at the quantum level, physical states satisfy

$$L_{m>0}|\text{phys}\rangle = 0 \ , \ (L_0 - a)|\text{phys}\rangle = 0 , \tag{4.44}$$

and similarly for the right-movers. Equations (4.44) are consistent with the classical constraints in the sense that $\langle\text{phys}'|L_m|\text{phys}\rangle = 0$ for all $m \neq 0$ by virtue of (4.39). Note that this situation is very similar to the one that occurs in the Gupta-Bleuler quantisation of electromagnetism. There, only the positive frequency part of the Lorentz gauge condition $\partial \cdot A = 0$ is imposed on physical states which suffices to get $\langle\text{phys}'|\partial \cdot A|\text{phys}\rangle = 0$.

Of course, there are other states in the Hilbert space which do not satisfy the constraints (4.44). There are even negative-norm states (or "ghosts") and null physical states due to the indefinite metric $\eta_{\mu\nu}$ appearing in the commutation relations

[6]Note that the normal ordering constant is the same in both the left and right sector and so the *level-matching* condition $(L_0 - \tilde{L}_0)|\text{phys}\rangle$ is automatic. However, as we shall see in section 4.1.3, it is possible for the normal ordering constant in the left and right sector to be different.

(4.22). A detailed analysis shows that the space of physical states modulo the space of null physical states is ghost-free and the degrees of freedom transverse in nature provided $a = 1$ and $D = 26$.

Finally, we note that the classical Hamiltonian for the closed string in conformal gauge is given by

$$\begin{aligned} H^{c} &= \int_0^{2\pi} d\sigma^2 (\partial_1 X \cdot P^1 - \mathcal{L}) \\ &= \tfrac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma^2 (\partial_+ X \cdot \partial_+ X + \partial_- X \cdot \partial_- X) \\ &= \tfrac{1}{2\pi} \int_0^{2\pi} d\sigma^2 (T_{++} + T_{--}) \\ &= L_0^c + \tilde{L}_0^c \ . \end{aligned} \tag{4.45}$$

Therefore, the quantum Hamiltonian on the cylinder is given by

$$H' = L_0' + \tilde{L}_0' = L_0 + \tilde{L}_0 - \frac{D}{12} \ . \tag{4.46}$$

Physical Spectrum

The relevant (normal-ordered) number operator for the left-movers is defined as

$$N_+^{(\alpha)} = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \ , \tag{4.47}$$

and acting on the Fock state (4.27) it gives

$$N_+^{(\alpha)} |N, \tilde{N}; k\rangle = N^{(\alpha)} |N, \tilde{N}; k\rangle \ , \quad N^{(\alpha)} = \sum_{\mu=0}^{D-1} \sum_{m=1}^{\infty} m N_{\mu m} \ . \tag{4.48}$$

The right-moving number operator $N_-^{(\alpha)}$ is similarly defined and the eigenvalue of the Fock space state (4.27) is $\tilde{N}^{(\alpha)} = \sum_{\mu=0}^{D-1} \sum_{m=1}^{\infty} m \tilde{N}_{\mu m}$.

The mass operator is given by

$$\begin{aligned} M^2 &= -\tfrac{1}{2}(p_L \cdot p_L + p_R \cdot p_R) \\ &= -\tfrac{1}{\alpha'}(\alpha_0 \cdot \alpha_0 + \tilde{\alpha}_0 \cdot \tilde{\alpha}_0) \ . \end{aligned} \tag{4.49}$$

Using (4.27), (4.37) and (4.44) we find that physical states have $N^{(\alpha)} = \tilde{N}^{(\alpha)}$ and are eigenstates of M^2 with eigenvalues

$$m^2 = -k^2 = \frac{4}{\alpha'}(N^{(\alpha)} - a) \ . \tag{4.50}$$

We see from this that the Fock vacuum $|0, \tilde{0}; k\rangle$ with $k^2 = 4a/\alpha'$ is physical and tachyonic (since $a = 1$). The first level, $N^{(\alpha)} = \tilde{N}^{(\alpha)} = 1$, has zero mass and consists of linear combinations of states of the form $|\alpha\rangle^{\mu\nu} = \tilde{\alpha}^{\mu}_{-1}\alpha^{\nu}_{-1}|0, \tilde{0}; k\rangle$ where k^μ is null. All other states are massive and not observable at low energies since $1/\alpha' \sim M_P^2$, where M_P is the Planck mass.

Let us now classify the different types of massless particle. To do so we multiply the state $|\alpha\rangle^{\mu\nu}$ by a polarisation tensor $\zeta_{\mu\nu}$ to form the state $|\alpha; \zeta\rangle$. Thus, there are D^2 independent polarisations initially. We can then work through the physical state conditions to find that they reduce to $k^\mu\zeta_{\mu\nu} = \zeta_{\mu\nu}k^\nu = 0$, with k^μ null. These conditions eliminate $2 \times D$ degrees of freedom. However, the states $a_\mu k_\nu|\alpha\rangle^{\mu\nu}$ and $k_\mu b_\nu|\alpha\rangle^{\mu\nu}$ are physical but null. Therefore, $\zeta_{\mu\nu} \cong \zeta_{\mu\nu} + a_\mu k_\nu + k_\mu b_\nu$, with $k \cdot a = b \cdot k = 0$. These latter conditions eliminate a further $2 \times (D-2)$ degrees of freedom. Therefore, in total there are $(D-2)^2$ independent polarisations, which is just the right amount of degrees of freedom to characterise a tensor of $SO(D-2)$. Since k^μ is null, we can take the momentum to lie along the light-cone parametrised by the $0, 1$ directions in the target space. The group $SO(D-2)$ is then the "little group" of $SO(1, D-1)$, where the $D-2$ refers to the directions transverse to the light-cone. Thus, we find that the massless particles in D dimensions are classified by their $SO(D-2)$ representation: a) the symmetric, traceless part (graviton, $G_{\mu\nu}$); b) the antisymmetric part (Kalb-Ramond field, $B_{\mu\nu}$) and c) the trace part (dilaton, Φ).

For open strings the right-movers are absent. The Hamiltonian on the strip is $L_0 - \frac{D}{24}$ and the mass operator is given by

$$M^2 = -p \cdot p = -\frac{1}{2\alpha'}\alpha_0 \cdot \alpha_0 \ , \tag{4.51}$$

and, therefore, the masses of the physical states are

$$m^2 = -k^2 = \frac{1}{\alpha'}(N^{(\alpha)} - a) \ . \tag{4.52}$$

Again, we see that the Fock vacuum $|0; k\rangle$ with $k^2 = a/\alpha'$, $a = 1$ is physical and a tachyon.

4.1.2 Conformal Field Theory

In what follows we will work on the (extended) complex plane so it is useful to recall some conformal field theory techniques. Furthermore, conformal field theory is relevant to string theory because the choice of conformal gauge does not fix the gauge completely; the string action in conformal gauge is still invariant under conformal transformations of $\mathbb{R}^{1,1}$ (or $\mathbb{R}^2$ after a Wick rotation, as described below). Conformal transformations are those diffeomorphisms that leave η_{ab} invariant up to a Weyl rescaling. Such transformations are of the form

$$\sigma^+ \to \sigma'^+ = f(\sigma^+) \ , \ \sigma^- \to \sigma'^- = g(\sigma^-) \ , \tag{4.53}$$

for arbitrary real functions f and g and are generated by the Virasoro generators, as we show below.

Wick Rotation of Worldsheet

To proceed further, we Wick rotate the string worldsheet from Lorentzian to Euclidean signature via $\sigma^1 \to \tilde{\sigma}^1 = i\sigma^1$. We then define complex coordinates on the string worldsheet by

$$i\sigma^+ \to w = \sigma^1 + i\sigma^2 \ , \ i\sigma^- \to \bar{w} = \sigma^1 - i\sigma^2 \ , \tag{4.54}$$

where we have dropped the tilde over σ^1. The transformations f and g of (4.53) become holomorphic and antiholomorphic transformations of w, respectively. We can now perform the conformal transformation

$$z = e^w = x + iy \ , \ \bar{z} = e^{\bar{w}} = x - iy \ , \tag{4.55}$$

in which the closed string's cylinder maps to the full complex plane, while the open string's strip maps to the upper half complex plane. In what follows we denote $\partial \equiv \partial_z$ and $\bar{\partial} \equiv \partial_{\bar{z}}$. Note that

$$\partial_+ \to i\partial_w = iz\partial \ , \ \partial_- \to i\partial_{\bar{w}} = i\bar{z}\bar{\partial} \ . \tag{4.56}$$

The Wick rotated form of the action (4.1) in conformal gauge is

$$S_X = \frac{1}{2\pi\alpha'} \int d^2z \, \partial X \cdot \bar{\partial} X \ , \tag{4.57}$$

where $d^2z = 2dxdy$. The closed and open string mode expansions (4.14), (4.15) and (4.24) respectively become:

$$X^\mu(z) = x_L^\mu - i\sqrt{\frac{\alpha'}{2}}\alpha_0^\mu \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{1}{n}\alpha_n^\mu z^{-n} , \tag{4.58}$$

$$\tilde{X}^\mu(\bar{z}) = x_R^\mu - i\sqrt{\frac{\alpha'}{2}}\tilde{\alpha}_0^\mu \ln \bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{1}{n}\tilde{\alpha}_n^\mu \bar{z}^{-n} , \tag{4.59}$$

$$X_{NN}^\mu(z,\bar{z}) = x^\mu - i\sqrt{\frac{\alpha'}{2}}\alpha_0^\mu \ln(z\bar{z}) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{1}{n}\alpha_n^\mu (z^{-n}+\bar{z}^{-n}) , \tag{4.60}$$

These, of course, solve the equation of motion $\partial\bar{\partial}X^\mu = 0$ and boundary conditions resulting from (4.57).

Correlation Functions and the Propagator

Up until now we have quantised the string using canonical quantisation. However, the path integral provides another, equivalent means of quantisation. Within the path integral formalism a natural way to define correlation functions on a worldsheet of given topology is via

$$\langle \ldots \rangle = \int DX \ldots e^{-S_X} . \tag{4.61}$$

While this is essentially correct, there are various subtleties such as the possible existence of a conformal anomaly and the question of how one treats different topologies (which correspond to different numbers of loops in field theory). Such technicalities are discussed when we consider the path integral in more detail in section 4.1.6.

For now, we proceed in a cavalier manner using the definition (4.61) and complete the square in the generating functional

$$\begin{aligned} Z[J] &= \int DX \exp\left(-\frac{1}{2\pi\alpha'}\int d^2z\, \partial X\cdot\bar{\partial}X + \int d^2z J\cdot X\right) \\ &= e^{\frac{1}{2}\int J\cdot\Delta\cdot J}\int DX \exp\left(-\frac{1}{2}\int (X-\Delta\cdot J)^\mu \Delta^{-1}_{\mu\nu}(X-\Delta\cdot J)^\nu\right) , \end{aligned} \tag{4.62}$$

where $\Delta^{-1}_{\mu\nu} = -\eta_{\mu\nu}\partial\bar{\partial}/\pi\alpha'$. Thus, we provide a heuristic calculation of the propagator:

$$\langle X^\mu(z',\bar{z}')X^\nu(z,\bar{z})\rangle = \frac{1}{Z[0]}\frac{\delta}{J_\mu(z',\bar{z}')}\frac{\delta}{J_\nu(z,\bar{z})}Z[J]\bigg|_{J^\mu=0}$$

$$
\begin{aligned}
&= \Delta^{\mu\nu}(z', \bar{z}'; z, \bar{z}) \\
&= -\frac{1}{2}\alpha' \eta^{\mu\nu} \ln |z' - z|^2 - \frac{1}{2}\alpha' D^{\mu\nu} \ln |z' - \bar{z}|^2, \qquad (4.63)
\end{aligned}
$$

where we have used the representation of the delta-function:

$$
\partial\bar{\partial} \ln |z|^2 = \partial \frac{1}{\bar{z}} = \bar{\partial} \frac{1}{z} = 2\pi\delta^2(z, \bar{z}) = \pi\delta(\sigma^1)\delta(\sigma^2) \ . \qquad (4.64)
$$

The harmonic part of the propagator involving $D^{\mu\nu} = D^{\nu\mu}$ is included so that $\Delta^{\mu\nu}$ satisfies the same boundary conditions as the fields X^μ.

For closed strings on the complex plane we can take $D^{\mu\nu} = 0$. Treating holomorphic and antiholomorphic components independently allows us to split (4.63) into two:

$$
\begin{aligned}
\langle X^\mu(z') X^\nu(z) \rangle &= -\frac{1}{2}\alpha' \eta^{\mu\nu} \ln(z' - z) \ , \qquad (4.65) \\
\langle \tilde{X}^\mu(\bar{z}') \tilde{X}^\nu(\bar{z}) \rangle &= -\frac{1}{2}\alpha' \eta^{\mu\nu} \ln(\bar{z}' - \bar{z}) \ , \qquad (4.66)
\end{aligned}
$$

For open strings on the upper half-plane, N-N directions satisfy $(\partial - \bar{\partial}) X^\mu_{NN} = 0$ on the real axis, as can be read off from (4.60). Hence we require

$$
(\partial - \bar{\partial})\Delta^{\mu\nu} = \frac{1}{2}\alpha' \left[\frac{\eta^{\mu\nu}}{z' - z} - \frac{D^{\mu\nu}}{z' - \bar{z}} - \frac{\eta^{\mu\nu}}{\bar{z}' - \bar{z}} + \frac{D^{\mu\nu}}{\bar{z}' - z} \right] = 0 \ , \qquad (4.67)
$$

on the real axis, $z = \bar{z}$. Hence, $D^{\mu\nu} = \eta^{\mu\nu}$.

Doubling Trick

At this point we introduce the *doubling trick* for open strings. We may write

$$
X^\mu_{NN}(z, \bar{z}) = X^\mu(z) + \tilde{X}^\mu(\bar{z}) \ , \qquad (4.68)
$$

where the x^μ mode is split evenly between X^μ and $\tilde{X}^\mu$. In particular,

$$
\begin{aligned}
X^\mu(z) &= \frac{1}{2}x^\mu - i\sqrt{\frac{\alpha'}{2}}\alpha_0^\mu \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}\backslash\{0\}} \frac{1}{n}\alpha_n^\mu z^{-n} \ , \qquad (4.69) \\
\tilde{X}^\mu(\bar{z}) &= \frac{1}{2}x^\mu - i\sqrt{\frac{\alpha'}{2}}\alpha_0^\mu \ln \bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}\backslash\{0\}} \frac{1}{n}\alpha_n^\mu \bar{z}^{-n} \ , \qquad (4.70)
\end{aligned}
$$

Due to the fact that $D^{\mu\nu}$ is non-zero we find cross-correlations between X^μ and $\tilde{X}^\mu$. Indeed, using (4.63) we find

$$\begin{aligned}
\langle X^\mu(z')X^\nu(z)\rangle &= -\frac{1}{2}\alpha'\eta^{\mu\nu}\ln(z'-z)\ , &(4.71)\\
\langle \tilde{X}^\mu(\bar{z}')\tilde{X}^\nu(\bar{z})\rangle &= -\frac{1}{2}\alpha'\eta^{\mu\nu}\ln(\bar{z}'-\bar{z})\ , &(4.72)\\
\langle X^\mu(z')\tilde{X}^\nu(\bar{z})\rangle &= -\frac{1}{2}\alpha' D^{\mu\nu}\ln(z'-\bar{z})\ , &(4.73)\\
\langle \tilde{X}^\mu(\bar{z}')X^\nu(z)\rangle &= -\frac{1}{2}\alpha' D^{\mu\nu}\ln(\bar{z}'-z)\ . &(4.74)
\end{aligned}$$

However, note that $X^\mu(\cdot)$ and $\tilde{X}^\mu(\cdot)$ are defined for their argument on the upper and lower half of the complex plane, respectively. So we extend $X^\mu(z)$ to the *whole* plane by defining

$$X^\mu(z) \ := \ D^\mu_\nu \tilde{X}^\nu(z) = \tilde{X}^\mu(z)\ , \qquad (4.75)$$

for $\mathrm{Im}\, z \leq 0$. We can then replace the four correlators (4.71-4.74) by the single correlator (4.71) defined in terms of the extended field provided the condition

$$D^\mu_\gamma D^\gamma_\nu = \eta^\mu_\nu \equiv \delta^\mu_\nu\ , \qquad (4.76)$$

holds. This condition is trivially met since $D^{\mu\nu} = \eta^{\mu\nu}$. We will use this doubling trick at various stages in what follows. (In particular, our motivation for working in terms of the matrix $D^{\mu\nu}$ rather than $\eta^{\mu\nu}$ will become clearer when we consider D-branes in section 4.2.)

Radial Ordering

Surfaces of equal time on the cylinder (strip) become circles (semi-circles) of equal radius on the complex plane. This means that the infinite past ($\sigma^1 = -\infty$) gets mapped to the origin of the plane ($z = 0$) and the infinite future becomes $z = \infty$. This leads naturally to the concept of radial ordering, in analogy with time ordering of ordinary quantum field theory:

$$R(A(z')B(z)) = \begin{cases} A(z')B(z) & |z'| > |z| \\ (-1)^F B(z)A(z') & |z| > |z'|\ , \end{cases} \qquad (4.77)$$

where F is $+1$ for two bosonic fields and -1 for two fermionic fields. The equal radius (anti-)commutator is then defined by

$$[A(z), B(z)] = \lim_{\delta \to 0} \left\{ (A(z')B(z))_{|z'|=|z|+\delta} - (-1)^F (B(z)A(z''))_{|z''|=|z|-\delta} \right\} . \quad (4.78)$$

Primary Fields

The basic objects of conformal field theory are the *primary fields* $\phi(z, \bar{z})$. These transform under a conformal transformation $z \to z' = f(z)$, $\bar{z} \to \bar{z}' = \bar{f}(\bar{z})$ as tensors:

$$\phi(z, \bar{z}) \to \phi'(z, \bar{z}) = \left(\frac{\partial z'}{\partial z} \right)^h \left(\frac{\partial \bar{z}'}{\partial \bar{z}} \right)^{\bar{h}} \phi(z'(z), \bar{z}'(\bar{z})) , \quad (4.79)$$

where the pair $(h, \bar{h})$ is the conformal weight of the field. The quantity $h + \bar{h}$ is called the *scaling dimension* of the field. Infinitesimally, we have $z' = z + \xi(z)$, $\bar{z}' = \bar{z} + \bar{\xi}(\bar{z})$ and (4.79) becomes

$$\begin{aligned} \delta_{\xi,\bar{\xi}} \phi(z, \bar{z}) &= \phi'(z, \bar{z}) - \phi(z, \bar{z}) \\ &= \left[(h \partial \xi + \xi \partial) + (\bar{h} \bar{\partial} \bar{\xi} + \bar{\xi} \bar{\partial}) \right] \phi(z, \bar{z}). \end{aligned} \quad (4.80)$$

Equation (4.79) implies that under the transformation (4.55) primary fields on the plane and cylinder (strip) are related according to:

$$\phi(z, \bar{z}) = z^{-h} \bar{z}^{-\bar{h}} \phi'(w(z), \bar{w}(\bar{z})) , \quad (4.81)$$

where $\phi'(w, \bar{w})$ is defined on the cylinder (strip) and $w = \ln z$. Consequently, the field $\phi(z, \bar{z})$ can be mode expanded as the Laurent series

$$\phi(z, \bar{z}) = \sum_{m,n} z^{-m-h} \bar{z}^{-n-\bar{h}} \phi_{m,n} , \quad (4.82)$$

where $\sum_{m,n} e^{-mw} e^{-n\bar{w}} \phi_{m,n}$ is the mode expansion on the cylinder (strip).

Finally, by considering how correlators of primary fields transform under the resricted conformal group **Möb**, consisting of the transformations of the form (3.16), it can be shown that two-point correlators for fields of the same conformal weight $(h, \bar{h})$ have the form

$$\langle \phi_1(z', \bar{z}') \phi_2(z, \bar{z}) \rangle = \frac{C_{12}}{(z' - z)^{2h} (\bar{z}' - \bar{z})^{2\bar{h}}} , \quad (4.83)$$

as $z' \to z$, where C_{12} is some constant. Furthermore, three-point correlators for holomorphic fields have the form

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_1+h_3-h_2}} \ , \tag{4.84}$$

where $z_{ij} = z_i - z_j$.

The Operator Product Expansion and Energy-Momentum Tensor

An alternative, but equivalent, definition of a primary field is in terms of the (anti-)commutator and operator product expansion (OPE). The basic idea of an operator product expansion is that if $\{\mathcal{O}_i\}$ is a complete set of local operators with definite scaling dimensions, then the product of two operators can be expanded as $z' \to z$ as[7]

$$\mathcal{O}_i(z')\mathcal{O}_j(z) = \sum_k C_{ijk}(z'-z)^{h_k-h_i-h_j}\mathcal{O}_k(z) \ , \tag{4.85}$$

where C_{ijk} are constants and h_i are the scaling dimensions of the fields which are not necessarily primary. This is an operator statement and, as such, should always be thought of as inserted into correlation functions. Note that (4.85) generally contains singular pole terms. For example, taking the derivative of the propagator (4.63) or (4.65) for the closed string we obtain the OPE

$$\partial' X^\mu(z')\partial X^\nu(z) = -\frac{1}{2}\alpha' \frac{\eta^{\mu\nu}}{(z'-z)^2} \ . \tag{4.86}$$

Comparing with (4.83) suggests that $\partial X^\mu(z)$ is a primary field of weight $(1,0)$. We will verify this below. It is conventional in writing OPEs to keep only the most singular terms and to disregard terms that are finite as $z' \to z$. The "=" sign appearing in the OPEs below should therefore be interpreted in this light.

Now, conformal invariance manifests itself in the tracelessness $T^a_a = 0$ of the energy momentum tensor. In addition, diffeomorphism invariance implies conservation, $\partial^a T_{ab}$. In complex coordinates these conditions translate into $T(z) \equiv T_{zz}(z,\bar z)$ being a holomorphic function of z and $\tilde T(\bar z) \equiv T_{\bar z\bar z}(z,\bar z)$ being an antiholomorphic function,

[7]For illustrative purposes we consider holomorphic fields only.

with $T_{z\bar{z}} = 0$. We define a conserved Noether charge

$$Q_{\xi,\bar{\xi}} = \frac{1}{2\pi i} \oint_{C_0} \left(dz \xi(z) T(z) - d\bar{z} \bar{\xi}(\bar{z}) \tilde{T}(\bar{z}) \right) , \tag{4.87}$$

where the contour integral is performed in the counterclockwise sense around some circle of fixed radius enclosing the origin. Clearly, $T(z)$ and $\tilde{T}(\bar{z})$ should be defined on the whole of the complex plane for the contour integral to make sense. We can now define the transformation of a primary field by

$$\begin{aligned} \delta_{\xi,\bar{\xi}} \phi(z,\bar{z}) &= [Q_{\xi,\bar{\xi}}, \phi(z,\bar{z})] \\ &= \frac{1}{2\pi i} \oint_{C_z} \left(dz' \xi(z') R(T(z')\phi(z,\bar{z})) - d\bar{z}' \bar{\xi}(\bar{z}') R(\tilde{T}(\bar{z}')\phi(z,\bar{z})) \right) . \end{aligned} \tag{4.88}$$

where the contour integral is taken anticlockwise about z. In writing the integral in the above form we have performed a deformation of the original contours involved in the definition of the commutator (4.78). The first contour encircles the origin and has $|z'| > |z|$. The second encircles the origin and has $|z'| < |z|$. Subtracting the second from the first gives the contour C_z. Using the Cauchy-Riemann formula

$$\oint_{C_z} \frac{dz'}{2\pi i} \frac{f(z')}{(z'-z)^{n+1}} = \frac{1}{n!} \partial^n f(z) , \tag{4.89}$$

and its complex conjugate, we see that (4.88) agrees with (4.80) provided T and $\tilde{T}$ have the following short-distance singularities with ϕ:

$$R(T(z')\phi(z,\bar{z})) = \frac{h}{(z'-z)^2} \phi(z,\bar{z}) + \frac{1}{z'-z} \partial \phi(z,\bar{z}) + \cdots , \tag{4.90}$$

$$R(\tilde{T}(\bar{z}')\phi(z,\bar{z})) = \frac{\bar{h}}{(\bar{z}'-\bar{z})^2} \phi(z,\bar{z}) + \frac{1}{\bar{z}'-\bar{z}} \bar{\partial} \phi(z,\bar{z}) + \cdots . \tag{4.91}$$

These operator product expansions serve as the alternative definition of a primary field of conformal weight $(h, \bar{h})$. As is conventional, we omit the R symbol and consider the OPE to be already radially ordered.

By considering the commutation properties of infinitesimal conformal transformations it is straightforward to show that, classically, $T(z)$ is a primary field of weight $(2, 0)$ while $\tilde{T}(\bar{z})$ is a primary field of weight $(0, 2)$. That is, we have

$$T(z')T(z) = \frac{2}{(z'-z)^2} T(z) + \frac{1}{z'-z} \partial T(z) , \tag{4.92}$$

and similarly for $\tilde{T}(\bar{z})$. Quantum mechanically, however, it is possible to add the further term

$$\frac{c/2}{(z'-z)^4} \ , \tag{4.93}$$

to right side of the OPE (4.92), where c is the c-number central charge. This is the so-called *conformal anomaly*. Its form, although not the value of c itself, is determined by analyticity, Bose symmetry and scale invariance (see Ginsparg's lectures). Indeed, scale invariance and the fact that T has scaling dimension 2 show that we cannot have a higher-order pole $\sim A(z)/(z'-z)^n$, $n > 4$ because the operator A would have to have negative scaling dimension. Such negative dimension operators are not allowed in a unitary conformal field theory. Now consider a cubic term $A(z)/(z'-z)^3$. The operator A would now have a scaling dimension of one and conformal dimension $(1,0)$. However, Bose symmetry requires the OPE to be symmetric under the interchange $z' \leftrightarrow z$, so the only possible choice is $A = 0$.

Using the above OPE we can derive how the energy-momentum tensor[8] transforms under infinitesimal conformal transformations:

$$\begin{aligned} \delta_{\xi,\bar{\xi}} T(z) &= [Q_{\xi,\bar{\xi}}, T(z)] \\ &= \tfrac{1}{2\pi i} \oint_{C_z} \xi(z') T(z') T(z) \\ &= (2\partial\xi + \xi\partial) T(z) + \tfrac{c}{12}\partial^3\xi(z) \ . \end{aligned} \tag{4.94}$$

This shows that, quantum mechanically, the stress tensor is *quasi-primary*, that is, it transforms as in (4.80) under those infinitesimal conformal transformations that are quadratic in nature, $\partial^3\xi(z) = 0$. Such transformations are just the restricted conformal transformations **Möb** generated by $L_0, L_{\pm 1}$. Equation (4.94) can be "exponentiated" to give the finite transformation

$$T'(z) = \left(\frac{\partial z'}{\partial z}\right)^2 T(z'(z)) + \frac{c}{12}\{z'(z), z\} \ , \tag{4.95}$$

where

$$\{z'(z), z\} = \frac{\partial z' \partial^3 z' - \frac{3}{2}(\partial^2 z')^2}{(\partial z')^2} \ , \tag{4.96}$$

[8]Henceforth, we consider only the holomorphic component of the energy-momentum tensor. A discussion similar to what follows applies to the antiholomorphic component.

is the *Schwarzian derivative.* Using (4.55) we find the relation between the energy-momentum tensors defined on the plane and on the cylinder (strip):

$$T(z) = z^{-2}(T'(w(z)) + \frac{c}{24}) \ , \tag{4.97}$$

where $T'(w)$ is defined on the cylinder (strip) and $w = \ln z$.

The Virasoro Algebra

The OPE (4.92,4.93) is equivalent to the Virasoro algebra (4.40). This can be seen from the mode expansion

$$T(z) = \sum_{m\in\mathbb{Z}} \frac{L_m}{z^{m+2}} \ , \ L_m = \oint_{C_0} \frac{dz}{2\pi i} z^{m+1} T(z) \ . \tag{4.98}$$

Then we have

$$\begin{aligned} [L_m, L_n] &= \oint_{C_0} \frac{dz}{2\pi i} \oint_{C_z} \frac{dz'}{2\pi i} z'^{m+1} z^{n+1} T(z') T(z) \\ &= (m-n) L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \ . \end{aligned} \tag{4.99}$$

Equation (4.97) also allows us to determine the relation

$$L_m = L'_m + \frac{c}{24}\delta_{m,0} \ , \tag{4.100}$$

between the Virasoro operators L_m defined on the plane and the corresponding operators L'_m defined on the cylinder (strip).

Conformal Normal Ordering

Now, as far as the bosonic string action (4.57) is concerned, the energy-momentum tensor on the plane is (classically) given by

$$T(z) = -\frac{1}{\alpha'}\partial X \cdot \partial X \ , \ \tilde{T}(\bar{z}) = -\frac{1}{\alpha'}\bar{\partial} X \cdot \bar{\partial} X \ . \tag{4.101}$$

This can be inferred from the Wick rotation (4.56) and the transformation (4.97) (ignoring the central charge term) applied to (4.12) and (4.13).

Quantum mechanically, however, the product of two operators at the same point is ill-defined so we adopt the conformal normal ordering prescription

$$\circ\!\!\!\circ\, A(z)B(z) \,\circ\!\!\!\circ = \lim_{\delta\to 0} \left((A(z')B(z))_{|z'|=|z|+\delta} - \text{poles} \right) \ , \tag{4.102}$$

where the pole terms to be subtracted are those arising from the operator product expansion of $A(z')B(z)$. In general, this form of normal ordering is *not* the same as the creation-annihilation normal ordering we introduced in section 4.1.1. However, for the bosonic string it leads to the same results, as we explicitly verify below. The quantum energy-momentum tensor is then defined as the conformally normal ordered form of (4.101).

Example: The Closed Bosonic String

For the closed string, it is straightforward to calculate

$$\begin{aligned} -\frac{1}{\alpha'}\partial' X(z',\bar{z}') \cdot \partial X(z,\bar{z}) &= \frac{1}{2}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} z'^{-m-1} z^{-n-1} \alpha_m^\mu \alpha_n^\nu \eta_{\mu\nu} \\ &= -\frac{1}{\alpha'} : \partial' X(z') \cdot \partial X(z) : + \frac{D}{2}(z'z)^{-1}\sum_{n=1}^{\infty} n\left(\frac{z}{z'}\right)^n , \end{aligned} \tag{4.103}$$

where

$$\begin{aligned} -\tfrac{1}{\alpha'} : \partial' X(z') \cdot \partial X(z) : = \ & \tfrac{1}{2}\textstyle\sum_{m\in\mathbb{Z}} \Big(\sum_{n\geq 0} z'^{-m+n-1} z^{-n-1} \alpha_{m-n}\cdot\alpha_n \\ & + \textstyle\sum_{n<0} z'^{-m+n-1} z^{-n-1} \alpha_n \cdot \alpha_{m-n}\Big) \ . \end{aligned}$$

For $|z'| > |z|$, the second term in (4.103) converges to

$$\frac{D}{2(z'-z)^2} , \tag{4.104}$$

which is a pole as $z' \to z$. It is consistent with the OPE (4.86) in this limit. Furthermore, it is clear that the first term in (4.103) contains no pole as $z' \to z$ and hence

$$\lim_{z'\to z} -\frac{1}{\alpha'} : \partial' X(z') \cdot \partial X(z) : \ = \ -\frac{1}{\alpha'} \circ\!\!\!\circ\, \partial X \cdot \partial X \,\circ\!\!\!\circ \equiv T(z) \tag{4.105}$$

$$= \sum_{m\in\mathbb{Z}} \frac{L_m}{z^{m+2}} , \tag{4.106}$$

where the operators L_m are given by (4.35) for $m \neq 0$ and (4.37) for $m = 0$.

As to be expected, if one calculates

$$T(z')T(z) = \frac{1}{\alpha'^2} \circ\!\!\!\circ\, \partial' X \cdot \partial' X \,\circ\!\!\!\circ\circ\!\!\!\circ\, \partial X \cdot \partial X \,\circ\!\!\!\circ , \tag{4.107}$$

using Wick's theorem, one finds a pole term $\frac{D}{2}(z'-z)^{-4}$, showing that the central charge is $c = D$. In addition, one can compute

$$T(z')\partial X^\mu = \frac{1}{(z'-z)^2}\partial X^\mu + \frac{1}{z'-z}\partial^2 X^\mu \ , \tag{4.108}$$

showing that ∂X^μ is a primary field of conformal weight $(1,0)$.

At this point we also introduce the operator $\circ\!\!\!\circ\, e^{ik\cdot X(z)} \,\circ\!\!\!\circ$. This field is primary with conformal weight $(\alpha' k^2/4, 0)$, as is easily verified by expanding the exponential and Wick contracting with $T(z)$. It will play a prominent role in the sequel.

A similar discussion applies for the antiholomorphic component $\tilde{T}(\bar{z})$ and gives the right-moving operators $\tilde{L}_m$ with the same central charge and the field $\bar{\partial}\tilde{X}^\mu(\bar{z})$ to be primary of conformal weight $(0,1)$.

For the case of the open string, the doubling trick is used to extend the domain of $X(z)$ and hence $T(z)$ from the upper half-plane to the full plane in order that the contour integrals be well-defined. Similar results to those above for the holomorphic sector of the closed string are obtained.

4.1.3 Free Superstrings

The bosonic string has many drawbacks, the two most important being that the spectrum contains a tachyon and that the theory does not possess space-time supersymmetry since there are no space-time fermions. Both of these drawbacks are overcome by adding D free fermions to the action (4.57) in order to make the world-sheet theory supersymmetric. Note that space-time supersymmetry is not manifest in this so-called RNS formalism, but it is achievable, as we shall see later. The introduction of space-time supersymmetry also removes the tachyon from the spectrum and reduces the value of D from 26 to 10.

The fermionic action is taken to be

$$S_\psi = \frac{1}{4\pi}\int d^2z \left(\tilde{\psi}\cdot\partial\tilde{\psi} + \psi\cdot\bar{\partial}\psi\,\right), \tag{4.109}$$

where the fields $\tilde{\psi}^\mu$ and ψ^μ are the anticommuting components of the two-dimensional Majorana spinor $\Psi^\mu = \begin{pmatrix} \tilde{\psi}^\mu \\ \psi^\mu \end{pmatrix}$ defined on the cylinder (strip).

The action above is written in *superconformal gauge.* Before gauge fixing and Wick rotation, it is necessary to also include the zweibein ${e_a}^\alpha(\sigma)$, defined by ${e_a}^\alpha(\sigma){e_b}^\beta(\sigma)\eta_{\alpha\beta} = g_{ab}(\sigma)$ and its superpartner, the gravitino. The full matter action $S_M = S_X + S_\psi$ then possesses on-shell local $(1,1)$ worldsheet supersymmetry which can be extended off-shell by the inclusion of auxiliary fields. The full action is also diffeomorphism-invariant and possesses two-dimensional Weyl and super-Weyl invariance as well as D-dimensional Poincaré invariance. Superconformal gauge is obtained by eliminating the auxiliary fields via their equations of motion and by using the symmetries to replace ${e_a}^\alpha(\sigma)$ by δ_a^α and to set the gravitino to zero. Finally, the fermions are rescaled, a Wick rotation is performed and the cylinder (strip) is exploded onto the complex plane (upper half-plane) for closed (open) strings. In analogy with the bosonic case, quantum mechanically the procedure is valid only locally unless $D = 10$. Details of the procedure may be found in [93].

The equations of motion resulting from (4.109) are

$$\partial\tilde{\psi}^\mu(z,\bar{z}) = 0 \ , \ \ \bar{\partial}\psi^\mu(z,\bar{z}) = 0 \ , \tag{4.110}$$

subject to boundary conditions to be considered below. Thus, ψ^μ is holomorphic and $\tilde{\psi}^\mu$ antiholomorphic. The X^μ equations of motion and boundary conditions are unaltered from those of the bosonic case.

The gauge fixing of the zweibein leads to the classical constraints

$$T(z) \ \equiv \ T_{zz} \equiv -\frac{1}{\alpha'}\partial X \cdot \partial X - \frac{1}{2}\psi \cdot \partial\psi = 0 \tag{4.111}$$

$$\tilde{T}(\bar{z}) \ \equiv \ T_{\bar{z}\bar{z}} \equiv -\frac{1}{\alpha'}\bar{\partial} X \cdot \bar{\partial} X - \frac{1}{2}\tilde{\psi} \cdot \bar{\partial}\tilde{\psi} = 0 \ , \tag{4.112}$$

after imposition of the equations of motion (4.110) to remove the $T_{z\bar{z}}$ component of the energy-momentum tensor. We see that the energy-momentum tensor is again conserved and traceless. Additionally, the gauge fixing of the gravitino leads to the (classical) vanishing of the *supercurrents*:

$$T_F(z) \ \equiv \ T_{Fz} \equiv i\sqrt{\frac{2}{\alpha'}}\,\psi(z) \cdot \partial X(z,\bar{z}) \tag{4.113}$$

$$\tilde{T}_F(\bar{z}) \ \equiv \ T_{F\bar{z}} \equiv i\sqrt{\frac{2}{\alpha'}}\,\tilde{\psi}(\bar{z}) \cdot \bar{\partial} X(z,\bar{z}) \ . \tag{4.114}$$

Furthermore, the supercurrent is conserved and traceless in the sense that $\rho^\alpha T_{F\alpha} = 0$, where ρ^α are the Dirac matrices of two-dimensional Euclidean space.

Using the partition function technique described earlier but adapted to take account of the Grassmanian variables, we find that the inverse fermionic propagators are given by

$$\Delta^{-1}_{\mu\nu} = \eta_{\mu\nu}\frac{\bar{\partial}}{2\pi} \ , \ \tilde{\Delta}^{-1}_{\mu\nu} = \eta_{\mu\nu}\frac{\partial}{2\pi} \ . \tag{4.115}$$

Hence, the propagators up to boundary conditions (and therefore OPEs, which are independent of boundary conditions) are

$$\langle\psi^\mu(z')\psi^\nu(z)\rangle = \frac{\eta^{\mu\nu}}{z'-z} \ , \ \langle\tilde{\psi}^\mu(\bar{z}')\tilde{\psi}^\nu(\bar{z})\rangle = \frac{\eta^{\mu\nu}}{\bar{z}'-\bar{z}} \ , \tag{4.116}$$

where we have used (4.64).

The full action in superconformal gauge is still invariant under conformal transformations generated by the stress tensor and under the superconformal transformations whose infinitesimal form is

$$\delta_{\eta,\bar{\eta}} X^\mu(z,\bar{z}) = \sqrt{\frac{\alpha'}{2}}\left(\eta(z)\psi^\mu(z) + \bar{\eta}(\bar{z})\tilde{\psi}^\mu(\bar{z})\right) \ , \tag{4.117}$$

$$\delta_{\eta,\bar{\eta}} \psi^\mu(z) = -\sqrt{\frac{2}{\alpha'}}\,\eta(z)\partial X^\mu(z,\bar{z}) \ , \tag{4.118}$$

$$\delta_{\eta,\bar{\eta}} \tilde{\psi}^\mu(\bar{z}) = -\sqrt{\frac{2}{\alpha'}}\,\bar{\eta}(\bar{z})\bar{\partial} X^\mu(z,\bar{z}) \ , \tag{4.119}$$

where the parameter $\eta(z)$ is anticommuting. That these transformations are symmetries of the gauge-fixed action follows from the fact that the supercurrents are (anti)holomorphic. They can be derived by first defining conserved charges analogous to (4.87) (without the factor i) and then using (4.88), the $(\partial X)X$ OPEs obtained from (4.63) and the OPEs (4.116).

The SuperVirasoro Algebra

Analogously to the bosonic case, the quantum energy-momentum tensor and supercurents are given by conformally normal ordering their classical counterparts[9].

[9] The normal ordering has no effect on the supercurrents.

Using Wick's theorem and the $(\partial\psi)\psi$ OPE calculated from (4.116) it is straightforward to calculate:

$$-\frac{1}{2} \circ\!\!\circ\, \psi(z') \cdot \partial'\psi(z') \,\circ\!\!\circ\, \psi^\mu(z) = \frac{\frac{1}{2}}{(z'-z)^2}\psi^\mu(z) + \frac{1}{z'-z}\partial\psi^\mu(z) \ . \qquad (4.120)$$

This shows that $\psi^\mu(z)$ is a primary field of weight $(\frac{1}{2}, 0)$. Similarly, $\tilde{\psi}^\mu(\bar{z})$ is a primary field of weight $(0, \frac{1}{2})$.

It is also straightforward to obtain the OPEs

$$T(z')T(z) \ = \ \frac{c/2}{(z'-z)^4} + \frac{2}{(z'-z)^2}T(z) + \frac{1}{z'-z}\partial T(z) \qquad (4.121)$$

$$T(z')T_F(z) \ = \ \frac{\frac{3}{2}}{(z'-z)^2}T_F(z) + \frac{1}{z'-z}\partial T_F(z) \ , \qquad (4.122)$$

$$T_F(z')T_F(z) \ = \ \frac{2c}{3(z'-z)^3} + \frac{2}{z'-z}T(z) \ , \qquad (4.123)$$

where $c = 3D/2$ is the central charge. Since the bosonic sector involving the X^μ's is unchanged and has central charge D, we see that each fermion ψ^μ weighs in with central charge $\frac{1}{2}$. In addition, (4.122) shows that $T_F(z)$ is a primary field of weight $(\frac{3}{2}, 0)$. Similar considerations apply for the antiholomorphic modes.

Boundary Conditions, Mode Expansions and Commutation Relations

Firstly, the boundary conditions and mode expansions of the X^μ's are as for the bosonic string.

For the fermions, the appropriate boundary conditions are[10]

$$(\tilde{\psi}\cdot\delta\tilde{\psi} - \psi\cdot\delta\psi)(\sigma^1, 0) \ = \ (\tilde{\psi}\cdot\delta\tilde{\psi} - \psi\cdot\delta\psi)(\sigma^1, 2\pi) \ , \qquad (4.124)$$

for closed strings and

$$\tilde{\psi}\cdot\delta\tilde{\psi} - \psi\cdot\delta\psi \ = \ 0 \qquad (4.125)$$

at each end of an open string. These follow from (4.109) by first transforming from the z to the w coordinate and then varying the action.

[10] For fermions, it is clearest to first consider the boundary conditions on the cylinder (strip) and then to transform to the plane. One also assumes $\delta\psi^\mu = 0 = \delta\tilde{\psi}^\mu$ at $\sigma^1 = \pm\infty$.

Closed Strings

The independence of the holomorphic and antiholomorphic spinors implies that (4.124) applies *separately* to each of ψ^μ and $\tilde{\psi}^\mu$. Thus we can have the following boundary conditions:

$$\text{Ramond (R)} \;:\; \psi^\mu(\sigma^1,0) = +\psi^\mu(\sigma^1,2\pi) \;, \tag{4.126}$$

$$\text{Neveu-Schwarz (NS)} \;:\; \psi^\mu(\sigma^1,0) = -\psi^\mu(\sigma^1,2\pi) \;, \tag{4.127}$$

and similarly for $\tilde{\psi}^\mu$. Therefore, there are four different sectors in the Hilbert space of the theory: NS-NS, R-R, NS-R, R-NS, where the first factor corresponds to the boundary condition on the holomorphic field ψ^μ.

The mode expansions on the cylinder are taken to be

$$\psi^\mu(w) = \sum_r \psi^\mu_r e^{-rw} \;,\; \tilde{\psi}^\mu(\bar{w}) = \sum_r \tilde{\psi}^\mu_r e^{-r\bar{w}} \;, \tag{4.128}$$

where $\psi^{\mu\dagger}_r = \psi^\mu_{-r}$ and where r ranges over the integers (including zero) for R boundary conditions and over $\mathbb{Z}+\frac{1}{2}$ for NS boundary conditions. Using the fact that ψ^μ and $\tilde{\psi}^\mu$ are primary fields of weights $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$, respectively, we obtain the expansions on the plane:

$$\psi^\mu(z) = \sum_{r\in\mathbb{Z}+\nu} \psi^\mu_r z^{-r-1/2} \;,\; \tilde{\psi}^\mu(\bar{z}) = \sum_{r\in\mathbb{Z}+\tilde{\nu}} \tilde{\psi}^\mu_r \bar{z}^{-r-1/2} \;, \tag{4.129}$$

where $\nu, \tilde{\nu}$ are zero in the Ramond sector and 1/2 in the Neveu-Schwarz sector. Note that there is a branch cut in the R sector. We can also expand the energy-momentum tensor and supercurrents as

$$T(z) = \sum_{m\in\mathbb{Z}} \frac{L_m}{z^{m+2}} \;,\; \tilde{T}(\bar{z}) = \sum_{m\in\mathbb{Z}} \frac{\tilde{L}_m}{\bar{z}^{m+2}} \;, \tag{4.130}$$

$$T_F(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{G_r}{z^{r+3/2}} \;,\; \tilde{T}_F(\bar{z}) = \sum_{r\in\mathbb{Z}+\tilde{\nu}} \frac{\tilde{G}_r}{\bar{z}^{r+3/2}} \;. \tag{4.131}$$

In terms of oscillators, these expansions are

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{m-n}\cdot\alpha_n: + \frac{1}{4}\sum_{r\in\mathbb{Z}+\nu}(2r-m):\psi_{m-r}\cdot\psi_r: + d(\nu)\delta_{m,0} \;, \tag{4.132}$$

$$G_r = \sum_{n\in\mathbb{Z}} \alpha_n\cdot\psi_{r-n} \;, \tag{4.133}$$

where $d(0) = c/24 = D/16$ and $d(\frac{1}{2}) = (c - 3D/2)/24 = 0$. Similar considerations apply to the antiholomorphic modes with the same values of d.

The expansions (4.129) can now be inserted into the OPEs (4.116) to yield the commutation relations

$$\{\psi_r^\mu, \psi_s^\nu\} = \{\tilde{\psi}_r^\mu, \tilde{\psi}_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0} \ , \tag{4.134}$$

which, of course, are consistent with those obtained using canonical quantisation. It is again easy to see that the oscillators with positive mode numbers are annihilation operators whereas oscillators with negative mode numbers are creation operators. We shall see below how the zero mode operators $\psi_0^\mu, \tilde{\psi}_0^\mu$ are to be interpreted.

The expansions (4.130) and (4.131) can be inserted into the OPEs (4.121-4.123) and yield, after the usual contour integration, the algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \ , \tag{4.135}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{12}(4r^2-1)\delta_{r+s,0} \ , \tag{4.136}$$

$$[L_m, G_r] = \frac{1}{2}(m-2r)G_{m+r} \ . \tag{4.137}$$

Open Strings

For the open string, the boundary condition (4.125) allows the possibilities

$$\psi^\mu(\sigma^1, 0) = e^{2\pi i\nu'}\tilde{\psi}^\mu(\sigma^1, 0) \ , \ \psi^\mu(\sigma^1, \pi) = e^{2\pi i\nu}\tilde{\psi}^\mu(\sigma^1, \pi) \ , \tag{4.138}$$

where ν' and ν are, independently, either 0 or 1/2. However, the redefinition $\tilde{\psi}^\mu \to e^{-2\pi i\nu'}\tilde{\psi}^\mu$ allows us to set $\nu' = 0$ without loss of generality. Therefore, there are two possibilities, namely the R sector ($\nu = 0$) and the NS sector ($\nu = 1/2$), with the mode expansions on the plane:

$$\psi^\mu(z) = \sum_{r\in\mathbb{Z}+\nu} \psi_r^\mu z^{-r-1/2} \ , \ \tilde{\psi}^\mu(\bar{z}) = \sum_{r\in\mathbb{Z}+\nu} \psi_r^\mu \bar{z}^{-r-1/2} \ . \tag{4.139}$$

As for the bosonic string, the previous conformal field theory analysis is not strictly valid for the open string since $\psi^\mu(z)$ is defined only on the upper half complex plane rather than the full plane (and, therefore, $\tilde{\psi}^\mu(\cdot)$ is defined on the lower half-plane). However, by use of the the doubling trick:

$$\psi^\mu(\sigma^1, -\sigma^2) := D^\mu_\nu\tilde{\psi}^\nu(\sigma^1, \sigma^2) = \tilde{\psi}^\mu(\sigma^1, \sigma^2), \ (0 \leq \sigma^2 < \pi), \tag{4.140}$$

$$\Longleftrightarrow \psi^\mu(z) \ := \ D^\mu_\nu \tilde{\psi}^\nu(z) = \tilde{\psi}^\mu(z) \ , \quad (\mathrm{Im}\, z \leq 0 \, , \, \arg z \neq -\pi) \ , \qquad (4.141)$$

we can extend $\psi^\mu(z)$ to be holomorphic over the full plane, $-\pi < \arg z \leq \pi$. The boundary condition $\nu' = 0$ at $\sigma^2 = 0$ is automatic, whereas R and NS boundary conditions at $\sigma^2 = \pi$ become, respectively, periodic and antiperiodic conditions on the extended field:

$$\psi^\mu(\sigma^1, -\pi) := D^\mu_\nu \tilde{\psi}^\nu(\sigma^1, \pi) = e^{-2\pi i \nu} D^\mu_\nu \psi^\nu(\sigma^1, \pi) = e^{-2\pi i \nu} \psi^\mu(\sigma^1, \pi) \ , \qquad (4.142)$$

Quantisation is then as for the holomorphic sector of the closed string above, leading to the anticommutation relations

$$\{\psi^\mu_r, \psi^\nu_s\} = \eta^{\mu\nu} \delta_{r+s,0} \ , \qquad (4.143)$$

and the single correlator

$$\langle \tilde{\psi}^\mu(z') \tilde{\psi}^\nu(z) \rangle = \eta^{\mu\nu} \frac{f(z', z)}{z' - z} \ , \qquad (4.144)$$

in terms of the extended field. The function $f(z', z)$ is 1 for NS boundary conditions and $\frac{1}{2}(\sqrt{z'/z} + \sqrt{z/z'})$ in the R sector. Note that $f(z', z) \to 1$ as $z' \to z$ and so (4.144) is consistent with the OPEs (4.116).

Fock Space, GSO Conditions and Physical Spectrum

In this section we consider the Fock space and spectrum generated by a single set of NS or R modes, corresponding to the open string.

Fermionic Fock Space

It is clear that the full Hilbert space is simply the tensor product of the bosonic Hilbert space times the fermionic Hilbert space. Therefore, in this section we consider only the fermionic Hilbert space.

The NS sector is simple. Since there are no fermionic zero modes, we take the Fock vacuum to be annihilated by all $r > 0$ modes:

$$\psi^\mu_r |0\rangle_{NS} = 0 \ , \quad r > 0 \ . \qquad (4.145)$$

A generic (non-normalised) state in the Fock space is then

$$|N\rangle = \prod_{\mu=0}^{D-1} \prod_{r \geq 0}^{\infty} (\psi^\mu_{-r})^{N_{\mu r}} |0\rangle_{NS} \ , \qquad (4.146)$$

where $N_{\mu r}$ is either 0 or 1. Such a state is an eigenstate of the number operator

$$N_+^{(\psi)} = \sum_{r>0}^{\infty} r\psi_{-r} \cdot \psi_r \ , \tag{4.147}$$

with eigenvalue $N^{(\psi)} = \sum_{\mu=0}^{D-1} \sum_{r>0}^{\infty} r N_{\mu r}$. From the space-time point of view all such states are bosonic.

The R sector is more involved. Firstly, one should note that zero modes are present and that they satisfy the $SO(1,9)$ Dirac algebra:

$$\Gamma^\mu = \sqrt{2}\psi_0^\mu \ , \quad \{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \ . \tag{4.148}$$

The Fock space is built from the Fock vacuum as in (4.146) and the number operator is defined as in (4.147). Note, however, that since $[\Gamma^\mu, N_+^{(\psi)}] = 0$ states of the same eigenvalue $N^{(\psi)}$ of $N_+^{(\psi)}$ are taken into each other under the action of Γ^μ. Therefore, the states should span a representation space for this gamma matrix algebra and so are space-time spinors. In particular, the Fock vacuum, which is annihilated by all positive modes, forms a 32-component spinor. The **32** is reducible to two Weyl representations **16**+**16**$'$, distinguished by their eigenvalue (± 1) under $\Gamma = \Gamma^0\Gamma^1 \cdots \Gamma^9$. We shall denote the vacuum by $|0;\alpha\rangle_R$, where α is a spinor index which shall often be omitted.

GSO Conditions

The classical constraints that the energy-momentum tensor and supercurrents vanish are implemented quantum mechanically by the constraints

$$(L_0 - a(\nu))|\text{phys}\rangle = 0 \ , \tag{4.149}$$

$$L_m|\text{phys}\rangle = 0 \ , \quad (m > 0) \ , \tag{4.150}$$

$$G_r|\text{phys}\rangle = 0 \ , \quad (r \geq 0) \ , \tag{4.151}$$

on physical states built from the α_{-m}^μ and ψ_{-r}^μ oscillators. In (4.149), we have absorbed the c-number $d(\nu)$ that is present in L_0 into the constant $a(\nu)$. Analogously to the bosonic string, it can be shown that the space of physical states modulo the space of null physical states is ghost-free and the degrees of freedom transverse in nature provided $D = 10$, $a(0) = 0$ and $a(\frac{1}{2}) = \frac{1}{2}$. Note that the G_0 condition in the R sector implies the L_0 condition with $a(0) = 0$ since (4.136) gives $G_0^2 = L_0 - \frac{D}{16}$. For the closed string, similar conditions hold for the right-movers with the same values of the normal ordering constant a.

To proceed further, we need to define the operator G, which is essentially the *worldsheet fermion number.* It is defined only mod 2 and counts whether states of the form (4.146) possess an even or or odd number of fermionic oscillators ψ^{μ}_{-r}. G therefore anticommutes with the oscillators. In the NS sector the vacuum $|0\rangle_{NS}$ is assigned the value $G = -1$ and so we can take

$$G = -(-1)^F \ , \ F = \sum_{r=1/2}^{\infty} \psi_{-r} \cdot \psi_r \ . \tag{4.152}$$

Hence, all states with an even number of oscillators have $G = -1$ and those with an odd number have $G = +1$. In the R sector G is taken to act as Γ on the vacuum $|0\rangle_R$. Therefore, we take

$$G = \Gamma(-1)^F \ , \ F = \sum_{r=1}^{\infty} \psi_{-r} \cdot \psi_r \ . \tag{4.153}$$

We can now introduce the GSO conditions. These are projections on the space of physical states as defined by the constraints (4.149-4.151). Specifically, they are

$$G|\text{phys}\rangle = +|\text{phys}\rangle \ , \ (NS) \ , \tag{4.154}$$

$$G|\text{phys}\rangle = \pm|\text{phys}\rangle \ , \ (R) \ , \tag{4.155}$$

where we choose one sign *for all* physical states in the R sector. Note that these conditions project out the NS vacuum and either the **16** or the **16**′ from the physical spectrum[11].

Physical Spectrum

The GSO conditions make the projected physical spectrum tachyon-free and space-time supersymmetric. We shall not prove this last statement in all generality but show it to be true at the massless level.

[11]Which sign is chosen and hence which projection occurs in the R sector is of no consequence. The spectrum built from the **16** is isomorphic to that built from the **16**′.

