Aspects of Chern–Simons Theory

by

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A thesis submitted to the School of Mathematics, Trinity College,

University of Dublin, for the degree of

Doctor of Philosophy

September 1999

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Declaration

This thesis is entirely my own work. No part of it has been previously submitted as an exercise for any degree at any university. This thesis is based on 4 published or in process of publication papers. They are fully integrated into the body of the thesis.

Emil M. Prodanov

Acknowledgements

I am sincerely grateful to my supervisor Prof. Siddhartha Sen for his help, advice and encouragement in both academic and non-academic aspects. I would like to thank Dr. David Adams for many useful discussions. Thanks also to Dr. Brian Dolan, Prof. Lochlain O'Raifeartaigh, Prof. Charles Nash. Thanks to Alan, Alex, Alfredo, Alison, Beatrice, Carlos, Colm, Conall, David, Fabian, James, Maurice, Oliver, Samik, Tom for the good time. The financial support of Trinity College Dublin and FORBAIRT is gratefully acknowledged. Thanks to my family for their patience and understanding.

Contents

General Introduction

This thesis is based on 4 papers resulting from my work during my stay in the School of Mathematics, Trinity College Dublin. Each of them forms an individual part of the thesis. The relation between these parts is emphasized throughout the thesis. The last, fifth, part is based on a somewhat exercise in operator formalism in Chern–Simons theory where I show, by analysis based on theta–functions, that the co-efficient must be even.

In each part of the thesis the published work is the last section (the last two sections for part III). All introductory material, together with some background and aspects of relations with other theories, is given in the sections preceeding the final one. I have tried to be most concise, avoiding unnecessary details. For instance, the definition of a p-form is given for the only purpose to state that we will avoid the numerical normalization factor in it and to set up a notation. As I am a physicist, whenever in the introductory sections a mathematical text seems unavoidable, I have tried to alternate it with physical (sometimes more hand–waving than necessary) arguments.

In the first part of the thesis I show that on three–dimensional Riemannian manifolds without boundaries and with trivial first real de Rham cohomology group (and in no other dimensions) scalar field theory and Maxwell theory are equivalent: the ratio of the partition functions is given by the Ray–Singer torsion of the manifold. In the presence of interaction with external currents, this equivalence persists provided there is a fixed relation between the charges and the currents.

In the next part, with ideology based on the first one, I explicitly obtain, using a group–invariant version of the Faddeev–Popov method, the partition functions of the Self–Dual Model and Maxwell–Chern–Simons theory. I show that their ratio coincides with the partition function of Abelian Chern–Simons theory to within a phase factor depending on the geometrical properties of the manifold.

Still in the same spirit, in part three I give an alternative evaluation of the partition function of Schwarz's topological field theory which results in 1. (The standard evaluation results in the Ray–Singer analytic torsion.) Mathematically, this amounts to a novel perspective on analytic torsion: it can be formaly written as a ratio of volumes of spaces of differential forms which is formally equal to 1 by Hodge duality.

In part four I show that when the induced parity breaking part of the effective action for the low–momentum region of U(1) $\times \ldots \times$ U(1) Maxwell gauge field theory with massive fermions in 2+1 dimensions is coupled to a ϕ^4 scalar field theory, it is not possible to eliminate the screening of the long-range Coulomb interactions and get external charges confined in the broken Higgs phase. This result is valid for non-zero temperature as well. This induced term, at zero temperature, is nothing else, but the eminent Chern–Simons term.

The papers which form this thesis are:

1. Emil M. Prodanov and Siddhartha Sen: *Abelian Duality*, hep-th/9906143, submitted to Physical Review **D**.

2. Emil M. Prodanov and Siddhartha Sen: *Equivalence of the Self-Dual Model and Maxwell–Chern–Simons Theory on Arbitrary Manifolds*, hep-th/9801026, Physical Review **D**59, 065019 (1999).

3. Emil M. Prodanov and David H. Adams: *A Remark on Schwarz's Topological Field Theory*, submitted to Letters in Mathematical Physics.

4. Emil M. Prodanov and Siddhartha Sen: *Absence of Cross–Confinement for Dynamically Generated Multi–Chern–Simons Theories*, hep-th/9810044, Physics Letters **B**445, 112–116 (1998).