Firstly, the mass-shell condition (4.149) leads to physical states having masses

$$m^2 = -k^2 = \frac{1}{\alpha'}(N^{(\alpha)} + N^{(\psi)} - a(\nu)) \ . \tag{4.156}$$

We see that the vacuum $|NS;k\rangle \equiv |0;k\rangle \otimes |0\rangle_{NS}$ with $k^2 = 1/2\alpha'$ is physical and a tachyon. However, it is projected out by the GSO conditions since it has $G = -1$. For any other value of k^2 such a state is not physical. Furthermore, since $a(0) = 0$, the lowest mass in the R sector is zero. Therefore, the projected spectrum is tachyon-free.

Secondly, the vacuum $|R;k\rangle \equiv |0;k\rangle \otimes |0\rangle_R$ with k^μ null is massless. The GSO conditions remove half the states, leaving the **16**, say. Now recall from our discussion of the bosonic string that massless particles in D space-time dimensions are classified by their representations under the little group $SO(D-2)$. Under $SO(1,9) \rightarrow SO(1,1) \times SO(8)$, the **16** decomposes into $\mathbf{8}_s + \mathbf{8}_c$, where $\mathbf{8}_s$ and $\mathbf{8}_c$ are the two spinor representations of $SO(8)$ of opposite chirality. The conditions (4.151) reduce to the G_0 condition, which is just the Dirac equation. This removes one of the **8**'s, leaving the $\mathbf{8}_s$ say[12], with $G = +1$. A Majorana condition can still be imposed to leave 8 real fermionic degrees of freedom[13]. The only other massless states are those of the form $|\psi\rangle^\mu = \psi^\mu_{-1/2}|NS;k\rangle$ with k^μ null. They are not projected out by the GSO conditions since they have $G = +1$. Furthermore, they transform as a vector (in the $\mathbf{8}_v$ representation) of $SO(8)$, giving 8 real bosonic degrees of freedom. Therefore, we see that the massless states fill the vector multiplet $\mathbf{8}_v + \mathbf{8}_s$ of $D = 10$, $\mathcal{N} = 1$ space-time supersymmetry.

Note that the GSO condition (4.154) ensures that the eigenvalues $N^{(\psi)}$ of the number operator (4.147) are half-odd-integers for projected states in the NS sector. Combined with the facts that: (a) $a(\frac{1}{2}) = \frac{1}{2}$, (b) $N^{(\psi)}$ is an integer in the R sector and (c) $a(0) = 0$, this implies that the masses of the projected open string states are

$$m^2 = \frac{n}{\alpha'} \ , \ n = 0, 1, \ldots \ . \tag{4.157}$$

[12]This implies that if one were to choose the opposite sign for the GSO projection in the R sector, the $\mathbf{8}_c$ of the $\mathbf{16}'$ would remain instead. We shall use this fact below when we consider closed strings.

[13]Majorana and Weyl conditions can be imposed simultaneously on spinors of $SO(p,q)$ if and only if $p - q = 0 \bmod 8$.

Finally, it is important to note that as they have been presented the GSO conditions seem very *ad hoc*. However, they are really *derived* conditions which are sufficient to ensure the modular invariance of the one-loop partition function. Modular transformations are "large" diffeomorphisms of the worldsheet, that is, transformations which are not continuously connected to the identity. Since the action is diffeomorphism-invariant then for consistency so too should be all path integrals with or without operator insertions. Modular transformations generically mix up different fermionic sectors of the theory. For example, for closed strings one-loop corresponds to a worldsheet with the topology of a torus and there are four sectors for each side (left and right): (NS, G) and (R, G) where $G = \pm 1$. The total partition function for each side is given by the sum over the partition functions for each sector. Requiring modular invariance of the sum restricts the number of allowable combinations. One such allowed combination results in (4.154) and (4.155). Other allowed combinations result in such undesirable features as tachyons or the absence of space-time fermions.

Closed Superstrings: Type II

For the closed string, similar GSO projections are imposed for the right-movers independently of the left-movers. In this case, one can choose the projection in the right-moving R sector to be opposite to or the same as that in the left-moving R sector. This leads to two distinct theories, Type IIA and Type IIB, respectively. Both theories are again tachyon-free. The masses of physical states are given by

$$m^2 = -k^2 = \frac{4}{\alpha'}(N^{(\alpha)} + N^{(\psi)} - a(\nu)) \ = \frac{4}{\alpha'}(\tilde{N}^{(\alpha)} + \tilde{N}^{(\psi)} - a(\tilde{\nu})) \ , \qquad (4.158)$$

The $SO(8)$ content of the massless states is

$$\begin{aligned} &\text{Type IIA:} \quad (\mathbf{8}_v + \mathbf{8}_s) \otimes (\mathbf{8}_v + \mathbf{8}_c) \ , \\ &\text{Type IIB:} \quad (\mathbf{8}_v + \mathbf{8}_s) \otimes (\mathbf{8}_v + \mathbf{8}_s) \ . \end{aligned} \qquad (4.159)$$

The NS-NS sector for both theories is as for the bosonic string:

$$\mathbf{8}_v \otimes \mathbf{8}_v = \mathbf{1} + \mathbf{28} + \mathbf{35} = \Phi + B_{\mu\nu} + G_{\mu\nu} \ , \qquad (4.160)$$

that is, a scalar (dilaton), an antisymmetric 2-form (Kalb-Ramond field), and a symmetric, tracefree second-rank tensor (graviton).

In the R-R sector we have

$$\begin{aligned} \text{Type IIA:} \quad & \mathbf{8}_s \otimes \mathbf{8}_c = [1] + [3] = \mathbf{8}_v + \mathbf{56}_t \ , \\ \text{Type IIB:} \quad & \mathbf{8}_s \otimes \mathbf{8}_s = [0] + [2] + [4]_+ = \mathbf{1} + \mathbf{28} + \mathbf{35}_+ \ . \end{aligned} \tag{4.161}$$

Here $[n]$ denotes the n-times antisymmetrised representation of $SO(8)$ and the subscript $+$ indicates self-duality. Note that the representations $[n]$ and $[8-n]$ are equivalent by Hodge duality[14]. Hence, IIA contains a 1-form and a 3-form antisymmetric tensor potential, whilst IIB contains a scalar, a 2-form potential and a 4-form potential with self-dual field strength. Therefore, these states are space-time bosons, as are all states in the R-R sector.

In the R-NS and NS-R sectors are the products:

$$\begin{aligned} \mathbf{8}_v \otimes \mathbf{8}_c &= \mathbf{8}_s + \mathbf{56}_c \ , \\ \mathbf{8}_v \otimes \mathbf{8}_s &= \mathbf{8}_c + \mathbf{56}_s \ . \end{aligned} \tag{4.162}$$

These correspond to two spinors of opposite (same) chirality and two vector-spinors (gravitini) of opposite (same) chirality for IIA (IIB). Since there are two gravitini there are also two supercharges. All states in these two sectors are space-time fermions.

Open Superstrings: Type I

The string theories we have been discussing are *oriented* theories. However, the type IIB closed theory, with the same chiralities in both the left and right sector, possesses the *worldsheet parity* symmetry $\Omega : \sigma^2 \to 2\pi - \sigma^2$ under which the left-movers and right-movers are interchanged. We can gauge this symmetry by projecting out from the physical spectrum all states not invariant under Ω [97]. In the NS-NS sector, the antisymmetric tensor is eliminated. Only the linear combination R-NS$\oplus$NS-R of the two fermionic sectors survives the projection. As a result, only one gravitino (and hence one supercharge) remains. Finally, in the R-R sector the $[2]$ is invariant since the $\mathbf{1} \oplus \mathbf{35}_+$ is in the symmetric product of $\mathbf{8}_s \otimes \mathbf{8}_s$, the $\mathbf{28}$ in the antisymmetric product and there is an extra minus sign in the exchange of the two fermions. The

[14]This duality will be shown in section 4.1.6.

projection results in a theory called *type I closed unoriented theory* which, however, turns out to be inconsistent unless open strings are added in a very precise fashion.

In order to add these open strings it is necessary to generalise the oriented open string theory discussed above. The endpoints of an open string are distinguished points at which non-dynamical Chan-Paton degrees of freedom can reside. These degrees of freedom manifest themselves as group theory factors tacked onto the string states:

$$|\phi\rangle \to \sum_{i,j=1}^{N} |\phi; ij\rangle \lambda^a_{ij} \ , \tag{4.163}$$

where λ^a are a basis of hermitian $N \times N$ matrices for the group and i and j label the two endpoints. For the open string, worldsheet parity $\Omega : \sigma^2 \to \pi - \sigma^2$ interchanges the two ends of the string. In the absence of Chan-Paton factors, the vector $|\psi\rangle^\mu$ would not survive the projection. However, when Chan-Paton factors are present it can be shown that Ω can have an action on the matrices λ more complicated than the naive $\lambda \to \lambda^T$. This action leads to the gauge group $SO(N)$ or $USp(N)$ if the vector is not to be eliminated by the projection. The superpartner of the vector is the gaugino and transforms in the $\mathbf{8}_s$ representation of $SO(8)$. The two together form the massless states of the unoriented open string.

The above unoriented open theory can now be added to the type I closed unoriented theory. It can be shown that the inconsistencies of the closed sector are removed only if the gauge group is $SO(32)$. The resulting theory is called *type I open plus closed string theory*.

4.1.4 Ghosts and Superghosts

Our presentation of string theory so far has been rather *ad hoc*, especially our discussion of the gauge fixing procedure and the determination of the normal ordering constants d and a. A more thorough treatment of the gauge fixing leads, in the usual fashion[15], to an additional sector, the *ghost* sector, in the worldsheet theory. This

[15]Some details are given in section 4.1.6.

sector is described in superconformal gauge by the action

$$S_g = \frac{1}{2\pi} \int d^2z (b\bar{\partial}c + \tilde{b}\partial\tilde{c} + \beta\bar{\partial}\gamma + \tilde{\beta}\partial\tilde{\gamma}) \ , \tag{4.164}$$

and should be added to the matter action S_M to form the full gauge-fixed action. In the above, b and c are the Faddeev-Popov ghosts (anticommuting worldsheet scalars) and are primary fields of conformal weights $(2, 0)$ and $(-1, 0)$, respectively. The superghosts β and γ are their superpartners (commuting worldsheet spinors) and are primary fields of weights $(\frac{3}{2}, 0)$ and $(-\frac{1}{2}, 0)$, respectively. Similar remarks apply to the tilded fields, which turn out to be antiholomorphic as a consequence of the equations of motion:

$$\bar{\partial}b = \bar{\partial}c = \bar{\partial}\beta = \bar{\partial}\gamma \ = \ 0 \ , \tag{4.165}$$

$$\partial\tilde{b} = \partial\tilde{c} = \partial\tilde{\beta} = \partial\tilde{\gamma} \ = \ 0 \ . \tag{4.166}$$

In the discussion below we concentrate on the holomorphic sector of the closed string.

OPEs, Mode Expansions, and SuperVirasoro Algebra

The partition function technique gives the OPEs

$$b(z')c(z) = \frac{1}{z'-z} \ , \quad c(z')b(z) = \frac{1}{z'-z} \ , \tag{4.167}$$

$$\beta(z')\gamma(z) = -\frac{1}{z'-z} \ , \quad \gamma(z')\beta(z) = \frac{1}{z'-z} \ . \tag{4.168}$$

The OPEs bb, etc. are non-singular. The mode expansions

$$b(z) \ = \ \sum_{m\in\mathbb{Z}} b_m z^{-m-2} \ , \tag{4.169}$$

$$c(z) \ = \ \sum_{m\in\mathbb{Z}} c_m z^{-m+1} \ , \tag{4.170}$$

$$\beta(z) \ = \ \sum_{r\in\mathbb{Z}+\nu} \beta_r z^{-r-3/2} \ , \tag{4.171}$$

$$\gamma(z) \ = \ \sum_{r\in\mathbb{Z}+\nu} \gamma_r z^{-r+1/2} \ , \tag{4.172}$$

where ν is 0 in the Ramond sector and $1/2$ in the Neveu-Schwarz sector, and subject to the hermiticity conditions

$$b_m^\dagger = b_{-m} \ , \ c_m^\dagger = c_{-m} \ , \ \beta_r^\dagger = -\beta_{-r} \ , \ \gamma_r^\dagger = \gamma_{-r} \ , \tag{4.173}$$

yield the (anti-)commutation relations

$$\{b_m, c_n\} = \delta_{m+n,0} \ , \ \ [\gamma_r, \beta_s] = \delta_{r+s,0} \ . \tag{4.174}$$

when inserted into the OPEs.

Additionally, the stress tensor and supercurrent

$$T^g(z) = \,:\!(\partial b)c - 2\partial(bc) + (\partial\beta)\gamma - \frac{3}{2}\partial(\beta\gamma)\!: \ , \tag{4.175}$$

$$T_F^g(z) = \,:\!-\frac{1}{2}(\partial\beta)c + \frac{3}{2}\partial(\beta c) - 2b\gamma\!: \ , \tag{4.176}$$

form a superconformal algebra of the form (4.121-4.123) with central charge $c^g = -15$ and have mode expansions on the plane

$$L_m^g = \sum_{n\in\mathbb{Z}}(m+n) : b_{m-n}c_n : + \sum_{r\in\mathbb{Z}+\nu}\frac{1}{2}(m+2r) : \beta_{m-r}\gamma_r : + d^g(\nu)\delta_{m,0}, \tag{4.177}$$

$$G_r^g = -\sum_{n\in\mathbb{Z}}\left[\frac{1}{2}(2r+n)\beta_{r-n}c_n + 2b_n\gamma_{r-n}\right] \ , \tag{4.178}$$

where $d^g(0) = -\frac{5}{8}$ and $d^g(\frac{1}{2}) = -\frac{1}{2}$. The normal ordering is as before except that b_0 and β_0 are included among the positive modes and c_0 and γ_0 among the negative modes. These generators should be added to the matter generators given in (4.132) and (4.133) to give the full generators of the theory. Note that $\beta(z)$ and $\gamma(z)$ have the same moding as $\psi^\mu(z)$. The full generators then form a superconformal algebra with central charge $c_{tot} = \frac{3}{2}D - 15$ and normal ordering constant $d_{tot}(0) = \frac{D-10}{16}$, $d_{tot}(\frac{1}{2}) = -\frac{1}{2}$.

One can now see why $D = 10$ for the superstring. Demanding that the full quantum theory is not conformally anomalous requires $c_{tot} = 0$ or, equivalently, $D = 10$. (A similar argument gives $D = 26$ for the bosonic string since the b, c system contributes -26 to c^g and -1 to $d^g(\nu)$.) In this case, the full Virasoro generators on the plane are also the generators on the cylinder (strip), as a consequence of (4.100).

Fock Space

The canonical Fock space of the ghost sector is built up from the vacuum in the usual way. For the holomorphic sector, the vacuum may be written as $|0^g\rangle_{NS} = |\downarrow\rangle \otimes |-1\rangle$

in the NS sector and as $|0^g\rangle_R = |\downarrow\rangle \otimes |-\frac{1}{2}\rangle$ in the R sector, where

$$\begin{aligned}
b_m|\downarrow\rangle &= 0\ , \quad m \geq 0 \quad , \quad c_m|\downarrow\rangle = 0\ , \quad m \geq 1\ , && (4.179)\\
\beta_r|-1\rangle &= 0\ , \quad r \geq \frac{1}{2} \quad , \quad \gamma_r|-1\rangle = 0\ , \quad r \geq \frac{1}{2}\ , && (4.180)\\
\beta_r|-\frac{1}{2}\rangle &= 0\ , \quad r \geq 0 \quad , \quad \gamma_r|-\frac{1}{2}\rangle = 0\ , \quad r \geq 1\ . && (4.181)
\end{aligned}$$

Consequently, the vacua satisfy

$$\begin{aligned}
L^g_m|0^g\rangle_{NS} &= d^g(1/2)\delta_{m,0}|0^g\rangle_{NS} \quad , \quad L^g_m|0^g\rangle_R = d^g(0)\delta_{m,0}|0^g\rangle_R\ , && (4.182)\\
G^g_r|0^g\rangle_{NS} &= 0 \quad , \quad G^g_r|0^g\rangle_R = 0\ , && (4.183)
\end{aligned}$$

for $m, r \geq 0$.

We can now better understand the physical state conditions (4.149-4.151) as

$$\begin{aligned}
L^f_m(|\text{phys}\rangle \otimes |\text{ghost}\rangle) &= 0\ , \quad (m \geq 0)\ , && (4.184)\\
G^f_r(|\text{phys}\rangle \otimes |\text{ghost}\rangle) &= 0\ , \quad (r \geq 0)\ , && (4.185)
\end{aligned}$$

where L^f_m and G^f_r are given by the sum of (4.132) and (4.177) and of (4.133) and (4.178), respectively. Only the matter part, $|\text{phys}\rangle$, of any physical state is non-trivial — the ghosts are in their ground states — and so $a(\nu) = -d_{tot}(\nu)$. The GSO projections can be similarly reinterpreted.

The discussion also applies to the antiholomorphic sector of the closed string. For the open superstring the doubling trick

$$b(z) := \tilde{b}(z) \quad (\text{Im}\, z \leq 0)\ , \tag{4.186}$$

and similarly for c, β, γ leads to the same results.

4.1.5 Spin Fields, Bosonisation, q-vacua and Inner Products

We have seen that the Ramond vacuum $|0\rangle_R$ transforms as a space-time spinor under $SO(1,9)$. It is appealing to find a primary field $S(z)$ of weight $(h, 0)$ that creates the Ramond vacuum from the Neveu-Schwarz vacuum $|0\rangle_{NS}$, that is,

$$|0\rangle_R = \lim_{z\to 0} S(z)|0\rangle_{NS}\ , \quad {}_R\langle 0| = (|0\rangle_R)^\dagger = \lim_{z'\to 0} {}_{NS}\langle 0|S^\dagger(z') = \lim_{z\to\infty} {}_{NS}\langle 0|S(z)z^{2h}\ , \tag{4.187}$$

where the hermitian conjugate field $S^\dagger$ is defined as the field S transformed under the conformal transformation $z \to z' = 1/z$, up to a phase factor. The value of h is determined below. Clearly, $S(z)$ should be a 32-component spinor constructed from the NS variables of the theory. Such an operator is called a *spin field.*

Construction of the Spin Field: Bosonisation

The construction involves the technique of *bosonisation.* Firstly, consider the OPE associated with the propagator (4.65)

$$X^\mu(z')X^\nu(z) = -\frac{1}{2}\alpha'\eta^{\mu\nu}\ln(z'-z) \ , \tag{4.188}$$

for the holomorphic part of the embedding fields $X^\mu(z,\bar{z})$. Let us introduce five holomorphic scalar fields $H^a(z)$, $a = 0,\ldots,4$ that have mode expansions as in (4.58). In addition, their zero modes are subject to the modified commutation relations (4.23). The fields $H^a(z)$ are taken to have OPEs similar to the spacelike components of (4.188), that is,

$$H^a(z')H^b(z) = -\frac{1}{2}\alpha'\delta^{ab}\ln(z'-z) \ . \tag{4.189}$$

Note that the indices a, b are "internal" indices as distinct from the Lorentz indices μ,ν.

Consider now exponentials of the form $e^{i\sqrt{\frac{2}{\alpha'}}k\cdot H(z)}$, where the "$\cdot$" denotes contraction of indices with the internal metric δ_{ab}. Using the relation

$$\circ\!\!\!\circ\, e^A \,\circ\!\!\!\circ\,\circ\!\!\!\circ\, e^B \,\circ\!\!\!\circ \quad = \quad e^{\langle AB\rangle}\,\circ\!\!\!\circ\, e^{A+B}\,\circ\!\!\!\circ \ , \tag{4.190}$$

we find

$$\circ\!\!\!\circ\, e^{i\sqrt{\frac{2}{\alpha'}}k_1\cdot H(z')} \,\circ\!\!\!\circ\,\circ\!\!\!\circ\, e^{i\sqrt{\frac{2}{\alpha'}}k_2\cdot H(z)} \,\circ\!\!\!\circ \quad = \quad (z'-z)^{k_1\cdot k_2}\,\circ\!\!\!\circ\, e^{i\sqrt{\frac{2}{\alpha'}}(k_1\cdot H(z')+k_2\cdot H(z))}\,\circ\!\!\!\circ \ , \tag{4.191}$$

valid for $|z'| > |z|$, and the associated OPE

$$\begin{aligned}\circ\!\!\!\circ\, e^{i\sqrt{\frac{2}{\alpha'}}k_1\cdot H(z')} \,\circ\!\!\!\circ\,\circ\!\!\!\circ\, e^{i\sqrt{\frac{2}{\alpha'}}k_2\cdot H(z)} \,\circ\!\!\!\circ &= (z'-z)^{k_1\cdot k_2}\,\circ\!\!\!\circ\, e^{i\sqrt{\frac{2}{\alpha'}}(k_1+k_2)\cdot H(z)}\,\circ\!\!\!\circ \\ &+(z'-z)^{k_1\cdot k_2+1}\,\circ\!\!\!\circ\, i\sqrt{\frac{2}{\alpha'}}k_1\cdot\partial H(z) e^{i\sqrt{\frac{2}{\alpha'}}(k_1+k_2)\cdot H(z)}\,\circ\!\!\!\circ \ ,\end{aligned} \tag{4.192}$$

obtained from a Taylor expansion inside the normal ordering. Note that $\circ\!\!\circ\, e^{i\sqrt{\frac{2}{\alpha'}}k\cdot H(z)} \,\circ\!\!\circ$ is a primary field of conformal weight $(\frac{k^2}{2},0)$.

Let us focus on one specific internal direction, say $a=1$, and consider $(k_1)_a = \pm\delta^1_a$ and $k_2 = \pm k_1$. The conformal weight of the normal-ordered exponential is then $(\frac{1}{2},0)$. For $k_1 = k_2$ it is clear from the above relations that $\circ\!\!\circ\, e^{i\sqrt{\frac{2}{\alpha'}}k_1 H^1(z')} \,\circ\!\!\circ$ and $\circ\!\!\circ\, e^{i\sqrt{\frac{2}{\alpha'}}k_2 H^1(z')} \,\circ\!\!\circ$ anticommute and have non-singular OPEs, while for $k_1 = -k_2$ they also anticommute but have the OPE

$$\circ\!\!\circ\, e^{i\sqrt{\frac{2}{\alpha'}}k_1 H^1(z')} \,\circ\!\!\circ\,\circ\!\!\circ\, e^{-i\sqrt{\frac{2}{\alpha'}}k_1 H^1(z)} \,\circ\!\!\circ \;=\; \frac{1}{(z'-z)} \,. \tag{4.193}$$

Now we form the complex combinations

$$\Psi^1(z) = \frac{1}{\sqrt{2}}(\psi^2(z) + i\psi^3(z)) \,, \quad \bar{\Psi}^1(z) = \frac{1}{\sqrt{2}}(\psi^2(z) - i\psi^3(z)) \,, \tag{4.194}$$

of the worldsheet spinors. The index on Ψ is again an internal index while those on ψ are Lorentz indices. Note that the central charge for this system is $2\times\frac{1}{2} = 1$. Using (4.116) we find the OPE

$$\Psi^1(z')\bar{\Psi}^1(z) = \frac{1}{z'-z} \,, \tag{4.195}$$

with the $\Psi\Psi$ and $\bar{\Psi}\bar{\Psi}$ OPEs non-singular. Comparing with (4.193), we are led to the tentative identifications[16]

$$\Psi^1(z) \;\cong\; \circ\!\!\circ\, e^{i\sqrt{\frac{2}{\alpha'}}H^1(z)} \,\circ\!\!\circ \,, \quad \bar{\Psi}^1(z) \;\cong\; \circ\!\!\circ\, e^{-i\sqrt{\frac{2}{\alpha'}}H^1(z)} \,\circ\!\!\circ \,, \tag{4.196}$$

$$\circ\!\!\circ\, \Psi^1(z)\bar{\Psi}^1(z) \,\circ\!\!\circ \;\cong\; i\sqrt{\frac{2}{\alpha'}}\partial H^1(z) \,, \tag{4.197}$$

where the "$\cong$" is interpreted as equality within correlators. In this way we have bosonised the two fermions $\psi^{2,3}$ to a single scalar H^1.

Taking $0,1$ as light-cone directions in the target space, this procedure is readily extended to the other transverse directions:

$$\Psi^a(z) \;\equiv\; \frac{1}{\sqrt{2}}(\psi^{2a}(z) + i\psi^{2a+1}(z)) \;\cong\; \circ\!\!\circ\, e^{i\sqrt{\frac{2}{\alpha'}}H^a(z)} \,\circ\!\!\circ\, C^+_a \,, \tag{4.198}$$

$$\bar{\Psi}^a(z) \;\equiv\; \frac{1}{\sqrt{2}}(\psi^{2a}(z) - i\psi^{2a+1}(z)) \;\cong\; \circ\!\!\circ\, e^{-i\sqrt{\frac{2}{\alpha'}}H^a(z)} \,\circ\!\!\circ\, C^-_a \,, \tag{4.199}$$

---

[16]Note that the conformal weights of the fermions and normal-ordered exponentials match. Both systems also have the same central charge.

for $a = 1, \ldots, 4$. The reason for the additional *cocycle factors*, $C_a^\pm$, is explained in the next paragraph. For the light-cone, we define

$$\Psi^0(z) \equiv \frac{1}{\sqrt{2}}(-\psi^0(z) + \psi^1(z)) \cong \,{}^\circ_\circ\, e^{i\sqrt{\frac{2}{\alpha'}}H^0(z)} \,{}^\circ_\circ\, C_0^+ , \quad (4.200)$$

$$\bar{\Psi}^0(z) \equiv \frac{1}{\sqrt{2}}(\psi^0(z) + \psi^1(z)) \cong \,{}^\circ_\circ\, e^{-i\sqrt{\frac{2}{\alpha'}}H^0(z)} \,{}^\circ_\circ\, C_0^- . \quad (4.201)$$

For consistency, we need to check that the right hand sides of the above identifications have the same anticommutation relations as the left hand sides. It is for this reason that the cocycle factors are included. Without them the identifications are inconsistent because we find, upon using (4.191), that ${}^\circ_\circ\, e^{i\sqrt{\frac{2}{\alpha'}}H^a(z')} \,{}^\circ_\circ$ commutes with $e^{\pm i\sqrt{\frac{2}{\alpha'}}H^b(z')} \,{}^\circ_\circ$ for $a \neq b$, whereas Ψ^a anticommutes with Ψ^b, $\bar{\Psi}^b$. As is easily verified and shown in [98, 99], the proper cocycle factors are

$$C_0^\pm = 1 \ , C_a^\pm = \exp\left(\pm\pi i \sum_{b<a} p^b\right) , \ a = 1, \ldots, 4 \ , \quad (4.202)$$

where p^a is the zero-mode momentum of H^a. Cocycle factors are usually not explicitly written (but are understood) in bosonisation formulae.

The spin field is now defined as

$$S_\alpha(z) = \,{}^\circ_\circ \exp\left(i\sqrt{\frac{2}{\alpha'}}\sum_{a=0}^{4} s_a H^a(z)\right) {}^\circ_\circ \, , \quad (4.203)$$

where $\alpha = (s_0, \ldots, s_4)$ is an index with each s_a being $\pm\frac{1}{2}$, giving $2^5 = 32$ components. That such a field does represent a spinor can be deduced from the fact α lies in one of the two spinorial conjugacy classes of the weight lattice of the $D_5 = so(10)$ Lie algebra. (One normally Wick rotates the X^0 direction so that the Lorentz group is $SO(10)$ and therefore $\psi^0(z) \to -i\psi^0(z)$.) The **16** corresponds to those α which have an even number of "+" signs (the S conjugacy class) and the **16′** to those α which have an odd number of "+" signs (the C class).

Some Correlators and OPEs of the Spin Field

Note that $S_\alpha(z)$ is a primary field of conformal weight $(\frac{5}{8}, 0)$. This can also be seen from the $1-1$ correspondence between primary fields $\phi(z)$ of weight $(h, 0)$ and highest weight states $|h\rangle = \phi(0)|0\rangle_{NS}$ with eigenvalue $L_0 = h$ and the fact that

$L_0|0\rangle_R = \frac{D}{16}|0\rangle_R = \frac{5}{8}|0\rangle_R$ in 10 dimensions. In addition, the OPE of a primary field with its conjugate includes the identity (as for any primary field in a unitary theory). Taking into account (4.83), we must have

$$\langle S^\alpha(z')S_\beta(z)\rangle = \frac{\delta^\alpha_\beta}{(z'-z)^{5/4}} \ , \quad S^\alpha(z')S_\beta(z) = \frac{\delta^\alpha_\beta}{(z'-z)^{5/4}} + \cdots \ , \tag{4.204}$$

where spinor indices are raised and lowered with the charge conjugation matrix[17]. Note that taking $z' \to \infty, z \to 0$ implies that the Ramond vacuum should be normalised as

$${}_R\langle 0;\alpha|0;\beta\rangle_R = C_{\alpha\beta} \ . \tag{4.205}$$

We also need to consider the correlator

$$\langle S^\alpha(z')\psi^\mu(z_1)S_\beta(z)\rangle = (F^\mu)^\alpha{}_\beta(z',z_1,z) \ . \tag{4.206}$$

Equation (4.84) determines F to be of the form

$$(F^\mu)^\alpha{}_\beta(z',z_1,z) = (f^\mu)^\alpha{}_\beta(z'-z_1)^{-1/2}(z_1-z)^{-1/2}(z'-z)^{-3/4} \ . \tag{4.207}$$

The coefficient is determined by again taking $z' \to \infty, z \to 0$. Using (4.187), we obtain

$$\lim_{\substack{z'\to\infty,\\ z\to 0}} z'^{5/4}(F^\mu)^\alpha{}_\beta(z',z_1,z) = {}_R\langle 0|\psi^\mu(z_1)|0\rangle_R = (\psi^\mu_0)^\alpha{}_\beta z_1^{-1/2} = \frac{1}{\sqrt{2}}(\Gamma^\mu)^\alpha{}_\beta z_1^{-1/2}. \tag{4.208}$$

Therefore, $(f^\mu)^\alpha{}_\beta = \frac{1}{\sqrt{2}}(\Gamma^\mu)^\alpha{}_\beta$. By taking the limit of (4.206) as $z_1 \to z$ and comparing with (4.204) we deduce the OPE

$$\psi^\mu(z')S_\beta(z) = (z'-z)^{-1/2}\frac{1}{\sqrt{2}}(\Gamma^\mu)^\beta{}_\alpha S_\beta(z) + \cdots \ . \tag{4.209}$$

Doubling of the Spin Field

In order to extend the above analysis to the open string one should double the spin field. After doubling the fermionic fields $\psi^\mu(z)$ according to (4.141) we find the scalars H^a are doubled thus: $H^a(z) = \tilde{H}^a(z) + \sqrt{2\alpha'}\pi n^a$, $(n^a \in \mathbb{Z})$ for $\mathrm{Im}\, z \leq 0$. Therefore, we find that the spin field is doubled as follows:

$$S_\alpha(z) := (-1)^{\sum_{a=0}^{4}\mathrm{sgn}\,(s_a)n^a}\tilde{S}_\alpha(z) \ , \quad \mathrm{Im}\, z \leq 0 \ . \tag{4.210}$$

[17]Spinor conventions are listed in appendix A.

The phase on the right side of this equation is just ± 1. An alternative but equivalent way to see this is to suppose that the doubling is via

$$S_\alpha(z) := (M)_\alpha{}^\beta \tilde{S}_\beta(z) \ , \quad (\mathrm{Im}\, z \leq 0) \ , \tag{4.211}$$

for some matrix M. Making this replacement and the substitution $\psi^\mu = D^\mu_\nu \tilde{\psi}^\nu$ in (4.209) and comparing with corresponding antiholomorphic OPE yields the relation

$$(M\Gamma^\mu) = D^\mu_\nu(\Gamma^\nu M) \ , \tag{4.212}$$

where we have used relation (4.76). Consistency with (4.204) and its antiholomorphic counterpart imposes the further restriction

$$(M)_\alpha{}^\gamma (M)_\beta{}^\delta C_{\gamma\delta} = C_{\alpha\beta} \ , \tag{4.213}$$

that is, $MC^{-1}M^T = C^{-1}$. For $D^{\mu\nu} = \eta^{\mu\nu}$, condition (4.212) and Schur's Lemma fix M to be bI, where I is the identity matrix. Condition (4.213) then fixes $b = \pm 1$ but it is conventional to choose $b = 1$. When we consider D-branes in section 4.2 we will see that the doubling matrix M assumes a more complicated form.

Bosonisation of Ghosts

The ghost fields can be bosonised by using scalar fields in the presence of a background charge vector, Q_μ. The background charge modifies the stress tensor $T(z)$ (cf. equation (4.101)) of D scalar fields by a term $\frac{1}{\sqrt{2\alpha'}} Q \cdot \partial^2 X$. Consequently, as is easily verified by taking the derivative of (4.65), the central charge of the system is also modified to the value $D + 3Q_\mu Q^\mu$. Fixing the size of Q_μ appropriately allows one to obtain a central charge of -26 or $+11$ which can be matched with the b, c or β, γ theory, respectively.

Bosonisation of the b, c system is given in terms of a single "timelike" scalar field by

$$b(z) \cong \,:\! e^{-\sqrt{\frac{2}{\alpha'}}H(z)} \!:\, , \quad c(z) \cong \,:\! e^{\sqrt{\frac{2}{\alpha'}}H(z)} \!:\, , \tag{4.214}$$

$$H(z')H(z) = \frac{1}{2}\alpha' \ln(z' - z) + \cdots \ . \tag{4.215}$$

The correct OPEs (4.167) are easily verified. In addition, it can be shown that $\circ\!\!\circ\, e^{q\sqrt{\frac{2}{\alpha'}}H(z)} \,\circ\!\!\circ$ is a primary field of conformal weight $(\frac{1}{2}q(q+Q), 0)$. Hence, $Q = -3$ if the conformal dimensions of the fields on both sides of the identification are to match. Furthermore, the central charge of the scalar system is $c = 1 - 3Q^2 = -26$.

On the other hand, bosonisation of the β, γ system is via a single "spacelike" scalar field:

$$\beta(z) \quad \cong \quad \circ\!\!\circ\, e^{-\sqrt{\frac{2}{\alpha'}}\phi(z)} \partial\xi(z) \,\circ\!\!\circ\,, \quad \gamma(z) \quad \cong \quad \circ\!\!\circ\, e^{\sqrt{\frac{2}{\alpha'}}\phi(z)} \eta(z) \,\circ\!\!\circ\,, \tag{4.216}$$

$$\phi(z')\phi(z) = -\frac{1}{2}\alpha' \ln(z'-z) + \cdots \,. \tag{4.217}$$

In these formulae η and ξ are holomorphic fermions of weights $(1,0)$ and $(0,0)$, respectively. They form a system of b, c type with the OPEs $\eta(z')\xi(z) = \xi(z')\eta(z) = (z'-z)^{-1} + \cdots$. Their introduction is necessary in order to accommodate Bose statistics. Again, the OPEs (4.168) are easily verified. Analogously to the b, c case, it can be shown that $\circ\!\!\circ\, e^{q\sqrt{\frac{2}{\alpha'}}\phi(z)} \,\circ\!\!\circ$ is a primary field of conformal weight $(-\frac{1}{2}q(q+Q), 0)$. The central charge of the scalar system is $c = 1 + 3Q^2$ and that of the η, ξ system is -2. This then requires $Q = 2$ for the identifications to be consistent. The η, ξ system can be bosonised in a similar fashion to the b, c system with background charge $Q = -1$.

The above bosonisation was for the holomorphic sector of the closed string but, as usual, it also applies for the open string after doubling of the fields H, ϕ, η, ξ as in (4.186).

Finally, it is important to note that the background charge can be related to the Riemann-Roch theorem. For the β, γ system one finds that there are no β and $Q = 2$ γ zero modes on the sphere. On the other hand, for the b, c system one finds that there are no b and $-Q = 3$ c zero modes, as is well known. Likewise, there are no η zero modes and $-Q = 1$ ξ zero mode. Consequently, the path integral over ϕ, η, ξ on the sphere (or extended complex plane) vanishes due to the Grassmann integral over the zero mode ξ_0 unless there is an insertion of $\xi(z)$ at some point (the specific point does not matter). However, (4.216) involves $\partial\xi$ rather than ξ; the zero mode ξ_0 never appears in the β, γ algebra. Therefore, it is possible to restrict to the "small" algebra with ξ_0 omitted and not integrated over in the path integral. An insertion is then not required. Similar remarks apply to the b, c system when integrating over c_0 (although there is no reduced algebra in this case). Three insertions of $c(z)$ are then required in the path integral on the sphere. We return to this point in section 4.1.6.

Ghost q-vacua and Inner Products

In order to calculate scattering amplitudes and to give correlation functions computed from the path integral an interpretation as expectation values in a Hilbert space, we have to define an inner product on that space. The inner product in terms of bras and kets in the matter sector is the usual one (apart from the normalisation of the Ramond vacuum as given in (4.205)) but in the ghost sector the story is a bit more subtle. For example, according to (4.179) the *Siegel vacuum* $|\downarrow\rangle$ is annihilated by b_0. But then so too is the adjoint $\langle\downarrow|$ since $b_0^\dagger = b_0$. This implies that

$$\langle\downarrow|\downarrow\rangle = \langle\downarrow|\{b_0, c_0\}|\downarrow\rangle = 0 \ , \tag{4.218}$$

which gives vanishing amplitudes if the inner product is naively defined in this way. The situation is remedied by the introduction of the ghost q-vacua.

In order to treat the ghosts and superghosts at the same time, we consider the *first order system*

$$S = \frac{1}{2\pi}\int d^2 z b\bar{\partial}c \ , \tag{4.219}$$

where $b(z)$ and $c(z)$ have conformal weights $(\lambda, 0)$ and $(1-\lambda, 0)$, respectively. If $\lambda = 2$ then this action corresponds to the holomorphic part of the b, c system, whereas if $\lambda = 3/2$ then it corresponds to the holomorphic part of the β, γ system.

The action (4.219) is invariant under the symmetry $\delta b = -i\eta b\,, \delta c = i\eta c$ which defines the current

$$j(z) \;=\; -\,\mathring{\circ}\, b(z)c(z)\,\mathring{\circ} \;=\; \epsilon\,\mathring{\circ}\, c(z)b(z)\,\mathring{\circ} = \sum_{m\in\mathbb{Z}} j_m z^{-m-1} \ , \tag{4.220}$$

where ϵ is $+1$ for $\lambda = 2$ and -1 for $\lambda = 3/2$. The modes are given by

$$j_m = -\sum_n : b_n c_{m-n} : +a\delta_{m,0} \ , \tag{4.221}$$

where the normal ordering constant a is $+1$ for the b, c system and -1 $(-1/2)$ in the NS (R) sector of the β, γ system. In addition, the stress tensor can be written as

$$T^g(z) \;=\; :\!(\partial b)c - \lambda\partial(bc)\!: \; . \tag{4.222}$$

This is consistent with (4.175). Therefore, we find the following OPEs:

$$\begin{aligned}
T^g(z')j(z) &= \frac{Q}{(z'-z)^3} + \frac{1}{(z'-z)^2}j(z) + \frac{1}{z'-z}\partial j(z) \;, && (4.223)\\
j(z')j(z) &= \frac{\epsilon}{(z'-z)^2} \;, && (4.224)\\
j(z')b(z) &= \frac{-1}{z'-z}b(z) \;, && (4.225)\\
j(z')c(z) &= \frac{1}{z'-z}c(z) \;, && (4.226)
\end{aligned}$$

where $Q \equiv \epsilon(1-2\lambda)$ has the same value as the background charge of the equivalent bosonised theory. The Q-dependent term in (4.223) means that $j(z)$ is not a true conformal field of weight $(1,0)$ as defined by (4.90). (Furthermore, the T^gT^g OPE gives a central charge of $c = \epsilon(1-3Q^2)$, that is, -26 for $\lambda = 2$ and 11 for $\lambda = 3/2$.) These OPEs translate into the commutation relations

$$\begin{aligned}
[L_m, j_n] &= -nj_{m+n} + \frac{Q}{2}m(m+1)\delta_{m+n,0} \;, && (4.227)\\
[j_m, j_n] &= \epsilon m\delta_{m+n,0} \;, && (4.228)\\
[j_m, b_n] &= -b_{m+n} \;, && (4.229)\\
[j_m, c_n] &= c_{m+n} \;, && (4.230)
\end{aligned}$$

The zero mode j_0 is called the *ghost number operator* and counts -1 for b and $+1$ for c.

Using the hermiticity properties (4.173), we find that $j_m^\dagger = -j_{-m}$ for all $m \neq 0$. The case $m = 0$ is delicate because of normal ordering ambiguities. One finds

$$j_0^\dagger = -[L_{-1}, j_1]^\dagger = -[L_1, j_{-1}] = -j_0 - Q \; . \tag{4.231}$$

We now introduce the q-vacua defined by the annihilation relations

$$\begin{aligned}
b_m|q\rangle &= 0, \quad m > \epsilon q - \lambda \;, && (4.232)\\
c_m|q\rangle &= 0, \quad m \geq -\epsilon q + \lambda \; . && (4.233)
\end{aligned}$$

where q is an integer for the the b, c system and the NS sector of the β, γ system and in $\mathbb{Z}+\frac{1}{2}$ for the Ramond sector. Using the expansions (4.177) and (4.221), it is easy to show[18] that $|q\rangle$ is an eigenstate of both j_0 and L_0^g

$$j_0|q\rangle = q|q\rangle \ , \ L_0^g|q\rangle = \frac{1}{2}\epsilon q(q+Q)|q\rangle \ , \tag{4.234}$$

and is annihilated by j_m and L_m for $m > 0$. It is easy to check that the b, c Siegel vacuum $|\downarrow\rangle$ defined in (4.179) corresponds to $q = 1$, while for the β, γ system the NS and R vacua $|-1\rangle$ and $|-\frac{1}{2}\rangle$, defined in (4.180) and (4.181), correspond to $q = -1$ and $q = -\frac{1}{2}$, respectively (which evidently explains the notation). The vacuum $|q = 0\rangle$ (which we shall denote simply by $|0\rangle$) is the unique $PSL(2,\mathbb{R})$-invariant vacuum because the relations (4.232,4.233) imply that it is annihilated by the $PSL(2,\mathbb{R})$ generators $L_0, L_{\pm 1}$. Note that for the case $\lambda = 2$, it is possible to go from one vacuum to another by the application of a finite number of creation or annihilation operators. For instance,

$$\begin{aligned} |\downarrow\rangle &= c_1|0\rangle = c(0)|0\rangle \Longleftrightarrow |0\rangle = b_{-1}|\downarrow\rangle \ , &(4.235)\\ |\uparrow\rangle &\equiv |q=2\rangle = c_0|\downarrow\rangle \ . &(4.236)\end{aligned}$$

However, this is not possible with the Bose statistics of the β, γ system.

Now, using (4.231), we find that if $\mathcal{O}_p$ is an operator with ghost charge p, that is, $[j_0, \mathcal{O}_p] = p\mathcal{O}_p$ then $p\langle q'|\mathcal{O}_p|q\rangle = \langle q'|[j_0, \mathcal{O}_p]|q\rangle = -(q'+q+Q)\langle q'|\mathcal{O}_p|q\rangle$. This implies that $\langle q'|\mathcal{O}_p|q\rangle$ is zero unless $q'+p+q = -Q$. In particular, taking $\mathcal{O}_p$ with $p = 0$ to be the identity operator we find that the q-vacua are normalised as follows

$$\langle q'|q\rangle = \delta_{q',-q-Q} \ , \tag{4.237}$$

which explains the result (4.218). (Similar remarks apply to the η, ξ system if one includes the zero mode ξ_0. However, if one restricts to the reduced algebra and does not include ξ_0 it is not necessary to neutralise the background charge $Q = -1$. In that

[18]One must remember to split the normal ordering constant $d^g(\nu)$ defined in (4.177) into its different components: -1 from the b, c system and $1/2$ $(3/8)$ from the NS (R) sector of the β, γ system.

case, one can take $\langle 0|0\rangle = 1$.) Equation (4.237) implies that the correlator receives a finite correction from the vacuum charge:

$$\langle c(z')b(z)\rangle_q \equiv \langle -q-Q|c(z')b(z)|q\rangle = \left(\frac{z'}{z}\right)^{\epsilon q}\frac{1}{z'-z} \ . \tag{4.238}$$

This is consistent with the OPEs (4.167) and (4.168).

The q-vacua for non-zero q are identical to the states $|q\rangle = \,:e^{q\sqrt{\frac{2}{\alpha'}}\chi(0)}:\,|0\rangle$, where $\chi(z)$ is either the scalar $H(z)$ or $\phi(z)$ as appropriate. This can be seen by bosonising the current:

$$j(z) = \epsilon\sqrt{\frac{2}{\alpha'}}\partial\chi \ , \tag{4.239}$$

and calculating the OPE/commutation relation

$$\begin{aligned} j(z')\,:e^{q\sqrt{\frac{2}{\alpha'}}\chi(z)}: &= \frac{q}{z'-z}\,:e^{q\sqrt{\frac{2}{\alpha'}}\chi(z)}: \\ &\Rightarrow\ [j_0, :e^{q\sqrt{\frac{2}{\alpha'}}\chi(z)}:] = q\,:e^{q\sqrt{\frac{2}{\alpha'}}\chi(z)}: \ . \end{aligned} \tag{4.240}$$

Alternatively, consider, for example, the case $\lambda = 3/2$. Then

$$\begin{aligned} b_m|q\rangle &= \frac{1}{2\pi i}\oint_{C_0} dz z^{m+\lambda-1} b(z)|q\rangle \\ &= \frac{1}{2\pi i}\oint_{C_0} dz z^{m+\lambda-1}\,:e^{-\sqrt{\frac{2}{\alpha'}}\phi(z)}\partial\xi(z)::e^{q\sqrt{\frac{2}{\alpha'}}\phi(0)}:|0\rangle \\ &= \frac{1}{2\pi i}\oint_{C_0} dz z^{m+\lambda+q-1}\,:e^{-\sqrt{\frac{2}{\alpha'}}\phi(z)}\partial\xi(z)e^{q\sqrt{\frac{2}{\alpha'}}\phi(0)}:|0\rangle \ , \end{aligned} \tag{4.241}$$

where we have used (4.190) and (4.217) in going from the second to the third line. Equation (4.232) now follows from regularity of the normal ordered product at the origin. We also see that when acting with an operator in the NS sector ($m+\lambda =$ integer) on a state with $q \in \mathbb{Z}+\frac{1}{2}$ we get a branch cut. Hence, such states belong to the R sector and the superghost spin field is therefore

$$\Sigma_q(z) \cong\, :e^{q\sqrt{\frac{2}{\alpha'}}\phi(z)}: \ , \quad q \in \mathbb{Z}+\frac{1}{2} \ . \tag{4.242}$$

Note that $\Sigma_{-1/2}$ has conformal weight $(\frac{3}{8},0)$.

4.1.6 Vertex Operators and Scattering Amplitudes

The notion of a *vertex operator* is one of the major ingredients in string theory and is needed in order to calculate the on-shell scattering amplitudes that describe physical processes. In conventional quantum field theory, there is on the one hand the space of states of the theory and on the other hand the set of local operators. In the text surrounding equation (4.204) we have already alluded to the *state-operator correspondence* which states that in a conformal field theory the set of states and the set of local operators are isomorphic. Let us consider this in more detail following [53].

Scattering amplitudes between in and out single-particle states are due to worldsheets with incoming and outgoing legs that are semi-infinite cylinders for the closed string or strips for the open string (see figure 4.1). Consider, for example, the sim-

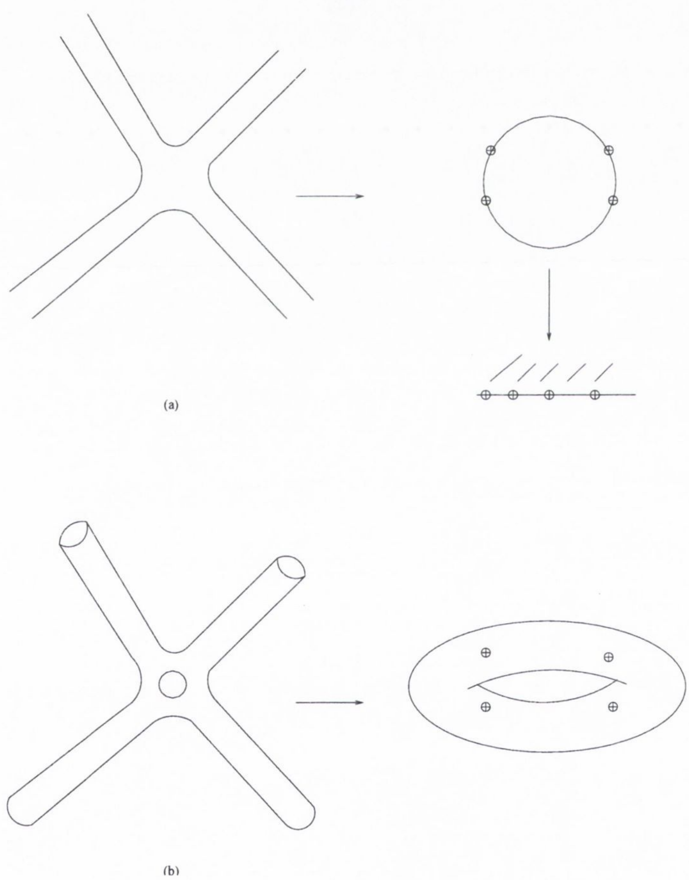

Figure 4.1: Typical amplitudes in string theory. Conformal transformations of the worldsheet map asymptotic states to finite points: (a) an open string tree amplitude in which the worldsheet is mapped onto the disc or upper half-plane with external states appearing as points on the boundary denoted by $\oplus$, (b) a one-loop closed string amplitude is mapped to the punctured torus.

plest case of a worldsheet with only one incoming and one outgoing closed string. It is described by the cylinder

$$w = \sigma^1 + i\sigma^2 \ , \ \sigma^2 \sim \sigma^2 + 2\pi \ , -\infty < \sigma^1 < \infty \ , \tag{4.243}$$

and can be mapped onto the plane by the conformal transformation $z = e^w$. To define the path integral in the w coordinate one must specify the boundary conditions on the fields as $\mathrm{Re}\, w \to \pm\infty$. In the z-coordinate this is equivalent to specifying the behaviour of the fields at the origin and at infinity. In effect, this defines local operators at the origin and at infinity associated with the incoming and outgoing states. Such operators are vertex operators. Note that the transformation to the plane is exactly Penrose's idea of conformal infinity. The metric on a cylinder of unit radius is $ds^2 = d(\sigma^1)^2 + d(\sigma^2)^2 = \frac{1}{r^2} d\tilde{s}^2$, where $d\tilde{s}^2 = dr^2 + r^2 d(\sigma^2)^2$ is the metric on the plane with $r = |z| = e^{\sigma^1}$. The conformal rescaling $ds^2 \to d\tilde{s}^2 = e^\phi ds^2, e^\phi = r^2$ then brings in the point $\sigma^1 = -\infty$ to the finite point $r = 0$. One can also bring in the point $z = +\infty$ to finite distance by modifying the conformal factor e^ϕ. To do this one maps to the sphere rather than the plane by defining $\sigma^1 = 2\ln(\tan\frac{1}{2}\theta)$, $0 < \theta < \pi$. The points $\sigma^1 = -\infty$ and $\sigma^1 = +\infty$ now map to the north and south poles, respectively. The conformal factor is now $e^\phi = \sin^2\theta$. Going in the other direction, the path integral on the sphere with vertex operator insertions $\mathcal{A}$ and $\mathcal{B}$ at the north and south pole maps to the path integral on the cylinder with specified initial and final state $|\mathcal{A}\rangle$ and $|\mathcal{B}\rangle$. For more complicated diagrams with more external legs (such as those in figure 4.1) the important point is that each leg L can be mapped to a finite point because it is only the asymptotic behaviour of e^ϕ far out on L that is relevant and this behaviour can be chosen independently for each L.

In conclusion, to determine a scattering amplitude between specified incoming and outgoing states, one calculates the path integral on a compact worldsheet (that is, a Riemann surface) of appropriate topology (eg., a torus for one-loop closed strings as in figure 4.1(b)) with appropriate local vertex operator insertions representing the states, that is, one calculates a correlator.

Form of the Correlator

We are interested in calculating tree amplitudes for closed or open strings. In the discussion below we concentrate first on bosonic closed strings for which the world-sheet has the topology of a sphere. We then move on to bosonic open strings and finally superstrings.

Bosonic Closed Strings

For the correlator of a product of vertex operators, the naive expression (4.61) is more precisely

$$\langle\prod_i V_i\rangle = \int_{\mathcal{M}} \frac{DXDg}{\mathrm{Vol(Diff)\,Vol(Weyl)}}\, e^{-S_X[X,g]} \prod_i V_i \ . \tag{4.244}$$

In this formula, $\mathcal{M}$ is the space of all metrics on the sphere and the factor Vol(Diff) Vol(Weyl) takes care of the infinite overcounting due to the gauge symmetry: if (X^μ, g) and (X'^μ, g') are related by a Diff $\times$ Weyl transformation they represent the same physical configuration. In writing (4.244), we have omitted a topological weight factor $e^{-\lambda\chi}$, where $\chi = 2 - 2g - b - c$ is the Euler number of the closed or open string worldsheet depending on its genus (g), its number of boundaries (b) and number of cross-caps (c). This weight factor is relevant when summing over different topologies (for the sphere $g = b = c = 0$) but will not be relevant to us[19]. The idea is now to use the Diff $\times$ Weyl symmetry to transform to conformal gauge, as outlined in the text preceding equation (4.8). In effect, this maps the sphere to the (extended) complex plane. As a result of this transformation, it can be shown that the path integral (4.244) reduces to

$$\langle\prod_i V_i\rangle = \int \frac{DXD'(c\tilde{c})D(b\tilde{b})}{\mathrm{Vol(CKG)}}\, e^{-S_X - S_g} \prod_i V_i \ , \tag{4.245}$$

[19]In fact, we see that higher topologies, which have larger $|\chi|$, are analogous to a higher number of loops in a field theory loop expansion. Increasing g by one adds a handle to the original Riemann surface and increases the amplitude by a factor $e^{2\lambda}$. Since adding a handle corresponds to emitting and reabsorbing a closed string, we see that the closed string coupling constant g_s is proportional to e^{λ}. Similarly, increasing b by one increases the amplitude by a factor e^{λ} and corresponds to emitting and reabsorbing an open string. The open string coupling constant is therefore given by $g_o \sim e^{\lambda/2}$.

where S_X is given by (4.57), S_g is the ghost part of the action (4.164) and the prime in the measure indicates that ghost zero modes are omitted.