I. Abelian Duality

1 Introduction

There has been recent interest in relating different theories and establishing their equivalence. Common to all applications of the different aspects of the notion of duality is the observation that when two different theories are dual to each other, then either the manifolds are changed or the fields and the coupling constants are related.

In this part ot the thesis we re-examine two simple systems — scalar field theory and Maxwell theory on three–dimensional Riemannian manifolds without boundaries and with trivial first homotopy group $\pi_1(\mathrm{M})$. We show the equivalence between these theories and we give the condition which must be satisfied by the charges and the external currents in order that the equivalence persists at the level of interactions. This is done by a direct calculation of the partition function of each theory paying particular attention to the fine structure of the zero–mode sector. In the spirit of Schwarz's method of invariant integration [1] we show that the ratio of the partition functions of the theories is equal to the square of the partition function of Chern–Simons theory (or the partition function of BF theory, that is, $\mathrm{U}(1) \times \mathrm{U}(1)$ Chern–Simons theory with purely off-diagonal coupling). Such equivalence between a scalar and vector theory is a novel form of duality which we call Abelian Duality. We show that when the coupling constants (overall scaling factors) are related as $\mathrm{R} \longleftrightarrow 1/\mathrm{R}$, then this Abelian Duality transforms into $\mathrm{R} \longleftrightarrow 1/\mathrm{R}$ duality. In this case the ratio of the partition functions is given by a topological invariant — the Ray–Singer torsion of the manifold [2]. We show how our results can be obtained by Schwarz's resolvent method [3] and we use a resolvent generated by the de Rham complex to comment on possibilities of equivalence between the theories in other dimensions. In our considerations we use zeta–regularised determinants.

The ingredients of our theory are as follows:

In this part of the thesis we shall be interested in homology 3-spheres, that is, 3-dimensional compact oriented connected Riemannian manifolds, without boundaries and with trivial first real de Rham cohomology group, e.g. $\mathbf{S}^3$ or the lens spaces $\mathrm{L}(p,0)$, $p = 0, 1, 2, \ldots$. Let us now explain why.

For Riemannian manifolds the metric (which is a symmetric matrix, due to the fact that the scalar product is commutative, and therefore has real eigenvalues only) has only positive eigenvalues. By a suitable diagonalization with an orthogonal matrix and rescaling of the basis vectors we can obtain Euclidean metric $\delta = \mathrm{diag}\,(1, \ldots, 1)$. We need Euclidean actions in order to avoid problems with convergence in the path integral of the theories considered (convergence in Minkowski's space depends on the fact that the integrand in the partition function is oscillating) — in Euclidean spaces the exponent in the integrand is negative definite and the integral converges.

The manifold we need should also be without a boundary. We would like to avoid problems with leftover after integration by parts. Our analysis would be applicable for the case when there are boundary terms present, but would also be more technical.

The reason for demanding non-trivial first real de Rham cohomology group is that we shall be dealing with Maxwell field theory and to be able to write the Maxwell tensor F as $d_1 A$ globally. The Maxwell equation $d_2 F = 0$ implies that F is an element in the second de Rham cohomology group* $\mathrm{H}^2_{\mathrm{dR}}(\mathrm{M})$. A is a one-form and therefore $d_1 A = 0$ in $\mathrm{H}^2_{\mathrm{dR}}(\mathrm{M})$, that is, if $F = d_1 A$ then the equivalence class $[F]$ is zero in $\mathrm{H}^2_{\mathrm{dR}}(\mathrm{M})$ (i.e. $[F] = [F'] \Longleftrightarrow F = F' + d_1 A$). When the second de Rham cohomology group is trivial, then $F = d_1 A$ is valid globally. In three dimensions $\mathrm{H}^2_{\mathrm{dR}}(\mathrm{M})$ is isomorphic to $\mathrm{H}^1_{\mathrm{dR}}(\mathrm{M})$ due to Hodge duality and $\mathrm{H}^1_{\mathrm{dR}}(\mathrm{M})$ being trivial means that the first homotopy group $\pi_1(\mathrm{M})$ is trivial (then $F = d_1 A$ globally). If $\pi_1(\mathrm{M})$ is non-trivial, then $F = d_1 A$ is valid only on contractible regions of the manifold. Our analysis is perfectly well suited to handle the case of manifolds with non-trivial homology — then Maxwell theory and scalar field theory would be patch–by–patch

*A summary of some basic notions of a manifold and of topological ideas are contained in the next section.

equivalent.