We briefly reason why this reduction should hold[20]. It comes about because under combined diffeomorphisms and Weyl rescalings metric deformations consist of two pieces: the Lie derivative part from the diffeomorphisms and the conformal part from the Weyl rescalings. These two parts can be split orthogonally (with respect to a certain inner product) into the symmetric traceless part and the trace part. Since on the sphere there are no metric deformations that cannot be absorbed by diffeomorphisms and Weyl rescalings (that is, there are no moduli), one trades in the integration over DX and Dg for one over DX, the conformal factor and globally defined vector fields V that are taken into symmetric traceless tensors under the action of the operator P defined by

$$(PV)_{\alpha\beta} \equiv \nabla_\alpha V_\beta + \nabla_\beta V_\alpha - (\nabla_\gamma V^\gamma) g_{\alpha\beta} \; . \tag{4.246}$$

The Jacobian of this trade-in consists of two factors. The first factor is the Faddeev-Popov determinant which is "exponentiated" to give the path integral over the ghosts in (4.245). The second factor is due to the Weyl anomaly. It stems from the fact that the measure $DXDg$ is not invariant under Weyl rescalings but involves the conformal field ρ. However, this second factor depends on the total central charge $c^X + c^g = D - 26$ and is unity in the critical dimension, that is, when $D = 26$. Therefore, we ignore this factor. The integrand is then independent of the conformal field and hence this field may be integrated over to cancel the Vol(Weyl) factor. Next, we note that P has zero modes on the sphere. These zero modes constitute the *conformal Killing group* (CKG) and are just the Möbius transformations (3.16) that we mentioned earlier. The group is thus $PSL(2, \mathbb{C})$ and is generated by the three vectors $\partial, z\partial, z^2\partial$ with complex coefficients. One can see this by solving $(PV) = 0$, noting that the resulting vectors must be globally defined and that such vectors generate the infinitesimal form of the Möbius transformations. The existence of a

[20]The technical analysis behind this reasoning is detailed and subtle. The reader is probably best referred to the literature for further details. See, for example, [93] or the original papers of Polyakov on the transformation properties of the path integral measure [100].

non-trivial CKG implies that diffeomorphisms orthogonally decompose as Diff $=$ Diff$^\perp \oplus$ CKG. Hence, Vol(Diff) $=$ Vol(Diff$^\perp$) Vol(CKG). The path integral over the vectors V is therefore over the non-zero modes Diff$^\perp$. However, the integrand does not depend on V; the integral may be performed to yield a factor Vol(Diff$^\perp$) and so we obtain (4.245).

Equation (4.245) is not yet in its final form. In the reduction we implicitly assumed that the vertex operators were (Diff $\times$ Weyl)-invariant, in particular they should be conformally invariant on the complex plane. Diff-invariance is obtained by integrating the vertex operators over the worldsheet, that is, we have

$$V_i = 2\int d^2\sigma\sqrt{g}\mathcal{V}_i \ , \tag{4.247}$$

where $\mathcal{V}_i$ is built from the X^μ but is independent of the ghosts. In conformal gauge this becomes

$$V_i = \int d^2 z_i \mathcal{V}_i(z_i, \bar{z}_i) \ , \tag{4.248}$$

and since d^2z transforms as a $(-1,-1)$ tensor under conformal transformations we find that $\mathcal{V}_i$ should be a primary field of conformal weight $(1,1)$. V_i is thus interpreted as the insertion of the vertex operator $\mathcal{V}_i$ at the point z_i summed over all possible insertion points. Assuming the correlator contains at least three vertex operators, we can trade in the integrations over z_i, $i = 1,2,3$ for an integration over the conformal Killing group. This is possible because given any two triplets of distinct points $z_i, z_i^0 \in \mathbb{C}_\infty$ there is a unique element $\zeta \in PSL(2,\mathbb{C})$ such $\zeta(z_i^0) = z_i$. In order to calculate the Jacobian of the transformation, we write the (infinitesimal) conformal Killing vectors as

$$V_0 = V_0^z\partial, \quad V_0^z = \alpha_0 + \alpha_1 z + \alpha_2 z^2 \ . \tag{4.249}$$

Then $z_i = \zeta(z_i^0) = e^{V_0} z_i^0 \Longrightarrow dz_i = \sum_{a=0}^2 d\alpha_a z_i^a$ and so

$$\prod_{i=1}^3 d^2 z_i = \left| \det \begin{pmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1^2 & z_2^2 & z_3^2 \end{pmatrix} \right|^2 \prod_{a=0}^2 d^2\alpha_a \ , \tag{4.250}$$

$$= |z_{12}z_{23}z_{13}|^2 \prod_{a=0}^2 d^2\alpha_a \ , \tag{4.251}$$

where $z_{ij} = z_i - z_j$. Integration over α_a then cancels the Vol(CKG) factor in (4.245). The points z_i, $i = 1, 2, 3$ are now fixed but arbitrary. We also note that in a Hilbert space interpretation the correlator up to overall normalisation is just the vacuum expectation value:

$$\langle \prod_i \mathcal{V}_i \rangle = \langle 0 | R(\prod_i \circ\, \mathcal{V}_i \circ) | 0 \rangle \ , \tag{4.252}$$

where $|0\rangle$ is the bosonic $PSL(2, \mathbb{C})$-invariant vacuum (see below) and R denotes radial ordering. We can use this correspondence to write the factor $|z_{12} z_{23} z_{13}|^2$ equivalently as a ghost insertion in the correlator/path integral:

$$\begin{aligned} \langle c(z_1)\tilde{c}(\bar{z}_1)c(z_2)\tilde{c}(\bar{z}_2)c(z_3)\tilde{c}(\bar{z}_3)\rangle &= \langle 0|c(z_1)\tilde{c}(\bar{z}_1)c(z_2)\tilde{c}(\bar{z}_2)c(z_3)\tilde{c}(\bar{z}_3)|0\rangle \ , \\ &= |\langle 0|c(z_1)c(z_2)c(z_3)|0\rangle|^2 \ , \\ &= |z_{12}z_{23}z_{13}|^2 \ , \end{aligned} \tag{4.253}$$

where we have assumed $|z_1| > |z_2| > |z_3|$ and used the explicit mode expansions (4.170). Therefore, we arrive at our final (standard) form for the correlator of $n \geq 3$ closed string vertex operators:

$$\begin{aligned} &\langle c\tilde{c}\mathcal{V}_1(z_1, \bar{z}_1)c\tilde{c}\mathcal{V}_2(z_2, \bar{z}_2)c\tilde{c}\mathcal{V}_3(z_3, \bar{z}_3)\prod_{i=4}^{n}\int d^2 z_i \mathcal{V}_i(z_i, \bar{z}_i)\rangle \\ &= \int DXD(c\tilde{c})D(b\tilde{b})\, e^{-S_X - S_g}\, c\tilde{c}\mathcal{V}_1(z_1, \bar{z}_1)c\tilde{c}\mathcal{V}_2(z_2, \bar{z}_2)c\tilde{c}\mathcal{V}_3(z_3, \bar{z}_3) \\ &\qquad \times \prod_{i=4}^{n}\int d^2 z_i \mathcal{V}_i(z_i, \bar{z}_i) \ . \end{aligned} \tag{4.254}$$

The important point is that (4.254) is now just a correlator in a free field theory and can be evaluated in the standard fashion à la (4.62) and (4.63) or by using the correspondence (4.252). The correlator vanishes if $n < 3$ because we cannot completely cancel the infinite factor Vol(CKG) from (4.245). The physical interpretations of these vanishing amplitudes are that the tree-level vacuum energy is zero ($n = 0$), there are no tree-level tadpoles ($n = 1$) and there are no mass corrections at tree-level ($n = 2$).

Bosonic Open Strings

For open bosonic strings the situation is similar. The worldsheet has the topology of a (unit) disc at tree level and can be mapped to the upper half-plane (figure 4.1(a)) via the map $z = i(1+v)/(1-v)$, $v = e^{i\theta}$. Diffeomorphism invariance is limited to changes that take the boundary of the worldsheet into itself. For the upper half-plane the boundary is $z = \bar{z}$ and so the conformal Killing group is $PSL(2, \mathbb{R})$, that is, the group generated by the vectors (4.249) with $\alpha_a = \bar{\alpha}_a$. There are no moduli. Vertex operators are inserted on the boundary so are of the form

$$V_i = \int dz_i \mathcal{V}_i(z_i) \ , \tag{4.255}$$

where $z_i \in \mathbb{R}$ and $\mathcal{V}_i$ is a primary field of conformal weight $(1, 0)$ that is independent of the ghosts. As for the closed string, three vertex operators may be fixed at arbitrary points with corresponding c-ghost insertions. The rest are integrated over. $PSL(2, \mathbb{R})$ does not, however, change the cyclic ordering of the operators on the boundary so in calculating amplitudes one sums over all cyclic orderings of the vertex operators. If Chan-Paton matrices are involved we assign to each vertex operator one of the matrices. However, since Chan-Paton factors are non-dynamical, the right-hand end of string 1 must be in the same state as the left-hand end of string 2, etc. The result is a trace over the corresponding cyclic ordering of the matrices.

We should note that the discussion in this section is of course consistent with our remarks in section 4.1.5 concerning the relationship of the background charge to the Riemann-Roch theorem and with our discussion around (4.237). This is because the Riemann-Roch theorem is also related to the degeneracy of the ghost zero modes: the number of (complex) moduli equals the number of b zero modes and the number of (complex) conformal Killing vectors equals the number of c zero modes on any Riemann surface. (Note that for open strings $\langle c(z_1)c(z_2)\tilde{c}(\bar{z}_3)\rangle$ is non-zero because of the doubling involved. This means that it is the *total* ghost number over both holomorphic and antiholomorphic sectors that should equal the background charge in order to get a non-vanishing amplitude. This contrasts with the closed string case where the ghost number in each sector should separately add up to $-Q$.)

Superstrings

The ghosts are treated in the same way as in the bosonic string above but we now also have to contend with the superghosts.

For closed strings the Riemann-Roch theorem tells us that there are no supermoduli (no β zero modes) and two conformal Killing spinors (two γ zero modes) on the sphere. It is usual to work with the bosonised versions of the superghosts, that is, with the fields ϕ, η, ξ. We will not consider the path integral here (see [101], for example, for further details) but simply quote the end result. The vertex operators $\mathcal{V}_i$ can now depend on the superghost fields. In particular, in the examples given below they depend on $e^{q^i_{sgh}\phi(z)}$ (in the holomorphic sector) and so have definite superghost number q^i_{sgh}. Therefore, in an obvious notation, the vertex operators are denoted $\mathcal{V}_i^{(q^i_{sgh},\tilde{q}^i_{sgh})}$ and are said to be in the *picture* $(q^i_{sgh}, \tilde{q}^i_{sgh})$. It is not necessary to have $q^i_{sgh} = \tilde{q}^i_{sgh}$ but the vertex operator should have weight $(1,1)$. To soak up the superghost number anomaly we must arrange to have $\sum_i q^i_{sgh} = -Q_\phi = -2$ and similarly in the antiholomorphic sector. One works in the small algebra with the zero modes $\xi_0, \tilde{\xi}_0$ excluded and not integrated over in the path integral. In this case, all vertex operators are independent of these zero modes and it is not necessary to neutralise the background charge $Q_{\eta,\xi} = -1$ by an insertion of $\xi(z)\tilde{\xi}(\bar{z})$. Amplitudes are then calculated in a similar fashion to the bosonic case. For open strings the situation is also similar after doubling.

Finally, it can be shown that the correlation functions are independent of how the total charge of -2 is distributed among the individual vertex operators. The proof involves passing back and forth between the small and the large algebra (in which ξ_0 is included) by inserting the identity $\int d\xi_0 \xi_0 = 1$ into the path integral. In the process (the precise details are not relevant) one comes across the (holomorphic) *picture changing operator*

$$\begin{aligned} P_{+1}(z) &= \circ\!\!\!\circ\, e^{\sqrt{\frac{2}{\alpha'}}\phi(z)} T_F(z) \,\circ\!\!\!\circ \\ &= i\sqrt{\frac{2}{\alpha'}} \,\circ\!\!\!\circ\, e^{\sqrt{\frac{2}{\alpha'}}\phi(z)} \psi(z)\cdot\partial X(z) \,\circ\!\!\!\circ\, , \end{aligned} \tag{4.256}$$

which acts on a vertex operator as

$$\mathcal{V}^{(q_{sgh}+1)}(z) = \circ\!\!\!\circ\, P_{+1}(z)\mathcal{V}^{(q_{sgh})}(z) \,\circ\!\!\!\circ\; . \tag{4.257}$$

Note that the operator $P_{+1}(z)$ has conformal weight zero and so $\mathcal{V}^{(q_{sgh}+1)}(z)$ and $\mathcal{V}^{(q_{sgh})}(z)$ have the same weight.

Physical Vertex Operators

The modern (and rigorous) way to derive vertex operators is to use BRST methods [53]. However, for our purposes this machinery is not required and so we shall take a more simple approach.

It can be shown that the $PSL(2,\mathbb{R})$-invariant vacuum[21]

$$|NS; k^\mu = 0\rangle \otimes |q_{gh} = 0\rangle \otimes |q_{sgh} = 0\rangle \ , \tag{4.258}$$

is associated with the unit operator, that is, to doing the path integral with no operator insertions. For the closed string, this vacuum represents only the holomorphic side and so should be tensored with that of the antiholomorphic side to give a $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R}) \cong PSL(2,\mathbb{C})$-invariant vacuum. We denote these vacua collectively by $|0\rangle$, as in the previous section.

Since $|0\rangle$ is associated with the unit operator, it follows that the state $|\mathcal{V}\rangle$ associated with the insertion of the vertex operator $\mathcal{V}$ at the origin is given by

$$|\mathcal{V}\rangle = \mathcal{V}(0)|0\rangle \ . \tag{4.259}$$

Therefore, for physical vertex operators $|\mathcal{V}\rangle$ should be a physical state. Since physical states are in the $|\downarrow\rangle = c(0)|0\rangle$ ghost vacuum[22], this means that such operators are in fixed form, that is, they include the ghost $c, \tilde{c}$ insertions. To get the integrated form one simply drops these terms.

Type II R-R Vertex Operator

Equation (4.259) implies that the spin fields $S_\alpha(z)$ and $\Sigma_q(z)$ of the last section are examples of vertex operators (although the states they create are not by themselves physical). Another example is the field $\circ\, e^{ik\cdot X(z)} \,\circ\circ\, e^{ik\cdot \tilde{X}(\bar{z})} \,\circ$ for the closed string (treating holomorphic and antiholomorphic components independently), the reason being that the bosonic vacuum $|0,\tilde{0};k\rangle = |0;k\rangle \otimes |\tilde{0};k\rangle$ is given by

$$|0,\tilde{0};k\rangle \ = \circ\, e^{ik\cdot X(0)} \,\circ\circ\, e^{ik\cdot \tilde{X}(0)} \,\circ\, |0,\tilde{0};k^\mu = 0\rangle = e^{ik\cdot(x_L + x_R)}|0,\tilde{0};k^\mu = 0\rangle \ , \tag{4.260}$$

[21]Recall that $|NS;k\rangle \equiv |0;k\rangle \otimes |0\rangle_{NS}$.

[22]Read $|\downarrow\downarrow\rangle = \tilde{c}(0)c(0)|0\rangle$ for the closed string.

and has momentum $p_L^\mu = p_R^\mu = p^\mu = k^\mu$. Now, since the Ramond vacuum of one side of the closed superstring is of the form $|0,k\rangle \otimes |0;\alpha\rangle_R \otimes |\downarrow\rangle \otimes |q_{sgh} = -\frac{1}{2}\rangle$ and is created from $|0\rangle$ by the operator

$$\mathcal{V}_\alpha^{(-1/2)}(z;k) \;=\; {}^\circ_\circ\, c(z)\Sigma_{-1/2}(z)S_\alpha(z)e^{ik\cdot X(z)} \,{}^\circ_\circ \;, \tag{4.261}$$

then the massless RR antisymmetric tensor fields of the type II string (cf. equation (4.161)) are created by the vertex operator

$$\mathcal{V}_{RR}^{(-1/2,-1/2)}(z,\bar{z};k) = V^{\alpha\beta}\mathcal{V}_\alpha^{(-1/2)}(z;k)\tilde{\mathcal{V}}_\beta^{(-1/2)}(\bar{z};k) \;, \tag{4.262}$$

where $k^2 = 0$ and $V^{\alpha\beta}$ is a polarisation bispinor encoding the physical state conditions and the GSO projections. Note that excluding the ghost fields $\mathcal{V}_{RR}^{(-1/2,-1/2)}(z,\bar{z};k)$ has conformal weight $(1,1)$.

Let us now determine the form of $V^{\alpha\beta}$. The antisymmetrised products $\Gamma^{\mu_1\dots\mu_k} = \Gamma^{[\mu_1}\dots\Gamma^{\mu_k]}$ of the gamma matrices[23] along with the identity I form a complete basis in which any bispinor can be expanded:

$$V_\alpha{}^\beta = (H_0)I_\alpha{}^\beta + \sum_{k=1}^{10}\frac{1}{k!}(H_k)_{\mu_1\dots\mu_k}(\Gamma^{\mu_1\dots\mu_k})_\alpha{}^\beta \;. \tag{4.263}$$

Let us consider one specific value of k in the sum (4.263). Ignoring the ghost sectors, the non-trivial physical state conditions $G_0 \sim \alpha_0\cdot\psi_0$ and $\tilde{G}_0 \sim \tilde{\alpha}_0\cdot\tilde{\psi}_0$ (cf. equations (4.133) and (4.151)) amount to two Dirac equations, one acting on the left-movers and the other on the right-movers:

$$(k\cdot\Gamma)_\alpha{}^\gamma(H_k)_{\mu_1\dots\mu_k}(\Gamma^{\mu_1\dots\mu_k})^{\alpha\beta} = (H_k)_{\mu_1\dots\mu_k}(\Gamma^{\mu_1\dots\mu_k})^{\alpha\beta}(k\cdot\Gamma)_\beta{}^\gamma = 0 \;. \tag{4.264}$$

These translate into

$$\Gamma^\nu\Gamma^{\mu_1\dots\mu_k}\partial_\nu(H_k)_{\mu_1\dots\mu_k} = \Gamma^{\mu_1\dots\mu_k}\Gamma^\nu\partial_\nu(H_k)_{\mu_1\dots\mu_k} = 0 \;, \tag{4.265}$$

after Fourier transformation. By use of the identities

$$\Gamma^\nu\Gamma^{\mu_1\dots\mu_k} \;=\; \Gamma^{\nu\mu_1\dots\mu_k} + k\eta^{\nu[\mu_1}\Gamma^{\mu_2\dots\mu_k]} \;, \tag{4.266}$$

$$\Gamma^{\mu_1\dots\mu_k}\Gamma^\nu \;=\; (-1)^k\Gamma^{\nu\mu_1\dots\mu_k} + (-1)^{k+1}k\eta^{\nu[\mu_1}\Gamma^{\mu_2\dots\mu_k]} \;, \tag{4.267}$$

[23] We define antisymmetrisation with the factorial factor included.

we find that equations (4.265) are equivalent to

$$dH_k = d * H_k = 0 \ . \tag{4.268}$$

These are just the Bianchi identity and field equation for an abelian k-form field strength[24]. Since $d^2 = 0$, we can write at least locally $H_k = dC_{k-1}$ where C_{k-1} is the $(k-1)$-form potential.

At this point we have not taken into account the GSO conditions which, according to (4.161), result in field strengths of rank 2 and 4 for type IIA and of rank $1, 3$ and 5 for type IIB, with the 5-form being self-dual. Firstly, for the left-movers, since $\Gamma S = S$ then

$$V_\alpha{}^\beta = -(\Gamma)_\alpha{}^\delta V_\delta{}^\beta \ . \tag{4.269}$$

We now use the identity

$$\Gamma\Gamma^{\mu_1\ldots\mu_k} = \frac{(-1)^{k(k+1)/2}}{(10-k)!}\epsilon^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_{10-k}}\Gamma^{\nu_1\ldots\nu_{10-k}} \ , \tag{4.270}$$

in (4.269) to find that $H_k, k \geq 5$ are related to $H_k, k \leq 5$ by Hodge duality:

$$-(-1)^{k(k+1)/2}(*H_k) = H_{10-k} \ , \tag{4.271}$$

and so are not independent fields. In particular, we find that H_5 is self-dual.

Therefore, we modify (4.263) to

$$V_\alpha{}^\beta = \sum_{k=1}^{5} \frac{1}{k!}(H_k)_{\mu_1\ldots\mu_k}(P_-\Gamma^{\mu_1\ldots\mu_k})_\alpha{}^\beta \ , \tag{4.272}$$

where P_- is the projection matrix $\frac{1}{2}(1-\Gamma)$ and accounts for the GSO projection of the right-movers. It is easy to show that

$$V^{\alpha\beta}S_\alpha\tilde{S}_\beta = \sum_{k=1}^{5} \frac{1}{k!}(H_k)_{\mu_1\ldots\mu_k}(\Gamma^{\mu_1\ldots\mu_k})^{\alpha\beta}S_\alpha(\frac{1}{2}[1-(-1)^k\Gamma]\tilde{S})_\beta \ , \tag{4.273}$$

and so odd forms get projected out if $\Gamma\tilde{S} = -\tilde{S}$ (type IIA) and even forms are projected out if $\Gamma\tilde{S} = +\tilde{S}$ (type IIB). Equation (4.273) makes it clear that the equations (4.268) are not modified by the introduction of P_-.

[24]Conventions for differential forms are given in appendix B.

Open String Tachyon Vertex Operator

The open superstring tachyon is given by $|NS;k\rangle\otimes|\downarrow\rangle\otimes|q_{sgh}=-1\rangle$ with $k^2=1/2\alpha'$ and so is created from $|0\rangle$ by the vertex operator

$$\mathcal{V}_T^{(-1)}(z,\bar{z};k) \;=\; ˚_˚\, \frac{c(z)+\tilde{c}(\bar{z})}{2} e^{-\sqrt{\frac{2}{\alpha'}}(\phi(z)+\tilde{\phi}(\bar{z}))/2} e^{ik\cdot X_{NN}(z,\bar{z})} \,˚_˚ \;, \quad (\mathrm{Im}\, z=0) \;. \tag{4.274}$$

After doubling this becomes

$$\mathcal{V}_T^{(-1)}(z;k) \;=\; ˚_˚\, c(z) e^{-\sqrt{\frac{2}{\alpha'}}\phi(z)} e^{2ik\cdot X(z)} \,˚_˚ \;, \quad (\mathrm{Im}\, z=0) \;, \tag{4.275}$$

Excluding the ghost, this operator has conformal weight $(1,0)$. After applying the picture changing operator (4.256) to (4.275), we find the vertex operator in the $q_{sgh}=0$ picture:

$$\mathcal{V}_T^{(0)}(z;k) = \sqrt{2\alpha'}\; ˚_˚\, c(z) k\cdot\psi(z) e^{2ik\cdot X(z)} \,˚_˚ \;, \quad (\mathrm{Im}\, z=0) \;. \tag{4.276}$$

This creates the state $|T\rangle = \sqrt{2\alpha'}k\cdot\psi_{-1/2}|NS;k\rangle\otimes|\downarrow\rangle\otimes|q_{sgh}=0\rangle$. To see that this state is another representation of the tachyon one notes that the normal ordering constant $a(\nu)=-d_{tot}(\nu)$ in the mass formula (4.156) is modified when the superghost vacuum is changed. In the NS sector we now have the mass formula

$$m^2=-k^2=\frac{1}{\alpha'}(N^{(\alpha)}+N^{(\psi)}-1-\frac{1}{2}q_{sgh}(q_{sgh}+2)) \;, \tag{4.277}$$

where the ghost contribution of -1 and that of the superghosts are a result of (4.184) and (4.234). Clearly, therefore, $|T\rangle$ has mass $m^2=-1/2\alpha'$.

4.1.7 Strings in Background Fields and Low-Energy Effective Actions

So far we have described the propagation of strings in either a $D=26$- or a 10-dimensional Minkowski target space with metric $\eta_{\mu\nu}$. We would like however to be able to describe *low-energy* string physics when the massless fields have non-trivial VEVs but the massive modes are not excited[25].

[25]Recall that the massive modes have energies $\sim 1/\sqrt{\alpha'}\sim M_P$.

Let us consider the NS-NS sector of the closed superstring in which the massless fields are the graviton, Kalb-Ramond field and dilaton. A natural generalisation of the Polyakov action (4.1) is to simply make the replacement $\eta_{\mu\nu} \rightarrow G_{\mu\nu}(X)$ in order to describe a string propagating in an arbitrary curved target space with general metric $G_{\mu\nu}$. Thus, we replace (4.1) by

$$S_X[G] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma(-g)^{1/2}g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) \ , \tag{4.278}$$

From the worldsheet point of view this still looks like a model of D bosonic fields X^μ, but with *field dependent* couplings given by the non-trivial metric $G_{\mu\nu}$.

One might question whether this procedure is consistent with the way in which strings in Minkowski space-time generate the graviton. To see the relation between the two approaches, consider the background metric to be close to flat:

$$G_{\mu\nu}(X) = \eta_{\mu\nu} - h_{\mu\nu}(X) \ , \tag{4.279}$$

where $h_{\mu\nu}$ is symmetric, traceless and small. One can then perturbatively expand the factor $e^{-S_X[G]}$ in the Euclidean path integral as

$$e^{-S_X[G]} = e^{-S_X[\eta]}\left(1 + \frac{1}{2\pi\alpha'}\int_\Sigma d^2 z h_{\mu\nu}(X)\partial X^\mu \bar{\partial} X^\nu + \cdots\right) \ , \tag{4.280}$$

where $S_X[\eta]$ is given by (4.57). If we take

$$h_{\mu\nu}(X) \propto \zeta^G_{\mu\nu} e^{ik\cdot X(z,\bar{z})} \ , \tag{4.281}$$

where $\zeta^G_{\mu\nu}$ is a symmetric, traceless polarisation tensor and $k^2 = k^\mu \zeta^G_{\mu\nu} = 0$, the term of order h in (4.280) is, up to an appropriate normalisation, just the bosonic part of the integrated graviton vertex operator in the $(0,0)$ picture (or the full integrated vertex operator in the case of the bosonic string):

$$V = \int d^2 z\, \zeta^G_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X(z,\bar{z})} + \cdots \ . \tag{4.282}$$

Indeed, for the bosonic string we find

$$\circ\!\!\!\circ\, \zeta^G_{\mu\nu}\partial X^\mu(0)\bar{\partial}X^\nu(0) e^{ik\cdot X(0,0)} \,\circ\!\!\!\circ\, |0,\tilde{0};k\rangle = \zeta^G_{\mu\nu}\tilde{\alpha}^\mu_{-1}\alpha^\nu_{-1}|0,\tilde{0};k\rangle \ , \tag{4.283}$$

which reproduces the graviton state that was discussed in the text below equation (4.50). Therefore, we see that the insertion of the full $G_{\mu\nu}(X)$ can be thought of as inserting an exponential of the graviton vertex operator, which is another way of saying that the curved background is a coherent state of gravitons. One may then go beyond this perturbation theory framework and assume that the gravitons form a non-trivial condensate in which the string propagates. In this case, $G_{\mu\nu}(X)$ is a more general metric that does not necessarily have the above weak-field expansion.

The above argument can be generalised to include a non-trivial Kalb-Ramond field ($B_{\mu\nu}$) and dilaton (Φ) with the result that to the action (4.278) we add

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \left(-\varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X) + \alpha'(-g)^{1/2} R^{(2)}(g)\Phi(X) \right) , \qquad (4.284)$$

where $\varepsilon^{ab} = g\epsilon^{ab}$ is totally antisymmetric with $\varepsilon^{12} = 1$ and $R^{(2)}(g)$ is the Ricci scalar of the worldsheet manifold (Σ, g). This is for a Lorentzian signature worldsheet; for Euclidean signature the minus sign in front of the first term becomes $+i$ after Wick rotation of the worldsheet. Note that for a constant dilaton field $\Phi(X) = \Phi_0$ the Euclidean path integral contains a factor

$$e^{-\Phi_0 \chi} \ , \ \chi = \frac{1}{4\pi} \int_\Sigma d^2\sigma \sqrt{g} R^{(2)}(g) \ , \qquad (4.285)$$

where χ is the Euler number of the worldsheet. This means that the constant λ introduced just after (4.244) can be shifted by choosing a different vacuum expectation value for the dilaton. Since λ determines the closed string coupling constant, we see that the string coupling is not a free parameter in the theory — different values of λ correspond not to different theories but to different backgrounds of the same theory.

The next step is to check that the modified *sigma-model action* (4.278,4.284) does not spoil Weyl invariance at the quantum level, that is, the trace of the energy momentum tensor should vanish. On dimensional and symmetry grounds it can be shown[26] that this trace (on a Euclidean signature worldsheet) can, in the critical dimension, be expanded as

$$\langle T \rangle = -\frac{1}{2\alpha'}\beta^G_{\mu\nu}\sqrt{g}g^{ab}\partial_a X^\mu \partial_b X^\nu - \frac{i}{2\alpha'}\beta^B_{\mu\nu}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\beta^\Phi \sqrt{g} R^{(2)} \ , \qquad (4.286)$$

with

$$\beta^G_{\mu\nu} = \alpha' \left(R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi - \frac{1}{4} H_{\mu\kappa\sigma} H_\nu{}^{\kappa\sigma} \right) + O(\alpha'^2) \ ,$$

[26]See D'Hoker's lectures [95] and references therein and also [102–104].

$$\beta^B_{\mu\nu} = \alpha'\left(-\frac{1}{2}\nabla^\kappa H_{\kappa\mu\nu} + \nabla^\kappa\Phi H_{\kappa\mu\nu}\right) + O(\alpha'^2) ,$$
$$\beta^\Phi = \alpha'\left(\nabla_\kappa\Phi\nabla^\kappa\Phi - \nabla^2\Phi - \frac{1}{4}R + \frac{1}{48}H_{\kappa\mu\nu}H^{\kappa\mu\nu}\right) + O(\alpha'^2) , \quad (4.287)$$

and where $H_{\mu\nu\kappa} \equiv \partial_\mu B_{\nu\kappa} + \partial_\nu B_{\kappa\mu} + \partial_\kappa B_{\mu\nu}$, $R_{\mu\nu}$ is the Ricci tensor constructed from $G_{\mu\nu}$, R the Ricci scalar and ∇_μ is the covariant derivative compatible with $G_{\mu\nu}$. The functions β are called *Weyl anomaly coefficients*[27] and they should vanish if Weyl invariance is not to be broken.

A key feature of the equations $\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0$ is that they can be derived from the *low-energy effective action*[28]:

$$S = \frac{1}{2\kappa_D^2}\int d^DX(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right] . \quad (4.288)$$

Note that the dilaton dependence is as expected for a closed string tree-level action on the sphere with Euler number $\chi = 2$. This action is written in the so-called *string frame*. However, by use of the redefinitions

$$G_{\mu\nu} = e^{2\omega}\tilde{G}_{\mu\nu} , \; \omega = \frac{2\Phi}{D-2} , \; \Phi = \frac{\kappa_D\sqrt{D-2}}{2}\tilde{\Phi} , \; B_{\mu\nu} = \sqrt{2}\kappa_D\tilde{B}_{\mu\nu} , \quad (4.289)$$

and the relation

$$R = e^{-2\omega}\left(\tilde{R} - 2(D-1)\tilde{\nabla}^2\omega - (D-1)(D-2)\tilde{\nabla}_\mu\omega\tilde{\nabla}^\mu\omega\right) , \quad (4.290)$$

we can rewrite the action in the *Einstein frame* in which the Einstein-Hilbert and dilaton kinetic terms are canonically normalised:

$$S = \int d^DX(-\tilde{G})^{1/2}\left[\frac{1}{2\kappa_D^2}\tilde{R} - \frac{1}{2}\partial_\mu\tilde{\Phi}\partial^\mu\tilde{\Phi} - \frac{1}{12}e^{-4\kappa_D\sqrt{D-2}\tilde{\Phi}}\tilde{H}_{\mu\nu\lambda}\tilde{H}^{\mu\nu\lambda}\right] . \quad (4.291)$$

---

[27]In the literature they are often called *beta functions* but the reader should be aware that they are different from, although closely related to, the familiar renormalisation group beta functions.

[28]The action is "low energy" for the following reason. The Weyl anomaly coefficients are expansions in powers of α'. In a target space with characteristic length $l_c^2 \sim 1/R$ the dimensionless expansion parameter is α'/l_c^2. Expanding to first order in α' implies $\sqrt{\alpha'}l_c^{-1} \ll 1$ which means that the characteristic energy scale of the target space is small compared to the string scale $1/\sqrt{\alpha'}$. This is consistent with ignoring the internal structure of the string and giving only the massless modes $G_{\mu\nu}, B_{\mu\nu}$ and Φ a non-trivial VEV.

In this action indices are raised and lowered using the Einstein metric $\tilde{G}_{\mu\nu}$. Equation (4.291) can also be deduced from an explicit calculation of various scattering amplitudes involving the graviton, Kalb-Ramond field and dilaton vertex operators[29]. It is precisely this fact that justifies our heretofore implicit identification of the spin-2 string mode (that is, (4.283) in the case of the bosonic string) as the graviton. Rather than review this derivation[30], we shall illustrate later in section 5.2.3 how one can infer an effective action from a direct evaluation of string scattering amplitudes.

This discussion can be generalised to the other sectors of the closed superstring with the result that the the low-energy effective action for type IIA/B theory is just the type IIA/B supergravity action[31].

The discussion is also generalised to the open string. We have seen that open string vertex operators are integrated along the boundary of the worldsheet, so we expect open string fields to appear as boundary terms in the sigma model action. Indeed, the massless vector $A^\mu \equiv |\psi\rangle^\mu$ in the NS sector gives rise to the term

$$-\int_{\partial\Sigma} A \ , \tag{4.292}$$

(in differential form notation) in the sigma model action. Again, this is written for a Lorentzian signature worldsheet; for Euclidean signature the minus sign becomes $+i$.

4.2 D-Branes

In section 4.1.1, we mentioned that open strings could satisfy Dirichlet boundary conditions as well as Neumann ones. Some consequences of this alternative choice will now be discussed. We follow [53, 96, 105–110].

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[29]The Kalb-Ramond field vertex operator is similar to the graviton vertex operator (4.282), the only difference being that the polarisation tensor $\zeta^B_{\mu\nu}$ is antisymmetric. A similar remark applies to the dilaton vertex operator with $\zeta^\Phi_{\mu\nu}$ transverse diagonal. In addition, fixing the relative normalisation between the field theory and the string theory results in the relation $2\kappa_{10}^2 = (2\pi)^7\alpha'^4$.

[30]For example, see [53, 92, 93] or any other standard textbook in this regard.

[31]There is a problem in the IIB case because the 5-form R-R field strength is self-dual and so no simple covariant action exists. This is overcome by writing down an action with self-duality imposed as an added constraint on the solutions to the equations of motion.

4.2.1 Bosonic Strings

Dirichlet boundary conditions at an endpoint of an open string imply that X^μ is fixed at that endpoint. Let us suppose that there are Dirichlet boundary conditions at both endpoints (D-D) for $D-(p+1)$ directions X^a: $X^a(\sigma^1,0)=x^a, X^a(\sigma^1,\pi)=y^a$ for $a=p+1,\ldots,D-1$, with N-N boundary conditions for each of the other $(p+1)$ directions $X^i(\sigma), i=0,\ldots,p$. Geometrically, this implies that in the target space manifold the string endpoints are each constrained to a flat $(p+1)$-dimensional hyperplane parametrised by $X^i(\sigma^1,\sigma^2)$, $\sigma^2=0,\pi$ (see figure 4.2). Such hyperplanes are called *Dp-branes*[32]. Clearly, Poincaré invariance is broken in the directions transverse to the branes (the D-D directions).

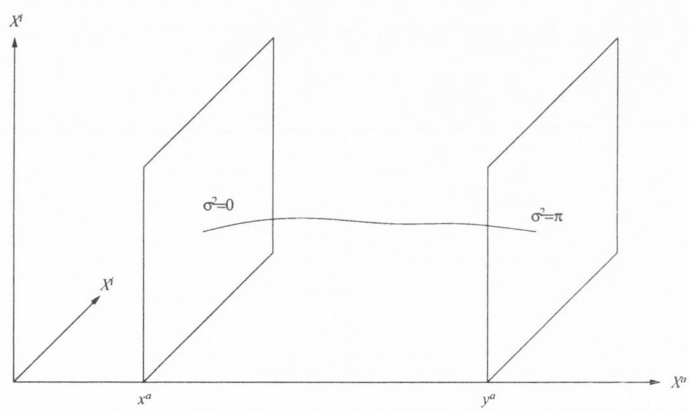


Figure 4.2: An open string stretched between two infinite, flat, parallel D-branes of the same dimension. It is possible for open strings to start and end on the same brane. It is also possible for two or more branes to be coincident with indistinguishable worldvolumes. In type II superstring theory closed strings propagate in the bulk in the region between the branes.

The mode expansion for a D-D direction is

$$X^a_{DD}(\sigma) = x^a + \frac{\Delta^a}{\pi}\sigma^2 - i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\backslash\{0\}}\frac{1}{n}\alpha^a_n e^{-in\sigma^-} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\backslash\{0\}}\frac{1}{n}\alpha^a_n e^{-in\sigma^+} , \quad (4.293)$$

[32]Note that our notation implicitly assumes $0\le p\le D-1$. However, it is also possible to have $p=-1$, in which case the X^μ directions are D-D for all $0\le\mu\le D-1$. This case is also called the *D-instanton* because the open string endpoints are localised in the timelike direction X^0.

where $\Delta^a = y^a - x^a$. In this case canonical quantisation yields[33]

$$[x^i, p^j] = i\eta^{ij} \ , \ [\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu} \ . \qquad (4.294)$$

We shall define $\alpha_0^\mu = (\sqrt{2\alpha'}p^i, \frac{\Delta^a}{\pi\sqrt{2\alpha'}})$ in this case. One can now rework section 4.1.1 and find that the analysis is unmodified (in particular $D = 26$ and $a = 1$), save for the definition of α_0^μ and a modified definition of the mass:

$$\begin{aligned} M^2 &= -p^i p_i \ , \\ \Rightarrow m^2 &= \frac{\Delta^2}{4\pi^2\alpha'^2} + \frac{1}{\alpha'}(N^{(\alpha)} - a) \ . \end{aligned} \qquad (4.295)$$

The $\Delta^2 = \Delta^a\Delta_a$ term above represents the tension of the open string stretched between the two Dp-branes.

The cases of a Neumann boundary condition at $\sigma^2 = 0$ and a Dirichlet one at $\sigma^2 = \pi$ (N-D) or *vice versa* (D-N) are referred to as *mixed* boundary conditions. The N-D mode expansion is similar to (4.24) except that the σ^1 term is absent and $n \in \mathbb{Z} + 1/2$. The D-N expansion is likewise half-integrally moded and is similar to (4.293) without the σ^2 term. These cases would arise, for example, if an open string ended on D-branes of different dimensions. They will not play a role in this thesis.

4.2.2 Doubling, Superstrings and Mixed Open-Closed Type II Amplitudes

In terms of the z coordinate we may rewrite (4.293) as

$$X^a_{DD}(z, \bar{z}) = X^a(z) + \tilde{X}^a(\bar{z}) \ , \qquad (4.296)$$

where

$$X^a(z) = \frac{1}{2}x^a - i\frac{\Delta^a}{2\pi}\ln z + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\backslash\{0\}}\frac{1}{n}\alpha^a_n z^{-n} \ , \qquad (4.297)$$

$$\tilde{X}^a(\bar{z}) = \frac{1}{2}x^a + i\frac{\Delta^a}{2\pi}\ln \bar{z} - i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\backslash\{0\}}\frac{1}{n}\alpha^a_n \bar{z}^{-n} \ . \qquad (4.298)$$

[33]Henceforth, indices $a, b, \ldots$ denote D-D directions, indices $i, j, \ldots$ denote N-N directions and indices $\mu, \nu, \ldots$ denote either case.

As for N-N directions, $X^a(\cdot)$ and $\tilde{X}^a(\cdot)$ are defined for their argument on the upper and lower half of the complex plane, respectively. Note that the boundary of the upper half-plane, $\mathrm{Im}\, z = 0$, corresponds to the locations of the D-branes. Doubling is via

$$X^a(z) \;:=\; x^a - \tilde{X}^a(z) \ , \tag{4.299}$$

$$\partial X^a(z) \;:=\; D^a_b \bar{\partial}\tilde{X}^b(z) = -\bar{\partial}\tilde{X}^a(z) \ , \tag{4.300}$$

for $\mathrm{Im}\, z \leq 0$. It is conventional to choose $x^a = 0$ which just amounts to choosing the origin in the target space to lie on one of branes. We can now rework (4.67) through (4.76). We find that the doubling matrix D^μ_ν is modified to

$$D^\mu_\nu = \begin{pmatrix} \delta^i_j & 0 \\ 0 & -\delta^a_b \end{pmatrix} . \tag{4.301}$$

Note that this matrix satisfies the conditions (4.76) and so in terms of the extended field we have the single correlator

$$\langle X^\mu(z')X^\nu(z)\rangle \;=\; -\frac{1}{2}\alpha'\eta^{\mu\nu}\ln(z'-z) \ . \tag{4.302}$$

For superstrings, preservation of worldsheet supersymmetry requires that the fermions be doubled according to

$$\psi^\mu(z) \;:=\; D^\mu_\nu\tilde{\psi}^\nu(z) \ , \quad (\mathrm{Im}\, z \leq 0) \ . \tag{4.303}$$

The ghosts and superghosts are doubled as before but the matrix M that we encountered (cf. equation (4.211)) in the doubling of the spin field $S_\alpha(z)$ is modified. Relations (4.212) and (4.213) still hold but due to (4.301) we now find [111]:

$$M = \begin{cases} \pm\Gamma^0\ldots\Gamma^p \ , & (p \text{ even}) \ , \\ \pm\Gamma^0\ldots\Gamma^p\Gamma \ , & (p \neq -1 \text{ odd}) \ , \\ \pm i\Gamma \ , & (p = -1) \ . \end{cases} \tag{4.304}$$

The overall sign is not physically relevant and is conventionally chosen as plus provided there is only one brane. However, in the presence of many branes the sign becomes significant. This is discussed in section 5.1.1.

The presence of D-branes naturally leads to a modification of type II closed superstring theory: closed strings propagate in the bulk while open strings are attached to the branes. There can be interaction between the two on the surface of a brane (see figure 4.3). For example, two open strings can join to form a closed string or the endpoints of a single open string can come together and form a closed string which is then emitted from the brane. The closed string fields are also subjected

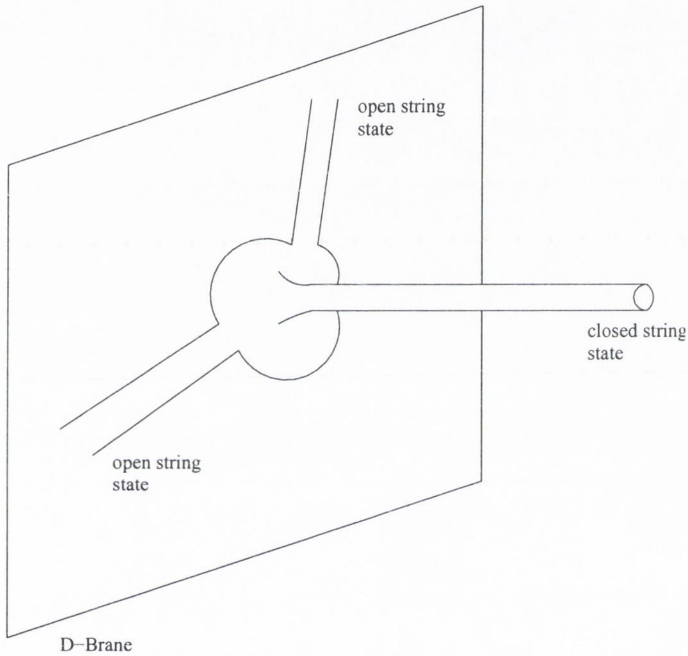


Figure 4.3: Two open strings and one closed string interacting on the surface of a D-brane. This diagram corresponds to an inelastic process where a closed string is absorbed by a D-brane, exciting its internal state by creating a pair of open strings or to the reverse process of spontaneous emission of a closed string by the brane. The amplitude for such a process is a disc (or upper half-plane) amplitude in which the open string vertex operators are located on the boundary and the closed string vertex operator in the interior.

to the doubling. More precisely, in order to calculate amplitudes corresponding to closed and open strings interacting on a D-brane one has to calculate a correlator on the unit disc consisting of a mixture of closed and open string vertex operators. To perform this calculation we invoke the following rules [111–117]:

(i) Map the disc to the upper half-plane.

(ii) For open strings use standard open string vertex operators living on the real axis.

(iii) Closed string vertex operators factorise into left and right components:

$$W_{\phi_{cl}}(z, \bar{z}; k) = V(\phi_{cl}(z); k)\tilde{V}(\tilde{\phi}_{cl}(\bar{z}); k) \ , \tag{4.305}$$

where ϕ_{cl} is a generic closed string field with $\phi_{cl}(z)$ and $\tilde{\phi}_{cl}(\bar{z})$ its holomorphic and antiholomorphic parts containing tilded and untilded oscillators, respectively. One then replaces the closed string tilded (untilded) fields by their open string tilded (untilded) counterparts. For example, in (4.262) one would replace (4.58) and (4.59) by (4.69) and (4.70), respectively. The untilded fields live on the interior of the upper half-plane and the tilded fields on the lower half.

(iv) Double all tilded fields as described above in order to write all vertex operators in terms of the extended untilded fields.

(v) The ghost number anomaly of -3 is taken care of by fixing the insertion points of one of the closed string and one of the open string vertex operators[34]; the others are in integrated form. Additionally, the sum of the superghost charges over all vertex operators should be -2.

According to this prescription, for the closed string we find

$$\Gamma S(0) \ := \ \begin{cases} -M\Gamma\tilde{S}(0) \ , & (p \text{ even}) \ , \\ M\Gamma\tilde{S}(0) \ , & (p \text{ odd}) \ . \end{cases} \tag{4.306}$$

Since the GSO projections have $\Gamma S(0) = +S(0)$, we must have $\Gamma\tilde{S}(0) = -\tilde{S}(0)$ for p even and $\Gamma\tilde{S}(0) = +\tilde{S}(0)$ for p odd. Consequently, Dp-branes with p even are associated with type IIA theory and those with p odd are associated with type IIB theory. This means that with the exception of the D9-brane and the 0-form field strength (and before Hodge dualising), the Dp-branes are in one-to-one correspondence with the R-R $(p+2)$-form field strengths (equivalently, the $(p+1)$-form potentials C_{p+1})

[34]The amplitudes considered in this thesis will always have at least one closed string and one open string vertex operator.

of type II string theory. This correspondence suggests that the 0-form field strength should be associated with a (-2)-brane but it is not clear how to interpret this. The D9-brane case is also pathological. The corresponding field strength would have to be of rank eleven and therefore would vanish in ten dimensions. However, the D9-brane does couple to a non-trivial ten-form potential. In type I theory, the closed string worldsheet parity projection of implies that (4.211) should be symmetric under the interchange $S_\alpha \leftrightarrow \tilde{S}_\alpha$. This leads to the extra condition $M^2 = 1$ when p is odd. This is true only for $p = 1, 5$ and 9.

To summarise, type IIA theory has Dp-branes for $p = 0, 2, 4, 6, 8$; type IIB has Dp-branes for $p = -1, 1, 3, 5, 7, 9$ and type I theory has Dp-branes for $p = 1, 5, 9$.

4.2.3 Action for a Single D-brane

In section 2.2 we mentioned that Leigh [57] derived that the low-energy dynamics on a single D-brane in bosonic string theory was governed by the *Born-Infeld action*. We now give some explicit details of this action[35]. We also introduce the so-called *Wess-Zumino term*, present in the superstring case but not the bosonic string case, which will be of more importance to us in the sequel.

The massless open string (bosonic) fields on a Dp-brane are a $U(1)$ vector A_i and $9 - p$ scalars Φ^a coming from decomposition of the 10-dimensional vector A^μ. The scalars (which should not be confused with the dilaton denoted Φ) are interpreted as collective coordinates for fluctuations of the brane worldvolume in the transverse directions, that is, a D-brane is a dynamical object. These fields interact on the brane with the closed string massless (bosonic) fields, namely the graviton, Kalb-Ramond field and dilaton of the NS-NS sector and the antisymmetric tensor fields of the R-R sector.

[35]We do not however consider the original derivation of Leigh which involved the calculation of the Weyl anomaly coefficients for the open string. Furthermore, we describe the bosonic sector of the supersymmetrised theory in $D = 10$ dimensions.

The Born-Infeld Action

Let us introduce coordinates $\xi^i, i = 0, \ldots, p$ on the brane. In what follows, we define the pull-back $\hat{\phi}$ of any target space tensor ϕ as:

$$\hat{\phi}_{i_1,\ldots,i_q}(\xi) = \frac{\partial f^{\mu_1}(\xi)}{\partial \xi^{i_1}} \cdots \frac{\partial f^{\mu_q}(\xi)}{\partial \xi^{i_q}} \phi_{\mu_1,\ldots,\mu_q}(X) \ , \tag{4.307}$$

where the fields $f^i(\xi), f^a(\xi) = 2\pi\alpha'\Phi^a(\xi)$ provide the embedding of the brane world-volume Σ_{p+1} into the target space $\mathcal{M}$.

The effective action for coupling to NS-NS fields is the Born-Infeld action:

$$S_{BI} = -T_p \int_{\Sigma_{p+1}} d^{p+1}\xi \, e^{-\Phi} \sqrt{-\det\left(\hat{G}_{ij} + \hat{B}_{ij} + 2\pi\alpha' F_{ij}\right)} \ , \tag{4.308}$$

where $F_{ij} = \partial_i A_j - \partial_j A_i$ is the gauge field strength. The brane tension is given by

$$\tau_p \equiv \frac{T_p}{g_s} = \frac{1}{g_s\sqrt{\alpha'}} \frac{1}{(2\pi\sqrt{\alpha'})^p} \ , \tag{4.309}$$

where $g_s = e^{\Phi_0}$ is the closed string coupling and Φ_0 the constant part of Φ. Note that the dilaton dependence is as expected for an open string tree-level action. The self-interactions of open string fields and their couplings to closed string fields arise first from the disc which has Euler number $\chi = 1$.

If we assume the background space-time is flat and the brane almost flat then we can go to so-called *static gauge*:

$$f^i = \xi^i \ , \ \Phi = \Phi_0 \ , \ \hat{G}_{ij} \approx \eta_{ij} + \partial_i f^a \partial_j f_a \ , \ B_{\mu\nu} = 0 \ . \tag{4.310}$$

If we further assume that $2\pi\alpha' F_{ij}$ and $\partial_i f^a$ are small and of the same order, the determinant in (4.308) can be expanded to quadratic order in the field strength:

$$S_{BI} = -\tau_p V_p - \frac{1}{4g_{YM}^2} \int_{\Sigma_{p+1}} d^{p+1}\xi \left(F_{ij}F^{ij} + 2\partial_i \Phi^a \partial^i \Phi_a\right) \tag{4.311}$$

where V_p is the (infinite) brane worldvolume and the Yang-Mills coupling is given by

$$g_{YM}^2 = \frac{g_s}{\sqrt{\alpha'}}(2\pi\sqrt{\alpha'})^{p-2} \ . \tag{4.312}$$

The second term in (4.311) is essentially just the action for a $U(1)$ theory in $(p+1)$-dimensions with $9-p$ scalar fields. In fact, after supersymmetrising [58], one finds that the action (4.311) is precisely the dimensional reduction to $p+1$ dimensions of the 10-dimensional $\mathcal{N}=1$ super Yang-Mills action with gauge group $U(1)$ and coupling constant given by (4.312). This fills in some of the details omitted in section 2.2 in the case of D3-branes.

The Wess-Zumino (or Chern-Simons) Term

Coupling of the D-brane to the antisymmetric tensor fields of the R-R sector of the closed superstring is via the Wess-Zumino term:

$$S_{WZ} = T_p \int_{\Sigma_{p+1}} \left\{ \hat{C} \wedge e^{\mathcal{F}} \wedge \sqrt{\frac{\hat{A}(R_T)}{\hat{A}(R_N)}} \right\}_{p+1} . \qquad (4.313)$$

This formula deserves some explanation:

1. $\hat{C} = \sum_q \hat{C}_q$ is the formal sum of pullbacks of all R-R potentials of the theory in question. For example, $q = 0, 2, 4, 6, 8, 10$ for type IIB.

2. We have defined $\mathcal{F} = \hat{B} + 2\pi\alpha' F$. Note that this combination of $\hat{B}$ and F also appears in the Born-Infeld action (4.308). The reason for the appearance of this combination is gauge invariance. The closed string Kalb-Ramond field B and the open string vector field A appear in the *string* worldsheet action (cf. (4.284) and (4.292)) as

$$-\frac{1}{2\pi\alpha'} \int_\Sigma \hat{B} - \int_{\partial\Sigma} A , \qquad (4.314)$$

where the hat denotes the pull-back to Σ. Associated with each field is a space-time gauge invariance which must be preserved for the consistency of the space-time theory. The ordinary gauge transformation $\delta_\lambda A = d\lambda$, where λ is a scalar, is an invariance of (4.314), the boundary term changing by the integral of a total derivative. The antisymmetric tensor variation $\delta_\Lambda B = d\Lambda$, where Λ is a 1-form, gives rise to the surface term

$$-\frac{1}{2\pi\alpha'} \int_{\partial\Sigma} \hat{\Lambda} , \qquad (4.315)$$

which can only be cancelled if A transforms as $\delta_\Lambda A = -\hat{\Lambda}/2\pi\alpha'$. Therefore, we have $\delta_\lambda F = d^2\lambda = 0 = \delta_\lambda \hat{B}$ and $\delta_\Lambda F = -d\hat{\Lambda}/2\pi\alpha'$ and so we conclude that only the combination $\mathcal{F} = \hat{B} + 2\pi\alpha' F$ is gauge invariant and so appears in the action.

3. R_T and R_N are the curvatures of the tangent and normal bundles to the brane, respectively.

4. $\hat{A}(R)$ is the the so-called "A-roof genus" or "Dirac genus" whose square root is given in terms of the Pontrjagin classes by

$$\sqrt{\hat{A}(R)} = 1 - \frac{(4\pi^2\alpha')^2}{48} p_1(R) + \cdots \ , \tag{4.316}$$

where $p_1(R) = -\frac{1}{8\pi^2} \mathrm{tr}\, R \wedge R$.

5. The notation $\{\ldots\}_{p+1}$ instructs us to expand the exponential and Dirac genera and then to extract the $(p+1)$-form part which can then be integrated over the brane worldvolume.

5. Strictly speaking, the R-R potentials cannot be globally defined in the presence of D-branes. So one should use the fact that $-1 + e^{\mathcal{F}} \wedge \sqrt{A(R_T)/A(R_N)}$ is an exact form to integrate by parts and express all terms in (4.313) except

$$T_p \int_{\Sigma_{p+1}} \hat{C}_{p+1} \ , \tag{4.317}$$

in terms of the R-R field strengths. The term (4.317) just means that Dp-branes are charged under the $(p+1)$-potential in the same way that a point particle (0-brane) is charged under a vector field [56].

The Wess-Zumino term is essential for the consistency of D-branes, in particular for the consistency of intersections of more than one brane[36]. Its inclusion can be justified by *anomaly inflow* arguments. Briefly, the theory on the brane (or on the intersection if more than one) can be anomalous due to the presence of chiral fermionic degrees of freedom. In order for the theory to be selfconsistent the anomaly must be cancelled by an equal and opposite contribution from the 10-dimensional bulk, that is, the gauge variation of the low-energy effective action on the brane is cancelled by

[36] We discuss the multiple brane case below.

a bulk term whose gauge variation is localised on the brane. A topological argument then constrains the form of the bulk term to be the push-forward of (4.313). Hence, the different terms involved in (4.313) are often called *anomalous couplings.* The reader is referred to the literature [118–121] for more complete details. Equation (4.313) can also be deduced from a direct string calculation (see [96] and references therein) using the *boundary state formalism* (see [113, 122–128] and, for a recent comprehensive review, [129]).

4.2.4 Multiple Type II D-Branes

Multiple D-branes have a natural interpretation in terms of Chan-Paton factors. Let us consider N non-coincident, parallel type II Dp-branes living in flat space. We label them by an index i running from 1 to N. There are massless gauge fields living on each D-brane worldvolume corresponding to a gauge theory with a total gauge group $U(1)^N$. In addition, we expect massive gauge fields to arise corresponding to strings stretching between each pair of branes. As well as standard Lorentz indices, these fields now carry a pair of Chan-Paton indices i, j indicating which branes are at the endpoints of the strings. Because the strings are oriented, there are $N^2 - N$ such fields (counting a vector $A_\mu = (A_i, \Phi_a)$ as a single field). The mass of a field corresponding to a string connecting branes i and j is proportional to the distance between these branes (cf. equation (4.295)). Witten [130] showed that as the D-branes approach each other and the stretched strings become massless, the fields arrange themselves precisely into the gauge field components and adjoint scalars of a supersymmetric $U(N)$ gauge theory in $p+1$ dimensions[37]. Generally, such a super Yang-Mills theory is described by the reduction to $p+1$ dimensions of a 10-dimensional non-abelian Yang-Mills theory where all fields are in the adjoint representation of $U(N)$.

The non-abelian extensions of (4.308) and (4.313) are highly non-trivial. The leading terms in the Born-Infeld action should correspond to the above SYM theory.

[37]Note that the gauge group is $U(N)$ here because we are dealing with the *oriented* strings of type II theory. For unoriented strings, $U(N)$ is reduced to $SO(N)$ or $USp(N)$ as discussed in section 4.1.3. The 32 of type I $SO(32)$ theory is therefore associated with 32 D9-branes.

Therefore, as a first approximation one simply includes a trace over the fundamental representation of the gauge group in (4.308). The same is true for the Wess-Zumino term. This procedure is not completely satisfactory, however. For example, due to the ordering ambiguities in the expansion of (4.308) introduced by the noncommutativity, the trace is ill-defined for the NNLO terms in α'. Unfortunately, the problem remains unsolved and the reader is referred to [131–136] (particularly the recent, concise review by Schwarz) and references therein for further discussion. For our purposes, we shall be content with the first approximation.

4.3 T-Duality

In this section we briefly introduce *T-duality*, an important concept which we shall use in section 5.1.3. The reader is referred to the comprehensive review [137] for more complete details.

4.3.1 T-Duality for Closed Strings

Let us first consider closed superstrings in a Minkowski background: $G_{\mu\nu} = \eta_{\mu\nu}, B_{\mu\nu} = 0, \Phi = \Phi_0$. Recall from (4.58) and (4.59) that the mode expansions for the embedding of the string into the target space are given by $X^\mu(z, \bar{z}) = X^\mu(z) + \tilde{X}^\mu(\bar{z})$, where

$$\begin{aligned} X^\mu(z) &= x_L^\mu - i\frac{\alpha'}{2}p_L^\mu \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n}\alpha_n^\mu z^{-n} , && (4.318) \\ \tilde{X}^\mu(\bar{z}) &= x_R^\mu - i\frac{\alpha'}{2}p_R^\mu \ln \bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n}\tilde{\alpha}_n^\mu \bar{z}^{-n} , && (4.319) \end{aligned}$$

with $p_L^\mu = p_R^\mu = p^\mu$ continuous. Now let us compactify one of the spacelike directions, X^9 say, on a circle of radius R. The strings can now wind around this circle so that X^9 is subjected to the modified periodicity condition

$$X^9(\sigma^1, 2\pi) = X^9(\sigma^1, 0) + 2\pi R w \ , \quad w \in \mathbb{Z} \ , \tag{4.320}$$

rather than the original condition (4.18). The integer w is called the *winding number*. In addition, requiring vertex operators (which contain $e^{ik\cdot X(z,\bar{z})}$) to be single-valued results in momenta along the X^9 direction being quantised in integer units of $1/R$. Consequently, the mode expansion for X^9 is now modified thus:

$$p_L^\mu = \frac{n}{R} + \frac{wR}{\alpha'} , \tag{4.321}$$

$$p_R^\mu = \frac{n}{R} - \frac{wR}{\alpha'} , \tag{4.322}$$

with $n \in \mathbb{Z}$. This modified expansion also results in a changed definition for the spectrum of the string masses. Instead of (4.50) we now have

$$\begin{aligned} m^2 &= -\frac{1}{2}\sum_{\mu=0}^{8}(k_L^\mu k_{L\mu} + k_R^\mu k_{R\mu}) \\ &= \frac{n^2}{R^2} + \frac{w^2R^2}{\alpha'^2} + \frac{2}{\alpha'}(N^{(\alpha)} + \tilde{N}^{(\alpha)} + N^{(\psi)} + \tilde{N}^{(\psi)} - a(\nu) - a(\tilde{\nu})) . \end{aligned} \tag{4.323}$$

T-duality stems from the observation that the mass spectrum is invariant under the simultaneous transformations

$$n \leftrightarrow w , \quad R \leftrightarrow \frac{\alpha'}{R} . \tag{4.324}$$

We note that these transformations take p_L^μ and p_R^μ to p_L^μ and $-p_R^\mu$, respectively. Therefore, we extend this action to the non-zero modes and define a T-dual transformation by $X^9(z) \rightarrow X^9(z)$ and $\tilde{X}^9(\bar{z}) \rightarrow -\tilde{X}^9(\bar{z})$. Note that the mass formula is still invariant under this extended action. Furthermore, it is clear that if we define the T-dual coordinate by $\bar{X}^9(z,\bar{z}) = X^9(z) - \tilde{X}^9(\bar{z})$ then (4.324) is implemented by everywhere replacing $X^9(z,\bar{z})$ by $\bar{X}^9(z,\bar{z})$. $\bar{X}^9(z,\bar{z})$ is therefore compactified on a circle of radius $\tilde{R} = \alpha'/R$ and has winding n.

Within the bosonic sector of type II theory the energy-momentum tensor, OPE and correlation functions do not change when we replace X^9 by $\bar{X}^9$ because signs always enter in pairs. In the fermionic sector we must have

$$\tilde{\psi}^9(\bar{z}) \rightarrow -\tilde{\psi}^9(\bar{z}) , \tag{4.325}$$

in order to preserve worldsheet supersymmetry. This has the the very important consequence that in the Ramond sector the zero mode $\tilde{\psi}_0^9$ changes sign and so the GSO projection (4.155) in this sector also flips sign. Thus, type IIA theory compactified on a circle of radius R has the same mass spectrum as type IIB theory compactified on a circle of radius $\tilde{R}$ and *vice versa*. In fact, this equivalence also extends to the fully interacting case [138].

Since type IIA and IIB theories have different R-R fields, T-duality must transform one set into the other. The action on the spin fields is of the form

$$S_\alpha(z) \to S_\alpha(z) \ , \ \tilde{S}_\alpha(\bar{z}) \to P_9 \tilde{S}_\alpha(\bar{z}) \ , \tag{4.326}$$

for some matrix P_9. To be consistent with (4.325) and the antiholomorphic equivalents of the OPEs (4.204) and (4.206), we find that P_9 is given by the matrix M of (4.304) with $p = 8$. Thus, $P_9 = \Gamma^9\Gamma$ up to a sign. Since the spin fields are of definite chirality the action of Γ just adds a sign. We can therefore consider the action of Γ^9 on the bispinor (4.263). We find that the dual bispinor is related to the original one via

$$\tilde{V}_\alpha{}^\beta = \pm(V\Gamma^9)_\alpha{}^\beta \ . \tag{4.327}$$

Use of the gamma matrix identity (4.267) then gives the relation between the k-forms of the original theory and the $(k \pm 1)$-forms of the dual theory. The effect is to add a 9-index if it is absent and to remove it if it is present. The same effect occurs for the potentials. Thus, for example, the C_9 component of the vector potential of type IIA theory maps to the IIB scalar C, whilst the other components $C_\mu \, , \mu \neq 9$ map to the components $C_{\mu 9}$ of the 2-form potential.

It is important to note that T-duality acts non-trivially on the given background fields according to the well-known *Buscher rules* [139]:

$$\tilde{G}'_{99} = \frac{1}{G'_{99}} = \frac{\tilde{R}^2}{\alpha'} \ , \ e^{2\tilde{\Phi}} = \frac{e^{2\Phi}}{G'_{99}} = \frac{e^{2\Phi}\alpha'}{R^2} \ , \tag{4.328}$$

where G'_{99} differs from $G_{99} = \eta_{99} = 1$ by a rescaling by R^2/α'. Clearly, the expression for $\tilde{G}'_{99}$ is consistent with the dual direction being compactified on a circle of radius $\tilde{R}$. In addition, since the closed string coupling is given by $g_s = e^{\Phi_0}$, we find

$$\tilde{g}_s = g_s \frac{\sqrt{\alpha'}}{R} \ . \tag{4.329}$$

Finally, this discussion trivially generalises to the compactification of d space-time dimensions on a torus T^d with radii $R_1, \ldots, R_d$. In particular, for $d = 2$ we can relate type IIA theory to itself and type IIB theory to itself with

$$\tilde{g}_s = g_s \frac{\alpha'}{R_1 R_2} \ . \tag{4.330}$$

4.3.2 T-Duality for Open Strings

Since D-branes are charged under R-R potentials, consistency with the above discussion implies that T-duality should transform even-p D-branes of type IIA into odd-p D-branes of type IIB and *vice versa.* This requires that Neumann directions should transform into Dirichlet directions and *vice versa* under T-duality. To see that this is actually the case, recall from (4.69) and (4.70) that the mode expansion for a Neumann direction is $X^i_{NN}(z, \bar{z}) = X^i(z) + \tilde{X}^i(\bar{z})$, where

$$X^i(z) = \frac{1}{2}x^i - i\alpha' p^i \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{1}{n}\alpha^i_n z^{-n} \ , \tag{4.331}$$

$$\tilde{X}^i(\bar{z}) = \frac{1}{2}x^i - i\alpha' p^i \ln \bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{1}{n}\alpha^i_n \bar{z}^{-n} \ . \tag{4.332}$$

Note that we can always add an arbitrary c-number $\frac{1}{2}c^i$ to X^i and subtract it from $\tilde{X}^i$ without altering X^i_{NN}. Therefore, let us define the dual coordinate $\bar{X}^i_{NN}(z, \bar{z}) = (X^i(z) + \frac{1}{2}c^i) - (\tilde{X}^i(\bar{z}) - \frac{1}{2}c^i)$. The coordinate $\bar{X}^i_{NN}(z, \bar{z})$ has the mode expansion

$$\bar{X}^i_{NN}(z, \bar{z}) = c^i - i\alpha' p^i \ln(z/\bar{z}) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{1}{n}\alpha^i_n (z^{-n} - \bar{z}^{-n}) \ , \tag{4.333}$$

which is formally similar to the D-D mode expansion (4.296-4.298) under the identifications $c^i \leftrightarrow x^i$, $p^i \leftrightarrow \Delta^i/2\pi\alpha'$. Therefore, we see that if the ith direction is the same as the one we are T-dualising then a Dp-brane extended along this direction is turned into a D$(p-1)$-brane which is localised in this direction. Conversely, if the ith direction is originally Dirichlet then it becomes Neumann under T-duality and so a Dp-brane becomes a D$(p+1)$-brane.

This concludes our discussion of strings and branes. In the next chapter, we shall use the concepts we have introduced to explain Ashoke Sen's work on unstable type

II D-branes and to infer a Wess-Zumino-like term for such a configuration from an evaluation of tachyon-R-R field scattering amplitudes calculated using the tachyon and R-R vertex operators introduced in section 4.1.6. We will also come across D$(p-2k)$-branes embedded within Dp-branes. By the above discussion, a T-duality transformation on each of the $2k$ relative transverse directions should transform these two branes (and their associated R-R charges) into one another.

Chapter 5

Unstable D-branes and R-R Couplings on Dp-$\overline{\mathrm{D}p}$ Systems

In this chapter, we briefly discuss part of A. Sen's work on unstable brane-antibrane systems in type II theory and then, in section 5.2, go on to derive a Wess-Zumino-like action for such systems. This action is inferred from a direct computation of certain open-closed string scattering amplitudes on the upper half-plane, for which much of the technical material of the last chapter is required. By now, several reviews of Sen's work have appeared in the literature [96, 140–142] and we shall follow them closely. Section 5.2 is based on work done in collaboration with A. Wilkins [143].

5.1 Type II Brane-Antibrane Systems

5.1.1 Branes, Antibranes and Space-time Supersymmetry

As discussed in the last chapter, the defining property of a Dp-brane is that it is a $(p+1)$-dimensional dynamical hyperplane in 10-dimensional space-time on which open strings can end and which acts as source for a R-R gauge field C_{p+1} with $+1$ unit of the corresponding R-R charge. An anti-Dp-brane, or $\overline{\mathrm{D}p}$-brane, is simply a Dp-brane with R-R charge -1. The fact that a brane and an antibrane have opposite R-R charge plays a crucial role in what follows. One such consequence is that a Dp-$\overline{\mathrm{D}p}$ system breaks all space-time supersymmetries. Let us examine this more closely following [111] and the lectures of Bachas.

On a closed type II worldsheet we can construct the space-time supercharges:

$$Q^{(-1/2)} = \epsilon^\alpha \oint \frac{dz}{z} \Sigma_{(-1/2)}(z) S_\alpha(z) \ , \quad \tilde{Q}^{(-1/2)} = -\tilde{\epsilon}^\alpha \oint \frac{d\bar{z}}{\bar{z}} \tilde{\Sigma}_{(-1/2)}(\bar{z}) \tilde{S}_\alpha(\bar{z}) \ , \tag{5.1}$$

where the integrands are simply the fermionic vertex operators at zero momentum (cf. equation (4.261)) and ϵ and $\tilde{\epsilon}$ are independent Majorana-Weyl polarisation spinors. Hence, there are 32 (real) supersymmetries since a Majorana-Weyl spinor in 10 dimensions has 16 real components. When D-branes are present we also need to define the action of the unbroken supercharges on the open strings. Suppose we have a single brane[1]. The corresponding integrals of the supersymmetry currents at fixed radial time σ^1 run over a semi-circle as in figure 5.1 below. Moving the integration

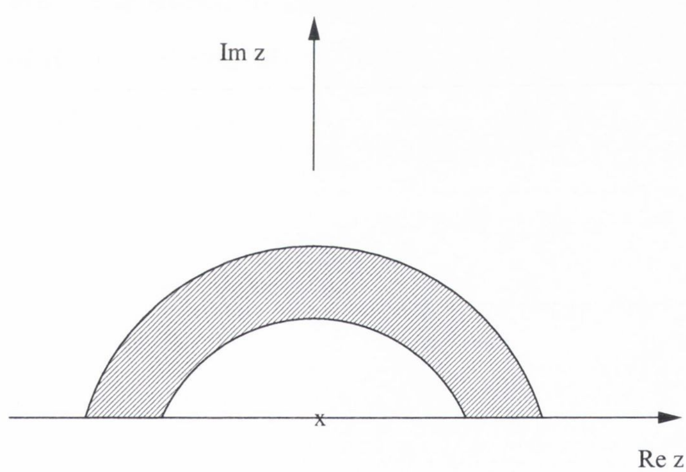


Figure 5.1: A semicircle in the upper half-plane represents an open string at a fixed radial time $\sigma^1 = \ln|z|$. A supercharge is conserved when its time variation can be expressed as a holomorphic plus anti-holomorphic contour integral around the shaded region. For this to hold, we demand that the contributions of the linear segments on the worldsheet boundary vanish. (Sketch courtesy of C. Bachas [95].)

to a later time is equivalent to conservation of the supercharges. However, this deformation is allowed only if the contributions of the worldsheet boundary vanish. This

[1]As remarked just after (4.298), this brane is located at $\text{Im}\, z = 0$. Recall from (4.54) and (4.55) that z is defined by $z = e^{\sigma^1 + i\sigma^2}$ where σ^1 is the Wick rotated timelike coordinate on the open string worldsheet and $0 \leq \sigma^2 \leq \pi$ is the spacelike coordinate.

is the case for the linear combination

$$Q^{(-1/2)} + \tilde{Q}^{(-1/2)} \sim \int_0^{\pi} d\sigma^2 \Sigma_{(-1/2)}(\epsilon^{\alpha} S_{\alpha} + \tilde{\epsilon}^{\alpha}\tilde{S}_{\alpha}) + \int d\sigma^1 \Sigma_{(-1/2)}(\epsilon^{\alpha} S_{\alpha} - \tilde{\epsilon}^{\alpha}\tilde{S}_{\alpha}) \ , \quad (5.2)$$

(where we have explicitly doubled the superghost field) provided we set

$$\epsilon^{\alpha}(M)_{\alpha}{}^{\beta} = \tilde{\epsilon}^{\beta} \ , \quad (5.3)$$

which follows upon using the doubling relation (4.211). Equation (5.3) implies that a single D-brane preserves only half of the 32 independent space-time supersymmetries.

Now let us suppose that within type II theory we have a single Dp-brane coincident with a single $\overline{\mathrm{D}p}$-brane such that their worldvolumes are indistinguishable[2]. This type of Dp-$\overline{\mathrm{D}p}$ system will be the focus of much of the rest of this chapter and is introduced in more detail below. It is shown in [111] (see also the lectures of Bachas) that the difference in sign of the R-R charge of the brane relative to the antibrane corresponds to the ambiguity in the sign of the doubling matrix M — for a brane we take the sign as plus while for an antibrane we take the sign as minus. As a result, the combined brane-antibrane system breaks *all* space-time supersymmetries because it is impossible to simultaneously satisfy (5.3) for the brane and the similar condition $-\epsilon^{\alpha}(M)_{\alpha}{}^{\beta} = \tilde{\epsilon}^{\beta}$ for the antibrane. This breaking of supersymmetry implies that the spectrum of open strings on the branes contains a tachyon and so the system is unstable as we shall show later in the chapter.

5.1.2 The Type II Dp-$\overline{\mathrm{D}p}$ System

Let us discuss the properties of the above-mentioned Dp-$\overline{\mathrm{D}p}$ system (see figure 5.2):

- The dynamics of the system is described by four kinds of open string beginning and ending on one or other of the branes. The four sectors are distinguished by their Chan-Paton (CP) factors:

$$(a): \ (\lambda^a)_{\bar{i}\bar{j}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ , \ (b): \ (\lambda^a)_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ ,$$

[2]Throughout this chapter we assume that the Kalb-Ramond field and dilaton are trivial, that is, $B = \Phi = 0$.

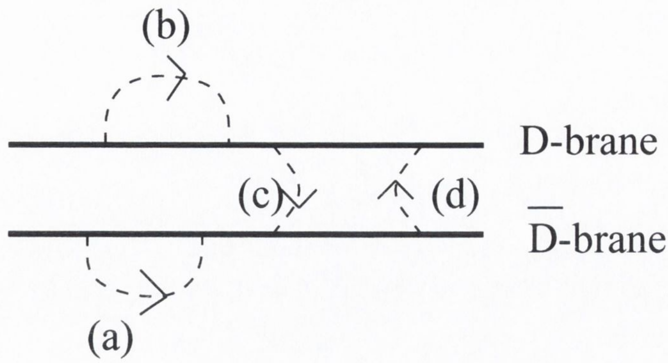


Figure 5.2: The four types of open string on a Dp-$\overline{\mathrm{D}p}$ system. The branes are shown to be separated for the sake of clarity but they are in fact coincident. The strings can either start and end on the same (anti)brane or start on a brane (antibrane) and end on an antibrane (brane). (Sketch courtesy of A. Sen, taken from [140].)

$$(c): \ (\lambda^a)_{i\bar{j}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ (d): \ (\lambda^a)_{\bar{i}j} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{5.4}$$

- The GSO projection[3] in each of the sectors (a) and (b) is the standard one with $G = +1$. In each sector, the bosonic fields of lowest mass are are the $U(1)$ vector A_i and scalars Φ_a resulting from the dimensional reduction of the 10-dimensional vector A_μ.

- The GSO projection in each of the sectors (c) and (d) is of the "wrong" sign with $G = -1$. It was argued that this should be so by Banks and Susskind [144] using the *open-closed string duality* result of Polchinski [56] (see figure 5.3). The amplitude for the closed string exchange between two (parallel) Dp-branes depicted in figure 5.3 consists of the sum of two contributions, one from the R-R sector and one from the NS-NS sector. After a modular transformation, the R-R part of the amplitude corresponds to a contribution from states in each of the NS and R sectors of the open string with the worldsheet fermion number operator, G, inserted. On the other hand, the NS-NS part corresponds to the same open string contributions but *without* the insertion of G. Thus, the amplitude only contains contributions from those open string states with

[3]The GSO projection acts trivially on CP factors in each sector.

$G = +1$, that is, from the standard GSO-projected states[4] . Polchinski showed that the total amplitude vanishes as a consequence of the famous "abstruse identity" of Jacobi. By contrast, Banks and Susskind considered a Dp-brane and a $\overline{\mathrm{D}p}$-brane. Since a brane and antibrane have opposite R-R charges, the sign of the R-R contribution to the closed string channel flips and the total amplitude no longer vanishes. More importantly, this means that the equivalent open string contributions now have an insertion of $-G$ rather than $+G$. Since the NS-NS sector is not affected, this implies that the GSO projection for open strings stretching between a brane-antibrane is $G = -1$ and so is opposite to that of the brane-brane and antibrane-antibrane cases.

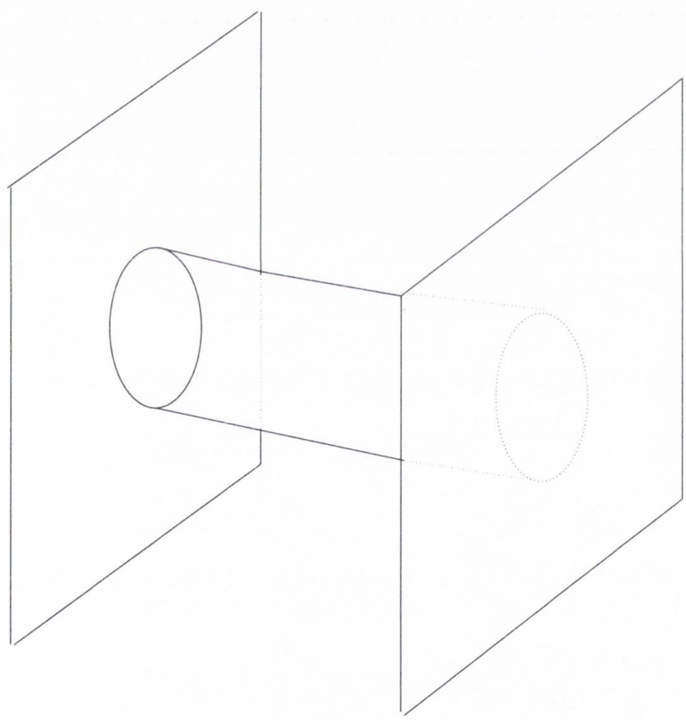


Figure 5.3: Two parallel branes interacting through the exchange of a closed string with worldsheet time running from one brane to the other. By a modular transformation, one can equivalently regard the diagram as an open string vacuum loop (the annulus diagram) in which worldsheet time runs around the the loop and the open string is stretched between the two branes. This equivalence was used in Polchinski's seminal result [56] concerning the force between two Dp-branes.

[4]This is precisely the situation in sectors (a) and (b).

- Since $G = -1$ in each of sectors (c) and (d), the tachyon $|NS; k\rangle$, $k^2 = 1/2\alpha'$ survives in each sector.

The bosonic open string fields of lowest mass on the shared worldvolume can be conveniently described by the *generalised gauge field*:

$$\hat{\mathcal{A}} = \begin{pmatrix} A & \bar{T} \\ T & A' \end{pmatrix} . \tag{5.5}$$

Here, A is the $U(1)$ gauge field associated with the Dp-brane and, likewise, A' is the $U(1)$ gauge field associated with the $\overline{\text{D}p}$-brane. $\bar{T}$ and T are the tachyons arising from sector (c) and (d), respectively. These tachyons are charged oppositely under both gauge fields due to the opposite orientation of the strings in sector (c) relative to (d). Equation (5.5) arises naturally via the CP factors (5.4).

There is a natural generalisation of the discussion to the multiple-brane case. Suppose we have N Dp-branes and an equal number of $\overline{\text{D}p}$-branes[5]. Then A and A' are $U(N)$ gauge fields whilst $\bar{T}$ and T are $N \times N$ matrices transforming in the bifundamental representations $(N, \bar{N})$ and $(\bar{N}, N)$ of $U(N) \times U(N)$, respectively. Thus, T is complex with $\bar{T}$ (also called the antitachyon) its complex conjugate. We shall make use of $\hat{\mathcal{A}}$ in the discussion below.

5.1.3 The Work of Ashoke Sen

The brane-antibrane system discussed above, as well as other configurations of unstable and non-supersymmetric D-branes in type II theory and in various type II orbifolds and orientifolds (such as type I theory), were studied in an important series of papers by Ashoke Sen [145–149].

[5]It is also possible to have unequal numbers of branes and antibranes but we shall not consider this case.

Motivation

Sen's main motivation for these works was the testing of various *duality conjectures* relating one string theory to another[6]. For example, suppose theory A at weak string coupling is conjectured to be related to theory B at strong coupling. This is the case for type I theory and the $SO(32)$ heterotic theory (which we have not discussed in this thesis). A partial understanding of this duality can be obtained by studying BPS objects. These objects have the property that their masses are fully determined by their charge. They have the further important property that supersymmetry ensures that they are stable and protected from quantum radiative corrections. One can therefore study their properties perturbatively in theory A at weak coupling, extrapolate these properties to strong coupling and reinterpret them in terms of non-perturbative objects in theory B. An example of such a BPS object is a single type II Dp-brane or $\overline{\text{D}p}$-brane, with p even for type IIA and odd for type IIB. Because of the role played by supersymmetry in the extrapolation of results from weak to strong coupling, it is not entirely clear whether or not the duality conjecture relating theories A and B holds beyond the BPS level. It is therefore of interest to study objects which are both non-BPS and stable and which carry certain conserved quantum numbers. Non-BPS objects are opposite to BPS objects — their mass is not fully determined by their charge, they break supersymmetry and they are not protected from quantum corrections so one cannot extrapolate their properties from weak to strong coupling with any degree of confidence. Nevertheless, if such theory-A objects are also stable and so unable to decay, they should have an interpretation in terms of similar stable non-BPS objects in theory B with the same conserved quantum numbers if the duality relation is correct. Identifying such objects in both theories A and B and thereby strengthening the conjecture relating the two theories was the topic of Sen's research.

[6]The relation of IIA theory to IIB theory through T-duality (see section 4.3) is one such conjecture. Explicit details of other conjectures will not be discussed here; see [55] for further details.

Tachyon Condensation on the Brane-Antibrane System

The combined Dp-$\overline{\text{D}p}$ system of section 5.1.2 is non-BPS but it is not stable. Nevertheless, such a system was a key ingredient in much of Sen's work. One of the things he argued was that a Dp-$\overline{\text{D}p}$ system with the tachyon at the minimum of its potential was equivalent to pure vacuum. We briefly discuss his reasoning following [107, 142, 147].

Let us consider the $N = 1$ D2-$\overline{\text{D2}}$ system of type IIA theory and compactify the two spatial directions of the brane-antibrane worldvolume on a rectangular torus with radii R_1 and R_2. The Wess-Zumino term of the D2-brane contains the coupling

$$T_2 \int_{\mathbb{R}\times T^2} \hat{C}_1 \wedge 2\pi\alpha' F = T_0 \int_{\mathbb{R}\times T^2} \hat{C}_1 \wedge \frac{F}{2\pi} , \tag{5.6}$$

where F is the $U(1)$ field strength and we have used the relation (4.309). This shows that the magnetic flux through the torus

$$\frac{1}{2\pi}\int_{T^2} F = k \in \mathbb{Z} , \tag{5.7}$$

is a source of C_1 charge (cf. equation (4.317)). That is, a D2-brane with magnetic flux carries D0-brane charge. Similar considerations apply to the $\overline{\text{D2}}$-brane but with an extra minus sign to account for the opposite 2-brane charge.

Suppose that the brane and anti-brane each have $+1$ unit of D0-brane charge and the field strengths are constant:

$$F_{12} = -F'_{12} = \frac{2\pi}{V} , \tag{5.8}$$

where $V = (2\pi R_1)(2\pi R_2)$ is the volume of the torus. In the limit $R_i \to \infty$, the magnetic field strength disappears and we recover the physics of the original brane-antibrane system.

To study the limit $R_i \to 0$, we make the T-duality transformation (see section 4.3) $R_i \to \tilde{R}_i = \alpha'/R_i$ to the T-dual torus $\tilde{T}^2$ with radii $\tilde{R}_1$ and $\tilde{R}_2$ and then study the limit $\tilde{R}_i \to \infty$. From (4.330), we know that the relation between the string coupling of the T-dual theory to that of the original theory is

$$\tilde{g}_s = g_s \frac{\alpha'}{R_1 R_2} . \tag{5.9}$$

Now, since T-duality interchanges Neumann and Dirichlet directions, we find that 0-branes and 2-branes (and their charges) get interchanged. The +1 fluxes that gave D0-brane charge in the original theory imply that the dual theory has two D2-branes (that is, both have charge +1) wrapped on $\tilde{T}^2$. The original D2 and $\overline{\text{D2}}$, with charges +1 and −1 respectively, give rise to a D0 and a $\overline{\text{D0}}$ respectively in the dual theory. When $\tilde{R}_1$ and $\tilde{R}_2$ are large, the dual system is described by a *supersymmetric* $U(2)$ gauge theory [130] with magnetic field strength

$$\tilde{F}_{12} = \frac{2\pi}{\tilde{V}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{5.10}$$

The original brane-antibrane system had a complex tachyon, signalling instability. In the dual system, this instability is reflected in the structure of the field strength (5.10) which gives rise to positive energy density on the D2-branes. The minimum energy configuration is given by $\tilde{F}_{12} = 0$, corresponding to annihilation of the D0 and $\overline{\text{D0}}$. Sen's non-trivial result, however, is that (5.10) is connected to $\tilde{F}_{12} = 0$ via a gauge transformation. Therefore, we are left with a BPS system of two D2-branes and no flux. In the original picture this corresponds to the tachyon condensing by rolling to a minimum of its potential.

Now, the mass of a D2-brane in the dual system is the brane's tension times its spatial volume:

$$\tilde{M}_2 = \frac{T_2}{\tilde{g}_s} \times (2\pi\tilde{R}_1)(2\pi\tilde{R}_2) = \frac{T_0}{g_s} = \tau_0 \ , \tag{5.11}$$

with T_2 and τ_0 given by (4.309). Thus, the total mass of the dual system after D0-$\overline{\text{D0}}$ annihilation is just

$$\tilde{M} = 2 \times \tilde{M}_2 = 2\tau_0 \ . \tag{5.12}$$

Since the dual system after 0-brane annihilation is BPS, we expect that the mass of the dual system to be unchanged when we continuously vary the radii $\tilde{R}_i$. In particular, we can study the small limit of $\tilde{R}_i$, equivalently the large R_i limit. Furthermore, a T-duality transformation does not change the mass, being just a new description of the same state. Therefore, the mass in the original system is $M = \tilde{M}$ and so the mass density M/V vanishes in the limit $R_i \to \infty$. But in this limit we should recover the physics of the D2-$\overline{\text{D2}}$ system without flux. The mass density of this configuration comes from the brane tensions and the energy density of the tachyonic condensate. Hence, we conclude

$$2\tau_2 + V_{min} = 0 \ , \tag{5.13}$$

where V_{min} is the minimum of the tachyon potential $V(T)$ in the low-energy effective action.

This reasoning is easily generalised to Dp-$\overline{\text{D}p}$ systems for $p \neq 2$. Therefore, we conclude that tachyon condensation on a brane-antibrane system gives a configuration of vanishing total energy density and thus is equivalent to pure vacuum.

Vortex Solutions

Let us again consider the above $N = 1$ D2-$\overline{\text{D2}}$ system in $\mathbb{R}^{1,9}$. Since the tachyon transforms in the $(\bar{1}, 1)$ representation of $U(1) \times U(1)$ this implies the gauge transformations:

$$T \to hTg^{-1} \ , \ \bar{T} \to g\bar{T}h^{-1} \ , \tag{5.14}$$

for $g, h \in U(1)$ and hence $T\bar{T}$ is gauge invariant. Since the tachyon potential is gauge invariant it can only depend on $|T|$. Although it is not known to all orders in $|T|$, the potential is known to assume a Mexican hat shape for weak fields [144, 150, 151], as shown in figure 5.4. Such a shape is well-known in quantum field theory for admitting vortex solutions — the shape is just that of the Higgs potential. The tachyon is thus likened to a Higgs field. It is assumed that the higher order terms do not change this basic Mexican hat shape and so we assume that the minimum of the potential occurs at $T = T_* e^{i\theta}$, with θ arbitrary.

Now, the tachyon kinetic term in the low energy effective is of the form (see [151]) $D^i T \overline{D_i T}$, where

$$D_i T = \partial_i T + iA'_i T - iTA_i \ , \tag{5.15}$$

$$\overline{D_i T} = \partial_i \bar{T} + iA_i \bar{T} - i\bar{T}A'_i \ . \tag{5.16}$$

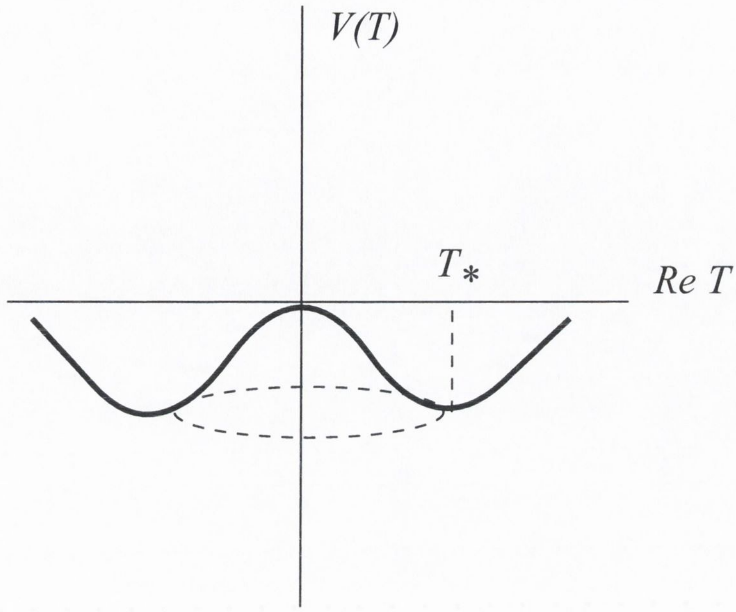


Figure 5.4: The tachyon potential for weak fields assumes a Mexican hat shape: $V(T) = -\frac{1}{2\alpha'}T\bar{T} + \frac{1}{4\alpha' T_*^2}(T\bar{T})^2$, with $T_* > 0$. The minimum of the potential is $V_{min} = -T_*^2/2$ and occurs for $T = T_* e^{i\theta}$, where θ is arbitrary.

Working in polar coordinates (r, θ) on the 2-branes, we take a static configuration whose asymptotic form as $r \to \infty$ is

$$T \sim T_* e^{i\theta} \ , \ \frac{\partial T}{\partial r} \sim \frac{1}{r^a} \ (a > 1) \ , \ A_\theta - A'_\theta \sim \frac{1}{r} \ , \ A_r \sim A'_r \sim 0 \ . \tag{5.17}$$

This ensures that

$$\begin{aligned} D_r T &= \frac{\partial T}{\partial r} - i(A_r - A'_r)T \to 0 \ , \\ D_\theta T &= \frac{1}{r}\frac{\partial T}{\partial \theta} - i(A_\theta - A'_\theta)T \to 0 \ , \\ V(T) &\to V_{min} \ , \end{aligned} \tag{5.18}$$

as $r \to \infty$ and so the energy density cancels for large r leaving a soliton in the small r region. Continuity requires that $T \to 0$ as $r \to 0$.

If we integrate the gauge fields over the circle C_∞ at infinity and use Stoke's theorem we find

$$\oint_{C_\infty} (A - A') \cdot d\ell = \int (F_{12} - F'_{12}) d\xi^1 d\xi^2 = \oint_{C_\infty} (A_\theta - A'_\theta) r d\theta = 2\pi \ , \tag{5.19}$$

where ξ^1, ξ^2 label the spatial directions of the D2-$\overline{\text{D2}}$ worldvolume. Hence, the vortex configuration has one unit of magnetic flux and so represents a D0-brane.

This result extends trivially to larger p: the D$(p-2)$-brane of type IIA/B string theory can be represented as a vortex solution on the Dp-$\overline{\text{D}p}$ system of type IIA/B theory. Our discussion in section 5.2 provides further evidence of this result.

5.1.4 Note on K-Theory and the Superconnection

In this section we briefly mention Witten's work on *K-theory* [152] in the absence of a B-field[7]. Although we shall not be using K-theory in the sequel, we introduce it because of its importance in the context of Sen's work. Of more importance for the next section is the *superconnection*, an object that we also introduce.

Firstly, we need to transcribe the brane-antibrane system of section 5.1.2 into fibre bundle language. We consider N flat Dp-branes coincident with an equal number of flat $\overline{\text{D}p}$ branes in a ten-dimensional Minkowski target space-time $\mathcal{M}$. We Wick rotate the timelike direction of $\mathcal{M}$ so that $\mathcal{M} = \mathbb{R}^{10}$. We can then decompose $\mathcal{M}$ as follows

$$\mathbb{R}^{10} = \Sigma_{p+1} \times \mathbb{R}^{9-p} \ , \tag{5.20}$$

where the first factor, $\Sigma_{p+1} = \mathbb{R}^{p+1}$, corresponds to the brane-antibrane worldvolume and the second factor to the transverse space. Recall from (5.5) that we can assemble the gauge fields and tachyon into the generalised gauge field

$$\hat{\mathcal{A}} = \begin{pmatrix} A & i\bar{T} \\ iT & A' \end{pmatrix} \ , \tag{5.21}$$

which is an $2N \times 2N$ matrix. Relative to (5.5), a factor i has been introduced in the off-diagonal blocks of (5.21) for later convenience. The fields A and A' can be regarded as connections on $U(N)$ complex vector bundles E and F, respectively. The tachyon T is then a section of the tensor product bundle $E^* \otimes F$ and $\bar{T}$ of $E \otimes F^*$. The worldvolume Σ_{p+1} is the base space of the bundles.

[7] Witten also showed how to incorporate the B-field but we shall not discuss this case as it would require a more detailed exposition of K-theory than that presented here.

K-Theory

We have seen that D-branes carry conserved charges that are sources for R-R gauge fields. Since the gauge fields are differential forms, it is natural to interpret the charges as cohomology classes — measured by integrating the differential forms over suitable cycles of the target space manifold. Minasian and Moore [153] showed that while this interpretation is approximately correct, one should regard the charges more precisely as elements of K-theory groups.

Briefly, K-theory is a sort of *generalised* cohomology theory. The word generalised refers to the fact that K-theory satisfies some but not all of the axioms of a cohomolgy theory (see [154] and references therein). Let us denote by $\mathrm{Vect}(X)$ the set of all complex vector bundles over compact X. The K-group $K(X)$ is the set of equivalence classes of pairs of elements of $\mathrm{Vect}(X)$, where we regard the pairs (E, F) and (G, H) as being equivalent if for some $J \in \mathrm{Vect}(X)$ the Whitney sum bundles $E \oplus H \oplus J$ and $F \oplus G \oplus J$ are isomorphic[8]. This equivalence relation thus allows us to consider the formal difference $E - F$ of two vector bundles and thus to complete $\mathrm{Vect}(X)$ into the abelian group $K(X)$. We can define another equivalence relation as follows. Two vector bundles E and F are equivalent if the addition of a trivial complex vector bundle I to E and I' to F renders them isomorphic [154–156]. Such bundles are called *stably equivalent.* The set of all stable equivalence classes of $\mathrm{Vect}(X)$ forms the reduced K-group $\widetilde{K}(X)$. If X is non-compact we define $K(X) = \widetilde{K}(\tilde{X})$, where $\tilde{X}$ is the compactification of X by adding a point at infinity. For compact X we define the higher reduced K-group $\widetilde{K}^{-1}(X)$ by $\widetilde{K}(S^1 \wedge X)$ where $S^1 \wedge X$ is the *smash product*

[8] We remind the reader of the definition of the Whitney sum bundle and of bundle isomorphism. Suppose we have two fibre bundles $E \equiv (M, \pi, \Sigma)$ and $E' \equiv (M', \pi', \Sigma)$, where π and π' are surjective maps (the *projections*) from the total spaces M and M' into a common base space Σ. For each $p \in \Sigma$ the pre-images $\pi^{-1}(p) \equiv F_p$ and $\pi'^{-1}(p) \equiv F'_p$ are called the *fibres* at p. Firstly, the Whitney sum $E \oplus E'$ is the triple $(M \oplus M', \sigma, \Sigma)$, where $M \oplus M'$ is the subspace of all $(u, u') \in M \times M'$ with $\pi(u) = \pi'(u)$ and $\sigma(u, u') = \pi(u) = \pi'(u)$. The fibre $\sigma^{-1}(p)$ is simply $F_p \times F'_p \subset M \times M'$. Now, given any two points $p, q \in \Sigma$, a *bundle map* $\bar{f} : M \to M'$ is a smooth map taking each fibre F_p onto F'_q. It naturally induces a smooth map $f : \Sigma \to \Sigma$ such that $f(p) = q$. Then E and E' are isomorphic if there exists a bundle map $\bar{f}$ such that f is the identity map on Σ and $\bar{f}$ is a diffeomorphism.

$(S^1 \times X)/((S^1 \times x_0) \cup (s_0 \times X))$, with (s_0, x_0) a base point in $S^1 \times X$. In particular, for $X = S^n$ it can be shown that $S^1 \wedge S^n \cong S^{n+1}$.

Using Moore and Minasian's observation, Witten reinterpreted much of Sen's work within the context of K-theory. For example, for type IIB theory he showed that the stable, supersymmetric Dp-brane charges are described by the K-group of the transverse space $X = \mathbb{R}^{9-p}$:

$$K(\mathbb{R}^{9-p}) = \widetilde{K}(S^{9-p}) = \begin{cases} \mathbb{Z} \ , & p \text{ odd} \ , \\ 0 \ , & p \text{ even} \ , \end{cases} \tag{5.22}$$

which agrees with the fact that p is odd.

The type IIA case is more subtle. It was conjectured by Witten and subsequently explained by Hořava [157] that the stable, supersymmetric Dp-brane charges are given by the higher reduced K-group $\widetilde{K}^{-1}(S^{9-p})$. However,

$$\widetilde{K}^{-1}(S^{9-p}) = \widetilde{K}(S^1 \wedge S^{9-p}) = \widetilde{K}(S^{10-p}) \ , \tag{5.23}$$

and so we find that p is even, as expected.

The type IIB vortex construction of the last section can also be reinterpreted in a similar K-theoretic light. One can imagine generalising the construction as follows. A D$(p-2)$-brane is created as a bound state of a Dp-$\overline{\mathrm{D}p}$ system. These can in turn each be created as a bound state of a D$(p+2)$-$\overline{\mathrm{D}(p+2)}$ pair. Continuing in this fashion, we see that we can create a Dp-brane for any p starting from an equal number of 9-branes and $\bar{9}$-branes. So let us consider consider N 9-branes and N $\bar{9}$-branes. The minima of the tachyon live on the vacuum manifold

$$\mathcal{V}(N) = \frac{U(N) \times U(N)}{U(N)_{\mathrm{diag}}} \cong U(N) \ . \tag{5.24}$$

That the gauge group is broken to the diagonal $U(N)$ subgroup was pointed out by Witten. It results from the possibility of separating the brane-antibrane pairs indicating that the eigenvalues of T_* are all equal. The form of T_* is then preserved by the gauge transformation (5.14) only if $h = g$, giving the unbroken group $U(N)_{\mathrm{diag}}$. In $\mathbb{R}^{10}$, a topological defect of dimension $p+1$ is stable provided the homotopy group

$$\pi_{8-p}(\mathcal{V}) = \pi_{8-p}(U(N)) \ , \tag{5.25}$$

is non-trivial. But the *Bott periodicity theorem* states that

$$\pi_{8-p}(U(N)) = \begin{cases} \mathbb{Z} \ , \ p \text{ odd} \ , \\ 0 \ , \ p \text{ even} \ , \end{cases} \tag{5.26}$$

for $N \geq (9-p)/2$. Therefore, combining (5.22) and (5.26) we have the following result: provided $N \geq (9-p)/2$, stable, supersymmetric Dp-branes can be formed as a bound state of N D9-branes and N D9-antibranes in type IIB theory. In particular, the result is true for $N = 1, p = 7$ which agrees with our treatment of the vortex solution as presented in the last section.

A similar but more involved generalised vortex construction holds for type IIA theory in which stable, supersymmetric Dp-branes are created as odd-codimension defects on a system of non-supersymmeric D9-branes. These latter objects were also considered by Sen but since we shall have no cause to use them in the sequel, we refer the reader to the extensive literature on the relationship between D-branes and K-theory. A by no means exhaustive list of references is contained in [158].

The Superconnection

The superconnection, which will be of relevance to us in the next section, was introduced by Quillen in his work on the Chern character of a K-class [159]. We briefly review some elements of his formalism following [160–163].

One regards the Whitney sum, $W = E \oplus F$, of the two $U(N)$ vector bundles E and F on the branes as a superbundle, that is, a bundle that carries a $\mathbb{Z}_2$-graded structure. The endomorphisms of W form a superalgebra with the following $\mathbb{Z}_2$-grading:

$$\deg(X) = \begin{cases} 0 \text{ if } X : E \to E \text{ or } F \to F \ , \\ 1 \text{ if } X : E \to F \text{ or } F \to E \ , \end{cases} \tag{5.27}$$

for $X \in End(W)$.

When considering differential forms on the base space Σ_{p+1} there is a natural $\mathbb{Z}$-grading corresponding to the degree of the forms. A r-form ω may be extended to $\omega \otimes X$ and thus it is the total $\mathbb{Z}_2$-grading, $r + \deg(X)$, that is relevant. These extended forms also form a superalgebra, defined by the following rule:

$$(\omega \otimes X) \wedge (\eta \otimes Y) = (-1)^{\deg(X) \cdot \deg(\eta)} (\omega \wedge \eta) \otimes XY \ . \tag{5.28}$$

The superconnection $\mathcal{D}$ on W is defined as an operator of odd degree satisfying the Leibnitz rule. Locally it is given by

$$\mathcal{A} = dI + \hat{\mathcal{A}} \ , \tag{5.29}$$

where I is the $2N \times 2N$ unit matrix and $\hat{\mathcal{A}}$ is given by (5.21). This can be broken down into the two components

$$\mathcal{D} = \begin{pmatrix} d+A & 0 \\ 0 & d+A' \end{pmatrix} , \tag{5.30}$$

which is an odd degree connection on W preserving the $\mathbb{Z}_2$-grading and

$$\mathcal{T} = \begin{pmatrix} 0 & i\bar{T} \\ iT & 0 \end{pmatrix} , \tag{5.31}$$

which is an odd degree endomorphism of W. Note that since $\mathcal{T}$ is odd, the tachyon T *anti*-commutes with one-forms $d\xi^i$, where ξ^i are coordinates on Σ_{p+1} [160, 163]. This can also be inferred from (5.28).

The supercurvature is calculated as

$$\mathcal{F} = \mathcal{A} \wedge \mathcal{A} = \begin{pmatrix} F - \bar{T}T & i\overline{DT} \\ iDT & F' - T\bar{T} \end{pmatrix} , \tag{5.32}$$

where $F = dA + A \wedge A$ and $F' = dA' + A' \wedge A'$ are the standard gauge field strengths of the associated gauge fields and

$$DT = d\xi^i D_i T \ , \ \overline{DT} = d\xi^i \overline{D_i T} \ , \tag{5.33}$$

with $D_i T$ and $\overline{D_i T}$ given by relations (5.15) and (5.16) but without the factor i:

$$\begin{aligned} D_i T &= \partial_i T + A'_i T - T A_i \ , \\ \overline{D_i T} &= \partial_i \bar{T} + A_i \bar{T} - \bar{T} A'_i \ . \end{aligned} \tag{5.34}$$

It is crucial that one takes T to anticommute with $d\xi^i$ in order to get the correct covariant derivatives. Therefore, the convention is that all differentials are written on the left, as above.

We can now introduce the Chern character. Let us first define the *supertrace*, Str, of matrices such as the supercurvature as the difference of the ordinary vector bundle traces in the fundamental representation of the upper left block and the lower right block. The Chern character is then defined as

$$\mathrm{ch}(\mathcal{F}) \equiv \mathrm{Str}\,\exp(\mathcal{F}) \ . \tag{5.35}$$

For future purposes, let us pick out the 2-forms contained in the first three terms of the Chern character. We have

$$\mathrm{ch}(\mathcal{F}) = \mathrm{Str}\,(1 + \mathcal{F} + \frac{1}{2}\mathcal{F} \wedge \mathcal{F}) + \cdots \ . \tag{5.36}$$

The first term is obviously zero. The second gives $\mathrm{tr}\, F - \mathrm{tr}\, F'$. For the third term we find that $\mathcal{F} \wedge \mathcal{F}$ is equal to

$$\begin{pmatrix} (F - \bar{T}T) \wedge (F - \bar{T}T) - \overline{DT} \wedge DT & * \\ * & (F' - T\bar{T}) \wedge (F' - T\bar{T}) - DT \wedge \overline{DT} \end{pmatrix} . \tag{5.37}$$

Therefore, we find

$$\begin{aligned} &\left\{\mathrm{Str}\,(1 + \mathcal{F} + \tfrac{1}{2}\mathcal{F} \wedge \mathcal{F})\right\}_2 \\ &\quad = \mathrm{tr}\,\left(F - F' - \tfrac{1}{2}\{F, \bar{T}T\} + \tfrac{1}{2}\{F', T\bar{T}\} - \tfrac{1}{2}\overline{DT} \wedge DT + \tfrac{1}{2}DT \wedge \overline{DT}\right) \\ &\quad = \mathrm{tr}\,\left(F - F' - \tfrac{1}{2}\{F, \bar{T}T\} + \tfrac{1}{2}\{F', T\bar{T}\} - \overline{DT} \wedge DT\right) \ , \end{aligned} \tag{5.38}$$

where, in going from the second to the third line, we have again used the fact that T anticommutes with $d\xi^i$.

Finally, we note that the superconnection obeys the *transgression formula* [161, p. 47]:

$$\mathrm{ch}(\mathcal{F}) - \mathrm{ch}(\mathcal{F}_0) = d \int_0^1 \mathrm{Str}\left(\frac{d\mathcal{A}_t}{dt} \exp \mathcal{F}_t\right) \ , \tag{5.39}$$

where $\mathcal{A}_t = \mathcal{D} + t\mathcal{T}$ is a differentiable one-parameter family of superconnections with $\mathcal{F}_t$ its corresponding curvature. In particular, we have

$$\mathcal{F}_0 = \begin{pmatrix} F & 0 \\ 0 & F' \end{pmatrix} . \tag{5.40}$$

5.2 R-R Couplings on Dp-$\overline{\text{D}p}$ Systems

In this section we present our proposal and calculation contained in [143].

5.2.1 The Proposal

As before, we consider N coincident Dp-$\overline{\text{D}p}$. It is clear from K-theory and Sen's vortex construction that we should be able to describe a supersymmetric D$(p-2k)$-brane as a bound state of the brane-antibrane system. To do this we trivially embed its worldvolume $\mathbb{R}^{p-2k+1}$ into the brane-antibrane worldvolume $\mathbb{R}^{p+1}$ and let the tachyon condense into a generalised vortex configuration. The precise form of T for $k > 1$ can be found in [152]. What is relevant is that the general case $N > 1$ and $k > 1$ has the same qualitative features as the $N = k = 1$ case we explicitly described in section 5.1.3. That is, T approaches its vacuum expectation value everywhere except close to the core Σ_{p-2k+1} where it vanishes and is such that the covariant derivative DT falls off sufficiently fast far from the core.

Our proposal is that the generalisation of the Wess-Zumino term (4.313) to the above situation is given by

$$S \sim \int_{\Sigma_{p+1}} \left\{ \hat{C} \wedge \mathrm{ch}(\mathcal{F}) \right\}_{p+1} , \tag{5.41}$$

where $\mathcal{F}$ is the supercurvature (5.32). We have written "$\sim$" rather than "$=$" because in our calculation below we will not be keeping track of overall numerical factors. We have also suppressed the A-roof genus contributions. The calculation does not explicitly check for their existence but a recent anomaly inflow argument of Schwarz and Witten [164] shows that they are present nonetheless. Furthermore, the calculation assumes $N = k = 1$ and checks only the terms

$$\begin{aligned} S &\sim \int_{\Sigma_{p+1}} \hat{C}_{p-1} \wedge \left\{ \mathrm{Str}\,(1 + \mathcal{F} + \frac{1}{2}\mathcal{F} \wedge \mathcal{F}) \right\}_2 \\ &\sim \int_{\Sigma_{p+1}} \hat{C}_{p-1} \wedge \left(F - F' - \frac{1}{2}\{F, \bar{T}T\} + \frac{1}{2}\{F', T\bar{T}\} - \overline{DT} \wedge DT \right) , \end{aligned} \tag{5.42}$$

where we have used (5.38) and dropped the trace since $N = 1$. We then conjecture that (5.41) is the full result.

Let us show that our proposal is at least plausible. If we integrate both sides of the transgression formula (5.39) over the $2k$-dimensional ball B^{2k} of large radius r surrounding the D$(p-2k)$-brane we find that the right hand side integrates to zero. This is because because DT vanishes on this space when the tachyon condenses and so all forms on this side are of odd degree integrated over an even-dimensional manifold. Hence, we conclude

$$\begin{aligned}\int_{B^{2k}} \mathrm{ch}(\mathcal{F}) &= \int_{B^{2k}} \mathrm{ch}(\mathcal{F}_0) \\ &= \int_{B^{2k}} \mathrm{tr}\, e^F - \int_{B^{2k}} \mathrm{tr}\, e^{F'} \ . \end{aligned} \tag{5.43}$$

That this is of the required form can be seen by taking $N = k = 1$. Then upon letting $r \to \infty$ we have

$$S \sim \int_{\Sigma_{p-1}} \hat{C}_{p-1} \wedge \int_{B^2} (F - F') = \int_{\Sigma_{p-1}} \hat{C}_{p-1} \wedge \oint_{C_\infty} (A - A') \cdot d\ell \ , \tag{5.44}$$

the form of which we see is consistent, up to normalisation, with the $p = 2$ analysis of section 5.1.3.