We will show the equivalence between Maxwell theory and scalar field theory for homology 3-spheres (the physically interesting manifolds). For the general case we would like to refer the reader to [4] where Witten has shown how to pass from scalar field theory to Maxwell theory and vice versa in two and three dimensions. As a compensation, we would like to offer many additional features emerging from our analysis of the equivalence between these two theories.

Even though we are considering two simple systems — scalar field theory and U(1) gauge field theory (Maxwell theory) which are free or have interactions with external currents, the manifolds will be arbitrary simply–connected and curved. The statement that the ratio of the partition functions of the two theories is a topological object is probably a generic feature of most "field theory equivalence" theorems. We certainly come across this feature in other examples. Scalar field theory and Maxwell theory play a fundamental role in the contemporary understanding of interactions and symmetries. We shall be dealing with systems with infinitely many degrees of freedom. Crucial role in such system is played by the symmetry of the model — either generating family of solutions by leaving the dynamical equations invariant, or leading to conservation of charges, energies, momenta, etc. These symmetries might be either geometrical transformations of space and time, or internal — not depending on space and time.

Another ingredients are the Chern–Simons theory [5] — a topological field theory of Schwarz type (to be introduced shortly) — and the calculation of its partition function (á priori a formal, mathematically ill-defined quantity) by formal manipulations which in the end lead to a mathematically meaningful result — the Ray–Singer torsion [2] of the manifold. (This analytic torsion will play a key role in the third part of the thesis.) The formal manipulations will be Shwarz's method of invariant integration [1] and Schwarz's resolvent method [3]. They both generalize the Faddeev–Popov trick [6] for the case when the group of gauge transformations does not act freely, that is, when ghosts–for–ghosts have to be included on the same basis as the ghosts themselves — to restict the gauge freedom by picking up only one representative of each orbit of the group of gauge transformations.

In the next few sections we give some background material.

2 Elements of Hodge-de Rham Theory

All operators entering our theory can be described by the following diagram:

$$\begin{array}{ccc} \Omega^p(\mathrm{M}) & \xrightarrow{\ d_p\ } & \Omega^{p+1}(\mathrm{M}) \\ \uparrow * & & \downarrow * \\ \Omega^{m-p}(\mathrm{M}) & \xrightarrow{\ d^\dagger_{m-p-1}\ } & \Omega^{m-p-1}(\mathrm{M}) \end{array} \tag{1}$$

where M is the manifold and $m = \dim$ M. The case of interest will be a three-dimensional manifold. Here $\Omega^p_q(\mathrm{M})$, or $\wedge^p \mathrm{T}^*_q(\mathrm{M})$, is the space of p-forms $\big(\mathrm{T}^*_q(\mathrm{M})$ is the dual space of the tangent space $\mathrm{T}_q(\mathrm{M})$ at point $q \in \mathrm{M}\big)$. (For an excellent introduction to geometry and topology see [7].) Let us accept the convention to ignore the numerical factor $\frac{1}{p!}$ from the definition of a p-form:

$$\Omega^p(\mathrm{M}) \ni \omega^p = \frac{1}{p!}\, \omega_{\mu_1\mu_2\cdots\mu_p}\, dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p}. \tag{2}$$

We will always write a subscript on the differential operator d (the exterior derivative) in order to keep track of the order of the forms it acts on (this is extremely important for our further analysis). In other words, we have:

$$\Omega^p(\mathrm{M}) \ni \omega^p \overset{d_p}{\longmapsto} \psi^{p+1} \in \Omega^{p+1}(\mathrm{M}), \tag{3}$$

where

$$\psi^{p+1} = d_p\, \omega^p = \Big(\frac{\partial}{\partial x^\nu}\omega_{\mu_1\mu_2\cdots\mu_p}\Big)\, dx^\nu \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p} \tag{4}$$

(the numerical factor is suppressed).