5.2.2 The Calculation

As we now show, the calculation of (5.41) (more precisely of (5.42)) involves inferring it from the direct computation of a two-point and a three-point string scattering amplitude on the upper half-plane. At this stage much of the technical material of the last chapter becomes relevant. In particular, the scattering process of interest is an inelastic one where a R-R boson annihilates onto the common worldvolume of a coincident Dp-brane and $\overline{\mathrm{D}p}$-brane to create some open strings (cf. figure 4.3). Therefore, we will make use of section 4.1.6 and the rules given in section 4.2.2 in order to evaluate these mixed open-closed amplitudes. Since we are dealing with infinite, flat branes we work in static gauge and ignore transverse fluctuations. All amplitudes are therefore calculated about a Minkowski background with $B_{\mu\nu} = \Phi = 0$.

For convenience, we list the vertex operators that will be needed:

Type II R-R Vertex Operator

From (4.242), (4.261), (4.262), (4.272) and the doubling rules we obtain the doubled

R-R vertex operators in fixed form[9]

$$\mathcal{V}_{RR}^{(-1)}(z,\bar{z};k) = (P_{-}\not{H}_{(k)})^{\alpha\beta}(M^{-1})_{\beta}{}^{\gamma} \circ\!\!\circ\, ce^{-\phi/2}S_{\alpha}e^{ik\cdot X}(z) \circ\!\!\circ \circ\!\!\circ\, ce^{-\phi/2}S_{\gamma}e^{i\tilde{k}\cdot X}(\bar{z}) \circ\!\!\circ \;, \quad (5.45)$$

where $\mathrm{Im}\, z > 0$, $\tilde{k}^{\mu} = D^{\mu}_{\nu}k^{\nu}$ with D^{μ}_{ν} given by (4.301) and

$$\not{H}_{(k)} = \frac{1}{k!}(H_k)_{\mu_1\ldots\mu_k}\Gamma^{\mu_1\ldots\mu_k} = \frac{1}{k!}(H_k)_{\mu_1\ldots\mu_k}\Gamma^{\mu_1}\Gamma^{\mu_2}\cdots\Gamma^{\mu_k} \;, \quad (5.46)$$

with $k = 2, 4$ for type IIA and $k = 1, 3, 5$ for type IIB. Hodge duality allows us to restrict the field strengths to $k \leq 5$ which correspond to Dq-branes with $q = k - 2$. We do not include $k = 0$ because, as noted in section 4.2.2, the existence of (-2)-branes is unclear. Each μ_i ranges over the indices $0, \ldots, 9$. The matrix M was given in (4.304) and it satisfies $M^{-1} = -(-1)^{p(p+1)/2}M$. Note that the vertex operator (5.45) has total superghost number -1. Recall that to obtain the integrated form of (5.45) one simply omits the c-ghost insertions.

Open String Tachyon Vertex Operator

The doubled open string tachyon vertex operators in fixed form in the -1 and 0 superghost pictures have been given explicitly in (4.275) and (4.276). As a reminder they are[10]

$$\mathcal{V}_T^{(-1)}(x;p) = \circ\!\!\circ\, ce^{-\phi}e^{2ip\cdot X}(x) \circ\!\!\circ \;, \quad (5.47)$$

$$\mathcal{V}_T^{(0)}(x;p) = 2 \circ\!\!\circ\, cp\cdot\psi e^{2ip\cdot X}(x) \circ\!\!\circ \;, \quad (5.48)$$

where x is on the real axis, $p^{\mu} = (p^i, \vec{0})$, $p^2 = 1/4$ is the tachyon momentum and we have suppressed the polarisation scalar. As discussed in [151], the fact that the momentum lies tangentially to the worldvolume directions is a consequence of the branes and antibranes all being coincident. Of course, the vertex operators for the antitachyon $\bar{T}$ are the same as those above — to distinguish between T and $\bar{T}$ the CP factors (5.4) should be understood.

[9] We follow [111] and set $\alpha' = 2$ throughout.

[10] The reader should not confuse p and k denoting the dimensionality of the various branes and field strengths with the same letters denoting momentum.

The Two-Point Amplitude

The amplitude containing one R-R field and one tachyon is

$$\mathcal{A}^{T,RR} = \int \frac{dx d^2 z}{\text{Vol(CKG)}} \langle V_T^{(-1)}(x;p) V_{RR}^{(-1)}(z,\bar{z};k)\rangle \ . \tag{5.49}$$

Note that the vertex operators are in integrated form and that their pictures have been chosen so that the total superghost number is -2, as we discussed section 4.2.2. On a similar note, we can trade in the integrations over x, z for one over the conformal Killing group in order to cancel the Vol(CKG) factor provided we also fix the vertex operators[11]. However, there is no need to perform any calculations here — the amplitude vanishes by virtue of the suppressed trace over the CP factors (5.4). An independent reason why the amplitude is zero is that the trace over the spinor indices vanishes, as outlined in appendix C.

The Three-Point Amplitude

The amplitude between one R-R field and two tachyonic particles is

$$\mathcal{A}^{T,T,RR} = \int \frac{dx dx' d^2 z}{\text{Vol(CKG)}} \langle V_T^{(0)}(x;p) V_T^{(-1)}(x';p') V_{RR}^{(-1)}(z,\bar{z};k)\rangle \ . \tag{5.50}$$

Again, the vertex operators are in integrated form and their pictures have been chosen so that the total superghost number is -2. In order to cancel the Vol(CKG) factor we follow [116] and fix the points $(x', z) = (-x, i)$ and the associated vertex operators $V_T^{(-1)}(x')$ and $V_{RR}^{(-1)}(z,\bar{z})$ and trade in the integrations over x', z for one over the conformal Killing group. The amplitude therefore simplifies to

$$\begin{aligned}\mathcal{A}^{T,T,RR} = & \int_0^\infty dx \langle V_T^{(0)}(x;p) V_T^{(-1)}(x';p') V_{RR}^{(-1)}(z,\bar{z};k)\rangle \\ & - \int_{-\infty}^0 dx \langle V_T^{(-1)}(x';p') V_T^{(0)}(x;p) V_{RR}^{(-1)}(z,\bar{z};k)\rangle \ . \end{aligned} \tag{5.51}$$

The interchange of the tachyonic vertex operators in the second term above relative to the first takes into account the two cyclic orderings of these operators on the real line. Note that the suppressed CP factors will also be interchanged. The sign in the second term above is negative and not positive for the following reason. We know

[11] Recall that this trade-in was discussed in section 4.1.6.

that the sign of the matrix M contained in $V_{RR}^{(-1)}$ is ambiguous (cf. (4.304)). It is conventional to take the sign as positive for a brane and negative for an antibrane. From (4.293) we know that the brane and antibrane are located at $\sigma^2 = 0, \pi$ in terms of the open string worldsheet coordinates (σ^1, σ^2) on the strip. The transformation to the upper half-plane is given by $x + iy = e^{\sigma^1 + i\sigma^2}$. Hence, the brane lies along the positive real axis and the antibrane along the negative real axis. Equation (5.51) therefore has the sign ambiguity built into it allowing us to take M with a positive sign, as in (C.2).

Now, given that amplitudes are independent of how the total superghost charge is distributed among the individual vertex operators, we may rewrite (5.51) as

$$\begin{aligned} \mathcal{A}^{T,T,RR} &= \int_0^\infty dx \langle V_T^{(0)}(x;p) V_T^{(-1)}(x';p') V_{RR}^{(-1)}(z,\bar{z};k)\rangle \\ &\quad - \int_{-\infty}^0 dx \langle V_T^{(0)}(x';p') V_T^{(-1)}(x;p) V_{RR}^{(-1)}(z,\bar{z};k)\rangle \; . \end{aligned} \tag{5.52}$$

The first term is given by

$$\begin{aligned} \mathcal{A}^{T,T,RR} &= 2\int_0^\infty dx \langle p\cdot\psi(x) e^{2ip\cdot X(x)} c(x') e^{-\phi(x')} e^{2ip'\cdot X(x')} c(z) e^{-\phi(z)/2} S_\alpha(w) e^{ik\cdot X(z)} \\ &\quad \times (P_- \not{H}_{(k)})^{\alpha\beta} (M^{-1})_\beta{}^\gamma c(\bar{z}) e^{-\phi(\bar{z})/2} S_\gamma(\bar{z}) e^{i\tilde{k}\cdot X(\bar{z})}\rangle \; . \end{aligned} \tag{5.53}$$

We find that the integrand factors into four independent pieces, one for each sector:

$$\text{ghosts:} \quad \langle c(x') c(z) c(\bar{z})\rangle \; , \tag{5.54}$$

$$\text{spin fields:} \quad (P_- \not{H}_{(k)})^{\alpha\beta} (M^{-1})_\beta{}^\gamma p_i \langle \psi^i(x) S_\alpha(z) S_\gamma(\bar{z})\rangle \; , \tag{5.55}$$

$$\text{bosons:} \quad \langle e^{2ip\cdot X(x)} e^{2ip'\cdot X(x')} e^{ik\cdot X(z)} e^{i\tilde{k}\cdot X(\bar{z})}\rangle \; , \tag{5.56}$$

$$\text{superghosts:} \quad \langle e^{-\phi(x')} e^{-\phi(z)/2} e^{-\phi(\bar{z})/2}\rangle \; . \tag{5.57}$$

All of these are well-known correlators. Indeed, (5.54) was evaluated in (4.253) to be

$$\text{ghosts:} \quad (x'-z)(x'-\bar{z})(z-\bar{z}) \; , \tag{5.58}$$

whilst the $\langle \psi S S\rangle$ correlator given in (4.206) determines (5.55) to be

$$\begin{aligned} \text{spin fields: } & \frac{1}{\sqrt{2}} (P_- \not{H}_{(k)})^{\alpha\beta} (M^{-1})_\beta{}^\gamma p_i (\Gamma^i)_{\alpha\gamma} (x-z)^{-1/2} (x-\bar{z})^{-1/2} (z-\bar{z})^{-3/4} \\ &= \frac{(-1)^{p(p+1)/2}}{\sqrt{2}} \mathrm{Tr}\,(P_- \not{H}_{(k)} M \Gamma^i) p_i (x-z)^{-1/2} (x-\bar{z})^{-1/2} (z-\bar{z})^{-3/4} \; , \end{aligned} \tag{5.59}$$

where we have used the spinor conventions listed in appendix A.

The bosonic correlator can be calculated by hand using the expansions (4.69), the $\langle XX\rangle$ correlator (4.302) and the relation (4.191). It is simpler to appeal to the general formula (which can be found in almost any conformal field theory text, see for example, Di Francesco *et al* [94]):

$$\begin{aligned}\langle\prod_i e^{ik_i\cdot X(z_i)}\rangle &= \left\{\prod_{i<j}(z_i-z_j)^{k_i\cdot k_j}\right\}\langle 0|e^{i\sum_i k_i\cdot x/2}|0\rangle \\ &= \left\{\prod_{i<j}(z_i-z_j)^{k_i\cdot k_j}\right\}\delta(\sum_i k_i/2)\ . \qquad (5.60)\end{aligned}$$

Hence, we find

$$\begin{aligned}\text{bosons:}\quad &(x-x')^{4p\cdot p'}(x-z)^{2p\cdot k}(x-\bar{z})^{2p\cdot\tilde{k}}(x'-z)^{2p'\cdot k}(x'-\bar{z})^{2p'\cdot\tilde{k}}(z-\bar{z})^{k\cdot\tilde{k}} \\ &\times\delta(p+p'+(k+\tilde{k})/2)\ . \qquad (5.61)\end{aligned}$$

We can use (5.60) and the substitutions $X\to\phi$, $k_i\to -iq^i_{sgh}$ and $x^\mu/2\to\phi(0)$ to determine

$$\begin{aligned}\langle\prod_i e^{q^i_{sgh}\cdot\phi(z_i)}\rangle &= \left\{\prod_{i<j}(z_i-z_j)^{-q^i_{sgh}q^j_{sgh}}\right\}\langle 0|e^{\sum_i q^i_{sgh}\phi(0)}|0\rangle \\ &= \left\{\prod_{i<j}(z_i-z_j)^{-q^i_{sgh}q^j_{sgh}}\right\}\langle 0|-Q_\phi\rangle \\ &= \left\{\prod_{i<j}(z_i-z_j)^{-q^i_{sgh}q^j_{sgh}}\right\}\delta_{Q_\phi,2}\ , \qquad (5.62)\end{aligned}$$

where $Q_\phi=-\sum_i q^i_{sgh}$ and we have used the superghost normalisation (4.237). In particular, we find

$$\langle e^{-\phi(x')}e^{-\phi(z)/2}e^{-\phi(\bar{z})/2}\rangle=(x'-z)^{-1/2}(x'-\bar{z})^{-1/2}(z-\bar{z})^{-1/4}\ . \qquad (5.63)$$

The second term in (5.52) is evaluated by making the substitutions $x\leftrightarrow x'$, $p\leftrightarrow p'$.

Using the fact that all momenta are on-shell: $p^2=p'^2=1/4$ and $k^2=0$, then putting everything together and after a little algebra we can express the amplitude as

$$\mathcal{A}^{T,T,RR}=\sqrt{8}(-1)^{t+p(p+1)/2}\int_0^\infty dx\left(\frac{(1+x^2)^2}{16x^2}\right)^{\frac{1}{2}+t}\frac{1}{1+x^2}\mathrm{Tr}\,(P_-\not{H}_{(k)}M\Gamma^i)(p_i-p'_i), \qquad (5.64)$$

where we have suppressed the delta-function and introduced the Mandelstam variable

$$t = -(p + p')^2 = -(k^i)^2 \ . \tag{5.65}$$

Note that given the fact that all momenta are on shell, we must have $t \geq 0$. In writing the amplitude in the form (5.64) we have used momentum conservation

$$2p + 2p' + k + \tilde{k} = 0 \ , \tag{5.66}$$

and various other relations such as

$$4p \cdot p' = -(1 + 2t) \ , \ -\frac{1}{2}k \cdot \tilde{k} = 2p \cdot k = 2p' \cdot k = 2p \cdot \tilde{k} = 2p' \cdot \tilde{k} = t \ , \tag{5.67}$$

and $\tilde{k}^2 = k^2$.

The integral in equation (5.64) can be expressed in the more suggestive form

$$4^{-2t-1} \times \int_0^\infty dx x^{2r-1}(1 + x^2)^{-r-s} \ , \tag{5.68}$$

where $r = s = -t$. For $t < 0$ this integral converges to $4^{-2t-1}B[-t, -t]/2$, where $B[r, s]$ is the beta function given in terms of the gamma function by $B[r, s] = \Gamma[r]\Gamma[s]/\Gamma[r + s]$. Use of the well-known identities

$$B[r, r]B[r + \frac{1}{2}, r + \frac{1}{2}] = \frac{\pi}{2^{4r-1}r} \ , \ \Gamma[r + 1] = r\Gamma[r] \ , \ , \tag{5.69}$$

then allow us to express the integral as

$$\frac{\pi}{2}\frac{\Gamma[-2t]}{\Gamma[\frac{1}{2} - t]^2} \ . \tag{5.70}$$

This may be written alternatively as

$$\frac{\sqrt{\pi}}{2} 2^{-2t-1} \frac{\Gamma[-t]}{\Gamma[\frac{1}{2} - t]} \ , \tag{5.71}$$

after using the *doubling formula*

$$\frac{\Gamma[2r]}{\Gamma[r + \frac{1}{2}]} = \frac{2^{2r-1}}{\sqrt{\pi}}\Gamma[r] \ . \tag{5.72}$$

This result is strictly valid for $t < 0$ but can be analytically continued for t non-negative (see next section). Therefore, we may write the three-point amplitude as

$$\mathcal{A}^{T,T,RR} \sim \sqrt{\pi}\, 2^{-2t-1} \frac{\Gamma[-t]}{\Gamma[\frac{1}{2} - t]} \operatorname{Tr}(P_- \not{H}_{(k)} M \Gamma^i)(p_i - p'_i) \ . \tag{5.73}$$

The trace over the spinor indices is calculated in appendix C with the result:

$$\mathrm{Tr}\,(P_- \not{H}_{(k)} M \Gamma^i)(p_i - p'_i) = 16i\,[\delta_{k,1}(H_k)^i + \delta_{10-k,1}(H_{10-k})^i](p_i - p'_i)\ , \tag{5.74}$$

for $p = -1$ and

$$\begin{aligned} &\mathrm{Tr}\,(P_- \not{H}_{(k)} M \Gamma^i)(p_i - p'_i) \\ &= \tfrac{16}{p!}(-1)^{p(p+1)/2}\,[\delta_{k,p}(H_k)^{a_1\ldots a_p}\epsilon_{i\ldots a_p} + \delta_{10-k,p}(H_{10-k})^{a_1\ldots a_p}\epsilon_{i\ldots a_p}](p^i - p'^i)\ , \end{aligned} \tag{5.75}$$

for $p \neq -1$.

Since $1 \leq k \leq 5$, only one of the two terms in (5.75) contributes to the amplitude depending on whether $p < 5$ or $p > 5$. When $p = 5$ the two terms contribute equally as a consequence of self-duality. Thus, without loss of generality we may consider $1 \leq p \leq 5$ and express the amplitude in the form

$$\mathcal{A}^{T,T,RR} \sim \sqrt{\pi}\,2^{-2t-1}\,\frac{\Gamma[-t]}{\Gamma[\frac{1}{2}-t]}\,\frac{1}{p!}\,(H_p)^{a_1\ldots a_p}\epsilon_{ia_1\ldots a_p}(p^i - p'^i)\delta_{k,p}\ , \tag{5.76}$$

Finally, a word concerning the suppressed trace over CP factors (5.4). Although we have denoted both tachyons generically by T a non-zero result is obtained only if one is the complex conjugate of the other. This may be observed by multiplying the matrices in (5.4) together. We find (symbolically)

$$c \times c = 0\ ,\ d \times d = 0\ ,\ c \times d = b\ ,\ d \times c = a\ , \tag{5.77}$$

where c and d are the CP factors for the antitachyon and tachyon, respectively. This indicates that if the two tachyons are of the same type then the amplitude is zero. Comparing the order of the matrices c and d with the order of the vertex operators in (5.51,5.52) implies that the tachyon T has momentum p' and that the antitachyon $\bar{T}$ has momentum p. We shall see below that (5.76) corresponds to a D$(p-2)$-brane embedded in the Dp-$\overline{\mathrm{D}p}$ system via the vortex construction as described earlier[12].

[12]The amplitude vanishes for $p = 0$ since we have excluded $k = 0$ (that is, (-2)-branes) from consideration. In addition, the amplitude takes a different form for $p = -1$ as a result of (5.74). This is to be expected since there are no (-3)-branes in type II theory. In the section below we only consider $p \geq 1$.

5.2.3 Inferring the Wess-Zumino Term

In this section we infer the Wess-Zumino term (5.41) from the amplitude (5.76).

Firstly, let us examine the prefactor in (5.76). The gamma function has simple poles at each non-positive integer. Therefore, the numerator has poles at $t = n$, $n = 0, 1, \ldots$, whilst the denominator has poles at $t = n/2$, $n = 1, 3, \ldots$. Hence, the amplitude has a pole for t a non-negative integer and a zero for t a positive half-odd-integer. In field theory, such poles correspond to resonances with mass-squared $m^2 = -(k^i)^2 = t = n = 2n/\alpha'$, $n = 0, 1, \ldots$, (see figure 5.4) and the zeros correspond to non-propagating modes[13]. In [116] it was shown that similar but reversed pole-zero behaviour occurs in the amplitude for one NS-NS string to decay into two massless open strings stuck to a D-brane.

At low energies, $-t = (k^i)^2 \sim 0$, we may Taylor expand the gamma functions as:

$$\begin{aligned} \Gamma[-t] &= -\tfrac{1}{t}\Gamma[1-t] \\ &= -\tfrac{\Gamma[1]}{t}\left(1 - t\psi[1] + \mathcal{O}(t^2)\right) , \\ \Gamma[\tfrac{1}{2} - t] &= \Gamma[\tfrac{1}{2}](1 - t\psi[\tfrac{1}{2}] + \mathcal{O}(t^2)) , \end{aligned} \tag{5.78}$$

where the psi-function is given by $\psi[r] = \Gamma'[r]/\Gamma[r]$. Given that: $\Gamma[1] = 1$, $\Gamma[\frac{1}{2}] = \sqrt{\pi}$, $\psi[1] = -\gamma$, $\psi[\frac{1}{2}] = -\gamma - 2\ln 2$, where γ is the Euler constant, we obtain the following form for the three-point amplitude in the low-energy regime:

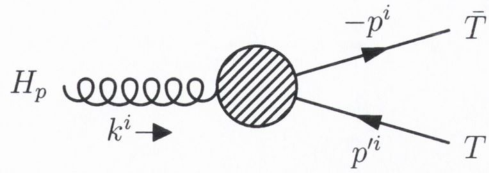
$$\mathcal{A}^{T,T,RR} \sim \left(-\frac{1}{2t} + 2\ln 2 + \mathcal{O}(t)\right)\frac{1}{p!}(H_p)^{a_1\ldots a_p}\epsilon_{ia_1\ldots a_p}(p^i - p'^i)\delta_{k,p} \; . \tag{5.79}$$

The Pole Term

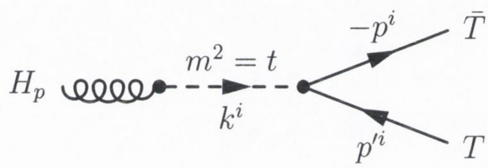
The first term in the amplitude (5.79) corresponds in field theory to an R-R boson decaying into the massless gauge field A (or A') which propagates (resulting in the pole) and then decays into two tachyons. The Feynman diagram for this process is shown in figure 5.5(a).

As we now show, the amplitude for this process can be derived from the low-energy

[13] Note that the poles and zeros together satisfy $m^2 = n/\alpha'$, $n = 0, 1, \ldots$, which is precisely the same equation for the masses of the (GSO-projected) *open* string modes as was given in (4.157).



(a)



(b)

Figure 5.4: (a) The general field theory kinematics corresponding to the two-tachyon—RR boson scattering process of the last section. Note that since the tachyon momenta are separately zero in the directions transverse to the brane, $p^a = p'^a = 0$, the momentum conservation relation (equation (5.66)) reduces to one for the longitudinal directions only. (b) The t-channel resonance in the amplitude (5.76) corresponds to the exchange of a particle of mass $m^2 = t = 2n/\alpha'$, $n = 0, 1, \ldots$ with propagator $\sim 1/((k^i)^2 + m^2)$.

effective Lagrangian

$$\mathcal{L}_1 = \lambda A \wedge H_p - \lambda A' \wedge H_p + \mathcal{L}_A + \mathcal{L}_{A'} - D^i T \overline{D_i T} \ , \tag{5.80}$$

where $\mathcal{L}_A$ and $\mathcal{L}_{A'}$ are the canonically normalised gauge field Lagrangians supplying the gauge field propagators, λ is some constant[14], and the tachyonic covariant derivatives are given by (5.34). The action is given by integrating $\mathcal{L}_1$ over the worldvolume Σ_{p+1} of the Dp-$\overline{\text{D}p}$ system.

The origins of the A–H and A'–H terms in this Lagrangian are well-known since

[14]Establishing a precise correspondence between string theory and field theory requires appropriate normalisation of the vertex operators so that the string theory amplitude agrees in magnitude (and not just in form) with the field theory amplitude, assuming that the kinetic terms in the field theory are canonically normalised. We shall be content to verify that the two amplitudes have the same form.

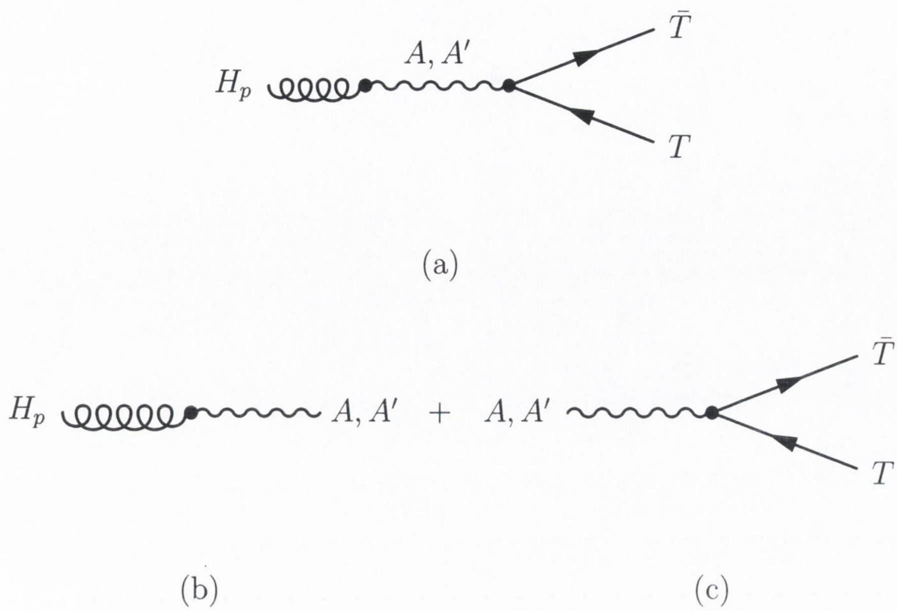


Figure 5.5: (a) Field theory Feynman diagram corresponding to the first term of three-point amplitude (5.79). This is built from the four vertices shown in (b) and (c) which arise from the first and second and last term of the Lagrangian (5.80) respectively.

they contribute to the standard Wess-Zumino term (4.313). For example, for the A–H string amplitude we note that the the doubled vertex operator for A in the -1 superghost picture is given by $\mathcal{V}_A^{(-1)i}(x;k) = c(x)A \cdot \psi(x)e^{-\phi(x)}e^{2ik\cdot X(x)}$ where $k^\mu = (k^i, \vec{0}), k^2 = 0$. A_i is the polarisation of the gauge field and satisfies $k \cdot A = 0$. Therefore, since the amplitude has no integrations (cf. (5.49)) and the trace over the spinor indices is equivalent to that in the $\mathcal{A}^{T,T,RR}$ amplitude we deduce

$$\mathcal{A}^{A,RR} \sim \frac{1}{p!}(H_p)^{a_1\ldots a_p}\epsilon_{ia_1\ldots a_p}A^i \ . \tag{5.81}$$

This is of the same form as the field theory amplitude coming from the A–H term in (5.80) with

$$\frac{-i\lambda}{p!}\epsilon_{ia_1\ldots a_p} \ , \tag{5.82}$$

as the vertex factor of the A–H interaction (see figure 5.5(b))[15]. Similarly, string

[15]To obtain the field theory vertex factor we have followed the rules in [165] where one examines $i\mathcal{L}_1$ and replaces ∂^i by $-ip^i$ when acting on a field with momentum p^i. Note that our metric is of opposite signature to that in [165].

theory gives

$$\mathcal{A}^{A',RR} \sim -\frac{1}{p!}(H_p)^{a_1\dots a_p}\epsilon_{ia_1\dots a_p}A'^i \ , \tag{5.83}$$

where the minus sign comes from the opposite sign of M on the antibrane relative to the brane. This accounts for the sign of the A'–H term in (5.80), leading to a vertex opposite in sign to (5.82).

The tachyon kinetic term in (5.80) can be deduced from a calculation of $\mathcal{A}^{T,T,A}$ and $\mathcal{A}^{T,T,A'}$. Indeed, we find

$$\begin{aligned}\mathcal{A}^{T,T,A} &= 2p_iA_j\langle c(x)\psi^i(x)e^{2ip\cdot X(x)}c(x')e^{-\phi(x')}e^{2ip'\cdot X(x')}c(y)\psi^j(y)e^{-\phi(y)}e^{2ik\cdot X(y)}\rangle \\ &\sim 2p\cdot A \\ &\sim (p-p')\cdot A \ ,\end{aligned} \tag{5.84}$$

where we have used relation (4.144): $\langle\psi^i(x)\psi^j(y)\rangle = \eta^{ij}/(x-y)$, momentum conservation and the on-shell condition $k\cdot A = 0$. Similarly, interchanging the order of the tachyonic vertex operators and redistributing the ghost charge as in (5.52), we find

$$\mathcal{A}^{T,T,A'} \sim 2p'\cdot A \sim -(p-p')\cdot A \ . \tag{5.85}$$

Note that (5.77) implies that $\bar{T}T$ couples to A and $T\bar{T}$ couples to A'. This is consistent with the definition of the covariant derivative in (5.34) and of the tachyon kinetic term which contains

$$-(\partial^iTA_i\bar{T} - TA_i\partial^i\bar{T}) - (A_i'T\partial^i\bar{T} - \partial^iT\bar{T}A_i') \ . \tag{5.86}$$

The vertex factor for the A–$\bar{T}$–T interaction depicted in figure 5.5(c) is therefore

$$(p^i - p'^i) \ , \tag{5.87}$$

where p'^i corresponds to T and where both p' and p are ingoing to the vertex. Clearly, equation (5.87) agrees with (5.84). Note that the A'–T–$\bar{T}$ interaction is of opposite sign to (5.87) and so agrees with (5.85). There is a subtlety here in that (5.86) is also contained in the alternative tachyon kinetic term

$$-(\partial^iT - TA^i)(\partial_i\bar{T} + A_i\bar{T}) - (\partial^iT + A'^iT)(\partial_i\bar{T} - \bar{T}A_i') \ . \tag{5.88}$$

However, an examination of the CP factors reveals that we can have crossed T–A–$\bar{T}$–A'-type interactions which are reproduced by the tachyon kinetic term in (5.80) but not by (5.88)[16].

We are now able to combine figure 5.5(b) and figure 5.5(c). We see that the field theory amplitude corresponding to figure 5.5(a) is

$$\begin{aligned}\mathcal{A} &\sim (H_p)^{a_1\ldots a_p} \times \frac{\mp i\lambda}{p!}\epsilon_{ia_1\ldots a_p} \times \frac{i\delta^i_j}{(k^i)^2} \times \pm(p^j - p'^j) \\ &\sim -\frac{\lambda}{t}\frac{1}{p!}(H_p)^{a_1\ldots a_p}\epsilon_{ia_1\ldots a_p}(p^i - p'^i) \ , \end{aligned} \tag{5.89}$$

where the gauge field propagator is given by $i\delta^i_j/(k^i)^2$ (in Feynman gauge, for example) and we have suppressed the tachyon polarisations. Therefore, we see that this amplitude is of the same form as the pole term in (5.79).

The Contact Term

The second term in (5.73) is a contact term corresponding to the field theory vertex depicted in figure 5.6.

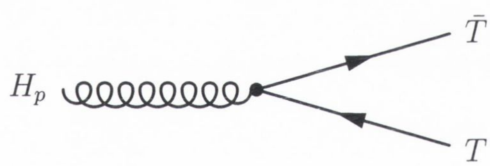


Figure 5.6: Field theory Feynman diagram corresponding to the second term of the three-point amplitude (5.79).

This vertex has a factor

$$\frac{\mu}{2 \times p!}\epsilon_{ia_1\ldots a_p}(p^i - p'^i) \ , \tag{5.90}$$

where μ is some constant. Comparing with (5.79) and (5.89) we determine that μ must be of the *same* sign as λ. After using momentum conservation and the Bianchi identity (4.268), we see that this vertex is reproduced by the Lagrangian

$$\mathcal{L}_2 = \mu \bar{T} dT \wedge H_p \ . \tag{5.91}$$

[16]In fact, it is the alternative kinetic term (5.88) that appears in the effective action if the brane and antibrane worldvolumes are considered to be distinguishable [151].

The Wess-Zumino Term

Consider the A–H and A'–H terms of (5.80) together with the contact term (5.91). After arranging covariance of the tachyon derivative in (5.91) and integrating by parts, the action for these terms can be written as

$$S \sim \int_{\Sigma_{p+1}} C_{p-1} \wedge \left(dA - dA' + \frac{\mu}{\lambda} d(\bar{T} DT) \right) , \tag{5.92}$$

where $H_p = dC_{p-1}$ and we have used the properties (B.3) concerning the commutativity of forms and the Leibnitz rule. Since λ and μ are of the same sign, we can rescale the tachyon and set $\mu/\lambda = 1$. Since we are assuming that the gauge fields are abelian (that is, $N = 1$) we have have $F = dA$, $F' = dA'$ and after a little algebra we can rewrite (5.92) as

$$S \sim \int_{\Sigma_{p+1}} C_{p-1} \wedge \left(F - F' - \frac{1}{2}\{F, \bar{T}T\} + \frac{1}{2}\{F', T\bar{T}\} + \overline{DT} \wedge DT \right) . \tag{5.93}$$

Note that since we are in static gauge and ignoring transverse fluctuations, the pull-back $\hat{C}_{p-1}$ is equal to C_{p-1}. Equation (5.93) is now to be compared with (5.42). At first sight there appears to be a discrepancy — the sign of the $\overline{DT} \wedge DT$ term above is opposite to that in (5.42). However, there is no inconsistency. This is because the tachyon in (5.42) is defined so as to anticommute with the differentials $d\xi^i$. But in (5.93) the tachyon is a standard field theory 0-form which commutes with $d\xi^i$. Recalling the convention that in working with the superconnection all differentials are written on the left, we find (5.93) to be entirely consistent with (5.42).

5.3 Conclusions and Further Work

In the last section we have seen by calculating some tree-level string amplitudes that in a coincident brane-antibrane system there is in addition to the usual Wess-Zumino term (4.313) the term

$$\int_{\Sigma_{p+1}} C_{p-1} \wedge d(\bar{T} DT) , \tag{5.94}$$

to order $\mathcal{O}(\bar{T}T)$ in static gauge. We have seen from (5.93) and (5.42) how this term along with the $\mathcal{O}(F)$ terms from the usual WZ action can be written in terms of the curvature of the superconnection for $U(1)$ gauge fields. Assuming that the non-abelian generalisation is given simply by including a trace over the fundamental representation of the gauge group as discussed in section 4.2.4, this prompts us to propose that the full result to all orders in the tachyon is given by (5.41):

$$S \sim \int_{\Sigma_{p+1}} \left\{ \hat{C} \wedge \mathrm{ch}(\mathcal{F}) \wedge \sqrt{\frac{\hat{A}(R_T)}{\hat{A}(R_N)}} \right\}_{p+1} . \tag{5.95}$$

Clearly, our 'derivation' of the above result has been somewhat heuristic. An obvious avenue for further work is to try to establish (5.95) in a more precise manner. Several papers along this line [162, 163, 166] using the techniques of boundary string field theory appeared in late December 2000 and have confirmed the result. For a recent review of string field theory in the context of tachyon condensation see Ohmori [167], for example. As mentioned just below (5.41), a very recent paper of Schwarz and Witten [164] has confirmed the existence of the A-roof genus terms using an anomaly inflow argument. Finally, tachyon condensation and other aspects of Ashoke Sen's work remain topics for current research [168].

Part III

Generalised Kaluza-Klein Theory

Chapter 6

(Brief) Review of Standard Kaluza-Klein Theory

In this chapter we review some of the essential features of five-dimensional abelian Kaluza-Klein theory following the treatment presented in [169–171].

6.1 The Kaluza-Klein Ansatz

Einstein's equations in five dimensions with no five-dimensional energy momentum tensor or cosmological constant are

$$\hat{G}_{AB} = 0 \ , \tag{6.1}$$

or, equivalently

$$\hat{R}_{AB} = 0 \ , \tag{6.2}$$

where $\hat{G}_{AB} = \hat{R}_{AB} - \hat{R}\hat{g}_{AB}/2$ is the Einstein tensor, $\hat{R}_{AB}$ and $\hat{R} = \hat{g}^{AB}\hat{R}_{AB}$ are the five-dimensional Ricci tensor and scalar respectively and $\hat{g}_{AB}$ is the five-dimensional metric tensor[1]. These equations may be obtained by varying the five-dimensional

[1]In this and subsequent chapters, we shall use r to denote the fifth direction; indices $A, B, ... = 0, 1, 2, 3, 4 \equiv r$ to denote five-dimensional tensor components; indices $i, j, ...$ will run over $0, 1, 2, 3$ and small Latin indices $a, b, ...$ will run over $1, 2, 3$. We use the mostly plus convention for the metric and define the Riemann tensor as $\hat{R}^{A}{}_{BCD} = \partial_C \hat{\Gamma}^{A}_{BD} + \hat{\Gamma}^{E}_{BD}\hat{\Gamma}^{A}_{EC} - (C \leftrightarrow D)$ and the Ricci tensor as $\hat{R}_{AB} = \hat{R}^{C}{}_{ACB}$.

Einstein-Hilbert action

$$S = \frac{1}{2\hat{\kappa}_5^2} \int d^4x \, dr \, \sqrt{-\hat{g}} \, \hat{R} \ , \tag{6.3}$$

with respect to the five-dimensional metric.

In order to solve (6.2) it is assumed that the fifth direction is compactified on a very small circle of radius r_0. One then chooses the following metric ansatz:

$$\hat{g}_{AB} = \begin{pmatrix} g_{ij} + 2\kappa^2\phi^2 A_i A_j & \sqrt{2}\kappa\phi^2 A_i \\ \sqrt{2}\kappa\phi^2 A_j & \phi^2 \end{pmatrix} \ , \tag{6.4}$$

where $\kappa^2 \equiv 8\pi G$, with G the four-dimensional Newton's constant. Note that the ansatz is just a 1+4 decomposition of the five-dimensional space-time in which the "lapse function" is the four-dimensional metric g_{ij} and the "shift vector" is the four-dimensional electromagnetic potential A_i. As discussed in the general introduction (see page xv), the 44-part of the metric is the square of the Brans-Dicke scalar ϕ.

One then assumes the so-called *cylinder condition*, which means dropping all derivatives with respect to the fifth coordinate. This is equivalent to expanding the fields in harmonics on S^1:

$$\begin{aligned} g_{ij}(x,r) &= \sum_{n=-\infty}^{n=\infty} g_{ij}^{(n)}(x) e^{inr/r_0} \ , \quad A_i(x,r) = \sum_{n=-\infty}^{n=\infty} A_i^{(n)}(x) e^{inr/r_0} \ , \\ \phi(x,r) &= \sum_{n=-\infty}^{n=\infty} \phi^{(n)} e^{inr/r_0} \ , \end{aligned} \tag{6.5}$$

(where $-\pi r_0 \leq r \leq \pi r_0$) and neglecting all but the zero modes $n = 0$. This low-energy approximation is justified if r_0 is very small because the four-dimensional masses n/r_0 of the non-zero modes will then be so large as to put them beyond experimental reach. The smallness of the extra dimension explains why it has not been directly observed.

With the cylinder condition, equations (6.2) may be decomposed as

$$\begin{aligned} G_{ij} &= \kappa^2\phi^2 T_{ij}^{EM} + \frac{1}{\phi}\left[\nabla_i\nabla_j\phi - g_{ij}\Box\phi\right] \ , \\ \nabla^i F_{ij} &= -3\,\frac{\partial^i\phi}{\phi} F_{ij} \ , \quad \Box\phi = -\frac{\kappa^2\phi^3}{2} F_{ij}F^{ij} \ , \end{aligned} \tag{6.6}$$

where $T_{ij}^{EM} \equiv F_i{}^k F_{jk} - g_{ij}F_{kl}F^{kl}/4$ is the electromagnetic energy-momentum tensor and $F_{ij} \equiv \nabla_i A_j - \nabla_j A_i$. G_{ij} is the four-dimensional Einstein tensor calculated from

g_{ij}, ∇_i is the corresponding compatible covariant derivative and $\Box \equiv \nabla_i \nabla^i$. Indices are lowered and raised with the metric g_{ij} and its inverse, respectively. We stress that equations (6.6) are satisfied by the massless ($n = 0$) modes only; for ease of reading we have dropped the superscript $^{(0)}$.

These equations may also be obtained by varying the $4d$ action

$$S = \int d^4x \sqrt{-g}\, \phi \left(\frac{R}{2\kappa^2} - \frac{1}{4}\phi^2 F_{ij}F^{ij} \right) . \tag{6.7}$$

It is the central result of Kaluza-Klein theory that this action may be obtained via dimensional reduction of the original $5d$ action (6.3). This may be seen by invoking the cylinder condition and ansatz (6.4) to expand $\hat{R}$ as

$$\hat{R} = R - \frac{\kappa^2}{2}\phi^2 F_{ij}F^{ij} - \frac{2}{\phi}\Box\phi , \tag{6.8}$$

integrating out the fifth dimension, neglecting the total derivative stemming from the third term in (6.8) and equating

$$\kappa^2 = \hat{\kappa}_5^2 / V_1 , \tag{6.9}$$

where $V_1 = 2\pi r_0$ is the volume of the fifth dimension.

If $\phi = 1$, then the first two of eqs. (6.6) are just Einstein's electrovac equations and Maxwell's equations respectively. However, $\phi = 1$ is only consistent with the third of the equations (6.6) if $F_{ij}F^{ij} = 0$ (cf. General Introduction, page xv). The ground state

$$< g_{ij} >= \eta_{ij} , \quad < A_i >= 0 , \quad < \phi >= 1 , \tag{6.10}$$

is therefore a solution of (6.6).

Consider now the effect of five-dimensional coordinate transformations $y^A \to y'^A \equiv y'^A(y^B)$, with $y^A \equiv (x^i, r)$, preserving the form of the line element

$$\begin{aligned} ds^2_{(0)} &= \hat{g}^{(0)}_{AB}\, dy^A dy^B \\ &= g_{ij}dx^i dx^j + \phi^2 (dr + \sqrt{2}\,\kappa A_i dx^i)^2 , \end{aligned} \tag{6.11}$$

constructed from the massless modes. Two such transformations are of note. Firstly, $\phi(x)$ and $A_i(x)$ are taken to transform as a scalar and a covariant vector field respectively under four-dimensional general coordinate transformations. Hence, $ds^2_{(0)}$ will be invariant under $x^i \to x'^i \equiv x'^i(x)$ with the r-coordinate remaining fixed. Secondly, the transformation

$$\begin{aligned} x^i &\to x'^i = x^i \ , \\ r &\to r' = r + \sqrt{2}\,\lambda(x) \ , \end{aligned} \tag{6.12}$$

requires the massless gauge field to transform as

$$A_i \to A_i' = A_i - \kappa^{-1}\,\partial_i\lambda \ , \tag{6.13}$$

in order that (6.11) be invariant. In this way one sees how $U(1)$ local gauge invariance is reinterpreted as an x-dependent coordinate transformation of the extra spatial coordinate r.

6.2 Extending Beyond $U(1)$

To extend the Kaluza- Klein approach to gauge groups more complicated than $U(1)$ requires more than five dimensions. Basically the idea is that the gauge group G obtained in $D = 4$ is identified with the isometry group of the $n > 1$ extra dimensions. These dimensions, in analogy with S^1, are taken to be compact and of Euclidean signature. For example, one could take as the internal space the homogeneous coset space G/H, where G is a compact Lie group with maximal proper subgroup H. However, a space-time in which the internal space is in general curved is not a solution to the higher-dimensional vacuum Einstein equations. In order to achieve a consistent compactification it is necessary to either (a) incorporate torsion [172–175], or (b) add higher-derivative terms (eg., $\hat{R}^2$) onto the higher-dimensional Einstein action [176], or (c) augment pure gravity with higher-dimensional matter fields, as shown by Cremmer *et al* [177]. The third of these alternatives has become the standard way to reconcile extra dimensions with the observed four-dimensionality of space-time in Kaluza-Klein theory. Unfortunately, due to the presence of the higher-dimensional matter, it abandons Kaluza's dream of a purely geometrical unified theory of nature. Such higher-dimensional KK theories, however, can only give rise to four-dimensional gauge *bosons*. If the $4d$ theory is to include *fermions* then these fields, at least, must be put in by hand. This led to the advent of KK supergravity with equal numbers of bosonic and fermionic degrees of freedom. Supergravity began as a four-dimensional theory in 1976 [178, 179] but was quickly extended to higher dimensions, such as $D = 11$ [180], following Nahm's discovery [181] that eleven was the maximum number of dimensions consistent with a single spin-2 graviton and Witten's argument [182] that eleven was the minimum number of dimensions required for a KK-theory to accommodate the $SU(3) \times SU(2) \times U(1)$ gauge group of the Standard Model. In an important contribution, Freund and Rubin [183] showed that $D = 11$ supergravity could be compactified to either four or seven dimensions only. Consequently, this supergravity became a leading candidate as the "theory of everything". It was not without its problems, however. Foremost amongst these problems were the facts that quantisation led to anomalies and that the theory was incompatible with the requirement of chirality in four dimensions. These obstacles led to consideration of theories in $D = 10$ where chirality was easier to obtain and many anomalies disappeared and inevitably pointed towards superstrings once it was realised that the $D = 10$ supergravities arose as low-energy approximations to the various superstring theories (recall section 4.1.7). The reader is referred to [9] for a concise historical overview of the development of KK-theory in more than five dimensions and to [10, 169–171] for calculational details.

For the purposes of the present work the key point to take from the higher-dimensional generalisations and one to which we shall return in the next chapter, is the relationship of the four-dimensional gravitational constant to the higher-dimensional one, namely

$$\kappa^2 = \hat{\kappa}^2_{4+n}/V_n \ , \tag{6.14}$$

where V_n is the volume of the internal compact space. Put differently, since the gravitational constant has dimensions $[\text{Length}]^{2+n}$ in $4 + n$ dimensions, the relationship between the *reduced* Planck scale $M_{Pl} = \kappa^{-1} = (8\pi G)^{-1/2} \sim 2.4 \times 10^{18}\,\text{GeV}$ and the higher-dimensional reduced mass scale $\hat{M}^{2+n} = \hat{\kappa}^{-2}_{4+n}$ is

$$M^2_{Pl} = \hat{M}^{2+n}\, V_n \ . \tag{6.15}$$

The compactification scale $2\pi R_c \equiv V_n^{1/n}$ is not fixed in these theories and is often taken as the string scale $\sqrt{\alpha'}$ so as to put the non-zero Kaluza-Klein modes beyond the Planck mass. Consequently, $\hat{M}$ must then be extremely large. If it is assumed that the higher-dimensional theory originates from some supergravity then $n \leq 7$.

Chapter 7

The Randall-Sundrum Models

In this chapter we build on the last and examine two exciting recent developments in particle physics that use so-called "warped" geometry (to be defined shortly) and might be termed "generalised Kaluza-Klein theory". The first development, described in sections 7.1 to 7.3, is an attempt to solve the age-old hierarchy problem that exists within particle physics while the second is an attempt to localise a theory of gravity with five non-compact dimensions to four dimensions and is described in sections 7.4 and 7.5.

7.1 The Hierarchy Problem

There are perhaps two fundamental energy scales in particle physics, the electroweak scale $m_{EW} \sim 10^3\,\mathrm{GeV}$ and the four-dimensional (reduced) Planck scale $M_{Pl} \sim 10^{18}\,\mathrm{GeV}$, where gravity becomes comparable in strength to gauge interactions. Explaining the smallness of the ratio $m_{EW}/M_{Pl} \sim 10^{-15}$ and the stability of the lighter scale from radiative corrections coming from the larger scale without fine-tuning of parameters is known as the "hierarchy problem" and has occupied the particle physics community since the earliest attempts to go beyond the Standard Model (SM) some thirty-odd years ago [184–186]. There is now a widely accepted picture of the basic nature of physics beyond the Standard Model. At the weak scale there is an effective field theory stabilising the hierarchy. Such a theory might be a softly broken supersymmetric theory, since the two scales may then be built into the tree-level effective potential and various non-renormalisation theorems ensure that loop radiative corrections do not destroy the hierarchy. However, although SUSY theories can accommodate the hierarchy they cannot explain its origin. At the Planck scale the leading candidate is string theory whilst in between the two scales a universally accepted effective theory is yet to emerge. Other important questions which are not answered by the SM, such as why there are three families of leptons or what is the origin of observed CP violation, are discussed in Wilczek's account [187] of physics beyond the SM and also in [188].

In early-to-mid 1998 a series of papers by Antoniadis, Arkani-Hamed, Dimopoulos and Dvali (ADD) [189] proposed a new solution to the hierarchy problem in which gravity and gauge interactions became united at the weak scale, and the observed weakness of gravity at long distances was due to the existence of *large* compact spatial dimensions. In other words, $\hat{M}$ in equation (6.15) was assumed to be of the order of $m_{EW} \sim 1\,\mathrm{TeV}$, yielding the typical size of the extra dimensions to be

$$R_c \sim 10^{\frac{30}{n}-17}\,\mathrm{cm} \times \left(\frac{1\,\mathrm{TeV}}{m_{EW}}\right)^{1+\frac{2}{n}} . \tag{7.1}$$

The stability of the weak scale is no longer an issue since it acts as a UV cutoff for the effective theory. The case $n = 1$, in which $R_c \sim 10^{13}\,\mathrm{cm}$ is of the order of solar system scales, is ruled out by the Cavendish experiment (see section 7.4). The large size of the extra dimensions is not ruled out by present-day experiments since such experiments have only accurately probed gravitational forces down to the $\sim 1\,\mathrm{cm}$ range (see section 7.4) whilst, for example, $n = 2$ has $R_c \sim 100\,\mu\mathrm{m} - 1\,\mathrm{mm}$ [189]. However, the gauge forces of the Standard Model have been accurately measured at weak scale distances and there is no indication of the presence of extra dimensions. Therefore, the scenario is phenomenologically viable if the Standard Model particles and forces are localised on a four-dimensional timelike hypersurface (a "three-brane") within the higher-dimensional space-time. The non-zero Kaluza-Klein modes do not present a problem as they simply escape into the extra dimensions because their wavelengths are much smaller than R_c [189]. Other phenomenological aspects of and astrophysical constraints on the ADD scenario can be found in the original papers [189] and also in [190]. For example, the constraint that the graviton luminosity should not exceed the gravitational binding energy of $\sim 10^{53}$ ergs released during the few seconds of the collapse of SN1987A imposes

$$\hat{M} \gtrsim 10^{\frac{15-4.5n}{n+2}} \text{ Tev} , \tag{7.2}$$

yielding $\hat{M} \gtrsim 30\,\text{TeV}$ for the minimal $n = 2$ case.

7.2 Warped v. Factorisable Geometry

Definition. The *warped product* $M = B \times_\phi F$ of two semi-Riemannian manifolds B and F is the product manifold $B \times F$ furnished with the metric tensor

$$\mathbf{g} = \pi^*(\mathbf{g_B}) + (\phi \circ \pi)^2 \sigma^*(\mathbf{g_F}) ,$$

where π and σ are the projections of $B \times F$ onto B and F respectively and $\phi > 0$ is a smooth function on B. The manifold M has a fibre bundle structure with B the base and F the fibre [191]. We shall say that "F is warped by B." In the special case $\phi = 1$, M is a direct or *factorisable* product (or a *trivial* fibre bundle.)

In the standard Kaluza-Klein ansatz (6.11) in five dimensions, the space-time is the warped product $M^4 \times_\phi S^1$ of four-dimensional Minkowski space M^4 and the internal S^1 with the Brans-Dicke scalar ϕ as the warping function. In higher-dimensional generalisations the space-time is actually a direct product $M^4 \times K^n$ with internal compact space K^n [169–171]. In either case, there is no warping of M^4 by K^n. In this sense, both standard Kaluza-Klein theory and the ADD scenario are based on the assumption of a factorisable geometry. We refer to KK-theory with M^4 warped by K^n as "generalised KK-theory" and we shall explore its consequences in the remainder of this thesis. One immediate consequence is apparent. Relation (6.15) will be modified because the decomposition $\hat{R} = R + \cdots$ (cf. (6.8)) will no longer hold and so integrating out the extra dimensions in dimensional reduction of the EH action will not simply give a volume factor. We shall return to this point in section 7.3.3.

7.3 The First Randall-Sundrum Model (RS1)

The ADD scenario contains a serious drawback. It does eliminate the hierarchy problem between the weak and the Planck scale but introduces a new one between the weak scale and the compactification scale $\mu_c \equiv 1/R_c$. This was noted by Randall and Sundrum (RS) [192] who proposed an alternative solution to the hierarchy problem based on the warping of M^4 by an internal one-dimensional space. For the remainder of this section we follow the treatment presented in the original paper [192].

7.3.1 The Set-Up

The RS set-up consists of an internal S^1 of radius r_0 together with the identification of the space-time points (x^i, r) and $(x^i, -r)$. The internal space is thus the orbifold $S^1/\mathbb{Z}_2$ with fixed points at $r = 0$, πr_0. Located at each fixed point is a flat Poincaré-invariant three-brane. The branes are embedded trivially in static-gauge in the higher-dimensional space-time, that is, if x^i are brane coordinates and y^A are coordinates in the bulk, then

$$
\begin{aligned}
y^i(x) &= x^i \ , \\
r(x) &= r' = const. \ ,
\end{aligned}
\tag{7.3}
$$

and the induced metric on the brane located at $r = r'$ is

$$
\begin{aligned}
g_{ij}(x) &= \hat{g}_{AB}(y) \frac{\partial y^A}{\partial x^i} \frac{\partial y^B}{\partial x^j} \ , \\
&= \hat{g}_{ij}(x, r') \ .
\end{aligned}
\tag{7.4}
$$

The "*visible*" brane at $r = 0$ is "our world" and supports the Standard Model (SM); the "*hidden*" brane at $r = \pi r_0$ is also capable of supporting a $(3+1)$-dimensional field theory but is invisible to observers in our world.

The classical action for the model is given by[1]:

$$
\begin{aligned}
S &= S_{gravity} + S_{vis} + S_{hid} \ , \\
S_{gravity} &= \frac{1}{2\hat{\kappa}_5^2}\int d^4x \int_{-\pi r_0}^{\pi r_0} dr \, \sqrt{-\hat{g}} \left(\hat{R} - 2\hat{\Lambda}\right) \ , \\
S_{vis} &= \int d^4x \, \sqrt{-g_{vis}} \left(\mathcal{L}_{vis} - V_{vis}\right) \ , \\
S_{hid} &= \int d^4x \, \sqrt{-g_{hid}} \left(\mathcal{L}_{hid} - V_{hid}\right) \ ,
\end{aligned} \tag{7.5}
$$

where g_{ij}^{vis} and g_{ij}^{hid} are the induced metrics on the visible and hidden branes, respectively. In contrast to standard KK-theory, a five-dimensional cosmological constant $\hat{\Lambda}$ is also included. Each brane Lagrangian is split into a "vacuum energy" part V which parametrises the tension of the brane and a part $\mathcal{L}$ containing various SM-type fields. The explicit form of $\mathcal{L}$ is not needed for determining the ground state of the $5d$ metric so we shall not comment further on it. The interested reader is referred to [193] for details.

Einstein's equations resulting from S are

$$
\begin{aligned}
\hat{G}_{AB} + \hat{\Lambda}\, \hat{g}_{AB} = &- \frac{\hat{\kappa}_5^2\, \delta_A^i \delta_B^j}{\sqrt{\hat{g}}} \left[V_{hid} \sqrt{-g_{hid}}\, g_{ij}^{hid}\, \delta(r - \pi r_0)\right. \\
&+ \left. V_{vis} \sqrt{-g_{vis}}\, g_{ij}^{vis}\, \delta(r)\right] \ ,
\end{aligned} \tag{7.6}
$$

where there is no sum over the indices i and j.

7.3.2 The Solution

An ansatz for the ground-state metric respecting four-dimensional Poincaré invariance in the x^i-directions is

$$
ds^2 = e^{2A(r)}\, \eta_{ij} dx^i dx^j + dr^2 \ . \tag{7.7}
$$

There are two main differences between this ansatz and the standard KK ground state (6.10). Firstly and most importantly, the exponential factor $e^{2A(r)}$ represents

[1] We work in the "upstairs" formalism, that is, on the *boundary-free* full circle $-\pi r_0 \leq r \leq \pi r_0$ with the two three-branes at $r = 0, \pi r_0$ and $\mathbb{Z}_2$ symmetry imposed on the fields. One could alternatively work in the "downstairs" formalism, that is, on the half-line, $0 \leq r \leq \pi r_0$, with boundaries (the two three-branes) at either end and appropriate boundary conditions imposed on the fields.

the warping of M^4 by the internal space. Secondly, the metric is required to be invariant under the $\mathbb{Z}_2$ action $r \to -r$. Therefore, $A(r)$ will be a function of $|r|$.

With this ansatz, equations (7.6) reduce to

$$\begin{aligned} 6A'^2 &= -\hat{\Lambda} , \\ 3A'' &= -\hat{\kappa}_5^2 \left[V_{vis}\, \delta(r) + V_{hid}\, \delta(r - \pi r_0)\right] . \end{aligned} \tag{7.8}$$

A solution to the first of these equations consistent with the orbifold symmetry is

$$A(r) = |r| \sqrt{-\hat{\Lambda}/6} , \tag{7.9}$$

where an overall additive constant has been omitted because it just amounts to an overall rescaling of the coordinates x^i. One sees that the solution only makes sense provided $\hat{\Lambda} < 0$. It is worth noting that with this solution for $A(r)$ the regions $0 < r < \pi r_0$ and $-\pi r_0 < r < 0$ are two regions of Lorentzian AdS_5 glued together across the visible brane at $r = 0$. This is easily seen if one recalls (cf. equation (2.9)) that the AdS_5 metric in Poincaré coordinates is

$$ds^2 = k^{-2} \frac{\eta_{ij} dx^i dx^j + du^2}{u^2} , \tag{7.10}$$

where k is the inverse of the AdS radius l (given by $\hat{R} = -20/l^2$), performs the transformation $u = e^{-k|r|}/k$ and equates $k = \sqrt{-\hat{\Lambda}/6}$.

It is understood that (7.9) is defined in the range $-\pi r_0 \leq r \leq \pi r_0$ and is extended to the whole real line by periodicity. Therefore,

$$A'' = 2k \left[\delta(r) - \delta(r - \pi r_0)\right] , \tag{7.11}$$

where we have used the relation $\frac{d^2}{dr^2}|r| = 2\delta(r)$ and the minus sign in the bracket $[\cdots]$ results from the periodicity and hence opposite gradient of A at $r = 0$ and $r = \pi r_0$. Comparison with the second of equations (7.8) yields

$$V_{vis} = -V_{hid} = -\,6k\, \hat{\kappa}_5^{-2} . \tag{7.12}$$

The final solution for the bulk metric is then

$$ds^2 = e^{2k|r|}\, \eta_{ij} dx^i dx^j + dr^2 . \tag{7.13}$$

One must bear in mind that it is an assumption of RS1 that the bulk curvature is smaller than the five-dimensional Planck scale, that is, $k < \hat{M}$ and that r_0 is small (but still larger than $1/k$).

7.3.3 How RS1 Solves the Hierarchy Problem

In this section we derive the four-dimensional Planck scale and the mass parameters of four-dimensional fields in terms of the five-dimensional scales $\hat{M}, k,$ and r_0 and show how the RS1 scenario solves the hierarchy problem. We again follow [192].

It is straightforward to compute the $4d$ Planck scale. One starts from the metric

$$ds^2 = e^{2A(r)}\,\bar{g}_{ij}(x)\,dx^i dx^j + dr^2 \;, \tag{7.14}$$

performs the transformation[2] $z = \int e^{-A(r)} dr$ to bring the metric to the conformal form $\hat{\mathbf{g}} = e^{2A(z)}\,\tilde{\mathbf{g}}$ and expands $\hat{R}$ as [18]:

$$\hat{R} = e^{-2A}\,[\,\tilde{R} - 8\,\tilde{\nabla}^2 A - 12\,(\tilde{\nabla} A)^2\,] \;, \tag{7.15}$$

where $\tilde{\nabla}$ is the covariant derivative compatible with $\tilde{\mathbf{g}}$ and indices on the right side are raised and lowered with $\tilde{\mathbf{g}}$. $\tilde{R} = \bar{R}$ by virtue of (6.8). The Einstein-Hilbert term in (7.5) can then be rewritten as

$$\frac{1}{2\hat{\kappa}_5^2} \int d^4x \int dz\, \sqrt{-\bar{g}}\, e^{3A}\,(\bar{R} + \cdots) \;, \tag{7.16}$$

from which one obtains

$$\kappa^{-2} = \hat{\kappa}_5^{-2} \int_{-\pi r_0}^{\pi r_0} dr\; e^{2A(r)} \;. \tag{7.17}$$

Substituting (7.9) for A yields

$$\begin{aligned} M_{Pl}^2 &= \frac{\hat{M}^3}{k}\,[e^{2\pi k r_0} - 1] \\ &\approx \frac{\hat{M}^3}{k}\, e^{2\pi k r_0} \;, \end{aligned} \tag{7.18}$$

where the approximation is valid if the exponential factor is large. This exponential factor is the modification of (6.15) alluded to in section 7.2. As we shall see momentarily, it allows electroweak scales to generate Planck scales.

[2] The transformation is defined so that $z = 0$ corresponds to $r = 0$.

The masses of four-dimensional fields can be obtained by considering, for example, a fundamental Higgs field on the visible brane

$$S_{vis} \supset \int d^4x \sqrt{-g_{vis}} \left(-g^{ij}_{vis}(D_iH)^\dagger D_jH - \lambda(|H|^2 - \hat{v}^2)^2\right) , \qquad (7.19)$$

where D_i is a gauge-covariant derivative. Such terms occur in the Lagrangian $\mathcal{L}_{vis}$ [193]. Using (7.4) and (7.14) one sees that $g^{vis}_{ij} = e^{2A(0)}\,\bar{g}_{ij}$. After canonically normalising the Higgs kinetic term on the visible brane via $H \to e^{-A(0)}\,H$ one obtains

$$S_{vis} \supset \int d^4x \sqrt{-\bar{g}} \left(-\bar{g}^{ij}D_iH^\dagger D_jH - \lambda(|H|^2 - e^{2A(0)}\,\hat{v}^2)^2\right) , \qquad (7.20)$$

and so, using (7.9),

$$v_{phys} = e^{A(0)}\,\hat{v} = \hat{v} . \qquad (7.21)$$

This result is completely general: any five-dimensional mass parameter $\hat{v}$ on a brane at $r = r'$ will correspond to a physical mass $v_{phys} = e^{A(r')}\,\hat{v}$ when measured with the metric $\bar{g}_{ij}$, which is the metric that appears in the effective Einstein action after integrating out r.

It can now be deduced how RS1 solves the hierarchy problem. Treating the electroweak scale as the fundamental scale we must have $\hat{v} = v_{phys} \sim m_{EW}$. Assuming all fundamental $5d$ mass parameters are of the same scale, we also have $\hat{M}, k \sim m_{EW} \sim 10^3\,\text{TeV}$. Generation of the $4d$ reduced Planck scale $M_{Pl} \sim 10^{18}\,\text{GeV}$ according to equation (7.18) then requires $e^{2\pi k r_0} \sim 10^{30}$ or $\pi k r_0 \approx 35$. Therefore, very large hierarchies among the fundamental parameters $\hat{v}, \hat{M}, k$ and $\mu_c = 1/r_0$ are not required.

7.4 Newton's Force Law

Newton's force law in $4+n$ flat space-time dimensions can be obtained by an application of Gauss' law to Poisson's equation

$$\nabla^2\varphi = S_{2+n}\,\hat{G}_{4+n}\,\rho , \qquad (7.22)$$

where φ is the gravitational potential per unit mass, ρ is the mass density of a matter source, ∇^2 is the $(3+n)$-dimensional spatial Laplacian, $\hat{G}_{4+n}$ is Newton's constant and $S_{2+n} = 2\pi^{\frac{3+n}{2}}/\Gamma(\frac{3+n}{2}) = \mathrm{Vol}\,(S^{2+n})$. For a point-particle source of mass M located at the origin, $\rho = M\,\delta(x)$. Integrating (7.22) over a $(3+n)$-dimensional ball B of radius r enclosing the origin yields

$$\begin{aligned}\int_B d^{3+n}x\,\nabla^2\varphi &= S_{2+n}\,\hat{G}_{4+n}\,M\int_B d^{3+n}x\,\delta(x)\\ &= S_{2+n}\,\hat{G}_{4+n}\,M\ .\end{aligned} \tag{7.23}$$

On the other hand, since $\partial B = S^{2+n}$, Gauss' law gives

$$\begin{aligned}\int_B d^{3+n}x\,\nabla^2\varphi &= \int_{S^{2+n}} d\mathbf{S}\cdot\nabla\varphi\\ &= r^{2+n}\,S_{2+n}\,\frac{\partial\varphi}{\partial r}\ .\end{aligned} \tag{7.24}$$

Equating the above two equations yields

$$\frac{\partial\varphi}{\partial r} = \frac{\hat{G}_{4+n}\,M}{r^{2+n}} \Rightarrow \varphi = -\frac{\hat{G}_{4+n}\,M}{(1+n)\,r^{1+n}}\ , \tag{7.25}$$

and thus we obtain the gravitational force (per unit mass) exerted by the source,

$$\mathbf{F}(\mathbf{r}) = -\nabla\varphi = -\frac{\hat{G}_{4+n}\,M}{r^{2+n}}\,\hat{\mathbf{r}}\ . \tag{7.26}$$

Remark. Poisson's equation can be derived as the Newtonian limit of the weak field expansion $\hat{\mathbf{g}} = \eta + \mathbf{h}$ to linear order in $\mathbf{h}$ of Einstein's equations $\hat{\mathbf{G}} = \hat{\kappa}^2_{4+n}\mathbf{T}$ in the presence of a weak, static dust source with stress tensor $T_{00} = \rho$ and all other components negligible. The linear approximation is justified if $\mathbf{h} = \mathcal{O}(\frac{1}{r^{1+n}})$ at spatial infinity in Cartesian coordinates. Calculational details can be found in any standard textbook on general relativity (see, for example, [19, 194, 195])[3]. The result is that $\varphi = -\frac{1}{2}h_{00} = -\frac{\hat{\kappa}^2_{4+n}}{(2+n)\,S_{2+n}}\frac{M}{r^{1+n}}$, where $M = \int d^{3+n}x\,T_{00}$ is the ADM mass. Comparing with (7.25), we see that $\hat{\kappa}^2_{4+n} \equiv \left(\frac{2+n}{1+n}\right)\,S_{2+n}\,\hat{G}_{4+n}$ is the reduced Newton's constant.

In 1798, Henry Cavendish, in his initial determination of Newton's constant of gravitation [196] using a torsion balance, mentioned that he had made a check of the inverse square law (meaning that $n = 0$ in (7.26) and that we live in four non-compact dimensions) but gave no details. Subsequent determinations of G in the

[3] A variant of this calculation is presented in the next section.

late 1800s did not directly check the law but the results from experiments in which distances between masses were different implied rough agreement with n being zero. Mackenzie [197], in his experiments on the attractions of crystals in 1895, placed masses at separations of 3.5, 5.5 and 7.4 cm and claimed that he had confirmed the inverse square law to 1/500 between 3.5 and 7.3 cm. Experiments of the 1970s and early 1980s implied that the inverse square law is followed to within 1 part in 10^{-4} at distances of ~ 10 cm. Therefore, it seems that we live in four non-compact dimensions. Additional dimensions can exist provided they are compact and their size does not exceed the aforementioned bound. Further details of experiments to determine G and to check the inverse square law may be found in [198].

7.5 The Second Randall-Sundrum Model (RS2)

In a second influential paper [199], Randall and Sundrum (RS) argued that the statement made in the preceding section regarding Newton's inverse square law and the number of non-compact dimensions implicitly assumed that the geometry was factorisable. They showed that if the geometry was warped then it was possible to have $4+n$ non-compact dimensions and still have compatibility with experimental results. This was because the leading corrections to the effective gravitational potential on the brane were sufficiently suppressed due to the presence of the warp factor. Of course, the concept of extra non-compact dimensions was not new. Earlier work studied the trapping of matter fields near a four-dimensional hypersurface [200–202] or studied finite volume but topologically non-compact extra dimensions [203, 204]. However, in the RS case gravity itself was trapped to be effectively four-dimensional and the extra-dimensional volume was infinite. We follow the original paper [199] and also [205–212] for the remainder of this section.

7.5.1 The Set-Up and Solution

The set-up of RS2 is the same as that of RS1 as previously discussed in section 7.3.

A second solution to equations (7.8) is

$$A(r) = -k|r| \ , \tag{7.27}$$

$$V_{vis} = -V_{hid} = 6k\,\hat{\kappa}_5^{-2} \ , \tag{7.28}$$

where $k = \sqrt{-\hat{\Lambda}/6}$. These differ from the RS1 solutions (7.9) and (7.12) by a change of sign. This sign stems from the fact that the first of equations (7.8) is quadratic in $A(r)$. The four-dimensional Planck mass (given by (7.17)) is

$$M_{Pl}^2 = \frac{\hat{M}^3}{k}\left[1 - e^{-2\pi k r_0}\right] \ . \tag{7.29}$$

One then takes the limit $r_0 \to \infty$ to send the hidden brane to infinity.

In summary, there is a single (visible) brane of tension $V_{vis} = 6k\,\hat{\kappa}_5^{-2}$ situated at the origin $r = 0$ of the extra dimension. The four-dimensional Planck mass is

$$M_{Pl}^2 = \hat{M}^3/k \tag{7.30}$$

and the bulk metric is

$$ds^2 = e^{-2k|r|}\,\eta_{ij}dx^i dx^j + dr^2 \ , \tag{7.31}$$

where $-\infty < r < \infty$.

Note that RS2 does not solve the hierarchy problem. It is an assumption of both RS1 and RS2 that $\hat{M}$ and k are of the same order. From (7.30) it is clear that one must have $\hat{M} \sim k \sim M_{Pl}$. Thus, one cannot generate M_{Pl} from m_{EW} as was done in the RS1 case.

7.5.2 The Kaluza-Klein Spectrum

To demonstrate conclusively that gravity is localised in the RS2 model one must determine the Kaluza-Klein spectrum of general linearised tensor fluctuations about the ground state and show that the non-zero modes do not lead to an unacceptably large deviation from Newton's inverse square law in four dimensions. As we shall show momentarily, it is the presence of the warp factor $e^{-2k|r|}$ that leads to the corrections to the Newtonian potential coming from the massive modes being highly suppressed and hence to a result not inconsistent with experiment.

To this end, we perturb the metric (7.7) according to

$$
\begin{aligned}
ds^2 = \hat{g}_{AB}\, dy^A dy^B &= (\hat{g}^{(0)}_{AB} + \hat{h}_{AB})\, dy^A dy^B \\
&= e^{2A(r)}\, (\eta_{ij} + h_{ij}(x,r))\, dx^i dx^j + dr^2 \ , \qquad (7.32)
\end{aligned}
$$

where we have defined $\hat{h}_{AB} = e^{2A(r)} h_{AB}$ and chosen the "axial gauge" constraint $h_{Ar} = 0$. The components h_{ij} are required to have even parity across the brane so that the metric remains invariant under the orbifolding $r \to -r$.

Axial gauge is not a total gauge fix; it still allows for some gauge freedom. To see this first define, as before, the coordinate z by $z = \int e^{-A(r)} dr$ so as to bring the unperturbed metric to the conformal form $\hat{g}^{(0)}_{AB}(x,z) = e^{2A(z)}\, \tilde{g}^{(0)}_{AB}(x,z)$. Then consider the infinitesimal coordinate transformation $y^A \to y'^A = y^A + \epsilon^A(y)$ (with $y^4 = z$) leaving the metric (7.32) invariant. The fluctuations transform as

$$
\hat{h}_{AB} \to \hat{h}'_{AB} = \hat{h}_{AB} + \hat{\nabla}^{(0)}_A \epsilon_B + \hat{\nabla}^{(0)}_B \epsilon_A \ . \qquad (7.33)
$$

After use of the identity (see [18]):

$$
\hat{\nabla}^{(0)}_A \epsilon_B = \tilde{\nabla}^{(0)}_A \epsilon_B - \frac{1}{2}\, \hat{g}^{(0)CD} \left(\tilde{\nabla}^{(0)}_A \hat{g}^{(0)}_{BD} + \tilde{\nabla}^{(0)}_B \hat{g}^{(0)}_{AD} - \tilde{\nabla}^{(0)}_D \hat{g}^{(0)}_{AB} \right) \epsilon_C \ , \qquad (7.34)
$$

we can recast (7.33) as

$$
h_{AB} \to h'_{AB} = h_{AB} + \partial_A \omega_B + \partial_B \omega_A + 2\, \eta_{AB} A'(z) \omega_z \ , \qquad (7.35)
$$

where $\epsilon_B \equiv \hat{g}^{(0)}_{BC}\, \epsilon^C = e^{2A} \omega_B$ (and therefore $\omega_B = \tilde{g}^{(0)}_{BC}\, \epsilon^C$) and we have used the fact that $\tilde{g}^{(0)}_{AB} = \eta_{AB}$. We see that the axial gauge is preserved provided

$$
\begin{aligned}
(\partial_z + A'(z))\, \omega_z &= 0 \ , \\
\partial_i \omega_z + \partial_z \omega_i &= 0 \ . \qquad (7.36)
\end{aligned}
$$

These are solved by

$$
\begin{aligned}
\omega_z(x,z) &= \alpha_\pm(x) e^{-A(z)} \\
\omega_i(x,z) &= \beta_i(x) - \partial_i \alpha_\pm(x) \int^z e^{-A(z')} dz' \ , \qquad (7.37)
\end{aligned}
$$

where the subscript '$\pm$' denotes the solution in the regions $z > 0$ and $z < 0$ respectively and $\alpha_+(x) = -\alpha_-(x)$. Note that the first of these implies $\epsilon^r \equiv \epsilon^z \frac{\partial r}{\partial z} = \alpha_+(x)$.

Therefore, if $\alpha_\pm \neq 0$ the brane will not stay fixed at $r = z = 0$ under coordinate transformations but will be relocated to $r = -\alpha_+(x)$. Setting $\alpha_\pm = 0$ leaves the residual $4d$ gauge freedom

$$h_{ij}(x,z) \to h'_{ij}(x,z) = h_{ij}(x,z) + \partial_i \beta_j(x) + \partial_j \beta_i(x) \ . \tag{7.38}$$

In this case it is in general impossible to impose auxiliary transverse and traceless constraints throughout the bulk space-time. To do so would require solutions for $\beta_i(x)$ to the equations

$$\begin{aligned} \Box \beta_i + \partial_i (\partial \cdot \beta) &= -\partial^j h_{ij} \ , \\ \partial \cdot \beta &= -\frac{1}{2} h \ , \end{aligned} \tag{7.39}$$

where $h = \eta^{ij}\, h_{ij}$ and $\Box = \eta^{ij} \partial_i \partial_j$. Since the components h_{ij} depend *a priori* on the coordinate z, solutions to these equations only exist on one hypersurface, say, on the brane at $z = 0$.

We now expand Einstein's equations (7.6) to linear order in h_{ij} and, after considerable algebra (more explicit details are given later in section 8.3.3), the following equations result:

$$\begin{aligned} -\frac{1}{2}\left(\Box + \partial_z^2 + 3A'(z)\,\partial_z\right) h_{ij} - \frac{1}{2}\,\eta_{ij}\, A'(z)\,\partial_z h & \\ -\frac{1}{2}\left(\partial_i \partial_j h - \partial_i \partial_k h^k_j - \partial_j \partial_k h^k_i\right) &= \hat{\kappa}_5^2\, \bar{t}_{ij} \ , \qquad (7.40) \\ -\frac{1}{2}\left(\partial_z^2 + A'(z)\,\partial_z\right) h &= \hat{\kappa}_5^2\, \bar{t}_{zz} \ , \qquad (7.41) \\ -\frac{1}{2}\,\partial_z \left(\partial_i h - \partial_k h^k_i\right) &= \hat{\kappa}_5^2\, \bar{t}_{iz} \ . \qquad (7.42) \end{aligned}$$

In the above, we have allowed for an intrinsic perturbation $\bar{t}_{AB} = \delta_I T_{AB} - \frac{1}{3}\,\hat{g}^{(0)}_{AB}\,\hat{g}^{(0)CD}\,\delta_I T_{CD}$ that is independent of but of the same order as the metric fluctuations h_{AB}[4]. The trace of (7.40) with respect to the metric η_{ij} yields the

[4]The stress tensor T_{AB} generally depends on the metric $\hat{g}^{(0)}_{AB}$ and on various other fields independent of this metric. To first order, the variation of the stress tensor therefore splits into a part $\delta_h T_{AB}$ depending on the metric perturbations h_{AB} and a second part $\delta_I T_{AB}$ due to the intrinsic perturbations of the other fields. The part $\delta_h T_{AB}$ has already been absorbed into the left side of the above equations. See section 8.3.3.

additional equation

$$\left(2\,\Box + \partial_z^2 + 7\,A'(z)\,\partial_z\right) h - 2\partial^i\partial_k h_i^k = -2\,\hat{\kappa}_5^2\,\bar{t}\ , \tag{7.43}$$

where $\bar{t} = \eta^{ij}\bar{t}_{ij}$. Using (7.41) we can recast this as

$$\left(\Box + 3\,A'(z)\,\partial_z\right) h - \partial^i\partial_k h_i^k = -\,\hat{\kappa}_5^2\,(\bar{t} - \bar{t}_{zz})\ . \tag{7.44}$$

Due to the even parity of h_{ij} (so that h_{ij} depends on z only through $|z|$) first derivatives (in z) will be discontinuous at zero[5] and second derivatives will contain delta functions. Integrating (7.40), (7.41) and (7.43) over the interval $z \in (-\varepsilon, \varepsilon)$ results in the jump conditions

$$\frac{1}{2}\,[\partial_z h_{ij}] = \partial_z h_{ij}(x, 0^+) = -\lim_{\varepsilon\to 0^+}\hat{\kappa}_5^2\int_{-\varepsilon}^{\varepsilon}\bar{t}_{ij}\,dz\ , \tag{7.45}$$

$$\frac{1}{2}\,[\partial_z h] = \partial_z h(x, 0^+) = -\lim_{\varepsilon\to 0^+}\hat{\kappa}_5^2\int_{-\varepsilon}^{\varepsilon}\bar{t}_{zz}\,dz\ , \tag{7.46}$$

$$\frac{1}{2}\,[\partial_z h] = \partial_z h(x, 0^+) = -\lim_{\varepsilon\to 0^+}\hat{\kappa}_5^2\int_{-\varepsilon}^{\varepsilon}\bar{t}\,dz\ , \tag{7.47}$$

where we have defined $[f] = f(0^+) - f(0^-)$ for any function f. It is easily verified that (7.46) and (7.47) are compatible provided $\delta_I T_{zz}$ is continuous across the brane. If $\delta_I T_{ij}$ is of the form[6] $\delta_I S_{ij}(x)\,\delta(z)/e^A = \delta_I S_{ij}(x)\,\delta(z)$, then the conditions (7.45) are equivalent to the perturbed Israel junction conditions.

In the sequel we shall consider a static dust source confined entirely within the brane:

$$\begin{aligned} \delta_I T_{Az} &= 0\ , \\ \delta_I T^i_j &= \delta(z)\,\mathrm{diag}(-\rho(x^a), 0, 0, 0)\ . \end{aligned} \tag{7.48}$$

Indices on $\delta_I T^i_j$ are raised and lowered with the unperturbed metric, so we have $\delta_I T_{ij} = \delta(z)\,\mathrm{diag}(\rho(x^a), 0, 0, 0)$ and hence $\delta_I S_{ij} = \mathrm{diag}(\rho(x^a), 0, 0, 0)$[7].

We now proceed to solve equations (7.40)–(7.44) in the region $z > 0$.

---

[5]More concretely, first derivatives will be opposite in sign either side of $z = 0$.

[6]The measure-invariant delta-function is $\delta(z)/\sqrt{\hat{g}^{(0)}_{zz}} = \delta(z)/e^A$. However, $A(0)$ is defined to be zero.

[7]We remind the reader that x^a are the three spatial coordinates on the brane.

Firstly, we find

$$\begin{aligned}\bar{t}_{iz} &= 0 , &(7.49)\\ \bar{t}_{ij} &= \delta(z)\,\rho(x^a)\,\mathrm{diag}(\frac{2}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}) , &(7.50)\\ \bar{t} &= \bar{t}_{zz} = \frac{1}{3}\,\delta(z)\,\rho(x^a) , &(7.51)\end{aligned}$$

and the jump conditions collapse to the Neumann boundary conditions

$$\partial_z h_{ij}(x,0^+) = -\hat{\kappa}_5^2\,\rho(x^a)\,\mathrm{diag}(\frac{2}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}) . \tag{7.52}$$

Inserting (7.49) into (7.42) we obtain

$$\partial_k h_i^k = \partial_i h + F_i(x) , \tag{7.53}$$

and so (7.44) becomes

$$3A'(z)\partial_z h(x,z) = \partial^i F_i(x) . \tag{7.54}$$

On the other hand, (7.41) is solved by

$$h(x,z) = G(x)\int^z e^{-A(z')}dz' + H(x) . \tag{7.55}$$

Now recall that in the RS2 model $A(z) = -k|r| = -\ln(1+k|z|)$. So $A'(z)\,e^{-A(z)} = -k$ in the region $z > 0$. Hence, using (7.52) and (7.54), we obtain

$$G(x) = -\frac{1}{3k}\,\partial^i F_i(x) = -\frac{\hat{\kappa}_5^2}{3}\,\rho(x^a) . \tag{7.56}$$

We are interested in static solutions to the 00-component of equations (7.40). This implies that $F_i(x)$ and $H(x)$ are time-independent. Using this fact along with (7.50), (7.52) and (7.56), we obtain

$$\left(\nabla^2 + \partial_z^2 + 3A'(z)\,\partial_z\right)\varphi = -\frac{1}{6}\,\kappa^2\rho(x^a) , \quad \partial_z\varphi(x^a,0^+) = \frac{1}{3}\,\hat{\kappa}_5^2\,\rho(x^a) , \tag{7.57}$$

where we have used (7.30) and defined $\varphi \equiv -\frac{1}{2}h_{00}$. It is evident that if we define $\varphi(x^a,z) = \varphi_0(x^a) + \varphi_1(x^a,z)$, then separation of variables leads to

$$\begin{aligned}\nabla^2\varphi_0 &= -\frac{1}{6}\,\kappa^2\rho(x^a) + c , &(7.58)\\ \left(\nabla^2 + \partial_z^2 + 3A'(z)\,\partial_z\right)\varphi_1 &= -c , \quad \partial_z\varphi_1(x^a,0^+) = \frac{1}{3}\,\hat{\kappa}_5^2\,\rho(x^a) , &(7.59)\end{aligned}$$

where c is the separation constant. We may set c to zero because the particular solutions $\varphi_0^{part} = -\varphi_1^{part} = cr^2/6$ cancel when φ_0 and φ_1 are recombined[8].

The solution to (7.58) is

$$\varphi_0(x^a) = -\frac{1}{6}\,\kappa^2 \int G(x^a, x^{a'})\,\rho(x^{a'})\,d^3x^{a'} \ . \tag{7.60}$$

where $G(x^a, x^{a'})$ is the Green's function of the $3d$ Laplacian which vanishes at infinity. It was shown in section 7.4 to be

$$G(x^a, x^{a'}) = -\frac{1}{4\pi\,|x^a - x^{a'}|} \ . \tag{7.61}$$

Defining the Fourier transform of φ_1 with respect to the brane coordinates by

$$\Psi^{(m)}(z) = \int d^3x\, e^{i\mathbf{m}\cdot\mathbf{x}}\,\varphi_1(x^a, z) \ , \ \ \varphi_1(x^a, z) = \int \frac{d^3m}{(2\pi)^3}\, e^{-i\mathbf{m}\cdot\mathbf{x}}\,\Psi^{(m)}(z), \tag{7.62}$$

we can recast (7.59) as

$$(\partial_z^2 + 3A'(z)\partial_z)\,\Psi^{(m)}(z) = m^2\,\Psi^{(m)}(z) \ , \tag{7.63}$$

subject to the boundary condition

$$\partial_z\Psi^{(m)}(0^+) = \frac{1}{3}\,\hat{\kappa}_5^2\,\tilde{\rho}(\mathbf{m}) \ , \tag{7.64}$$

where $\tilde{\rho}(\mathbf{m})$ is the Fourier transform of ρ and $m = |\mathbf{m}|$.

For the remainder of this section we focus on the Kaluza-Klein spectrum defined by equation (7.63). From (7.40) we see that this equation is common to all components h_{ij}, that is, all components admit a decomposition similar to (7.62).

Firstly, notice that $\tilde{\rho}(0) = \int d^3x\rho(x^a) = M$, where M is the mass of the source. Hence, the zero mode solution of (7.63) is given by

$$\Psi^{(0)}(z) = \frac{\hat{\kappa}_5^2 M}{12k}(1 + kz)^4 + c \ , \tag{7.65}$$

where c is an arbitrary constant.

[8] We have defined r as the radial distance from the origin on the three-bane, $r = |x^a|$. We trust that the reader will not be confused with the previous definition of r as the fifth direction (here denoted by the conformal coordinate z).

Secondly, by use of the transformation $\Psi^{(m)}(z) = e^{-3A/2}\,\psi^{(m)}(z)$ we can recast (7.63) into the form of a Schrödinger equation

$$\left(-\partial_z^2 + \frac{9}{4}A'(z)^2 + \frac{3}{2}A''(z)\right)\psi^{(m)}(z) = -m^2\,\psi^{(m)}(z)\ , \tag{7.66}$$

for the massive modes. Substituting in $A(z) = -\ln(kz+1)$ (in the region $z > 0$) this equation becomes

$$\left(-\partial_z^2 + \frac{\alpha(\alpha+1)}{(z+1/k)^2}\right)\psi^{(m)}(z) = -m^2\,\psi^{(m)}(z)\ , \tag{7.67}$$

where $\alpha = 3/2$. The general solution of this differential equation is

$$\psi^{(m)}(z) = (mu)^{1/2}\,[A_m\,I_2(mu) + B_m\,K_2(mu)]\ . \tag{7.68}$$

In the above we have defined $u \equiv z + 1/k$ and I_2 and K_2 are standard modified Bessel functions of the first and second kind respectively. The KK modes thus form a continuum.

By use of the recurrence relations

$$\partial_u(u^\nu I_\nu(u)) = u^\nu I_{\nu-1}(u)\ ,\ \ \partial_u(u^\nu K_\nu(u)) = -u^\nu K_{\nu-1}(u)\ , \tag{7.69}$$

one finds that the boundary condition (7.64) becomes

$$\frac{m^{3/2}}{k^{1/2}}\,[A_m\,I_1(m/k) - B_m\,K_1(m/k)] = \frac{1}{3}\,\hat{\kappa}_5^2\,\tilde{\rho}(\mathbf{m})\ . \tag{7.70}$$

To extend the solution $\psi^{(m)}(z)$ to all z we simply replace z by $|z|$ in (7.68).

7.5.3 How RS2 Localises Gravity

In order to obtain a unique solution to (7.63) we need to impose another boundary condition which we take to be

$$\Psi^{(m)}(x^a, z)\,|_{z\to\infty} = 0\ . \tag{7.71}$$

Such a condition is clearly desirable if the linearisation is to be justified. Clearly, the zero mode does not satisfy the condition and so $m > 0$. From the asymptotic expansions

$$(mu)^{1/2} I_\nu(mu) \sim \frac{e^{mu}}{\sqrt{2\pi}}\ \ ,\ \ (mu)^{1/2} K_\nu(mu) \sim \sqrt{\frac{\pi}{2}}\,e^{-mu}\ , \tag{7.72}$$

valid for large mu, we see that the solution (7.68), taking account of (7.70), becomes

$$\psi^{(m)}(z) = -\frac{\hat{\kappa}_5^2\, k^{1/2}}{3}\,\tilde{\rho}(\mathbf{m})\,\frac{(mu)^{1/2}}{m^{3/2}}\,\frac{K_2(mu)}{K_1(m/k)}\ . \tag{7.73}$$

Putting together (7.60) and (7.73) we obtain the solution to (7.57) in the form

$$\begin{aligned}\varphi(x^a, z) = &- \frac{1}{6}\kappa^2 \int G(x^a, x^{a'})\,\rho(x^{a'})\,d^3x^{a'} \\ &- \frac{\hat{\kappa}_5^2\, k^2}{3}\,u^2\!\int \frac{d^3m}{(2\pi)^3} e^{-i\mathbf{m}\cdot\mathbf{x}}\frac{\tilde{\rho}(\mathbf{m})}{m}\,\frac{K_2(mu)}{K_1(m/k)}\ .\end{aligned} \tag{7.74}$$

We may omit the internal space dependence on z since all point particles on the brane can be taken to have $z = 0$. Noting the recurrence relation $K_2(u) = \frac{2}{u}K_1(u) + K_0(u)$, we find

$$\begin{aligned}\varphi(x^a, 0) = &- \frac{1}{6}\kappa^2 \int G(x^a, x^{a'})\,\rho(x^{a'})\,d^3x^{a'} \\ &- \frac{2\kappa^2}{3}\lim_{z\to 0^+}\int \frac{d^3m}{(2\pi)^3} e^{-i\mathbf{m}\cdot\mathbf{x}}\frac{\tilde{\rho}(\mathbf{m})}{m^2}\,\frac{K_1(mu)}{K_1(m/k)} \\ &- \frac{\kappa^2}{3k}\lim_{z\to 0^+}\int \frac{d^3m}{(2\pi)^3} e^{-i\mathbf{m}\cdot\mathbf{x}}\frac{\tilde{\rho}(\mathbf{m})}{m}\,\frac{K_0(mu)}{K_1(m/k)}\ ,\end{aligned} \tag{7.75}$$

where we have used (7.30) again.

We now specialise to the case of a point source located at the origin, in which case $\rho(x^{a'}) = M\,\delta(x^{a'})$, $\tilde{\rho}(\mathbf{m}) = \tilde{\rho}(0) = M$. In the second integral of (7.75) we can take the limit inside the integral. Noting that

$$\int \frac{d^3m}{(2\pi)^3}\frac{e^{-i\mathbf{m}\cdot\mathbf{x}}}{m^2} = -G(x^a, 0)\ , \tag{7.76}$$

we obtain

$$\begin{aligned}\varphi(x^a, 0) &= \frac{\kappa^2 M}{2}G(x^a, 0) - \frac{\kappa^2 M}{3k}\lim_{z\to 0^+}\int \frac{d^3m}{(2\pi)^3}\frac{e^{-i\mathbf{m}\cdot\mathbf{x}}}{m}\,\frac{K_0(mu)}{K_1(m/k)} \\ &= -\frac{\kappa^2 M}{4\pi r}\left(\frac{1}{2} + \frac{2}{3\pi k r}\lim_{z\to 0^+}\int_0^\infty ds\,\sin s\,\frac{K_0(su/r)}{K_1(s/kr)}\right)\ ,\end{aligned} \tag{7.77}$$

where $s = mr$. We may not take the limit inside the above integral because of the divergence for large s. The non-zero z acts as a regulator of the integral for large s. In RS2 it is assumed that $m \ll k$ or, equivalently, $kr \gg 1$. We may then use the asymptotic expansion

$$K_1(s/kr) \sim (s/kr)^{-1}\ , \tag{7.78}$$

for small m/k to approximate the above integral as

$$\frac{1}{kr}\lim_{z\to 0^+}\int_0^\infty ds\, s\sin s\, K_0(su/r) = \frac{\pi}{2kr}\left(\frac{1}{k^2r^2}+1\right)^{-\frac{3}{2}} \sim \frac{\pi}{2kr} \ . \tag{7.79}$$

Thus, we obtain the final form for the effective $4d$ gravitational potential:

$$\varphi(x^a,0) \sim -\frac{GM}{r}\left(1+\frac{2}{3k^2r^2}\right) \ , \tag{7.80}$$

where we have substituted $\kappa^2 = 8\pi G$. The leading term is the usual Newtonian potential; the KK modes generate an extremely suppressed correction term for k taking the expected value of order the fundamental Planck scale and r of the size tested with gravity $(1-10\,\mathrm{cm})$.

To show that we have localised gravity and thus have obtained Newton's law on the brane we must consider the motion of a massive test particle on the brane. There are two possibilities for this motion. Firstly, we could assume that the particle is free to move in five dimensions, that is, that it will follow a timelike geodesic in five-space. As shown in [213], this case leads to an extra force being present in four dimensions due to a non-zero velocity component along the extra dimension. The second possibility, which is the one we consider, is that the particle is constrained to move along the brane by some as yet unknown non-gravitational mechanism. Potential shortcuts via the fifth dimension are then not allowed and the dynamics is determined by the induced metric on the brane only.

The four-dimensional geodesic equation is

$$\frac{d^2x^i}{d\lambda^2} + \hat{\Gamma}^i_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = 0 \ , \tag{7.81}$$

subject to

$$\hat{g}_{ij}(x^i,0)\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = -1 \ , \tag{7.82}$$

where $\hat{g}_{ij}(x^i,0)$ is the perturbed metric (7.32) on the brane at $r=0$ in the presence of the matter source (7.48), $\hat{\Gamma}^i_{jk}$ the corresponding Christoffel symbols and λ an affine parameter along the geodesic. Assuming that velocities $dx^a/dt \ll 1$ are negligible (compared to the speed of light), equation (7.82) gives

$$\frac{dt}{d\lambda} \approx e^{-A(0)} = 1 \ , \tag{7.83}$$

while (7.81) becomes

$$\frac{d^2x^a}{dt^2} \approx -\hat{\Gamma}^a_{00} = \frac{1}{2}\partial_a h_{00}(x^a, 0) = -\partial_a \varphi(x^a, 0) \ , \tag{7.84}$$

with $\varphi(x^a, 0)$ given by (7.80). Thus, we obtain Newton's $1/r^2$ force law (cf. equation (7.26)) plus suppressed correction terms.

Finally, although we have concentrated exclusively on the 00-component of equations (7.40), it is of course possible to consider the other components. The details do not concern us here. We simply note that the equations can be solved consistently and refer the reader to [208, 211, 212] for explicit calculations.

Chapter 8

Cosmological Generalisations of the RS Models

In both of the RS models discussed in the last chapter the metrics (7.13) and (7.31) depend only on the coordinate r of the extra dimension and not on the brane coordinates. In particular, they are static as they do not depend on the time coordinate t. It is a logical next step to study the cosmology of these models by allowing such time dependence in the metrics. The first section of this chapter details a non-static generalisation of the RS scenario with a static fifth dimension. We then consider a further generalisation with a non-static fifth dimension.

8.1 The BDL Model

Binétruy, Deffayet and Langlois (BDL) [214] considered the cosmological evolution of a thin four-dimensional brane-like universe embedded in five dimensions in which brane matter was modelled as a perfect fluid source in the five-dimensional Einstein equations. As will be shown momentarily, the authors found that the theory exhibited non-conventional cosmology in that the Hubble parameter H was proportional to the density $\tilde{\rho}$ on the brane instead of the usual $H \sim \sqrt{\tilde{\rho}}$ of standard big bang cosmology. This was an important result since the successes of standard cosmology such as nucleosynthesis and the common understanding of subsequent evolution rely critically on the assumption $H \sim \sqrt{\tilde{\rho}}$. Some ideas as to how the standard behaviour of H could be recovered include the thickening of the brane and the cancellation of bulk and brane cosmological constants. These and other cosmological aspects of "braneworlds" have been considered in [215–224]. In the remainder of this section we follow BDL [214] and also [217].

8.1.1 The Set-Up

The BDL model is similar to the RS1 model considered in section 7.3. We again have five dimensions with the fifth dimension forming an $S^1/\mathbb{Z}_2$ orbifold and a three-brane situated at each of the fixed points. The action is

$$\begin{aligned} S &= S_{gravity} + S_{bulk} + S_{branes} \ , \\ S_{gravity} &= \frac{1}{2\hat{\kappa}_5^2}\int d^4x \int_{-\pi r_0}^{\pi r_0} dr\,\sqrt{-\hat{g}}\,\hat{R} \ , \\ S_{bulk} &= \int d^4x \int_{-\pi r_0}^{\pi r_0} dr\,\sqrt{-\hat{g}}\,\mathcal{L}_{bulk} \ , \\ S_{branes} &= \int d^4x\,\sqrt{-g_{vis}}\,\mathcal{L}_{vis} + \int d^4x\,\sqrt{-g_{hid}}\,\mathcal{L}_{hid} \ , \end{aligned} \tag{8.1}$$

where g_{ij}^{vis} and g_{ij}^{hid} are the induced metrics on the visible and hidden branes at $r = 0$ and $r = \pi r_0 \equiv r'$, respectively.

The ansatz for the metric is

$$ds^2 = \hat{g}_{AB}dy^A dy^B = -n^2(t,r)dt^2 + a^2(t,r)\delta_{ab}dx^a dx^b + b^2(t,r)dr^2 \ , \tag{8.2}$$

and clearly contains the RS ansatz (7.7) as a special case. In addition, it assumes flat spatial three-sections on the brane.

The matter on each brane is assumed to be in the form of a perfect fluid at rest in the above coordinate system. The total stress-tensor may then be written as

$$\hat{T}^A_B = \check{T}^A_B\,|_{bulk} + \tilde{T}^A_B\,|_{branes} \ , \tag{8.3}$$

where

$$\tilde{T}^A_B\,|_{branes} = \frac{\delta(r)}{b}\,\mathrm{diag}(-\tilde{\rho},\tilde{p},\tilde{p},\tilde{p},0) + \frac{\delta(r-r')}{b}\,\mathrm{diag}(-\tilde{\rho}_*,\tilde{p}_*,\tilde{p}_*,\tilde{p}_*,0) \ , \tag{8.4}$$

and $\tilde{\rho}$ ($\tilde{\rho}_*$) and $\tilde{p}$ ($\tilde{p}_*$) are the density and pressure on the visible (hidden) brane respectively. Note that in the RS1 model one has $\tilde{\rho} = -\tilde{p} = V_{vis}$ and similarly $\tilde{\rho}_* = -\tilde{p}_* = V_{hid}$. The exact constitution of the bulk stress tensor, as we shall see, is not needed and so remains unspecified.

Einstein's equations are then $\hat{G}^A_B = \hat{\kappa}_5^2 \hat{T}^A_B$, where the Einstein tensor is that for the ansatz (8.2):

$$\hat{G}^0_0 = -\frac{3}{n^2}\left\{\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}\right) - \frac{n^2}{b^2}\left(\frac{a''}{a}+\frac{a'}{a}\left(\frac{a'}{a}-\frac{b'}{b}\right)\right)\right\}, \tag{8.5}$$

$$\begin{aligned}\hat{G}^a_b &= \frac{\delta^a_b}{b^2}\left\{\frac{a'}{a}\left(\frac{a'}{a}+2\frac{n'}{n}\right) - \frac{b'}{b}\left(\frac{n'}{n}+2\frac{a'}{a}\right) + 2\frac{a''}{a}+\frac{n''}{n}\right\} \\ &+\frac{\delta^a_b}{n^2}\left\{\frac{\dot{a}}{a}\left(-\frac{\dot{a}}{a}+2\frac{\dot{n}}{n}\right) - 2\frac{\ddot{a}}{a}+\frac{\dot{b}}{b}\left(-2\frac{\dot{a}}{a}+\frac{\dot{n}}{n}\right)-\frac{\ddot{b}}{b}\right\},\end{aligned} \tag{8.6}$$

$$\hat{G}^0_r = -\frac{3}{n^2}\left(\frac{n'}{n}\frac{\dot{a}}{a}+\frac{a'}{a}\frac{\dot{b}}{b}-\frac{\dot{a}'}{a}\right), \tag{8.7}$$

$$\hat{G}^r_0 = \frac{3}{b^2}\left(\frac{n'}{n}\frac{\dot{a}}{a}+\frac{a'}{a}\frac{\dot{b}}{b}-\frac{\dot{a}'}{a}\right), \tag{8.8}$$

$$\hat{G}^r_r = \frac{3}{b^2}\left\{\frac{a'}{a}\left(\frac{a'}{a}+\frac{n'}{n}\right) - \frac{b^2}{n^2}\left(\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a}-\frac{\dot{n}}{n}\right)+\frac{\ddot{a}}{a}\right)\right\}. \tag{8.9}$$

In the above expressions a prime stands for a derivative with respect to r and a dot for a derivative with respect to t. (The other components of the Einstein tensor vanish identically.)

8.1.2 Non-standard Cosmology from BDL Model

In what follows we shall concentrate on the visible brane at $r = 0$. As in the last chapter, the metric is required to be continuous $r = 0$ and to have even parity under $r \to -r$. Consequently, it depends on r only through $|r|$. Therefore, first derivatives in r will be opposite in sign either side of $r = 0$ and second derivatives will contain delta-functions.

We can thus derive jump conditions across the visible brane in much the same way as was done in section 7.5.2. It is clear that a'' may be written as

$$a'' = \left.\frac{\partial^2 a(t,u)}{\partial u^2}\right|_{u=|r|} + [a']\,\delta(r)\ , \tag{8.10}$$

where $[a'] = a'(t, 0^+) - a'(t, 0^-) = 2a'(t, 0^+)$, and similarly for n''. The jump conditions resulting from (8.5) and (8.6) can easily be deduced by inspection:

$$3\frac{[a']}{a_0 b_0^2} = -\frac{\hat{\kappa}_5^2}{b_0}\tilde{\rho} ,$$
$$2\frac{[a']}{a_0 b_0^2} + \frac{[n']}{n_0 b_0^2} = \frac{\hat{\kappa}_5^2}{b_0}\tilde{p} ,$$

or, equivalently,

$$\frac{[a']}{a_0 b_0} = -\frac{\hat{\kappa}_5^2}{3}\tilde{\rho} , \quad (8.11)$$
$$\frac{[n']}{n_0 b_0} = \frac{\hat{\kappa}_5^2}{3}(3\tilde{p} + 2\tilde{\rho}) , \quad (8.12)$$

where the subscript 0 indicates that the functions a, b, n are valued at $r = 0$. If we take the jump of either (8.7) or (8.8) across $r = 0$ we obtain

$$\frac{[n']}{n_0}\frac{\dot{a}_0}{a_0} + \frac{[a']}{a_0}\frac{\dot{b}_0}{b_0} - \frac{\dot{[a']}}{a_0} = 0 ,$$

or, upon using (8.11) and (8.12),

$$\dot{\tilde{\rho}} + 3\frac{\dot{a}_0}{a_0}(\tilde{p} + \tilde{\rho}) = 0 . \quad (8.13)$$

This is the usual four-dimensional conservation of energy condition[1]. Finally, taking the mean value across $r = 0$ of (8.9), using (8.11) and (8.12), and recalling that $[a'] = 2a'(t, 0^+)$ (similarly for $[n']$) we obtain the Friedmann-type equation

$$\frac{1}{n_0^2}\left(\frac{\ddot{a}_0}{a_0} + \frac{\dot{a}_0^2}{a_0^2} - \frac{\dot{a}_0}{a_0}\frac{\dot{n}_0}{n_0}\right) = -\frac{\hat{\kappa}_5^4}{36}\tilde{\rho}(\tilde{\rho} + 3\tilde{p}) - \frac{\hat{\kappa}_5^2 \check{T}_{rr}|_{r=0}}{3b_0^2} .$$

If one defines "cosmic time" (that is, the proper time as measured by comoving observers on the brane) via $d\tau = n_0 dt$, then (8.13) and the above equation respectively become

$$\dot{\tilde{\rho}} + 3\frac{\dot{a}_0}{a_0}(\tilde{p} + \tilde{\rho}) = 0 , \quad (8.14)$$

[1]It should be stressed that equations (8.11–8.13) are only valid provided the bulk stress-energy $\check{T}_{AB}$ is continuous across the brane, which need not actually be the case once the global solutions of Einstein's equations are found. In particular, if $\check{T}_{0r}$ is discontinuous across the brane then (8.13) will not hold.

$$\frac{\ddot{a}_0}{a_0} + \frac{\dot{a}_0^2}{a_0^2} = -\frac{\hat{\kappa}_5^4}{36}\tilde{\rho}(\tilde{\rho} + 3\tilde{p}) - \frac{\hat{\kappa}_5^2 \check{T}_{rr}\,|_{r=0}}{3b_0^2} \ , \tag{8.15}$$

where the dot is now differentiation with respect to τ. These are the two master equations of the BDL scenario.

There are two important features of (8.15). Firstly, it is independent of the energy and pressure of the hidden brane. This is due to the fact that the delta function $\delta(r - r')$ does not contribute at $r = 0$. Secondly, it should be contrasted with the conventional Friedmann equation

$$\frac{\ddot{a}_0}{a_0} + \frac{\dot{a}_0^2}{a_0^2} = \frac{\kappa^2}{6}(\tilde{\rho} - 3\tilde{p}) \ . \tag{8.16}$$

Therefore, we see the evidence of the non-standard cosmology with (8.15) depending quadratically on $\tilde{\rho}$, whereas the usual equation, (8.16), depends only linearly on $\tilde{\rho}$.

A consequence of (8.15) is that it can lead to slower evolution than usual. To see this, suppose that the bulk pressure $\check{T}_{rr}\,|_{r=0}$ is of the same order as the bulk energy density and that the condition

$$\tilde{\rho}^2 \gg \frac{\rho_{bulk}}{\hat{\kappa}_5^2} \ , \tag{8.17}$$

is satisfied. Then we may neglect the last term in (8.15). In so far as this approximation is justified, it is not necessary to solve Einstein's equations for the whole bulk in order to determine a_0. Assuming the standard equation of state $\tilde{p} = \omega\tilde{\rho}$, with ω constant, we can integrate (8.14) to find

$$\tilde{\rho} \propto a_0^{-3(1+\omega)}. \tag{8.18}$$

If we then look for power law solutions to (8.15) of the form

$$a_0(\tau) \sim \tau^q \ , \tag{8.19}$$

we find that

$$q = \frac{1}{3(1+\omega)} \ , \tag{8.20}$$

as opposed to $q_{standard} = 2/(3(1+\omega))$ for solutions to (8.16). Furthermore, defining the Hubble parameter as per usual by $H = \dot{a}_0/a_0$, we see that $H \sim 1/\tau$ so that

$$H \sim \tilde{\rho} \ , \tag{8.21}$$

in the BDL case, in contrast to $H \sim \sqrt{\tilde{\rho}}$ in the standard case.

A related consequence is that cooling of the early radiation-dominated epoch ($\tau \lesssim 4 \times 10^{10}\,\text{sec}$) is slower than usual. To see this we use the usual expressions for the density and pressure of a dilute, weakly-interacting gas of particles of mass m, with g internal degrees of freedom and in thermal equilibrium at temperature T:

$$\tilde{\rho} = \frac{g}{(2\pi)^3}\int d^3p\, f(\mathbf{p})E(\mathbf{p}) \;, \tag{8.22}$$

$$\tilde{p} = \frac{g}{(2\pi)^3}\int d^3p\, f(\mathbf{p})\frac{|\mathbf{p}|^2}{3E} \;, \tag{8.23}$$

where $E(\mathbf{p}) = \sqrt{|\mathbf{p}|^2 + m^2}$ and $f(\mathbf{p})$ is the familiar Fermi-Dirac or Bose-Einstein distribution:

$$f(\mathbf{p}) = (\exp((E-\mu)/T) \pm 1)^{-1} \;, \tag{8.24}$$

where μ is the chemical potential and the plus sign is for fermions and the negative sign for bosons. In the relativistic limit $T \gg m, \mu$ these are evaluated to be

$$\tilde{\rho} = \begin{cases} (\pi^2/30)gT^4 & \text{(Bosons)} \;, \\ (7/8)(\pi^2/30)gT^4 & \text{(Fermions)} \;, \end{cases} \tag{8.25}$$

$$\tilde{p} = \tilde{\rho}/3 \;. \tag{8.26}$$

Thus, $\omega = 1/3$ and $\tilde{\rho} \sim T^4$. Using $H \sim 1/\tau$ and (8.21) we obtain $T \sim 1/\tau^{1/4}$, as opposed to the conventional case $T \sim 1/\tau^{1/2}$. Thus, the cooling is slower than that predicted by (8.16).

Other phenomenological consequences, such as the implications for nucleosynthesis, are beyond the scope of the present work. The interested reader is referred to [214–224] for further discussion.

8.2 Time-dependence of Extra Dimensions

The proper radius of the fifth dimension is defined as

$$R(t) = \frac{1}{2\pi}\int_{-\pi r_0}^{\pi r_0} dr\,\sqrt{\hat{g}_{rr}(t,r)} = R(t) = \frac{1}{2\pi}\int_{-\pi r_0}^{\pi r_0} dr\, b(t,r) \;. \tag{8.27}$$

If the internal dimensions change with time, then the four-dimensional fundamental constants also do [225–227]. In particular, the Fermi constant G_F will vary. Adapting (7.19)-(7.21) to the case when b depends only on time, we see that the physical four-dimensional Higgs mass is given by

$$v_{phys} = \frac{\hat{v}}{\sqrt{b}} \ . \tag{8.28}$$

The Fermi constant is then given by $G_F = \frac{1}{\sqrt{2}\, v_{phys}^2} \propto b$.

The successful predictions of light element abundances in the standard model of big bang nucleosynthesis (SBBN) place stringent limits on the time variation of the fundamental constants and, consequently, on the time variability of the extra dimensions. This can be demonstrated by considering the primordial abundance of ^{4}He. The abundance is primarily determined by the neutron-to-proton ratio at *freeze-out*[2], which is given by the non-relativistic equilibrium condition (see [225]):

$$(n/p)_f \approx \exp(-Q/T_f) \tag{8.29}$$

where T_f is the freeze-out temperature and Q the neutron-proton mass difference. This ratio is slightly altered by free neutron decay on a time scale of 10^3 sec between T_f and the onset of nucleosynthesis at $T_N \sim 0.1$ Mev. The mass fraction of ^{4}He is

$$Y_P \approx \frac{2(n/p)_N}{1+(n/p)_N} \ , \tag{8.30}$$

where it is assumed that all neutrons are incorporated into ^{4}He and $(n/p)_N$ is the neutron-proton ratio at the time synthesis finally takes place, that is, at temperature T_N. Now, changes in G_F will change the masses of fermions and so will alter Q. Therefore, we see that small changes in the Fermi constant G_F due to time variability will induce large changes in Y_P. Conversely, observationally inferred limits on Y_P for ^{4}He:

$$Y_P = 0.232 \pm 0.008 \ , \tag{8.31}$$

[2]Freeze-out occurs when the interactions of a particle species decouple, that is, when the species becomes very long lived compared to the age of the universe ($\Gamma \lesssim H$, where Γ is the decay rate). For the reaction $n \leftrightarrow p$, it can be shown (see [225]) that $\Gamma \sim G_F^2 T^5$ in the limit $T \gg m_e, Q \equiv m_n - m_p$. This leads to a freeze–out temperature $T_f \sim 0.7 - 0.8$ Mev for SBBN. (The BDL scenario gives $T_f \sim 2 - 3$ Mev because of the different dependence of H on T. See [214] for details.)