Let us also suppress the numerical factors from the definition of the Hodge star operator

$$* \,\omega^p = \frac{\sqrt{|g|}}{p!(m-p)!}\,\omega_{\mu_1\cdots\mu_p}\,\epsilon^{\mu_1\cdots\mu_p}{}_{\nu_{p+1}\cdots\nu_m}\,dx^{\nu_{p+1}}\wedge\ldots\wedge dx^{\nu_m} \tag{5}$$

and the sign from the identity map (for Riemannian manifolds)

$$* * \,\omega^p = (-1)^{p(m-p)}\,\omega^p. \tag{6}$$

Thus the adjoint exterior derivative will be given, modulo possible sign, by:

$$d^\dagger_{m-p-1} = * \, d_p \, * \,. \tag{7}$$

We will write (note the difference in the notation):

$$\Omega^{p+1}(\mathrm{M}) \ni \omega^{p+1} \overset{d_p^\dagger}{\longmapsto} \psi^p \in \Omega^p(\mathrm{M}). \tag{8}$$

With this notation, the Laplacian acting on p-forms is given by:

$$\Delta_p = d_p^\dagger d_p + d_{p-1} d^\dagger_{p-1}. \tag{9}$$

Both d and $d^\dagger$ are nilpotent:

$$d_{p+1}\, d_p = 0 = d^\dagger_p \, d^\dagger_{p+1}. \tag{10}$$

The product of two p-forms is symmetric:

$$\omega \wedge *\psi = \psi \wedge *\omega \tag{11}$$

and thus the inner product, defined via

$$\langle\omega,\,\psi\rangle=\int\limits_{\mathrm{M}}\omega\wedge*\psi=\int\limits_{\mathrm{M}}\sqrt{|g|}\,\omega_{\mu_1\ldots\mu_p}\,\psi^{\mu_1\ldots\mu_p}\,dx^1\ldots dx^m, \tag{12}$$

is symmetric: $\langle\omega,\,\psi\rangle=\langle\psi,\,\omega\rangle$.

Note that $\omega\wedge*\psi$ is an m-form and its integral over the manifold is well defined. For Riemannian manifolds the inner product is always positive definite: $\langle\psi,\,\psi\rangle\geq 0$ ("=" when $\psi=0$).

Let us now consider integration by parts for closed manifolds. Stokes theorem yields: $\int\limits_{\mathrm{M}} d\omega=\int\limits_{\partial\mathrm{M}}\omega$. Therefore, for closed manifolds, $\int\limits_{\mathrm{M}} d\omega=0$ for all $\omega\in\Omega^{m-1}(\mathrm{M})$. If we now take $\omega=\alpha^p\wedge\beta^q$ (with $p+q+1=m$), we then get:

$$\int\limits_{\mathrm{M}}(d_p\alpha^p)\wedge\beta^q=(-1)^{p+1}\int\limits_{\mathrm{M}}\alpha^p\wedge d_q\beta^q. \tag{13}$$

If M is not closed, this formula holds only if $\alpha^p\wedge\beta^q$ vanishes on ∂M.

With the listed definitions and properties it is easy to prove that

$$\langle d_p\alpha^p,\,\beta^q\rangle=\langle\alpha^p,\,d^\dagger_{q-1}\beta^q\rangle \tag{14}$$

(with p+q+1=m).

Following [7], we would like to give some more theorems and definitions: On Riemannian manifolds a p-form ω^p is harmonic, that is, $\Delta_p\omega^p=0$, if, and only if, ω^p is closed (that is, $d_p\,\omega^p=0$) and co-closed (that is, $d^\dagger_{p-1}\,\omega^p=0$). The set of harmonic p-forms is denoted by $\mathcal{H}^p(\mathrm{M})$. The set of closed p-forms is called the p^{th} cocycle group and is denoted by $Z^p(\mathrm{M})$. A p-form ω^p is exact (co-exact) if it can be globally written as $\omega^p = d_{p-1}\,\psi^{p-1}$ $(\omega^p = d_p^\dagger\,\chi^{p+1})$. The set of exact p-forms is called the p^{th} coboundary group and is denoted by $B^p(\mathrm{M})$. Both $Z^p(\mathrm{M})$ and $B^p(\mathrm{M})$ are vector spaces with real co-efficients. Since $d^2 = 0$, $Z^p(\mathrm{M}) \supset B^p(\mathrm{M})$.