(see [228]), mean that by the time of primordial nucleosynthesis the size of any internal dimension was already very close to the size that it is today.

The above paragraph explains why both of the RS models discussed in the last chapter have the fifth dimension independent of time ($g_{rr} = 1$). This time-independence is also implicitly assumed in the BDL scenario of the last section (that is, $b(t,r)$ in (8.2) is a function of r only), although the derivation of (8.14) and (8.15) does not rely on this assumption.

All theories with extra dimensions must have some mechanism to keep the internal dimensions (almost) static, otherwise they would expand or contract too fast and so be at odds with observational data.

One such mechanism within the RS1 framework is the Goldberger-Wise (GW) mechanism [229]. Recall that r_0 was essentially a free parameter that was not determined by the dynamics of the model. For solution of the hierarchy problem we required $\pi k r_0 \approx 35$. The GW mechanism generates a potential for r_0 so that it can be stabilised at the minimum of this potential. It does this by adding to the RS action (7.5) the bulk scalar action

$$S_\Phi = \frac{1}{2}\int d^4x \int_{-\pi r_0}^{\pi r_0} dr\, \sqrt{-\hat{g}}\, \left(-\hat{g}^{AB}\partial_A\Phi\partial_B\Phi - m^2\Phi^2\right) \,. \tag{8.32}$$

In addition, the tensions V_{vis} and V_{hid} gain Φ dependent parts. GW choose

$$V_{vis} \to V_{vis}(\Phi) = V_{vis} + \lambda_{vis}(\Phi^2 - v_{vis}^2)^2 \,, \tag{8.33}$$

and similarly for V_{hid}. These terms on the branes cause Φ to develop a r-dependent vacuum expectation value $\Phi(r)$ which is determined classically by solving the equation of motion for Φ in the RS1 background (7.13). The mechanism thus neglects the back-reaction of the scalar field on the background[3]. After solving for $\Phi(r)$, inserting the solution back into the action and integrating over the extra dimension we obtain the effective potential in the large λ_{vis}, λ_{hid} and small m/k limit:

$$V_{eff}(r_0) = 4k\left(v_{vis} - v_{hid}e^{-\epsilon\pi k r_0}\right)^2 + \mathcal{O}(\epsilon) \,, \tag{8.34}$$

[3]Neglect of back-reaction from the scalar field is justified if $\hat{\kappa}_5^2 v_{vis}^2$ and $\hat{\kappa}_5^2 v_{hid}^2$ are small. See [229].

where $\epsilon = m^2/4k^2$. Ignoring the linear and higher order terms in ϵ, this potential has a minimum at

$$\pi k r_0 = \frac{4k^2}{m^2} \ln\left(\frac{v_{hid}}{v_{vis}}\right) , \tag{8.35}$$

and thus

$$\frac{4k^2}{m^2} \ln\left(\frac{v_{hid}}{v_{vis}}\right) \approx 35 . \tag{8.36}$$

One might worry that the mechanism relies on unnaturally large values of λ_{vis} , λ_{hid}. However, using a linearisation about the large λ solution given above and considering the leading $1/\lambda$ correction to the potential, GW showed that the location of the minimum is relatively insensitive to the precise values of λ_{vis} , λ_{hid}. The radius of the extra dimension is thus stabilised at this minimum value.

The GW mechanism can be extended to the BDL scenario (see [230, 231]), in which case the bulk scalar becomes dynamical.

8.3 A Scenario with a Non-static Extra Dimension

In this section we investigate cosmological solutions of five-dimensional gravity coupled to a bulk scalar field sigma-model with indefinite metric in which we allow the scalars to depend on time as well as the fifth dimension. We also include a bulk *a priori* anisotropic fluid[4] with energy-momentum tensor $\check{T}^A_B(\rho) = \mathrm{diag}(-\rho, p, p, p, P)$ and equations of state $P = \tilde{\omega}\rho$, $p = \omega\rho$. The extra dimension is assumed to be infinite in extent and time-dependent. From the discussion in the last section regarding stabilisation of the extra dimension, it is clear that if such a scenario is ever physically relevant it must be at early times before nucleosynthesis. We will show that the fluid exists provided $\tilde{\omega} = \omega = 1$ and that it is possible to obtain standard cosmology on the brane.

[4]In the RS and BDL models it is only an *assumption* that the fluid matter resides within the brane. In the absence of a well-defined non-gravitational mechanism which confines matter to the brane, it is of interest to consider what happens when the fluid leaks from the brane and fills the entire extra dimension.

It may appear somewhat unnatural to have an indefinite target space metric since some of the scalars then have "wrongly-signed" (that is, negative) kinetic terms, which can lead to a violation of the energy conditions[5]. However, such scalars have been considered before in the literature. Within the context of $d+1$ gravity they are descended from vector fields after dimensional reduction along a timelike direction of a higher dimensional "two-time" theory [49, 232], whilst in $d+0$ dimensions they are interpreted as axions after dualisation of a $(d-1)$-form field strength [233–235].

This section is based on [236].

8.3.1 The Set-Up

We consider a single, thin "visible" brane at $r=0$ in $(4+1)$-dimensions, as in the RS2 model. The action for the gravity and scalar part of the model is an amalgam of the RS2, BDL and GW actions:

$$\begin{aligned} S &= S_{gravity} + S_{\Phi} + S_{brane} \ , \\ S_{gravity} &= \frac{1}{2\hat{\kappa}_5^2} \int d^4x dr \, \sqrt{-\hat{g}} \, \hat{R} \ , \\ S_{\Phi} &= \int d^4x dr \, \sqrt{-\hat{g}} \left(-\frac{1}{2} \hat{g}^{AB} \partial_A \Phi^{\mu} \partial_B \Phi^{\nu} \, G_{\mu\nu}(\Phi) - U(\Phi) \right) \ , \\ S_{brane} &= \int d^4x \sqrt{-g_{vis}} \, (-V_{vis}(\Phi)) \ , \end{aligned} \tag{8.37}$$

where the tension of the brane is, as previously, parametrised by $V_{vis}(\Phi)$. $G_{\mu\nu}(\Phi)$ is the sigma-model metric and for simplicity we shall consider two scalar fields and take $G_{\mu\nu}(\Phi) = \mathrm{diag}(1,-1)$. The "correctly-signed" scalar, Φ^1, may be interpreted as a "dilaton" and the "wrongly-signed" scalar, Φ^2, as an "axion".

We assume a separable metric with flat spatial three-sections on the brane:

$$\begin{aligned} ds^2 &= \hat{g}_{AB} dy^A dy^B \\ &= e^{2A(r)} \left(-dt^2 + g(t) \delta_{ab} dx^a dx^b \right) + f(t) dr^2 \ . \end{aligned} \tag{8.38}$$

This is a natural generalisation of the $4d$ flat Friedmann-Robertson-Walker (FRW) metric to a RS context and is a special case of the BDL ansatz (8.2) with $n = e^A$, $a = e^A g^{1/2}$, $b = f^{1/2}$.

[5]The null energy condition is discussed later in section 8.4.

Given the above ansatz, it is not unreasonable to assume scalars of the form

$$\Phi^\mu(t,r) = a^\mu\,\phi(t) + b^\mu\,\chi(r)\ . \tag{8.39}$$

Since Φ^μ can be considered as coordinates on the target space-time we must require them to be linearly independent. This imposes the condition

$$\det\begin{pmatrix} a^1 & b^1 \\ a^2 & b^2 \end{pmatrix} \neq 0\,. \tag{8.40}$$

From the relation

$$(a\cdot b)^2 = (a\cdot a)(b\cdot b) + (a^1b^2 - a^2b^1)^2\ , \tag{8.41}$$

we see that the Schwarz inequality

$$(a\cdot a)(b\cdot b) < (a\cdot b)^2 \tag{8.42}$$

follows as a corollary.

We shall also make the ansatz that both the potentials $U(\Phi)$ and $V_{vis}(\Phi)$ are of Liouville type (see, for instance, [237]):

$$\begin{aligned} U(\Phi) &= U_0 e^{\alpha\cdot\Phi}\ , \\ V_{vis}(\Phi) &= V_{vis} e^{\beta\cdot\Phi}\ , \end{aligned} \tag{8.43}$$

where U_0 and V_{vis} are constants.

The stress-tensor for the scalar fields is easily computed to be

$$\hat{T}^A_B = \check{T}^A_B\,|_{bulk} + \tilde{T}^A_B\,|_{brane}\ , \tag{8.44}$$

where

$$\check{T}^A_B\,|_{bulk} = \check{T}^A_B(\Phi) = \partial^A\Phi\cdot\partial_B\Phi - \delta^A_B\Big(\frac{1}{2}\partial^C\Phi\cdot\partial_C\Phi + U(\Phi)\Big)\ , \tag{8.45}$$

and

$$\tilde{T}^A_B\,|_{brane} = \tilde{T}^A_B(\Phi) = -\frac{1}{f^{1/2}}V_{vis}(\Phi)\,\delta(r)\,g^{vis}_{ij}\,\hat{g}^{iA}\,\delta^j_B\ , \tag{8.46}$$

where there is no sum over the indices i and j.

The bulk fluid has the stress-tensor

$$\check{T}^A_B(\rho) = \mathrm{diag}(-\rho, p, p, p, P) \tag{8.47}$$

in the comoving coordinates y^A. Here ρ is the density and p and P the pressures in the three spatial directions on the brane and in fifth dimension, respectively.

8.3.2 The Solutions

We now proceed to solve Einstein's equations given the above ansatze.

If we take a linear combination of the 00- and 11-components of Einstein's equations and use (8.5) and (8.6) then the following equation results:

$$\frac{1}{4}\frac{\dot{f}}{f}\frac{\dot{g}}{g} + \frac{\dot{g}^2}{g^2} + \frac{1}{4}\frac{\dot{f}^2}{f^2} - \frac{1}{2}\frac{\ddot{f}}{f} - \frac{\ddot{g}}{g} - \hat{\kappa}_5^2 a \cdot a\dot{\phi}^2 - \hat{\kappa}_5^2 e^{2A}(\rho + p) = 0 \ . \tag{8.48}$$

Therefore, we see that ρ and p must be of the form

$$\rho(t,r) \ = \ e^{-2A(r)}\left(\tilde{\rho}(t) + F(t,r)\right) \ , \tag{8.49}$$

$$p(t,r) \ = \ e^{-2A(r)}\left(\tilde{p}(t) - F(t,r)\right) \ , \tag{8.50}$$

for arbitrary $F(t,r)$. However, it is normal to assume the equation of state $p = \omega\rho$, where ω is constant in the range $-1 \leq \omega \leq 1$. In the generic case $\omega \neq -1$ this implies that F should be zero. We shall assume this also to be so in the special case $\omega = -1$. Furthermore, we shall also assume $P = \tilde{\omega}\rho$. Equation (8.48) then reduces to

$$\frac{1}{4}\frac{\dot{f}}{f}\frac{\dot{g}}{g} + \frac{\dot{g}^2}{g^2} + \frac{1}{4}\frac{\dot{f}^2}{f^2} - \frac{1}{2}\frac{\ddot{f}}{f} - \frac{\ddot{g}}{g} - \hat{\kappa}_5^2 a \cdot a\dot{\phi}^2 - \hat{\kappa}_5^2 (1+\omega)\tilde{\rho} = 0 \ . \tag{8.51}$$

Given $F = 0$, it is not unreasonable to assume separation of the 00-component of Einstein's equations into

$$\frac{3}{4}\frac{\dot{f}}{f}\frac{\dot{g}}{g} + \frac{3}{4}\frac{\dot{g}^2}{g^2} - \frac{\hat{\kappa}_5^2}{2} a \cdot a\dot{\phi}^2 - \hat{\kappa}_5^2\,\tilde{\rho} = 0 \ , \tag{8.52}$$

$$\frac{3}{2}(4A'^2 + 2A'') + \frac{\hat{\kappa}_5^2}{2} b \cdot b\chi'^2 + \hat{\kappa}_5^2\, f\, U + \hat{\kappa}_5^2\, f^{1/2}\, V\,\delta(r) = 0 \ , \tag{8.53}$$

where we have set the separation constant to zero for simplicity.

The rr-equation (using (8.9)) also splits in two:

$$\frac{3}{2}\frac{\ddot{g}}{g} + \frac{\hat{\kappa}_5^2}{2} a \cdot a\dot{\phi}^2 + \hat{\kappa}_5^2\,\tilde{\omega}\,\tilde{\rho} = 0, \tag{8.54}$$

$$6A'^2 - \frac{\hat{\kappa}_5^2}{2} b \cdot b\chi'^2 + \hat{\kappa}_5^2\, f\, U = 0 \ , \tag{8.55}$$

where we have again set the separation constant to zero.

Finally, upon using (8.7) the $0r$-equation becomes

$$\frac{3}{2} A' \frac{\dot{f}}{f} = \hat{\kappa}_5^2 \, a \cdot b \, \dot{\phi} \chi' \ . \tag{8.56}$$

In addition, the equations of motion for the scalar fields

$$\hat{\nabla}^2 \Phi_\mu - \frac{\partial U(\Phi)}{\partial \Phi^\mu} - \frac{\sqrt{-g_{vis}}}{\sqrt{-\hat{g}}} \frac{\partial V_{vis}(\Phi)}{\partial \Phi^\mu} \delta(r) = 0 \ , \tag{8.57}$$

result in the bulk equations

$$\partial_t \left(f^{1/2} g^{3/2} \dot{\phi} \right) = 0 \ , \tag{8.58}$$

$$b_\mu \left(-4A'\chi' - \chi'' \right) + \alpha_\mu \, f \, U = 0 \ , \tag{8.59}$$

along with the jump condition

$$b_\mu \left[\chi' \right] = \beta_\mu \, f^{1/2} \, V_{vis}(\Phi(t,0)) \ . \tag{8.60}$$

Note that the scalar field equations of motion imply that $\hat{\nabla}^A \check{T}_{AB}(\Phi) = 0$ (and conversely off the brane only). This, in turn, implies that the fluid equations of motion $\hat{\nabla}^A \check{T}_{AB}(\rho) = 0$ are automatically satisfied (because $\hat{\nabla}^A \hat{G}_{AB} = 0$ identically) and so it is not necessary to consider them.

The Warp Factor

We are primarily interested in solutions with $\dot{f} \neq 0$ and a non-trivial warp factor A. From (8.56) we see that this implies $a \cdot b \neq 0$. We shall assume that $a \cdot b$ is non-zero even in the case of constant f. Equation (8.56) then implies that we can make the following choice when $\dot{f} \neq 0$:

$$\hat{\kappa}_5 \, \chi'(r) = \epsilon \sqrt{12} \, A'(r) \, , \tag{8.61}$$

$$\hat{\kappa}_5 \, \dot{\phi}(t) = \epsilon \frac{\sqrt{3}}{4} \frac{1}{a \cdot b} \frac{\dot{f}(t)}{f(t)} \, , \tag{8.62}$$

where $\epsilon = \pm 1$. The solution for Φ^μ is therefore

$$\Phi^\mu(t,r) = \begin{cases} \epsilon_+ \frac{a^\mu}{a \cdot b} \frac{\sqrt{3}}{4 \, \hat{\kappa}_5} \ln f(t) + \epsilon_+ \, b^\mu \frac{\sqrt{12}}{\hat{\kappa}_5} A(r) & r > 0 \ , \\ \epsilon_- \frac{a^\mu}{a \cdot b} \frac{\sqrt{3}}{4 \, \hat{\kappa}_5} \ln f(t) + \epsilon_- \, b^\mu \frac{\sqrt{12}}{\hat{\kappa}_5} A(r) & r < 0 \ , \end{cases} \tag{8.63}$$

where ϵ_+ and ϵ_- are (as yet) independent. On the other hand, if f is constant we have ϕ constant with χ' undetermined. Clearly, however, it is not inconsistent to impose (8.61) in this case. Therefore, we shall assume (8.61) and (8.62) (and hence (8.63)) hold also in the case $\dot{f} = 0$.

If we now substitute (8.61) into (8.55) we obtain U in the form

$$U = -\frac{6}{\hat{\kappa}_5^2}\frac{1}{f}A'^2(1 - b \cdot b). \tag{8.64}$$

We can also express the brane potential $V_{vis}(\Phi)\,\delta(r)$ as $V_{vis}(\Phi)\,\delta(r) = V_{vis}\,f(t)^{-1/2}\,\delta(r)$. Equation (8.53) can then be recast as

$$A'' + 4b \cdot bA'^2 + \frac{\hat{\kappa}_5^2}{3}V_{vis}\,\delta(r) = 0 \;, \tag{8.65}$$

yielding the following options for $A(r)$ and V_{vis}:

1. If $b \cdot b = 0$, we find $A(r) = \sigma k|r|$, where $\sigma = \pm 1$. Then $V_{vis} = -6\sigma k\hat{\kappa}_5^{-2}$. We see that $\sigma = +1$ corresponds to the RS1 solution and $\sigma = -1$ corresponds to the RS2 solution, as described in the last chapter.

2. If $b \cdot b \neq 0$, we find $A(r) = \frac{1}{4\,b\cdot b}\ln(k|r| + 1)$ and $V_{vis} = -\frac{3\,k\,\hat{\kappa}_5^{-2}}{2\,b\cdot b}$. As observed in [238] and [239], if $k < 0$ there are naked singularities at $|r| = -1/k$ whose interpretation is of some debate [240, 241].

In the above we have defined $A(0) = 0$. The scalars Φ^μ should be continuous across the brane. This implies that $\epsilon_+ = \epsilon_- = \epsilon$ if $\dot{f} \neq 0$. The above forms for $U(\Phi)$ and $V(\Phi)$ are then consistent with the ansatz (8.43) if $\beta_\mu = \frac{\alpha_\mu}{2} = -\frac{2\,\hat{\kappa}_5\,\epsilon\,b_\mu}{\sqrt{3}}$ and $U_0 = -6\,\hat{\kappa}_5^{-2}\,A'^2(0)\,(1 - b \cdot b)$. Note that $A'^2(0)$ is well-defined even though $A'(0)$ is not. It is now easily verified that (8.59) is equivalent to (8.65) in the bulk, whilst (8.60) yields no further information.

On the other hand, if f is a constant ($= 1$) continuity of Φ^μ does not relate ϵ_+ and ϵ_-. In this case we have either:

(A) $b \cdot b = 1$. Then we have $U(\Phi) = U_0 = 0$ from (8.64). From (8.60) we find $\beta_\mu = -\frac{\hat{\kappa}_5\,(\epsilon_+ + \epsilon_-)\,b_\mu}{\sqrt{3}}$. The case $\epsilon_+ = -\epsilon_-$ is an example of the self-tuning solution (solution (I)) of Kachru, Schulz and Silverstein [242], whilst the case $\epsilon_+ = \epsilon_-$ is solution (II) of [242].

(B) $b \cdot b \neq 1$. In this case consistency requires $\epsilon_+ = \epsilon_- = \epsilon$ and α_μ, β_μ, and U_0 to be of the same form as in the $\dot{f} \neq 0$ case above. This is solution (III) of [242].

The Cosmology

If we substitute (8.62) into the equation of motion (8.58) we find that f and g are actually related:

$$\frac{\dot{f}(t)}{f(t)^{1/2}} = \mu\, g(t)^{-3/2} \,, \tag{8.66}$$

where μ is a constant of dimension $[\mathrm{Time}]^{-1}$. Clearly, $\mu = 0$ iff f is constant and in this case g is not determined by (8.66).

Adding equations (8.52) and (8.54) gives:

$$\dot{g}^2 + 2g\ddot{g} + \frac{\dot{f}}{f}\,\dot{g}g + \frac{4}{3}\,\hat{\kappa}_5^2\, g^2(\tilde{\omega} - 1)\,\tilde{\rho} = 0 \;. \tag{8.67}$$

On the other hand, using (8.52) and (8.66) in (8.51) we obtain:

$$\dot{g}^2 + 2g\ddot{g} + \frac{\dot{f}}{f}\,\dot{g}g + 2\,\hat{\kappa}_5^2 g^2\,(\omega - 1)\,\tilde{\rho} = 0 \;. \tag{8.68}$$

Consequently, we deduce the relation

$$\omega = \frac{1}{3}\,(1 + 2\tilde{\omega}) \;. \tag{8.69}$$

From (8.52) and (8.62), $\tilde{\rho}$, which is actually the density on the brane since we have defined $A(0) = 0$, is given by

$$\tilde{\rho}(t) = \frac{3}{4\hat{\kappa}_5^2}\left(\frac{\dot{f}}{f}\,\frac{\dot{g}}{g} + \frac{\dot{g}^2}{g^2} - \frac{a \cdot a}{8(a \cdot b)^2}\,\frac{\dot{f}^2}{f^2}\right) , \tag{8.70}$$

so that (8.67) may be alternatively expressed as:

$$\tilde{\omega}\,\frac{\dot{g}^2}{g^2} + 2\frac{\ddot{g}}{g} + \tilde{\omega}\,\frac{\dot{f}}{f}\,\frac{\dot{g}}{g} + (1 - \tilde{\omega})\,\frac{a \cdot a}{8(a \cdot b)^2}\,\frac{\dot{f}^2}{f^2} = 0 \;. \tag{8.71}$$

Taken together with (8.66), equation (8.71) defines the cosmology.

(I) $\dot{f} = 0$

For constant f equations (8.66) and (8.71) reduce to the single equation

$$\tilde{\omega}\,\frac{\dot{g}^2}{g^2} + 2\frac{\ddot{g}}{g} = 0 \,, \tag{8.72}$$

with the solution

$$g \sim (\gamma\, t)^{2/(2+\tilde{\omega})} \sim (\gamma\, t)^{2q_{standard}} \; . \tag{8.73}$$

where $q_{standard} = 2/(3(1+\omega))$ and γ is an arbitrary positive constant of dimension $[\text{Time}]^{-1}$.

Consequently, from (8.70) we see that the density $\tilde{\rho}$ is positive and goes like $\sim 1/t^2$. Furthermore, the Hubble parameter $H = \dot{a}/a \sim 1/t$ and so we obtain conventional cosmology $H \sim \sqrt{\tilde{\rho}}$ on the brane with evolution at the standard rate.

Of particular note is the case of a radiation-dominated fluid on the brane which results in vanishing pressure in the fifth direction:

$$\check{T}^A_B(\rho) = e^{-2A(r)}\, \tilde{\rho}(t)\, \text{diag}\,(-1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0) \; , \tag{8.74}$$

with $q_{standard} = 1/2$.

The case of an isotropic fluid $P = p$ is also interesting. In this case the fluid is "stiff" ($\omega = \tilde{\omega} = 1$) and $q_{standard} = 1/3$. Here, stiff reflects the fact that the velocity of sound in the fluid is equal to the velocity of light.

Our solutions contradict the claim made in [243] that when f is constant the only solution is that of a stiff, isotropic fluid. This is because the authors of [243] assumed *a priori* that the fluid was isotropic in all four spatial directions.

It should be stressed that when f is constant conventional cosmology is obtained irrespective of whether one considers an indefinite sigma-model metric or a more conventional positive definite metric.

(II) $\dot{f} \neq 0$

In this case we seek either power law, $f = (\gamma\, t)^q$, or exponential, $f = e^{\gamma t}$, solutions of (8.71). From (8.66) we obtain the corresponding solutions for $g(t)$: $g \sim (\gamma\, t)^{(2-q)/3}$ and $g \sim e^{-\gamma t/3}$ respectively. The exponents q and γ are non-zero and we take $\gamma > 0$ in the power law case. The power law solutions are Kasner-like in the sense that the sum of the exponents over the four spatial directions is equal to 2. However, unlike the Kasner case, the sum of the squares of the exponents is not equal to 4 unless $q = -1$ or $q = 2$.

There are two cases to consider: $\tilde{\omega} = 1$ and $\tilde{\omega} \neq 1$.

(a) $\tilde{\omega} \neq 1$

Solution of (8.71) leads to the following relations:

$$\frac{a \cdot a}{(a \cdot b)^2} = h(q) \equiv \frac{16}{9} \frac{(2-q)(1+q)}{q^2} , \tag{8.75}$$

in the power law case and,

$$\frac{a \cdot a}{(a \cdot b)^2} = -\frac{16}{9} , \tag{8.76}$$

in the exponential case. If we now insert the above relations and the corresponding forms for f and g into (8.70) we find that $\tilde{\rho}$ vanishes identically. Therefore, the fluid does not exist for $\tilde{\omega} \neq 1$.

(b) $\tilde{\omega} = 1$

In this case we find that $\tilde{\rho}$ is non-zero and hence the fluid exists and is both "stiff" and isotropic. The density is positive in the power law case provided $\frac{a \cdot a}{(a \cdot b)^2} < h(q)$ and in the exponential case provided $\frac{a \cdot a}{(a \cdot b)^2} < -\frac{16}{9}$. Standard cosmology $H \sim \sqrt{\tilde{\rho}}$ is again obtained in both of these cases. Note that if one were to assume a conventional sigma-model with $a \cdot a > 0$ and a positive density $\tilde{\rho}$ then one would exclude the exponential solutions and restrict q to either $-1 < q < 0$ or $0 < q < 2$.

8.3.3 Perturbation Analysis of Solutions

In this section we examine the equations of linearised transverse, traceless fluctuations about the backgrounds presented in the last section. This section is similar to section 7.5.2 and we shall give some details which were omitted there.

We will again choose axial gauge and consider metric perturbations

$$\begin{aligned} ds^2 = \hat{g}_{AB}\, dy^A dy^B &= (\hat{g}^{(0)}_{AB} + \hat{h}_{AB})\, dy^A dy^B \\ &= e^{2A(r)}\, (\bar{g}_{ij}(t) + h_{ij}(x,r))\, dx^i dx^j + f(t) dr^2 , \end{aligned} \tag{8.77}$$

where we have defined $\hat{h}_{AB} = e^{2A(r)} h_{AB}$ and $\bar{g}_{ij}(t) = \mathrm{diag}(-1, g(t), g(t), g(t))$. The components h_{ij} are required to have even parity across the brane so that the metric remains invariant under the orbifolding $r \to -r$. Again, it will be simplest to work in terms of the conformal coordinate z, given by $z = \int e^{-A(r)}dr$, so as to bring the unperturbed metric to the conformal form $\hat{g}^{(0)}_{AB}(x,z) = e^{2A(z)}\,\tilde{g}^{(0)}_{AB}(x,z)$, where

$$\tilde{g}^{(0)}_{AB}(x,z) = \begin{pmatrix} \bar{g}_{ij}(t) & 0 \\ 0 & f(t) \end{pmatrix} . \tag{8.78}$$

The linearisation is more complicated than that of section 7.5.2 because the metric $\tilde{g}^{(0)}_{AB}(x,z)$ is no longer flat.

The Einstein Tensor

The linearisation of the Einstein tensor is standard and can be found in [244], for example. A valuable source in which the linearisation of the Christoffel symbols, Riemann tensor, etc., as well as second variations of the Einstein-Hilbert action and higher derivative terms, are detailed is the paper by Barth and Christensen [245].

The first step is to remove the conformal factor from $\hat{\mathbf{g}}$ and write $\hat{\mathbf{G}}$ in terms of $\tilde{\mathbf{G}}$ and covariant derivatives of A with respect to $\tilde{\mathbf{g}}$. Thus one expands $\hat{\mathbf{G}}$ as [18, 210]:

$$\hat{G}_{AB} = \tilde{G}_{AB} + 3\left[\tilde{\nabla}_A A \tilde{\nabla}_B A - \tilde{\nabla}_A \tilde{\nabla}_B A + \tilde{g}_{AB}\left(\tilde{\nabla}^2 A + (\tilde{\nabla} A)^2\right)\right] , \tag{8.79}$$

where indices on the right are raised with $\tilde{g}^{AB}$. Note that $\tilde{g}^{AB} \approx \tilde{g}^{(0)AB} - h^{AB}$ to linear order, where indices on h_{AB} are raised with the unperturbed metric $\tilde{g}^{(0)AB}$.

The second step is to expand $\tilde{\mathbf{G}}$ to linear order in $\mathbf{h}$. We use the expansions (to linear order) given in [245]:

$$\tilde{R}_{AB} = \tilde{R}^{(0)}_{AB} + \frac{1}{2}\left(\tilde{\nabla}^{(0)}_C \tilde{\nabla}^{(0)}_A h^C_B + \tilde{\nabla}^{(0)}_C \tilde{\nabla}^{(0)}_B h^C_A - \tilde{\nabla}^{(0)2} h_{AB} - \tilde{\nabla}^{(0)}_A \tilde{\nabla}^{(0)}_B h\right) , \tag{8.80}$$

$$\tilde{R} = \tilde{R}^{(0)} - \tilde{R}^{(0)}_{AB} h^{AB} + \left(\tilde{\nabla}^{(0)}_A \tilde{\nabla}^{(0)}_B h^{AB} - \tilde{\nabla}^{(0)2} h\right) , \tag{8.81}$$

where we have defined $h = h^A_A$ and all indices on the right are raised with the unperturbed metric $\tilde{g}^{(0)AB}$. The relation

$$\tilde{\nabla}^{(0)}_C \tilde{\nabla}^{(0)}_A h^C_B = \tilde{\nabla}^{(0)}_A \tilde{\nabla}^{(0)}_C h^C_B + h^C_D \tilde{R}^{(0)D}{}_{BAC} + h^C_B \tilde{R}^{(0)}_{CA} , \tag{8.82}$$

proves useful for commuting the covariant derivatives involved in (8.80).

Thus, $\tilde{G}_{AB} = \tilde{R}_{AB} - \frac{1}{2}\tilde{g}_{AB}\tilde{R}$ is given to first order in $\mathbf{h}$ by

$$\tilde{G}_{AB} = \tilde{R}_{AB} - \frac{1}{2}\tilde{g}^{(0)}_{AB}\tilde{R} - \frac{1}{2}h_{AB}\tilde{R}^{(0)} , \tag{8.83}$$

where $\tilde{R}_{AB}$ and $\tilde{R}$ are as in (8.80) and (8.81), respectively.

The final step is to expand the covariant derivatives of A in (8.79) in terms of covariant derivatives with respect to the unperturbed metric and terms linear in $\mathbf{h}$. Firstly, note that $\tilde{\nabla}_B A = \partial_B A = \tilde{\nabla}^{(0)}_B A$. Thus,

$$\tilde{\nabla}_A A \tilde{\nabla}_B A = \tilde{\nabla}^{(0)}_A A \tilde{\nabla}^{(0)}_B A , \tag{8.84}$$

$$\tilde{g}_{AB}(\tilde{\nabla} A)^2 = \tilde{g}^{(0)}_{AB}(\tilde{\nabla}^{(0)} A)^2 - \tilde{g}^{(0)}_{AB} h^{CD} \tilde{\nabla}^{(0)}_C A \tilde{\nabla}^{(0)}_D A + h_{AB}(\tilde{\nabla}^{(0)} A)^2 . \tag{8.85}$$

Next, we expand the Christoffel symbol:

$$\tilde{\Gamma}^A_{BC} = \tilde{\Gamma}^{(0)A}{}_{BC} - h^{AD} S_{DBC}[\tilde{\mathbf{g}}^{(0)}] + \tilde{g}^{(0)AD} S_{DBC}[\mathbf{h}] , \tag{8.86}$$

where we have defined $S_{DBC}[\mathbf{g}] = \frac{1}{2}(g_{DB,C} + g_{DC,B} - g_{BC,D})$ for any metric $\mathbf{g}$. In turn, this allows us to expand

$$\tilde{\nabla}_A \tilde{\nabla}_B A = \tilde{\nabla}^{(0)}_A \tilde{\nabla}^{(0)}_B A + h^{CD} \partial_D A \, S_{CAB}[\tilde{\mathbf{g}}^{(0)}] - (\partial^C A) \, S_{CAB}[\mathbf{h}] , \tag{8.87}$$

$$\begin{aligned} \tilde{g}_{AB} \tilde{\nabla}^2 A &= \tilde{g}^{(0)}_{AB} \tilde{\nabla}^{(0)2} A + h_{AB} \tilde{\nabla}^{(0)2} A - \tilde{g}^{(0)}_{AB} h^{CD} \tilde{\nabla}^{(0)}_C \tilde{\nabla}^{(0)}_D A \\ &+ \tilde{g}^{(0)}_{AB} \tilde{g}^{(0)DE} h^{CF} \partial_F A \, S_{CDE}[\tilde{\mathbf{g}}^{(0)}] - \tilde{g}^{(0)}_{AB} \tilde{g}^{(0)DE} (\partial^C A) \, S_{CDE}[\mathbf{h}]. \end{aligned} \tag{8.88}$$

All indices on the right sides of (8.85), (8.87) and (8.88) are raised with the unperturbed metric $\tilde{g}^{(0)AB}$.

Piecing together (8.79)–(8.88) we obtain a rather lengthy expression (which we shall not explicitly write down) for the first variation $\delta\hat{\mathbf{G}} = \hat{\mathbf{G}} - \hat{\mathbf{G}}^{(0)}$.

The Bulk Scalar Stress Tensor

The unperturbed bulk scalar stress tensor is given by (8.45). To first order, the variation of the stress tensor splits into a part $\delta_h T_{AB}$ depending on the metric perturbations h_{AB} and a second part $\delta_I T_{AB}$ due to the intrinsic perturbations of the scalar fields. The metric perturbation is easily calculated to be

$$\begin{aligned}\delta_h \check{T}_{AB}(\Phi) &= \frac{1}{2}\,\tilde{g}^{(0)}_{AB} h^{CD} \partial_C \Phi \cdot \partial_D \Phi \\ &- h_{AB}\left(\frac{1}{2}\tilde{g}^{(0)CD} \partial_C \Phi \cdot \partial_D \Phi + e^{2A} U(\Phi)\right) ,\end{aligned} \tag{8.89}$$

while the intrinsic perturbation is

$$\begin{aligned}\delta_I \check{T}_{AB}(\Phi) &= \partial_A \tilde{\Phi} \partial_B \Phi + \partial_A \Phi \partial_B \tilde{\Phi} \\ &- \tilde{g}^{(0)}_{AB}\left(\tilde{g}^{(0)CD} \partial_C \tilde{\Phi} \cdot \partial_D \Phi + e^{2A} \tilde{\Phi} \cdot \frac{\partial U(\Phi)}{\partial \Phi}\right) ,\end{aligned} \tag{8.90}$$

where the perturbations $\tilde{\Phi}^{\mu}$ are of the same order as the perturbations h_{AB}.

Note that the **h**-dependent bulk scalar stress tensor can be re-expressed as

$$\begin{aligned}\delta_h \check{T}_{AB}(\Phi) = [\check{T}_{AC}(\Phi) &+ \tilde{T}_{AC}(\Phi)]\, h^C_B + F_{AB} \\ &+ \frac{1}{f^{1/2}} V_{vis}(\Phi)\, \delta(z)\, \bar{g}_{ij}\, \delta^i_A h^j_B ,\end{aligned} \tag{8.91}$$

where

$$F_{AB} = \frac{1}{2}\tilde{g}^{(0)}_{AB}\, \partial_C \Phi \cdot \partial_D \Phi\, h^{CD} - \partial_A \Phi \cdot \partial_C \Phi\, h^C_B , \tag{8.92}$$

$\tilde{T}_{AB}(\Phi)$ is the unperturbed brane scalar stress tensor given by (8.46) and there is a sum over j but not over i.

The Brane Scalar Stress Tensor

Noting that $\delta_h \sqrt{-\tilde{g}} = \frac{1}{2}\sqrt{-\tilde{g}}\, h$, we find

$$\delta_h \tilde{T}_{AB}(\Phi) = \frac{1}{f^{1/2}} V_{vis}(\Phi)\, \delta(z) \left(\frac{1}{2} h^z_z\, \tilde{g}^{(0)}_{ij} - h_{ij}\right) \delta^i_A\, \delta^j_B , \tag{8.93}$$

$$\delta_I \tilde{T}_{AB}(\Phi) = -\frac{1}{f^{1/2}}\, \delta(z)\, \tilde{\Phi} \cdot \frac{\partial V_{vis}(\Phi)}{\partial \Phi}\, \tilde{g}^{(0)}_{ij}\, \delta^i_A\, \delta^j_B . \tag{8.94}$$

The Fluid

In this section before deriving the perturbations we need to examine more closely the kinematics and thermodynamics of the fluid [246, 247].

We can write the stress-tensor (8.47) in the form [248, 249]:

$$\check{T}_{AB}(\rho) = (\rho + p)\, U_A U_B + p\, \hat{g}^{(0)}_{AB} + (P - p)\, n_A n_B , \tag{8.95}$$

where $U^A = (e^{-A}, 0, 0, 0, 0)$ is the tangent vector to the flow of the fluid and $n^A = (0, 0, 0, 0, e^{-A})$ in the coordinates $y^A = (x^i, z)$. Note that U^A is timelike and normalised: $U^A U^B \hat{g}^{(0)}_{AB} = -1$. On the other hand, n^A is spacelike and normalised and orthogonal to U^A.

Equation (8.95) can be re-expressed in the more standard notation:

$$\check{T}_{AB}(\rho) = \rho U_A U_B + \bar{p} P_{AB} + \pi_{AB} \ , \tag{8.96}$$

where $\bar{p} = (3p + P)/4$ is the mean kinetic pressure, P_{AB} is the projection tensor $\hat{g}^{(0)}_{AB} + U_A U_B$ and

$$\pi_{AB} = (P - p) \left(n_A n_B - \frac{1}{4} P_{AB} \right) \ , \tag{8.97}$$

is the anisotropy (shear) tensor that satisfies $\pi^A_A = \pi_{AB} U^B = 0$.

We should also note that the fluid is assumed to satisfy the Gibbs equation

$$d\epsilon + \hat{p} dv = T dS \ , \tag{8.98}$$

where ϵ is the specific internal energy density of the fluid, $v = 1/\mu$ is the specific volume (μ is the rest-mass density per particle as measured by an observer travelling with 5-velocity U^A and is given by $\mu = \rho/(1 + \epsilon)$), $\hat{p}$ is the mean static pressure, T is the temperature and S is the entropy.

Note that $\bar{p}$ and $\hat{p}$ are logically distinct. However, they are related via

$$\bar{p} = \hat{p} - \xi \theta \ , \tag{8.99}$$

where ξ is the coefficient of bulk viscosity and

$$\theta \equiv \hat{\nabla}^{(0)}_A U^A = e^{-A} \left(\frac{3}{2} \frac{\dot{g}}{g} + \frac{1}{2} \frac{\dot{f}}{f} \right) \tag{8.100}$$

is the expansion rate of the flow lines. θ is zero for the exponential solutions but equal to e^{-A}/t for the power law solutions.

For simplicity we shall consider only *isentropic* fluid flow, that is, flow at constant entropy. Equation (8.98) then implies $\hat{p} = \mu^2 (d\epsilon/d\mu)$. It is shown in [246, 247] that such flow is possible only when $\pi_{AB} = 0$ and $\bar{p} = \hat{p}$. In turn, this implies $\bar{p} = P = p = \mu^2 (d\epsilon/d\mu)$ and that the flow is "perfect".

We now come to the calculation of the metric perturbations. The fluid current vector is defined by $j^A = \mu\, U^A$ and is conserved:

$$\hat{\nabla}^{(0)}_A j^A = \frac{1}{\sqrt{-\hat{g}^{(0)}}}\frac{\partial}{\partial y^A}(\sqrt{-\hat{g}^{(0)}}j^A) = 0 \ . \tag{8.101}$$

This implies that the quantity $\sqrt{-\hat{g}^{(0)}}\,\mu\, U^A$ is unchanged when the metric is varied [250]. Therefore, we calculate

$$\begin{aligned} \delta_h\mu &= -\frac{1}{2}\,\mu\,(e^{2A}\,U^AU^Bh_{AB}+h)\ , &(8.102)\\ \delta_h\rho &= \delta_h\mu\left(1+\epsilon+\mu\,\frac{d\epsilon}{d\mu}\right) \\ &= -\frac{1}{2}\,(\rho+p)\,(e^{2A}\,U^AU^Bh_{AB}+h) &(8.103)\end{aligned}$$

and

$$\delta_h U_A = U_A\frac{\delta_h(\mu\sqrt{-\hat{g}^{(0)}})}{\mu\sqrt{-\hat{g}^{(0)}}} = -\frac{1}{2}\,e^{2A}\,U_AU^CU^Dh_{CD}\ . \tag{8.104}$$

Given that the fluid is also stiff (that is, $p=\rho$) we finally find

$$\delta_h\check{T}_{AB}(\rho) = -\tilde{\rho}\,(4h_{00}+2h)\,\delta_{A0}\delta_{B0} - \tilde{\rho}\,(h_{00}+h)\,\tilde{g}^{(0)}_{AB} + \tilde{\rho}\,h_{AB}\ . \tag{8.105}$$

This can be re-expressed as

$$\delta_h\check{T}_{AB}(\rho) = \check{T}_{AC}(\rho)h^C_B + H_{AB}\ , \tag{8.106}$$

where

$$H_{AB} = \begin{pmatrix} -\tilde{\rho}\,(h_{00}+h) & -2\,\tilde{\rho}\,h^0_a & -2\,\tilde{\rho}\,h^0_z \\ 0 & -\tilde{\rho}\,(h_{00}+h)\,g(t)\,\delta_{ab} & 0 \\ 0 & 0 & -\tilde{\rho}\,(h_{00}+h)\,f(t) \end{pmatrix}. \tag{8.107}$$

Combining with (8.91) we find

$$\delta_h\check{T}_{AB}(\Phi) + \delta_h\check{T}(\rho) = \hat{\kappa}_5^{-2}\hat{G}^{(0)}_{AC}\,h^C_B + F_{AB} + H_{AB} + \frac{1}{f^{1/2}}\,V_{vis}(\Phi)\,\delta(z)\,\bar{g}_{ij}\,\delta^i_A h^j_B. \tag{8.108}$$

The intrinsic fluid perturbations take the form [251]:

$$\delta_I\check{T}_{AB}(\rho) = \begin{pmatrix} \delta\tilde{\rho} & -2\tilde{\rho}\,e^{-A}\,\delta U^a & -2\tilde{\rho}\,e^{-A}\,\delta U^z \\ -2\tilde{\rho}\,e^{-A}\,\delta U^a & \delta\tilde{\rho}\,g(t)\,\delta_{ab} & 0 \\ -2\tilde{\rho}\,e^{-A}\,\delta U^z & 0 & \delta\tilde{\rho}\,f(t) \end{pmatrix}. \tag{8.109}$$

where we have neglected the possibility of any anisotropic stress terms[6].

[6]For a static, isotropic, non-stiff perturbation we would instead find $\delta_I\check{T}_{AB}(\rho) = \mathrm{diag}\,(\delta\tilde{\rho}, \delta\tilde{p}\,g(t)\,\delta_{ab}, \delta\tilde{p}\,f(t))$. A dust source then has $\delta\tilde{p}=0$ and so we reproduce (7.48).

Transverse, Traceless Modes

We are now able to write down the equations for linearised perturbations:

$$\begin{aligned}\delta\hat{G}_{AB} - \hat{G}^{(0)}_{AC}\, h^C_B - F_{AB} \;-\; & H_{AB} - \delta_h\tilde{T}_{AB}(\Phi) - \frac{1}{f^{1/2}}\, V_{vis}(\Phi)\,\delta(z)\,\bar{g}_{ij}\,\delta^i_A h^j_B \\ = \;& \delta_I\check{T}_{AB}(\Phi) + \delta_I\tilde{T}_{AB}(\Phi) + \delta_I\check{T}_{AB}(\rho)\ , \end{aligned} \tag{8.110}$$

where we have set $\hat{\kappa}_5^2 = 1$ for simplicity. The left side of this equation contains all the metric perturbations while the right side contains all the intrinsic perturbations. Note that if we were to set $g = f = 1$, put the scalars to zero and consider an arbitrary intrinsic perturbation $\delta_I\check{T}_{AB}(\rho)$, the above equations would exactly reproduce (7.40)–(7.42) of section 7.5.2.

Clearly, when combined with the perturbed scalar equations of motion (which we shall not derive), these equations form an extremely complicated set of coupled partial differential equations and it has proven impossible to solve them in all generality. Indeed, even in the simpler case of $g = f = 1$, a single scalar depending on z only and no fluid [207], the equations have not been solved in general.

The equations do simplify, however, if we consider the *transverse, traceless* components of the metric perturbations. These components are defined with respect to the maximally symmetric subspace of our manifold. This subspace is just given by the three spatial directions x^a. Then h_{ab} can be decomposed in the following way [252]:

$$h_{ab} = \bar{h}_{ab} + 2\partial_{(a}E_{b)} + \partial_a\partial_b E + C\delta_{ab}\ , \tag{8.111}$$

where C and E are scalar perturbations, E_a is a vector perturbation and $\bar{h}_{ab}$ is a transverse, traceless tensor perturbation: $\bar{h}^a_a = \partial^a\bar{h}_{ab} = 0$. Additionally, h_{00} and h_{0a} decompose as

$$h_{00} = A \tag{8.112}$$

$$h_{0a} = \partial_a B + \bar{B}_a \tag{8.113}$$

where A and B are scalar perturbations and $\bar{B}_a$ is a vector perturbation. Each of the fifteen equations contained in (8.110) then splits separately into three: scalar, vector and tensor. In the sequel we shall consider only the tensor perturbations $\bar{h}_{ab}$.

From (8.90) and the fact that Φ^μ is independent of x^a, we see that $\delta_I \check{T}_{ab}(\Phi)$ is proportional to δ_{ab} and will therefore not couple to $\bar{h}_{ab}$. Likewise, $\delta_I \tilde{T}_{ab}(\Phi)$ and $\delta_I \check{T}_{ab}(\rho)$ are proportional to δ_{ab}, as can be verified from (8.94) and (8.109). Therefore, the equations for the transverse, traceless fluctuations $\bar{h}_{ab}$ completely decouple from the intrinsic perturbations. Furthermore, from (8.92), F_{ab} is proportional to δ_{ab}, as are H_{ab} (cf. (8.107)) and the h^z_z part of $\delta_h \tilde{T}_{ab}(\Phi)$ (cf. (8.93)). In addition, the final term on the left side of (8.110) cancels the remaining term of $\delta_h \tilde{T}_{ab}(\Phi)$.

Hence, we end up with the following equations for the components $\bar{h}_{ab}$:

$$[\delta \hat{G}_{ab} - \hat{G}^{(0)}_{aC}\, h^C_b]^{TT} = 0 \ , \tag{8.114}$$

where ^{TT} denotes the transverse, traceless part. We have not as yet imposed the axial gauge constraint which is now used in order to simplify these equations even further. In this gauge we find that (8.114) reduces to

$$\frac{1}{2}[\tilde{\nabla}^{(0)2} h_{ab}]^{TT} + \frac{3}{2f}\partial_z A \partial_z \bar{h}_{ab} - [h^i_j \tilde{R}^{(0)j}{}_{abi} + \frac{1}{2} h^i_a \tilde{R}^{(0)}_{ib} - \frac{1}{2} h^i_b \tilde{R}^{(0)}_{ia}]^{TT} = 0. \tag{8.115}$$

It is easily verified by hand (or, alternatively, using Mathematica[7]) that

$$\tilde{R}^{(0)}_{0a} = 0 \ , \ \tilde{R}^{(0)}_{ab} \propto \delta_{ab} \ , \tag{8.116}$$

and

$$h^i_j \tilde{R}^{(0)j}{}_{abi} = \frac{\dot{g}^2}{4\,g^2} h_{ab} - \frac{h_{00}}{2}\left(\frac{\dot{g}^2}{2\,g^2} - \ddot{g}\right)\delta_{ab} \ . \tag{8.117}$$

It should also be noted that $\tilde{\nabla}^{(0)2}$ acting on h_{ab} is *not* the scalar D'Alembertian of the unperturbed metric (8.78). Instead, we find

$$\frac{1}{2}[\tilde{\nabla}^{(0)2} h_{ab}]^{TT} = \left[\frac{1}{2}\Box^{(0)} + \frac{\dot{g}}{g}\partial_t + \frac{1}{2}\left(\frac{\dot{f}\dot{g}}{2fg} + \frac{\ddot{g}}{g}\right)\right]\bar{h}_{ab} \ , \tag{8.118}$$

where $\Box^{(0)}$ is the scalar D'Alembertian of the unperturbed metric. Upon using (8.71) with $\tilde{\omega} = 1$, we arrive at the final form for the equations of transverse, traceless fluctuations:

$$\Box^{(0)} \bar{h}_{ab} + \frac{2\,\dot{g}}{g}\,\partial_t \bar{h}_{ab} + \frac{3}{f}\partial_z A \partial_z \bar{h}_{ab} - \frac{\dot{g}^2}{g^2}\bar{h}_{ab} = 0 \ . \tag{8.119}$$

[7] My thanks to Pekka Jahunen of the Finnish Meteorological Institute for providing an invaluable Mathematica program for calculating the Einstein tensor [253].

Let us now proceed to solve these equations. It has been noted in [254] that in general this equation is non-separable and therefore must be solved perturbatively order by order. However, since we have assumed a separable ansatz for the metric we find that the equations are separable. Let us first note that

$$\Box^{(0)} = -\partial_t^2 - \frac{1}{2}\frac{\dot{f}}{f}\partial_t - \frac{3}{2}\frac{\dot{g}}{g}\partial_t + \frac{1}{g}\Delta + \frac{1}{f}\partial_z^2 \ , \tag{8.120}$$

where Δ is the scalar Laplacian for the three spatial directions on the brane. Therefore, we see that the perturbations can be decomposed into the Fourier modes:

$$\bar{h}_{ab} = g(t)\,\Upsilon_m^{(p)}(t)\,\Psi^{(p)}(z)\,e^{-i\mathbf{m}\cdot\mathbf{x}}\,e_{ab} \tag{8.121}$$

where e_{ab} is a transverse, traceless polarisation tensor and where Ψ and Υ satisfy the following equations:

$$\left(\partial_z^2 + 3\,A'(z)\,\partial_z\right)\Psi^{(p)}(z) = p^2\,\Psi^{(p)}(z) \ , \tag{8.122}$$

$$\left(\partial_t^2 + \frac{1}{2}\frac{\dot{f}}{f}\partial_t + \frac{3}{2}\frac{\dot{g}}{g}\partial_t - \frac{2\,\dot{g}}{g}\,\partial_t + \frac{\dot{g}^2}{g^2} + \frac{m^2}{g} - \frac{p^2}{f}\right) g\,\Upsilon_m^{(p)}(t) = 0 \ , \tag{8.123}$$

where $m = |\mathbf{m}|$. Note that $\Upsilon_m^{(p)}(t)\,\Psi^{(p)}(z)\,e^{-i\mathbf{m}\cdot\mathbf{x}}\,e_{ab}$ is the transverse, traceless perturbation of the metric δ_{ab}. Note also that (8.122) has the same form as equation (7.63). Indeed, the perturbations studied in the last chapter were static with $g = f = 1$. From (8.123) we see that this requires $p^2 = m^2 \geq 0$. However, in the time-dependent case it is also possible to have $p^2 < 0$.

The modes $\Psi^{(p)}(z)$ are subject to the boundary condition

$$\partial_z \Psi^{(p)}(0^+) = 0 \ , \tag{8.124}$$

as can be deduced from (8.119).

The Modes $\Psi^{(p)}(z)$

Firstly, the zero mode of (8.122) satisfying the boundary condition (8.124) is given by $\Psi^{(0)}(z) = constant$.

For the non-zero modes we employ the transformation

$$\Psi^{(p)}(z) = e^{-3A/2}\,\psi^{(p)}(z) \ , \tag{8.125}$$

as in section 7.5.2, to put (8.122) into the form

$$\left(-\partial_z^2 + \frac{9}{4}A'(z)^2 + \frac{3}{2}A''(z)\right)\psi^{(p)}(z) = -p^2\,\psi^{(p)}(z)\ . \tag{8.126}$$

We shall concentrate on the case $b \cdot b < 0$, since the case $b \cdot b = 0$ was presented in the original RS2 paper [199] and the case $b \cdot b > 0$ was discussed in a different setting in [238]. (Our analysis, however, is also valid for $b \cdot b > 0$.) Furthermore, we shall take $k > 0$ so as not to have singularities in the bulk. In this case we find

$$A(z) = \frac{1}{\beta}\ln(\bar{k}|z| + 1)\ , \tag{8.127}$$

where $\beta = 4b \cdot b - 1$ and $\bar{k} = k\left(1 - \frac{1}{4b\cdot b}\right)$.

Therefore, in the region $z > 0$ (8.126) becomes

$$\left(-\partial_z^2 + \frac{\alpha(\alpha+1)}{(z+1/\bar{k})^2}\right)\psi^{(p)}(z) = -p^2\,\psi^{(p)}(z)\ , \tag{8.128}$$

where $\alpha = -\frac{3}{2\beta}$. This is similar to equation (7.67). Indeed, (7.67) is obtained from the above equation by formally setting $\beta = -1$ and $\bar{k} = k$.

The boundary condition (8.124) becomes

$$\partial_z\psi^{(p)}(0^+) = -\alpha\,\bar{k}\,\psi^{(p)}(0^+)\ . \tag{8.129}$$

The solution to the above equation is

$$\psi^{(p)}(z) = \begin{cases} (pu)^{1/2}\left[I_{-\nu}(pu) - \frac{I_{-\nu+1}(p/\bar{k})}{I_{\nu-1}(p/\bar{k})}I_\nu(pu)\right] & p^2 > 0\ , \\ (pu)^{1/2}\left[J_{-\nu}(pu) + \frac{J_{-\nu+1}(p/\bar{k})}{J_{\nu-1}(p/\bar{k})}J_\nu(pu)\right] & p^2 < 0\ , \end{cases} \tag{8.130}$$

where $p = \sqrt{|p^2|}$, $u = z + 1/\bar{k}$ and $\nu = \alpha + 1/2$. These solutions assume that ν is not an integer. If ν is an integer we instead find

$$\psi^{(p)}(z) = \begin{cases} (pu)^{1/2}\left[K_\nu(pu) + \frac{K_{\nu-1}(p/\bar{k})}{I_{\nu-1}(p/\bar{k})}I_\nu(pu)\right] & p^2 > 0\ , \\ (pu)^{1/2}\left[Y_\nu(pu) - \frac{Y_{\nu-1}(p/\bar{k})}{J_{\nu-1}(p/\bar{k})}J_\nu(pu)\right] & p^2 < 0\ . \end{cases} \tag{8.131}$$

Here, J_ν and Y_ν are Bessel functions of the first and second kind respectively and I_ν and K_ν are as defined in section 7.5.2. In verifying that these solutions satisfy the boundary condition (8.129) it is necessary to use the recurrence relations (7.69) as well as

$$\partial_u(u^\nu I_{-\nu}(u)) = u^\nu I_{-\nu+1}(u)\ ,\ \partial_u(u^\nu J_{-\nu}(u)) = -u^\nu J_{-\nu+1}(u)\ , \tag{8.132}$$

$$\partial_u(u^\nu Y_\nu(u)) = u^\nu Y_{\nu-1}(u) \ , \ \partial_u(u^\nu J_\nu(u)) = u^\nu J_{\nu-1}(u) \ . \tag{8.133}$$

To extend the solution $\psi^{(m)}(z)$ to all z, we simply replace z by $|z|$.