The definition of the p^{th} de Rham cohomology group is [7]:

$$\mathrm{H}^p_{\mathrm{dR}}(\mathrm{M}) = Z^p(\mathrm{M}) / B^p(\mathrm{M}) \tag{15}$$

(M is a differentiable manifold). Stokes' theorem provides the duality between the cohomology group and the homology group. We would like to refer the reader to [7] for introduction to boundary operators, co-boundary operators, cycle groups, boundary groups, and homology groups.

An extremely important theorem is the Hodge decomposition theorem [7]: Let (M, g) be a compact orientable Riemannian manifold without a boundary. Then $\Omega^p(\mathrm{M})$ is uniquely decomposed as:

$$\Omega^p(\mathrm{M}) = d_{p-1}\,\Omega^{p-1}(\mathrm{M}) \oplus d_p^\dagger\,\Omega^{p+1}(\mathrm{M}) \oplus \mathcal{H}^p(\mathrm{M}), \tag{16}$$

that is, any p-form ω^p can be globally written as

$$\omega^p = d_{p-1}\,\alpha^{p-1} + d_p^\dagger\,\beta^{p+1} + \gamma^p, \tag{17}$$

where $\alpha^{p-1} \in \Omega^{p-1}(\mathrm{M})$, $\beta^{p+1} \in \Omega^{p+1}(\mathrm{M})$, and γ^p is harmmonic.

Any element in $\mathrm{H}^p_{\mathrm{dR}}(\mathrm{M})$ can be uniquely written as $\omega^p = d_{p-1}\,\alpha^{p-1} + \gamma^p$.

If $\omega \in Z^p(\mathrm{M})$, then $[\omega] \in \mathrm{H}^p_{\mathrm{dR}}(\mathrm{M})$ is the equivalence class $\{\omega' \in Z^p(\mathrm{M}) \mid$

$\omega' = \omega + d\psi,\ \ \psi \in \Omega^{p-1}(\mathrm{M})\}$. Two forms, which differ by an exact form, are called cohomologous.

It should be clear by now why we need homology 3-spheres: The Maxwell equation $d_2F = 0$ implies that F is an element in $\mathrm{H}^2_{\mathrm{dR}}(\mathrm{M})$. A is a one-form and therefore $d_1A = 0$ in $\mathrm{H}^2_{\mathrm{dR}}(\mathrm{M})$, that is, if $F = d_1A$ then the equivalence class $[F]$ is zero in $\mathrm{H}^2_{\mathrm{dR}}(\mathrm{M})$ (i.e. $[F] = [F'] \Longleftrightarrow F = F' + d_1A$). When the second de Rham cohomology group is trivial, then $F = d_1A$ is valid globally. In three dimensions $\mathrm{H}^2_{\mathrm{dR}}(\mathrm{M})$ is isomorphic to $\mathrm{H}^1_{\mathrm{dR}}(\mathrm{M})$ due to Hodge duality and $\mathrm{H}^1_{\mathrm{dR}}(\mathrm{M})$ being trivial means that the first homotopy group* $\pi_1(\mathrm{M})$ is trivial (then $F = d_1A$ globally). If $\pi_1(\mathrm{M})$ is non-trivial, then $F = d_1A$ is valid only on contractible regions of the manifold.

Finally, we would like to introduce another piece of the ingredients of our theory — the Betti numbers. Hodge's theorem states [7] that on a compact orientable Riemannian manifold (M, g), $\mathrm{H}^p_{\mathrm{dR}}(\mathrm{M})$ is isomorphic to $\mathcal{H}^p(\mathrm{M})$:

$$\mathrm{H}^p_{\mathrm{dR}}(\mathrm{M}) \cong \mathcal{H}^p(\mathrm{M}). \tag{18}$$

In particular, $\dim\Big(\mathrm{H}^p_{\mathrm{dR}}(\mathrm{M})\Big) = \dim\Big(\mathcal{H}^p(\mathrm{M})\Big) = b^p$, where b^p is the p^{th} Betti number.