The Modes $\Upsilon_m^{(p)}(t)$

In order to solve (8.123) we rescale the four-dimensional metric $\bar{g}_{ij}$ (defined in (8.77)) by a factor $f^{1/2}$ and define a new time coordinate τ by $\tau = \pm \int f^{1/4} dt$. Then (8.123) becomes

$$\left(\partial_\tau^2 + \frac{3}{2}\frac{\dot{\tilde{g}}}{\tilde{g}}\partial_\tau - \frac{2\,\dot{g}}{g}\,\partial_\tau + \frac{\dot{g}^2}{g^2} + \frac{m^2}{\tilde{g}} - \frac{p^2}{f^{3/2}} \right) g\,\Upsilon_m^{(p)}(\tau) = 0 \ , \tag{8.134}$$

where $\tilde{g} = f^{1/2} g$ and the dot now denotes differentiation with respect to τ.

<u>(I) $\dot{f} = 0$</u>

In this case we take $f = 1$ so that $\tau = t$ and $\tilde{g} = g$. Since we are considering the case of a stiff, perfect fluid then from (8.73) we find $g \sim (\gamma\,\tau)^{2/3}$. The coordinate τ ranges from 0 (where there is a singularity) to ∞. In this case large t obviously corresponds to large τ.

Equation (8.134) reduces to

$$\left(\tau^2 \partial_\tau^2 + \tau \partial_\tau + \tilde{m}^2 \tau^{4/3} - p^2 \tau^{3\delta} \right) \Upsilon_m^{(p)}(\tau) = 0 \ , \tag{8.135}$$

where $\delta = 2/3$ and $\tilde{m}^2 = m^2/\gamma^{2/3}$.

For the zero modes $p^2 = 0$, the solutions of (8.135) are

$$\Upsilon_m^{(0)}(\tau) = \begin{cases} J_0(\frac{3}{2}\,\tilde{m}\,\tau^{2/3}) \ , \\ Y_0(\frac{3}{2}\,\tilde{m}\,\tau^{2/3}) \ . \end{cases} \tag{8.136}$$

There are no closed-form solutions to (8.135) when $p^2 \neq 0$; equation (8.135) must be solved numerically. However, we can study the small τ and large τ limits, where one of the last two terms in (8.135) dominates over the other. The small τ behaviour is the same as that of the zero modes while for large τ we find

$$\Upsilon_m^{(p)}(\tau) = \begin{cases} I_0(p\,\tau) \ , \\ K_0(p\,\tau) \ , \end{cases} \tag{8.137}$$

when $p^2 > 0$ and

$$\Upsilon_m^{(p)}(\tau) = \begin{cases} J_0(p\,\tau) \ , \\ Y_0(p\,\tau) \ , \end{cases} \tag{8.138}$$

when $p^2 < 0$. In both of these cases we have defined, as before, $p = \sqrt{|p^2|}$.

(II) $\dot{f} \neq 0$

(A) *Exponential Solutions*:

For the exponential solutions we find

$$\frac{|\gamma|}{4}\,\tau = e^{(\gamma/4)\,t} \tag{8.139}$$

and therefore

$$g \sim \tau^{-4/3} \ , \ \tilde{g} = \left(\frac{|\mu|}{4}\right)^{2/3} \tau^{2/3} \ . \tag{8.140}$$

Since μ is arbitrary, we choose $|\mu| = 4\,|\gamma|$. Note that $-\infty < t < \infty$, so τ ranges from 0 to ∞. If the fifth dimension is expanding ($\gamma > 0$) then large t corresponds to large τ. On the other hand, if the fifth dimension is contracting ($\gamma < 0$) then large t corresponds to small τ.

Equation (8.134) then becomes

$$\left(\tau^2\partial_\tau^2 + \tau\partial_\tau + \tilde{m}^2\tau^{4/3} - \tilde{p}^2\tau^{3\delta}\right)\Upsilon_m^{(p)}(\tau) = 0 \ , \tag{8.141}$$

where $\delta = -4/3$, $\tilde{m}^2 = m^2/|\gamma|^{2/3}$ and $\tilde{p}^2 = p^2/(|\gamma|/4)^6$.

For the zero modes $p^2 = 0$, the solutions of (8.141) are

$$\Upsilon_m^{(0)}(\tau) = \begin{cases} J_0(\frac{3}{2}\,\tilde{m}\,\tau^{2/3}) \ , \\ Y_0(\frac{3}{2}\,\tilde{m}\,\tau^{2/3}) \ . \end{cases} \tag{8.142}$$

Again, we find that there are no closed-form solutions to (8.141) when $p^2 \neq 0$. The small τ behaviour is

$$\Upsilon_m^{(p)}(\tau) = \begin{cases} I_0(\frac{1}{2}\,\bar{p}\,\tau^{-2}) \ , \\ K_0(\frac{1}{2}\,\bar{p}\,\tau^{-2}) \ , \end{cases} \tag{8.143}$$

when $p^2 > 0$ and

$$\Upsilon_m^{(p)}(\tau) = \begin{cases} J_0(\frac{1}{2}\,\bar{p}\,\tau^{-2}) \ , \\ Y_0(\frac{1}{2}\,\bar{p}\,\tau^{-2}) \ , \end{cases} \tag{8.144}$$

when $p^2 < 0$. In both of these cases we have defined $\bar{p} = \sqrt{|\tilde{p}^2|}$. The large τ behaviour is that of the zero modes.

(B) *Power Law Solutions with* $q \neq -4$:

For the power law solutions with $q \neq -4$ we may take

$$|\gamma\,\zeta(q)|\,\tau = (\gamma\,t)^{\zeta(q)}\ , \tag{8.145}$$

where $\zeta(q) = 1 + q/4$. Therefore, we find

$$g \sim \tau^{\delta(q)}\ ,\ \tilde{g} = \left(\left|\frac{\mu\,\zeta(q)}{q}\right|\right)^{2/3}\tau^{2/3}\ , \tag{8.146}$$

where $\delta(q) = \frac{(2-q)}{3\zeta(q)}$. We can choose $|\mu| = |q\,\gamma/\zeta(q)|$. The coordinate t ranges from the singularity at 0 to ∞, as does τ. We find that large t corresponds to large τ if $q > -4$ and to small τ if $q < -4$.

Equation (8.134) then becomes

$$\left(\tau^2\partial_\tau^2 + \tau\partial_\tau + \tilde{m}^2\tau^{4/3} - \tilde{p}(q)^2\tau^{3\delta(q)}\right)\Upsilon_m^{(p)}(\tau) = 0\ , \tag{8.147}$$

where, as before, $\tilde{m}^2 = m^2/\gamma^{2/3}$ and we have also defined $\tilde{p}(q)^2 = p^2/|\gamma\,\zeta(q)|^{3q/2\zeta(q)}$. Note that if we take the limit $q \to 0$ in the above equation we recover (8.135). Alternatively, if we take the limit $q \to \infty$ and simultaneously rescale γ and m according to $\gamma \to q^{-1}\gamma$ and $m \to q^{-1/3}m$ (so that $\tilde{m}$ remains constant) we recover (8.141).

For the zero modes $p^2 = 0$, the solutions of (8.147) are

$$\Upsilon_m^{(0)}(\tau) = \begin{cases} J_0(\frac{3}{2}\,\tilde{m}\,\tau^{2/3})\ , \\ Y_0(\frac{3}{2}\,\tilde{m}\,\tau^{2/3})\ . \end{cases} \tag{8.148}$$

For general q there are no closed-form solutions to (8.147) when $p^2 \neq 0$. However, when $3\,\delta(q) = 4/3$, that is, $q = 1/2$ (which implies an isotropic five-dimensional universe: $f = g = (\gamma\,t)^{1/2}$), we find

$$\Upsilon_m^{(p)}(\tau) = \begin{cases} J_0(\frac{3}{2}\sqrt{\tilde{m}^2 - \tilde{p}(\frac{1}{2})^2}\,\tau^{2/3})\ , \\ Y_0(\frac{3}{2}\sqrt{\tilde{m}^2 - \tilde{p}(\frac{1}{2})^2}\,\tau^{2/3})\ , \end{cases} \tag{8.149}$$

when $\tilde{m}^2 > \tilde{p}(\frac{1}{2})^2$ and

$$\Upsilon_m^{(p)}(\tau) = \begin{cases} I_0(\frac{3}{2}\sqrt{|\tilde{m}^2 - \tilde{p}(\frac{1}{2})^2|}\,\tau^{2/3})\ , \\ K_0(\frac{3}{2}\sqrt{|\tilde{m}^2 - \tilde{p}(\frac{1}{2})^2|}\,\tau^{2/3})\ , \end{cases} \tag{8.150}$$

when $\tilde{m}^2 < \tilde{p}(\frac{1}{2})^2$.

There are also closed-form solutions in the special case $\delta(q) = 0$, or $q = 2$:

$$\Upsilon_m^{(p)}(\tau) = J_{\pm\nu}(\frac{3}{2}\,\tilde{m}\,\tau^{2/3}) \ , \ \nu = \frac{3}{2}\,\tilde{p}(2) \ , \tag{8.151}$$

where $\tilde{p}(2) = \sqrt{\tilde{p}(2)^2}$ can be either real or imaginary. (If ν is an integer $J_{-\nu}$ is replaced by Y_ν. In particular, this is the case for the zero modes.)

For the range $-4 < q < 1/2$, for which $3\,\delta(q) > 4/3$, we find that the small τ behaviour is that of the zero modes, while the large τ behaviour is

$$\Upsilon_m^{(p)}(\tau) = \begin{cases} I_0(\frac{2}{3|\delta(q)|}\,\bar{p}(q)\,\tau^{3\delta(q)/2}) \ , \\ K_0(\frac{2}{3|\delta(q)|}\,\bar{p}(q)\,\tau^{3\delta(q)/2}) \ , \end{cases} \tag{8.152}$$

when $p^2 > 0$ and

$$\Upsilon_m^{(p)}(\tau) = \begin{cases} J_0(\frac{2}{3|\delta(q)|}\,\bar{p}(q)\,\tau^{3\delta(q)/2}) \ , \\ Y_0(\frac{2}{3|\delta(q)|}\,\bar{p}(q)\,\tau^{3\delta(q)/2}) \ , \end{cases} \tag{8.153}$$

when $p^2 < 0$. In both of these cases we have defined $\bar{p}(q) = \sqrt{|\tilde{p}(q)^2|}$. Outside the aforementioned range of q (but $q \neq 2$), the small τ and large τ behaviours are interchanged.

(C) *Power Law Solution with* $q = -4$:

For the power law solution with $q = -4$ we find

$$e^{\gamma\,\tau} = \gamma\,t \ , \ g \sim e^{2\gamma\,\tau} \ , \ \tilde{g} = \left(-\frac{4\,\gamma}{\mu}\right)^{-2/3} \ . \tag{8.154}$$

Again, the coordinate t ranges from 0 to ∞, as does τ, with large t corresponding to large τ.

Choosing $\mu = -4\gamma$, (8.134) becomes

$$\left(\partial_\tau^2 + m^2 - p^2 e^{6\gamma\tau}\right)\Upsilon_m^{(p)}(\tau) = 0 \ . \tag{8.155}$$

The zero mode solutions are

$$\Upsilon_m^{(0)}(\tau) = e^{\pm i m \tau} \tag{8.156}$$

In this case closed-form solutions for $p^2 \neq 0$ exist:

$$\Upsilon_m^{(p)}(\tau) = \begin{cases} I_{\pm\nu}(\frac{e^{3\gamma\tau}p}{3\gamma}) \ , & p^2 > 0 \ , \\ J_{\pm\nu}(\frac{e^{3\gamma\tau}p}{3\gamma}) \ , & p^2 < 0 \ , \end{cases} \tag{8.157}$$

where $\nu = \frac{i\,m}{3\gamma}$.

8.3.4 Stability of the Transverse, Traceless Modes

In this section we discuss the stability of the transverse, traceless modes found in the last section by considering the norm of the fluctuations $\psi^{(p)}(z)$ and the late time behaviour of $\Upsilon^{(p)}_m(t)$ as $t \to \infty$.

Normalisation of the Modes $\psi^{(p)}(z)$

According to [210], the correct measure to be used in the normalisation of the modes $\psi^{(p)}(z)$ is determined by examining the expansion of the Einstein-Hilbert term to second order in the fluctuations $\hat{h}_{AB}$. The complete expansion is given in [245]. However, for our purposes we need only perform the calculation symbolically.

Firstly, we use (7.15) to remove the conformal factor e^{2A} from the Ricci scalar:

$$\hat{R} = e^{-2A}(\tilde{R} + \cdots) \ . \tag{8.158}$$

The Einstein-Hilbert term is then

$$S_{gravity} = \frac{1}{2\hat{\kappa}_5^2} \int d^4x \int dz \sqrt{-\tilde{g}^{(0)}}\, e^{3A} \left(\tilde{R} + \cdots\right) \ , \tag{8.159}$$

where $\tilde{g}^{(0)}_{AB}$ is as defined in (8.78).

The Christoffel symbol expansion was given in (8.86) and we see that symbolically

$$\delta\Gamma \sim \tilde{g}^{(0)-1}\partial h + \cdots \ . \tag{8.160}$$

Since $\tilde{R} \sim \partial\Gamma + \Gamma^2$ we find

$$\delta\tilde{R} \sim \tilde{\nabla}^{(0)}_A h_{CD} \tilde{\nabla}^{(0)A} h^{CD} + \cdots \ , \tag{8.161}$$

where we have invoked covariance to rewrite the partial derivatives as covariant ones. Performing a transverse, traceless projection we find

$$\delta\tilde{R} \sim \frac{1}{g^3}\, \partial_a \bar{h}_{cd} \partial_a \bar{h}_{cd} + \cdots \ . \tag{8.162}$$

Combining the above expansion with (8.159) we find

$\delta^2 S_{gravity}$

$$\delta^2 S_{gravity} \sim \int d^3x\, dt\, dz\, g^{-3/2} f^{1/2}\, e^{3A} \left(\partial_a \bar{h}_{cd} \partial_a \bar{h}_{cd} + \cdots\right) \ . \tag{8.163}$$

Recalling the Fourier decomposition (8.121) and the field redefinition (8.125), we find that the modes $\psi^{(p)}(z)$ are normalised with the flat space measure dz.

We have seen in the last section that the zero mode is given by $\Psi^{(0)}(z) = constant\,(=1)$, or equivalently, $\psi^{(0)}(z) = e^{3A(z)/2}$. Therefore, the zero mode is normalisable provided

$$\int_{-\infty}^{\infty} e^{3A(z)}dz = \int_{-\infty}^{\infty} (\bar{k}|z|+1)^{3/\beta}dz < \infty \ , \tag{8.164}$$

which implies $3/\beta < -1$, or equivalently, $-\frac{1}{2} < b \cdot b < 0$. This also implies that the four-dimensional Planck mass (in the Einstein frame[8]) is then given by

$$M_{Pl}^2 = \hat{M}^3 \frac{2|\beta|/\bar{k}}{3-|\beta|} \ . \tag{8.165}$$

Note that if we formally set $|\beta| = 1$ and $\bar{k} = k$ we recover (7.30).

As far as the non-zero modes are concerned, we see from the asymptotic expansions (7.72) that the $p^2 > 0$ modes diverge as $|z| \to \infty$ and are therefore non-normalisable. On the other hand, from the large pu expansions

$$(pu)^{1/2} J_{\pm\nu}(pu) \sim \sqrt{\frac{2}{\pi}} \cos(pu \mp \frac{\pi}{2}\nu - \frac{\pi}{4}) \ , \tag{8.166}$$

$$(pu)^{1/2} Y_{\pm\nu}(pu) \sim \sqrt{\frac{2}{\pi}} \sin(pu \mp \frac{\pi}{2}\nu - \frac{\pi}{4}) \ , \tag{8.167}$$

we see that the $p^2 < 0$ modes asymptote to plane waves as $|z| \to \infty$. They are therefore plane-wave normalisable over a period as $|z| \to \infty$:

$$||\Psi^{(p)}||^2 \sim \frac{2}{p}\left[1 + \left(\frac{J_{-\nu+1}(p/\bar{k})}{J_{\nu-1}(p/\bar{k})}\right)^2\right] \ , \tag{8.168}$$

for the $p^2 < 0$ modes of (8.130) and

$$||\Psi^{(p)}||^2 \sim \frac{2}{p}\left[1 + \left(\frac{Y_{\nu-1}(p/\bar{k})}{J_{\nu-1}(p/\bar{k})}\right)^2\right] \ , \tag{8.169}$$

for the $p^2 < 0$ modes of (8.131).

---

[8]The four-dimensional metric $\bar{g}_{ij} = \text{diag}(-1, g(t), g(t), g(t))$ is defined in the so-called Jordan (or string) frame with t the comoving time. The rescaling $\bar{g}_{ij} \to f^{1/2}\bar{g}_{ij}$, $t \to \tau$ transforms the Jordan frame to the Einstein frame with τ the comoving time. The $4d$ Planck mass is time-dependent in Jordan frame but time-independent in Einstein frame.

Decay of the Modes $\Upsilon_m^{(p)}(t)$

In this section we shall concentrate on the decay of the modes $\Upsilon_m^{(p)}(t)$ (with $p^2 \leq 0$ since we have just seen above that the z-dependence makes the $p^2 > 0$ modes non-normalisable) as the comoving time $t \longrightarrow \infty$. It is these modes that determine, for instance, the amplitude of anisotropies seen in the cosmic microwave background by observers who live on the brane[9].

As noted in the last section, the large t limit generically corresponds to the the large τ limit. In the two exceptional cases of an exponentially contracting fifth dimension and a power law contracting fifth dimension with $q < -4$ the large t limit corresponds instead to the small τ limit.

We have seen previously how the case of $\dot{f} = 0$ (case I) and the $\dot{f} \neq 0$ exponential case (case IIA) can be considered as the $q \longrightarrow 0$ and $q \longrightarrow \infty$ limits respectively of the general power law case. Therefore, we shall consider cases IIB and IIC only.

The Zero Modes $\Upsilon_m^{(0)}(t)$

For case IIB (power law solutions with $q \neq -4$), using (8.148), (8.166), (8.167) and the small v expansions

$$J_\nu(v) \sim \frac{1}{\Gamma(\nu+1)} \left(\frac{v}{2}\right)^\nu \ , \quad Y_0(v) \sim \frac{2}{\pi} \ln \frac{v}{2} \ , \tag{8.170}$$

we find the amplitude:

$$|\Upsilon_m^{(0)}(\tau)| \sim \begin{cases} \tau^{-1/3} & \tau \longrightarrow \infty \quad (J_0, Y_0) \ , \\ 1 & \tau \longrightarrow 0 \quad (J_0) \ , \\ \ln \tau & \tau \longrightarrow 0 \quad (Y_0) \ , \end{cases} \tag{8.171}$$

Therefore, for power law solutions with $q > -4$, the amplitude of the J_0 mode decays to zero at late times from a constant value initially. The amplitude of the Y_0 mode also decays to zero at late times from an initially divergent value. This same behaviour occurs if the fifth dimension is exponentially expanding.

On the other hand, for $q < -4$ or an exponentially contracting fifth dimension the amplitude of the J_0 mode grows from zero initially to a constant value at late

[9] My thanks to David Wands for a discussion on this point. See also [255].

times, while the amplitude of the Y_0 mode grows from zero initially but diverges at late times.

For case IIC (power law solutions with $q = -4$), we see from (8.156) that the zero modes $\Upsilon_m^{(0)}(\tau)$ are just plane waves. Therefore, the modes oscillate at constant amplitude for all times.

The Non-Zero Modes $\Upsilon_m^{(p)}(t)$ with $p^2 < 0$

For case IIB with $-4 < q < 1/2$, we find (using (8.153)) that the amplitude has the large τ behaviour:

$$|\Upsilon_m^{(p)}(\tau)| \sim \tau^{-3\delta(q)/4} \quad (J_0, Y_0) \ , \tag{8.172}$$

while the small τ behaviour is that of the zero modes. For this range of q we find $3\delta(q) > 4/3$. Therefore,

$$|\Upsilon_m^{(p)}(\tau)| \sim \begin{cases} \tau^{-\lambda(q)} & \tau \to \infty \quad (J_0, Y_0) \ , \\ 1 & \tau \to 0 \quad (J_0) \ , \\ \ln \tau & \tau \to 0 \quad (Y_0) \ , \end{cases} \tag{8.173}$$

where $\lambda(q) = 3\delta(q)/4 > 1/3$. The interpretation is the same as that above for the zero modes with $q > -4$, albeit with decay at a more rapid rate.

For $1/2 \leq q < 2$, the large τ behaviour is that of the zero modes while the small τ behaviour is that of (8.149) or (8.153). Therefore, we find:

$$|\Upsilon_m^{(p)}(\tau)| \sim \begin{cases} \tau^{-1/3} & \tau \to \infty \quad (J_0, Y_0) \ , \\ 1 & \tau \to 0 \quad (J_0) \ , \\ \ln \tau & \tau \to 0 \quad (Y_0) \ . \end{cases} \tag{8.174}$$

The interpretation is again the same as that above for the zero modes with $q > -4$, this time with the decay being at the same rate.

For $q = 2$ we obtain (using (8.151)):

$$|\Upsilon_m^{(p)}(\tau)| \sim \begin{cases} \tau^{-1/3} & \tau \to \infty \quad (J_{\pm\nu}) \ , \\ |\tau^{2\nu/3}| = 1 & \tau \to 0 \quad (J_\nu) \ , \\ |\tau^{-2\nu/3}| = 1 & \tau \to 0 \quad (J_{-\nu}) \ , \end{cases} \tag{8.175}$$

where $\nu = \frac{3i}{2}\tilde{p}(2)$. Hence, the modes decay from a constant amplitude initially to zero at late times at the same rate as the zero modes.

For $q < -4$ or $q > 2$, we must remember that $\delta(q)$ is negative. The large τ behaviour is that of the zero modes while the small τ behaviour is that of the modes (8.153). However, because $\delta(q)$ is negative we need to use the *large* asymptotic expansions (8.166) and (8.167) in the small τ limit. Therefore, we obtain:

$$|\Upsilon_m^{(p)}(\tau)| \sim \begin{cases} \tau^{-1/3} & \tau \to \infty \quad (J_0, Y_0) \ , \\ \tau^{3|\delta(q)|/4} & \tau \to 0 \quad (J_0, Y_0) \ . \end{cases} \tag{8.176}$$

Hence, these modes initially grow from zero, reach a peak and then decay back to zero at late times. This is also true of the exponential solutions ($\delta(\infty) = -4/3$) whether the fifth dimension is expanding or contracting.

Finally, for case IIC ($q = -4$) we find (using (8.157)):

$$|\Upsilon_m^{(p)}(\tau)| \sim \begin{cases} e^{-3\gamma\tau/2} & \tau \to \infty \quad (J_{\pm\nu}) \ , \\ |e^{3\gamma\nu\tau}| = 1 & \tau \to 0 \quad (J_\nu) \ , \\ |e^{-3\gamma\nu\tau}| = 1 & \tau \to 0 \quad (J_{-\nu}) \ , \end{cases} \tag{8.177}$$

where $\nu = \frac{im}{3\gamma}$. Therefore, the modes decay from a constant amplitude initially to zero at late times.

Summary

If we characterise the stability of the space-time by a gradual decay of the normalisable perturbations from early times to late times, we conclude from the above analysis that only those space-times where the fifth dimension expands or contracts according to the power law $(\gamma t)^q$ with $-4 < q \leq 2$ are stable.

In the next section we shall narrow down this range of q even further by requiring that the null energy condition on the brane not be violated.

8.4 Energy Conditions on the Brane

In this section we examine the null energy condition both in the bulk and on the brane.

The energy conditions, encoded in the evolution of the expansion scalar governed by Raychaudhuri's equation [18, 250], are fundamental to classical general relativity. They are used in the proofs of the singularity theorems (guaranteeing, under certain circumstances, gravitational collapse), in the positive mass theorem, in the proof of the zeroth law of black hole thermodynamics (the constancy of the surface gravity over the event horizon) and in a host of other powerful mathematical theorems (see [256] and references therein). Therefore, space-times violating these conditions can have serious repercussions for physics. There are five such energy conditions: trace, strong, weak, dominant and null [257]. The weakest of these is the null energy condition (NEC) and it is usually the easiest to work with and to analyse. The standard lore is that all reasonable forms of matter should at least satisfy the NEC at least classically (quantum effects can lead to violation). Visser [258] has analysed the energy conditions and their implications for FRW space-times. (He calls matter violating the NEC "exotic".) His analysis has recently been extended to RS-inspired brane-world scenarios in [259, 260].

In section 8.3 we noted that an indefinite sigma-model metric, which gives rise to negative kinetic terms for some of the scalar fields, could lead to a violation of the energy conditions. By requiring the NEC to be satisfied on the brane, we eliminate the exponential solutions for the scale factors and restrict the range of q to $-1 \leq q \leq 2$ for the power law solutions, as we now show.

The NEC is a point-wise condition which states that

$$\hat{T}^{(0)}_{AB}\zeta^A\zeta^B \geq 0 \ , \tag{8.178}$$

for arbitrary null ζ^A (that is, $\hat{g}^{(0)}_{AB}\zeta^A\zeta^B = 0$), where $\hat{T}^{(0)}_{AB}$ is the *total* energy-momentum tensor in the background space-time. Using Einstein's equations and the fact that ζ^A is null, this condition becomes

$$\hat{R}^{(0)}_{AB}\zeta^A\zeta^B \geq 0 \ , \tag{8.179}$$

where $\hat{R}^{(0)}_{AB}$ is the Ricci tensor for the background metric $\hat{g}^{(0)}_{AB}$ defined in (8.38). Upon calculating the Ricci tensor (using Mathematica) and using the equations of motion (8.66) and (8.71) with $\tilde{\omega} = 1$, we find that the NEC becomes

$$-3\,\frac{\ddot{g}}{g}\,(\zeta^t)^2 - 3\,A''(r)\,(\zeta^r)^2 + 3\,A'(r)\,\frac{\dot{f}}{f}\,\zeta^t\zeta^r \geq 0 \ . \tag{8.180}$$

Firstly, the RS1 and RS2 models both have $g = f = 1$. Therefore, the NEC is violated unless $A''(r) \leq 0$. Since RS1 has $A = k|r|$ and RS2 has $A = -k|r|$ (k is positive in both cases) and $\frac{d^2|r|}{dr^2} = 2\delta(r)$ in the neighbourhood of $r = 0$, it follows that the NEC is not violated in the bulk (where the delta function vanishes) in either model. However, for RS1 it is violated on the brane at $r = 0$ where the delta function has support. This violation of the NEC on the brane is a serious drawback of the RS1 model whose consequences have not been fully explored in the literature to date.

Secondly, for $b \cdot b \neq 0$, if we use equation (8.65) we can rewrite (8.180) as

$$-3\frac{\ddot{g}}{g}(\zeta^t)^2 + 12b \cdot bA'(r)^2(\zeta^r)^2 + \hat{\kappa}_5^2 V_{vis}\delta(r)(\zeta^r)^2 + 3A'(r)\frac{\dot{f}}{f}\zeta^t\zeta^r \geq 0 \ , \tag{8.181}$$

where, according to section 8.3,

$$A(r) = \frac{1}{4\,b \cdot b}\ln(k|r|+1) \ , \ V_{vis} = -\frac{3k\hat{\kappa}_5^{-2}}{2\,b \cdot b} \ . \tag{8.182}$$

We are interested in the case $-1/2 < b \cdot b < 0$ with $k > 0$, so V_{vis} is positive. If we take $\zeta^t = 0$ it is easy to see that the NEC is violated in the bulk[10]. However, it is more important from the point of view of observers living on the brane that the NEC is not violated on the brane. Therefore, *on the brane* the condition becomes

$$-3\frac{\ddot{g}}{g}(\zeta^t)^2 + \hat{\kappa}_5^2 V_{vis}\delta(r)(\zeta^r)^2 \geq 0 \ , \tag{8.183}$$

where it is legitimate to drop the other two terms in comparison to the (non-vanishing) delta function. Since we already have $V_{vis} > 0$, we are left with the condition $\ddot{g} \leq 0$. This immediately rules out the exponential solutions $g \sim e^{-\gamma\, t/3}$, while for the power law solutions $g \sim (\gamma\, t)^{(2-q)/3}$, it restricts q to the range $-1 \leq q \leq 2$ (including the case $q = 0$).

Furthermore, as we have seen in the last section, for the above range of q the normalisable perturbations decay to zero at late times. In addition, if we exclude the two Kasner solutions $q = -1, 2$ by choosing

$$0 < \frac{a \cdot a}{(a \cdot b)^2} < h(q) \equiv \frac{16}{9}\frac{(2-q)(1+q)}{q^2} \ , \tag{8.184}$$

then the Schwarz inequality (8.42) is automatically satisfied[11].

[10]Some of the other RS-type scenarios not discussed in this thesis, for example, the Gregory-Rubakov-Sibiryakov model [261] also violate the NEC in the bulk [262, 263].

[11]This choice ensures that after integrating over the fifth dimension the dimensionally reduced sigma-model action becomes that of a single scalar field with a positive kinetic term.

8.5 Conclusions and Further Work

We have seen in sections 8.3 and 8.4 that a scalar field sigma-model with indefinite metric can be coupled to gravity and a bulk perfect fluid in such a way that

- the warp factor is given by

$$e^{2A(r)} = (k|r| + 1)^{2/b\cdot b} , \tag{8.185}$$

  where $-1/2 < b \cdot b < 0$, $k > 0$. It therefore decreases from a cusp on the brane at $r = 0$ to zero as $|r| \to \infty$;

- the bulk metric is given by

$$ds^2 = (k|r| + 1)^{2/b\cdot b} \left(-dt^2 + (\gamma\, t)^{(2-q)/3}\delta_{ab}dx^a dx^b\right) + (\gamma\, t)^q dr^2 , \tag{8.186}$$

  where $-1 < q < 2$ and γ is an arbitrary positive constant;

- there are no singularities in the bulk since $k > 0$;
- the tension of the brane is given by $V_{vis} = -\frac{3k\hat{\kappa}_5^{-2}}{2\, b\cdot b}$ and is positive;
- the fluid has positive density;
- conventional cosmology is obtained — the square of the Hubble parameter on the brane is proportional to the density of the fluid;
- the null energy condition on the brane is not violated;
- the zero mode $\psi^{(0)}(z) = e^{3A(z)/2}$ is normalisable with flat space measure dz;
- the four-dimensional reduced Planck mass in the Einstein frame is related to the five-dimensional one by

$$M_{Pl}^2 = \hat{M}^3 \frac{2|\beta|/\bar{k}}{3 - |\beta|} , \tag{8.187}$$

  where $|\beta| = 4|b \cdot b| + 1$ and $\bar{k} = k|\beta|/4|b \cdot b|$;

- the dimensionally-reduced sigma-model has a positive kinetic term;

- the normalisable transverse, traceless fluctuations on the brane decay to zero as $t \to \infty$.

The above points seem to suggest that gravity might be localised on the brane. Indeed, naively, the normalisability of the zero mode wave function is equivalent to localisation of gravity [210, 241] (at least classically — quantum effects might cause delocalisation; such quantum effects are a topic for future research). However, to really prove that gravity is localised we need to examine the h_{00} fluctuation in much the same way as we did in section 7.5.2. Unfortunately, as we noted in section 8.3.3, the scalar perturbations couple to the intrinsic perturbations of the scalar fields and the fluid and, therefore, we must solve a complicated set of coupled differential equations. Such an investigation deserves further attention.

We have seen that other power law solutions and exponentially inflating/deflating solutions exist for the expansion factor $g(t)$. However, these solutions generically violate the null energy condition on the brane. Such violations are often associated with weird physics (e.g. wormholes) and might be worthy of further investigation.

From equations (8.51)–(8.60), it is clear that under the transformation $f \to -f$ the potentials U and V change sign but otherwise the analysis is unmodified. Thus, it is possible to make the extra dimension timelike rather than spacelike. Such a possibility was alluded to in [171, 264] and its physical implications might be interesting.

Finally, it would be interesting to see if our model can be embedded in a five-dimensional supergravity, as has recently been done for the RS models [265, 266] (see also [267] and references therein).

Appendices

Appendix A

Spinor Conventions

With C the charge conjugation matrix, we adopt the following conventions for raising and lowering spinor indices:

$$\lambda^\alpha = C^{\alpha\beta}\lambda_\beta \ , \ \ \lambda_\alpha = C_{\alpha\beta}\lambda^\beta \ , \ \ C_{\alpha\beta}C^{\beta\gamma} = \delta_\alpha^\gamma \ , \tag{A.1}$$

and for both $SO(1,9)$ and $SO(10)$ we have the following:

$$C^T = C^{-1} = -C \ , \ \ \Gamma^{\mu T} = -C\Gamma^\mu C^{-1} \ . \tag{A.2}$$

With these conventions, C is given by $C^{\alpha\beta}$, C^{-1} by $C_{\alpha\beta}$ and the index structure on the gamma matrices is $(\Gamma^\mu)_\alpha{}^\beta$. Therefore, $-\Gamma^\mu C^{-1}$ is given by $(\Gamma^\mu)_{\alpha\beta} = C_{\beta\gamma}(\Gamma^\mu)_\alpha{}^\gamma = (\Gamma^\mu)_{\beta\alpha}$ and $\Gamma^{\mu T}$ is given by $(\Gamma)^\alpha{}_\beta$. Note also that $(\Gamma^\mu)_\alpha{}^\beta\lambda_\beta = -(\Gamma^\mu)_{\alpha\beta}\lambda^\beta$. Furthermore, the gamma matrices act on the Ramond vacuum of the superstring as $(\Gamma^\mu)_\alpha{}^\beta|0;\beta\rangle_R$. We define $\Gamma = \Gamma^0\cdots\Gamma^9$. It satisfies $\Gamma^2 = 1, (\Gamma)_{\alpha\beta} = (\Gamma)_{\beta\alpha}$ and $\Gamma^T = -C\Gamma C^{-1}$.

Appendix B

Differential Forms

We give here some conventions on k-forms in a 10-dimensional space-time of Lorentzian signature.

A k-form is defined by

$$H_k = \frac{1}{k!}(H_k)_{\mu_1\dots\mu_k} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k} , \tag{B.1}$$

where $(H_k)_{\mu_1\dots\mu_k}$ is a totally antisymmetric tensor and $\wedge$ the standard wedge product: $dx^{\mu_1} \wedge dx^{\mu_2} = -dx^{\mu_2} \wedge dx^{\mu_1}$. The exterior derivative d (which satisfies $d^2 = 0$) takes a k form to a $(k+1)$-form:

$$(dH_k)_{\nu\mu_1\dots\mu_k} = (k+1)\partial_{[\nu}(H_k)_{\mu_1\dots\mu_k]} , \tag{B.2}$$

and the following properties hold

$$H_k \wedge H_l = (-1)^{kl} H_l \wedge H_k , \quad d(H_k \wedge H_l) = dH_k \wedge H_l + (-1)^k H_k \wedge dH_l . \tag{B.3}$$

The Hodge dual is the $(10-k)$-form with components

$$(*H_k)_{\nu_1\dots\nu_{10-k}} = \frac{\sqrt{|G|}}{k!}(H_k)_{\mu_1\dots\mu_k} \epsilon^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_{10-k}} , \tag{B.4}$$

In the above formula, $\epsilon_{\mu_1\dots\mu_{10}}$ is the totally antisymmetric Levi-Civita symbol with $\epsilon_{01\dots9} = 1$. The indices on ϵ are raised with the inverse metric $G^{\mu\nu}$ (equal to $\eta^{\mu\nu}$ on page 103) and acting on a k-form we have $** = (-1)^{k(10-k)+1}$. A 5-form is selfdual if $H_5 = *H_5$ and anti selfdual if $H_5 = - * H_5$.

We define integration of a k-form over a k-dimensional orientable manifold M as

$$\int_M H_k \equiv \int_M d^k x (H_k)_{1\ldots k} \ , \tag{B.5}$$

where $x^1, \ldots, x^k$ form a right-handed coordinate system on M. Finally, for a manifold with boundary we have Stoke's theorem:

$$\int_M dH_{k-1} = \int_{\partial M} H_{k-1} \ . \tag{B.6}$$

Appendix C

Dirac Matrix Algebra for Section 5.2.2

In this appendix we outline the Dirac algebra involved in calculating the amplitudes of section 5.2.2.

C.1 Trace for the Two-Point Amplitude

We have seen that the two-point amplitude $\mathcal{A}^{T,RR}$ trivially vanishes because of the trace over the CP factors (5.4). There is, in fact, a separate and independent reason why the amplitude is zero — the trace over the spinor indices vanishes, as we now show[1].

Explicit consideration of the two-point amplitude results in it being proportional to

$$\mathrm{Tr}\,(P_- \not{H}_{(k)} M). \tag{C.1}$$

Let us write the matrix M defined in (4.304) in the alternative form:

$$M = \begin{cases} \frac{1}{(p+1)!}\epsilon_{a_0 \ldots a_p}\Gamma^{a_0}\ldots\Gamma^{a_p} \ , & (p \text{ even}) \ , \\ \frac{1}{(p+1)!}\epsilon_{a_0 \ldots a_p}\Gamma^{a_0}\ldots\Gamma^{a_p}\Gamma \ , & (p \neq -1 \text{ odd}) \ , \\ i\Gamma \ , & (p = -1) \ , \end{cases} \tag{C.2}$$

[1]This section is required in order to calculate a similar trace for the three-point amplitude (see section C.2).

where a_i ranges over the indices $0, \ldots, p$ and $\epsilon_{a_0 \ldots a_p}$ is totally antisymmetric with $\epsilon_{01 \ldots p} = 1$. Recall from section 4.2.2 and section 5.1.1 that there is an ambiguity in the sign of M. However, this sign is of no consequence here and without loss of generality we have chosen it to be positive. Now, since $\Gamma P_- = -P_- = -(1-\Gamma)/2$, we have

$$\mathrm{Tr}\,(P_- \not{H}_{(k)} M) \propto \begin{cases} \frac{1}{k!}\mathrm{Tr}\,((1-\Gamma)\Gamma^{\mu_1} \cdots \Gamma^{\mu_k}\Gamma^{a_0} \cdots \Gamma^{a_p})(H_k)_{\mu_1 \ldots \mu_k}\epsilon_{a_0 \ldots a_p}, & (p \neq -1), \\ \frac{1}{k!}\mathrm{Tr}\,((1-\Gamma)\Gamma^{\mu_1} \cdots \Gamma^{\mu_k})(H_k)_{\mu_1 \ldots \mu_k}, & (p = -1). \end{cases} \tag{C.3}$$

We see that the trace effectively splits into two parts. The first part is proportional to

$$\begin{aligned} &\mathrm{Tr}\,(\Gamma^{\mu_1} \cdots \Gamma^{\mu_k}\Gamma^{a_0} \cdots \Gamma^{a_p})(H_k)_{\mu_1 \ldots \mu_k}\epsilon_{a_0 \ldots a_p} \ , \quad (p \neq -1) \ , \\ &\mathrm{Tr}\,(\Gamma^{\mu_1} \cdots \Gamma^{\mu_k})(H_k)_{\mu_1 \ldots \mu_k} \ , \quad (p = -1) \ , \end{aligned} \tag{C.4}$$

whilst the second is proportional to

$$\begin{aligned} &-(-1)^{k(k+1)/2}\mathrm{Tr}\,(\Gamma^{\nu_1} \cdots \Gamma^{\nu_{10-k}}\Gamma^{a_0} \cdots \Gamma^{a_p})(*H_k)_{\nu_1 \ldots \nu_{10-k}}\epsilon_{a_0 \ldots a_p} \ , \quad (p \neq -1) \ , \\ &-(-1)^{k(k+1)/2}\mathrm{Tr}\,(\Gamma^{\nu_1} \cdots \Gamma^{\nu_{10-k}})(*H_k)_{\nu_1 \ldots \nu_{10-k}} \ , \quad (p = -1) \ , \end{aligned} \tag{C.5}$$

where we have used the gamma matrix identity (4.270) and the definition of the Hodge dual, equation (B.4).

Step One

Since the trace of any antisymmetrised product of gamma matrices is zero, it is clear that both parts vanish separately for $p = -1$ and $1 \leq k \leq 5$.

Step Two

Consider (C.4) for $p \neq -1$. Repeated use of $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$ yields

$$\begin{aligned} &\mathrm{Tr}\,(\Gamma^{\mu_1} \cdots \Gamma^{\mu_k}\Gamma^{a_0} \cdots \Gamma^{a_p})\,(H_k)_{\mu_1 \ldots \mu_k}\epsilon_{a_0 \ldots a_p} \\ &= (-1)^p[2\eta^{\mu_1 a_0}\mathrm{Tr}\,(\Gamma^{\mu_2} \cdots \Gamma^{\mu_k}\Gamma^{a_1} \cdots \Gamma^{a_p}) \\ &\qquad -\mathrm{Tr}\,(\Gamma^{\mu_1}\Gamma^{a_0}\Gamma^{\mu_2} \cdots \Gamma^{\mu_k}\Gamma^{a_1} \cdots \Gamma^{a_p}]\,(H_k)_{\mu_1 \ldots \mu_k}\epsilon_{a_0 \ldots a_p} \\ &= (-1)^p[2\eta^{\mu_1 a_0}\mathrm{Tr}\,(\Gamma^{\mu_2} \cdots \Gamma^{\mu_k}\Gamma^{a_1} \cdots \Gamma^{a_p}) - 2\eta^{\mu_2 a_0}\mathrm{Tr}\,(\Gamma^{\mu_1}\Gamma^{\mu_3} \cdots \Gamma^{\mu_k}\Gamma^{a_1} \cdots \Gamma^{a_p}) \\ &\qquad +\mathrm{Tr}\,(\Gamma^{\mu_1}\Gamma^{\mu_2}\Gamma^{a_0}\Gamma^{\mu_3} \cdots \Gamma^{\mu_k}\Gamma^{a_1} \cdots \Gamma^{a_p}](H_k)_{\mu_1 \ldots \mu_k}\epsilon_{a_0 \ldots a_p} \ . \end{aligned} \tag{C.6}$$

The first two terms in the third line above are equal due to the antisymmetry of the components of H_k. Continuing in this way we can return Γ^{a_0} to its original position by commuting it with all the Γ^{μ_i}. The overall sign is then $(-1)^{p+k}$ and must be negative if we are to get a non-zero result since the trace of an odd number of gamma matrices is zero. The final term can then be brought over to the left hand side and so we obtain

$$\begin{aligned}&\mathrm{Tr}\left(\Gamma^{\mu_1}\cdots\Gamma^{\mu_k}\Gamma^{a_0}\cdots\Gamma^{a_p}\right)(H_k)_{\mu_1\ldots\mu_k}\epsilon_{a_0\ldots a_p}\\&\quad=(-1)^p k\eta^{\mu_1 a_0}\mathrm{Tr}\left(\Gamma^{\mu_2}\cdots\Gamma^{\mu_k}\Gamma^{a_1}\cdots\Gamma^{a_p}\right)(H_k)_{\mu_1\ldots\mu_k}\epsilon_{a_0\ldots a_p}\ .\end{aligned} \tag{C.7}$$

We now do the same with Γ^{a_1} on the right hand side of (C.7) and obtain

$$\begin{aligned}&\mathrm{Tr}\left(\Gamma^{\mu_1}\cdots\Gamma^{\mu_k}\Gamma^{a_0}\cdots\Gamma^{a_p}\right)(H_k)_{\mu_1\ldots\mu_k}\epsilon_{a_0\ldots a_p}\\&\quad=(-1)^{p(p-1)}k(k-1)\eta^{\mu_1 a_0}\eta^{\mu_2 a_1}\mathrm{Tr}\left(\Gamma^{\mu_3}\cdots\Gamma^{\mu_k}\Gamma^{a_2}\cdots\Gamma^{a_p}\right)(H_k)_{\mu_1\ldots\mu_k}\epsilon_{a_0\ldots a_p}\ .\end{aligned} \tag{C.8}$$

We keep going until all the Γ^{a_i} or Γ^{μ_i} are used up. However, since the trace of any antisymmetrised product of gamma matrices is zero, it is clear that we get a non-zero result only if $k = p+1$. Therefore, we end up with

$$\begin{aligned}&\mathrm{Tr}\left(\Gamma^{\mu_1}\cdots\Gamma^{\mu_k}\Gamma^{a_0}\cdots\Gamma^{a_p}\right)(H_k)_{\mu_1\ldots\mu_k}\epsilon_{a_0\ldots a_p}\\&\quad=(-1)^{p(p+1)/2}k!\delta_{k,p+1}\eta^{\mu_1 a_0}\eta^{\mu_2 a_1}\cdots\eta^{\mu_{k-1}a_{p-1}}\mathrm{Tr}\left(\Gamma^{\mu_k}\Gamma^{a_p}\right)(H_k)_{\mu_1\ldots\mu_k}\epsilon_{a_0\ldots a_p}\\&\quad=32(-1)^{p(p+1)/2}(p+1)!\delta_{k,p+1}(H_k)^{a_0\ldots a_p}\epsilon_{a_0\ldots a_p}\ ,\end{aligned} \tag{C.9}$$

where the factor 32 results from the fact that the gamma matrices are of dimension 32×32 in 10 space-time dimensions.

Now consider (C.5) for $p\neq -1$. It is clear that a calculation similar to that above results in

$$\begin{aligned}&-(-1)^{k(k+1)/2}\mathrm{Tr}\left(\Gamma^{\nu_1}\cdots\Gamma^{\nu_{10-k}}\Gamma^{a_0}\cdots\Gamma^{a_p}\right)(*H_k)_{\nu_1\ldots\nu_{10-k}}\epsilon_{a_0\ldots a_p}\\&\quad=-32(-1)^{p(p+1)/2}(-1)^{k(k+1)/2}(p+1)!\delta_{10-k,p+1}(*H_k)^{a_0\ldots a_p}\epsilon_{a_0\ldots a_p}\\&\quad=32(-1)^{p(p+1)/2}(p+1)!\delta_{10-k,p+1}(H_{10-k})^{a_0\ldots a_p}\epsilon_{a_0\ldots a_p}\ ,\end{aligned} \tag{C.10}$$

where we have used the duality relation (4.271).

Why the Two-Point Amplitude Vanishes

From (C.9) and (C.10) we see that the trace (C.3) vanishes unless there exists a field strength of degree $p+1$. However, no such field strength exists in type IIA/B theory for p even/odd. Therefore, we conclude that $\mathcal{A}^{T,RR}=0$.

C.2 Trace for the Three-Point Amplitude

The trace over spinor indices involved in (5.73) is:

$$\begin{aligned}
&\mathrm{Tr}\,(P_- \not{H}_{(k)} M \Gamma^i) \\
&= \begin{cases} \frac{1}{2}\frac{1}{k!(p+1)!}\mathrm{Tr}\,((1-\Gamma)\Gamma^{\mu_1}\cdots\Gamma^{\mu_k}\Gamma^{a_0}\cdots\Gamma^{a_p}\Gamma^i)(H_k)_{\mu_1\ldots\mu_k}\epsilon_{a_0\ldots a_p}\ , & (p\neq -1), \\ \frac{i}{2}\frac{1}{k!}\mathrm{Tr}\,((1-\Gamma)\Gamma^{\mu_1}\cdots\Gamma^{\mu_k}\Gamma^i)(H_k)_{\mu_1\ldots\mu_k}\ , & (p=-1). \end{cases}
\end{aligned} \tag{C.11}$$

Since i and a_j both range over the indices $0,\ldots,p$, the matrix Γ^i will annihilate one of the matrices Γ^{a_j} when $p\neq -1$. Therefore, according to our discussion of the trace in the case of the two-point amplitude, we should expect (C.11) to vanish unless $k=p$ or $10-k=p$ whenever $p\neq -1$. This is in fact true, as we now show.

Step One

For $p=-1$, we can use the results (C.9) and (C.10) directly by putting $p=0$ and substituting the index i for a_0. We obtain

$$\mathrm{Tr}\,(P_- \not{H}_{(k)} M \Gamma^i) = 16i\,[\delta_{k,1}(H_k)^i + \delta_{10-k,1}(H_{10-k})^i]\ . \tag{C.12}$$

Step Two

For $p\neq -1$, let us consider the first part of (C.11), that is, the part not containing Γ. It is proportional to

$$\mathrm{Tr}\,(\Gamma^{\mu_1}\cdots\Gamma^{\mu_k}\Gamma^{a_0}\cdots\Gamma^{a_p}\Gamma^i)(H_k)_{\mu_1\ldots\mu_k}\epsilon_{a_0\ldots a_p}\ . \tag{C.13}$$

Clearly, this is non-vanishing only if $k+p+1$ is odd. By commuting Γ^i with each Γ^{μ_j} we obtain in much the same way as we obtained (C.6),

$$\begin{aligned}
&\mathrm{Tr}\,(\Gamma^{\mu_1}\cdots\Gamma^{\mu_k}\Gamma^{a_0}\cdots\Gamma^{a_p}\Gamma^i)(H_k)_{\mu_1\ldots\mu_k}\epsilon_{a_0\ldots a_p} \\
&\quad = [2k\eta^{\mu_1 i}\mathrm{Tr}\,(\Gamma^{\mu_2}\cdots\Gamma^{\mu_k}\Gamma^{a_0}\cdots\Gamma^{a_p}) \\
&\qquad +(-1)^k\mathrm{Tr}\,(\Gamma^{\mu_1}\cdots\Gamma^{\mu_k}\Gamma^i\Gamma^{a_0}\cdots\Gamma^{a_p})](H_k)_{\mu_1\ldots\mu_k}\epsilon_{a_0\ldots a_p}\ .
\end{aligned} \tag{C.14}$$

We now commute Γ^i with each Γ^{a_j} until Γ^i ends up in its original position with a overall sign $(-1)^{k+p+1}=-1$. Bringing this final term over to the left hand side and using the antisymmetry of $\epsilon_{a_0\ldots a_p}$ we therefore find

$$\begin{aligned}
&\mathrm{Tr}\,(\Gamma^{\mu_1}\cdots\Gamma^{\mu_k}\Gamma^{a_0}\cdots\Gamma^{a_p}\Gamma^i)(H_k)_{\mu_1\ldots\mu_k}\epsilon_{a_0\ldots a_p} \\
&\quad = [k\eta^{\mu_1 i}\mathrm{Tr}\,(\Gamma^{\mu_2}\cdots\Gamma^{\mu_k}\Gamma^{a_0}\cdots\Gamma^{a_p}) \\
&\qquad +(-1)^k(p+1)\eta^{a_0 i}\mathrm{Tr}\,(\Gamma^{\mu_1}\cdots\Gamma^{\mu_k}\Gamma^{a_1}\cdots\Gamma^{a_p})](H_k)_{\mu_1\ldots\mu_k}\epsilon_{a_0\ldots a_p}\ .
\end{aligned} \tag{C.15}$$

Now we use the result (C.9) in each of the two terms on the right hand side above to obtain

$$
\begin{aligned}
&\mathrm{Tr}\,(\Gamma^{\mu_1}\cdots\Gamma^{\mu_k}\Gamma^{a_0}\cdots\Gamma^{a_p}\Gamma^{i})(H_k)_{\mu_1\ldots\mu_k}\epsilon_{a_0\ldots a_p}\\
&=32[(-1)^{p(p+1)/2}k!\delta_{k-1,p+1}\eta^{\mu_1 i}\eta^{\mu_2 a_0}\cdots\eta^{\mu_k a_p}\\
&\qquad +32(-1)^{k}(-1)^{(p-1)p/2}(p+1)!\delta_{k,p}\eta^{a_0 i}\eta^{\mu_1 a_1}\cdots\eta^{\mu_k a_p}](H_k)_{\mu_1\ldots\mu_k}\epsilon_{a_0\ldots a_p}\\
&=32(-1)^{p(p+1)/2}[(p+2)!\delta_{k,p+2}(H_k)^{i a_0\ldots a_p}\epsilon_{a_0\ldots a_p}+(p+1)!\delta_{k,p}(H_k)^{a_1\ldots a_p}\eta^{a_0 i}\epsilon_{a_0\ldots a_p}].
\end{aligned}
\tag{C.16}
$$

The first term in the last line above is zero because i and a_j both range over the indices $0,\ldots,p$. Hence, we finally obtain

$$
\begin{aligned}
&\mathrm{Tr}\,(\Gamma^{\mu_1}\cdots\Gamma^{\mu_k}\Gamma^{a_0}\cdots\Gamma^{a_p}\Gamma^{i})(H_k)_{\mu_1\ldots\mu_k}\epsilon_{a_0\ldots a_p}\\
&\quad=32(-1)^{p(p+1)/2}(p+1)!\delta_{k,p}(H_k)^{a_1\ldots a_p}\epsilon_{a_0\ldots a_p}\eta^{a_0 i}\ .
\end{aligned}
\tag{C.17}
$$

Consider now the second part of (C.11) for $p\neq -1$, that is, the part containing Γ. It is clear that a similar calculation to that above results in

$$
\begin{aligned}
&-\tfrac{1}{k!}\mathrm{Tr}\,(\Gamma\Gamma^{\mu_1}\cdots\Gamma^{\mu_k}\Gamma^{a_0}\cdots\Gamma^{a_p}\Gamma^{i})(H_k)_{\mu_1\ldots\mu_k}\epsilon_{a_0\ldots a_p}\\
&\quad=-32(-1)^{p(p+1)/2}(-1)^{k(k+1)/2}\tfrac{(p+1)!}{(10-k)!}\delta_{10-k,p}(*H_k)^{a_1\ldots a_p}\epsilon_{a_0\ldots a_p}\eta^{a_0 i}\\
&\quad=32(-1)^{p(p+1)/2}\tfrac{(p+1)!}{(10-k)!}\delta_{10-k,p}(H_{10-k})^{a_1\ldots a_p}\epsilon_{a_0\ldots a_p}\eta^{a_0 i}\ ,
\end{aligned}
\tag{C.18}
$$

where we have used the duality relation (4.271).

Therefore, we arrive at the final form for the trace involved in the three-point amplitude (5.73) for $p\neq -1$:

$$
\begin{aligned}
&\mathrm{Tr}\,(P_{-}\not{H}_{(k)}M\Gamma^{i})\\
&\quad=\tfrac{16}{p!}(-1)^{p(p+1)/2}\,[\delta_{k,p}(H_k)^{a_1\ldots a_p}\epsilon_{a_0\ldots a_p}+\delta_{10-k,p}(H_{10-k})^{a_1\ldots a_p}\epsilon_{a_0\ldots a_p}]\eta^{a_0 i}\ .
\end{aligned}
\tag{C.19}
$$

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