*Once again, consult [7] for homotopy groups.

3 Topological Quantum Field Theory and Chern–Simons Theory

Topological Quantum Field Theory is a way of formally constructing metric–independent Quantum Field Theory on some manifold M. Using Topological Quantum Field Theory it has been shown how new topologcal information regarding M or information of topological structures present in M can be obtained. For example, new knot invariants have been discovered and the equivalence of Donaldson and Seiberg–Witten invariants on four–dimensional manifolds has been conjectured (for a review see [8] and the references there-in).

The modern interaction between quantum physics and geometry can be traced to the work of Chern [9]. He showed the importance of the notion of a vector bundle with a connection over a manifold: the relation between the gauge potential and the connection form and between the Faraday tensor and the curvature form. The formulation of the fundamental theories of physics as "gauge theories" then had an immediate geometric interpretation. Even if all curvatures vanish, gauge theories have non-trivial global features (in contrast with classical field theories, like Einstein's theory of relativity where the gravitational force is interpreted in terms of curvature). This led to rapid developments at quantum level with electromagnetism as the prototype of all gauge theories. Subsequently, this U(1) gauge theory was replaced by non-abelian gauge theory; the gauge summetry was broken (Higgs model); magnetic monopoles (introduced to "heal" the asymmetry in Maxwell equations) were studeied by means of topology; etc.

By definition, Topological Quantum Field Theory is a Quantum Field Theory in which the vacuum expectation values of some set of operators are invariant under variation of the metric of the background Riemannian manifold. i.e. this is a theory which does not depend on any background geometry. There are two distinct types [8] of Topological Quantum Field Theories (based on two different ways of achieving the independece of the vacuum expectation values on the metric variation) — of Schwarz type and of Witten type. The former are based on a metric–independent action and the observables are constructed out of gauge invariant operators which do not contain the metric. Witten type theory is based on a symmetry of the model which leaves invariant the action, the measure in the path integral and each of the operators in the above mentioned set. We will confine our attention to Schwarz type theories.

The classical example for a theory of Schwarz type is the Chern–Simons theory [5]. It is an intrinsically odd–dimensional theory and on m-dimensional manifolds (m — odd) it is given by the integral of the m^{th} Chern–Simons form (for a review see [7] and the references therein). Chern–Simons forms are:

$$\begin{aligned}
Q_1(A) &= \frac{i}{2\pi}\,\mathrm{tr}\,A, \\
Q_3(A) &= \frac{1}{2}\left(\frac{i}{2\pi}\right)^2 \mathrm{tr}\Big(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\Big), \\
Q_5(A) &= \frac{1}{6}\left(\frac{i}{2\pi}\right)^3 \mathrm{tr}\Big(A\wedge dA\wedge dA \\
&\quad + \frac{3}{2}A\wedge A\wedge A\wedge dA + \frac{3}{5}A\wedge A\wedge A\wedge A\wedge A\Big), \qquad (19)
\end{aligned}$$

and so on. Here A is the connection on the trivial G-bundle over the manifold M and G is a compact Lie group.

In 1978 Schwarz showed [3] how to evaluate the partition function of a theory with a quadratic action functional. In particular, Schwarz introduced the resolvent method to determine the partition function of Abelian Chern–Simons theory. He showed that the partition function was related to the analytic torsion [2]: a well known topological invariant of the manifold. This remarkable paper suggested that by using Topological Quantum Field Theory, topological invariants of the manifold M could be discovered. The idea to probe topology using Quantum Field Theory (with particular application to the non-abelian Chern–Simons theory) was resurrected by Witten. In 1988 he constructed Topological Quantum Field Theory [10] as Quantum Field Theory representation of the Donaldson's study [11] of the topolgy of low dimensional manifolds. He showed how such theory extracts numerical invariants from the background odd-dimensional closed manifolds. (Topological invariants are not only numbers — they can be algebraic structures (groups), compactness, connectedness, etc.) In 1989 Witten showed [12] how to calculate the partition function of non-abelian Chern–Simons theory by using connections between three–manifolds and the closely associated with them vacuum expectation values of Wilson loops. Witten used Hamiltonian quantization and exploited the method of surgery on three–manifolds.

Chern–Simons theory has not a quadratic kinetic term, but a term linear in momentum. Therefore the Hamiltonian is zero and this is a theory with no dynamical degrees of freedom. However, there are many physical situations in which Chern–Simons theory is relevant. The degrees of freedom of Chern–Simons theory can be shown to be related to topology. A spectacular example of this was Witten's work on knot theory unisng the non-abelian Chern–Simons theory [12]. Even the abelian Chern–Simons theory has interesting and curious features. Adding it to the Maxwell term results in topologically massive electrodynamics [13] which describes a new form of gauge field mass generation. It could also be coupled to other dynamical matter fields (scalars or fermions) to describe anyons. It plays important role in gravity, e.g. $d = 11$, $N = 1$ supergravity, where Chern–Simons term enters with a particular co-efficient dictated by the supersymmetry.

Important for our further analysis is the fact that the Chern–Simons term has the same properties with respect to $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ discrete symmetries as the fermion mass term in 2+1 dimensions — invariant under charge conjugation and changing sign under parity transformation and time reversal. This leads to the idea to get the Chern–Simons term as a dynamically generated term from the parity–breaking part of a theory with massive fermions — we will investigate the application of this phenomenon to confinement at finite temperature in part IV of the thesis.

4 Faddeev–Popov Ghosts

The action of gauge field theories is invariant with respect to local gauge transformations. This huge symmetry degenerates the Lagrangian [1] — the generalized velocities cannot be uniquely expressed in terms of the generalized momenta. To proceed, we will follow a method introduced by Faddeev and Popov [6]. This method is based on the development of a formal path integral approach which eliminates the infinite volume factors present in the partition function due to the fact that we integrate over all gauge fileds — even those, which are equivalent (that is, related by a gauge transformation or, in other words, having the same action) with this resulting in overcounting. We have to somehow restrict the gauge freedom by passing from the space of all possible gauge configurations to the moduli space of the gauge theory, that is, to the space of all possible gauge configurations quotiened by the action of the group of gauge transformations. The space of all possible gauge configurations is partitioned nicely by the action of the group of gauge transformations into a set of non-intersecting "strings" — the orbits of the group. Any two elements from a given orbit can be related together by the action of the group of gauge transformations. Such elements belong to the same equivalence class and in this sense they do not contribute to the physics — they only cause divergence of the partition function. A way to factor them out is to intersect each orbit by a hyperplane. This hyperplane, called a gauge fixing condition*, selects only one representative of each orbit of the group of gauge transformations. The integration is then performed over this hyperplane. The final result must be strictly independent on the choice of this gauge fixing condition. Technically, it amounts to the appearance of a delta–function of this gauge fixing condition and the Faddeev–Popov ghost determinant inside the path integral. Strictly speaking, these arguments are for a finite–dimensional manifold with compact group of isometries, but they can easily be extended to the infinite–dimensional case with a non-compact group. However, when the group of gauge transformations does not act freely, we run into difficulties: the intersection with a hyperplane would not be enough to guarantee that we select only one representative of each equivalence class — the gauge fixing condition does not fix the gauge uniquely.

*We are ignoring the global difficulties related to gauge fixing.

In this case we have non-trivial stabilizers of the group of gauge transformations, for example, gauge transformation of the potential A_μ by a constant function. After such gauge transformation each selected representative of the group orbit will be mapped into itself and thus the gauge freedom will be only partially restricted and still leading to divergence of the partition function. Therefore, we have to introduce an analogue of the gauge fixing condition — this time for the ghosts themselves, not for the fields. This will result in the appearance of an additional ghost–for–ghost determinant with statistics opposite to that of the original ghosts. For higher dimensional manifolds, it is likely to get even more ghost determinants — all of which with alternating statistics and each ghost "healing" the residual divergence left by the previous one. In principle, for theories with quadratic action functionals, restriction of the gauge freedom by selection of only one representative of each equivalence class is equivalent to extraction of the zero mode sector of the theory: the volume of the kernel of the action functional is exactly equal to the ghost determinant times the ghost–for–ghost determinant, etc. The method for extraction of the zero mode sector of the theory was originally introduced by Schwarz [3] and it became an exremely powerful technique for calculation of the partition functions of theories with quadratic action functionals. We will give details in one of the following sections and we shall be heavily using it in the remainder of the thesis. We would like to mention here that Schwarz's resolvent method also breaks down when the cohomology of the space is not vanishing — in this case the gauge fixing conditon does not fix the gauge uniquely. We would like to refer the reader to [14] where the problem of non-vanishing cohomology groups is solved.

Let us go back to the case when the group of gauge transformations does not act freely. There are two ways of overcoming this difficulty. We can replace the group of all gauge transformation by the group of all gauge trasnsformations arising from functions which are equal to 1 for any fixed point. We can take this fixed point at infinity for Euclidean path integrals and we will thus get a free action of the group. Alternatively, we can select a group–invariant version of the Faddeev–Popov trick [1]. This method is very powerful not only because it is able to spot the finer structure of the zero mode sector and overcome the problem with the free action of the group. It also works when the theory is quantized around a reducible classical solution (in that case the Faddeev–Popov procedure breaks down again). The problems associated with the reducible classical solutions are in the fact that the ghost propagator is ill-defined (the gauge is not fixed uniquely). The method of invariant integration is not dealing with intersection of the orbits with a hyperplane, but with the space of orbits itself. Exploiting the symmetry of the model we will reduce the integration from integration over the Riemannian manifold to integration over the space of the orbits. When doing this we have to include in the path integral the volume of each orbit of the group with respect to the Riemannian metric of M. This is the subject of our next section.

5 Method of Invariant Integration

In this section we will review, following [3], the method of reducing an integral of a function with some symmetries over some space to an integral over a lower-dimensional space.

Take M to be a Riemannian manifold and G — a compact group. Let $W = {}^{\mathrm{M}}/_{G}$ denote the space of orbits.

Using the Riemannian metric, there are no problems in defining volume elements on W and M.

Let $\lambda(x)$ be the volume of the orbit Gx with respect to the Riemannian metric on M. $\lambda(x)$ is G-invariant (since $\lambda(gx) = \lambda(x)$ for $g \in G$). Therefore $\lambda(x)$ is a function on $W = {}^{\mathrm{M}}/_{G}$.

Let $f(x)$ be G-invariant function on $W = {}^{\mathrm{M}}/_{G}$.

Hence:

$$\int\limits_{\mathrm{M}} f(x) d\mu = \int\limits_{{}^{\mathrm{M}}/_{G}} f(x) \lambda(x) dv. \tag{20}$$

Define the linear operator $\mathcal{T}_x : \mathrm{Lie}(G) \longrightarrow \mathrm{T}_x(\mathrm{M})$, where $\mathrm{T}_x(\mathrm{M})$ is the tangent space to M at x and $\mathrm{Lie}(G)$ is the Lie algebra of the group G.

Let H_x be the stabilizer of the group G at x, i.e.:

$$\mathrm{H}_x\, x = x. \tag{21}$$

Therefore:

$$\mathcal{H}_x \equiv \mathrm{Lie}(\mathrm{H}_x) = \ker(\mathcal{T}_x). \tag{22}$$

Consider the linear operator $\widetilde{\mathcal{T}_x} : \mathrm{Lie}(G)/_{\mathrm{Lie}(\mathrm{H}_x)} \longrightarrow \mathrm{T}_x(\mathrm{M})$.

The operator $\mathcal{T}_x^{\dagger}\mathcal{T}_x$ is non-degenerate if, and only if, G acts with discrete stabilizers. The operator $\widetilde{\mathcal{T}_x}^{\dagger}\widetilde{\mathcal{T}_x}$ is always non-degenerate. The quotient $G/_{\mathrm{H}_x}$ is homeomorphic to the orbit Gx under the map $g \mapsto gx$ for $g \in G$. The differential of this map at the identity coincides with the operator $\widetilde{\mathcal{T}_x}$. Therefore:

$$\mathrm{vol}\,(Gx) \;=\; \mathrm{vol}\left(G/_{\mathrm{H}_x}\right)\,|\det \widetilde{\mathcal{T}_x}| = \mathrm{vol}\left(G/_{\mathrm{H}_x}\right)\,\det\,(\widetilde{\mathcal{T}_x}^{\dagger}\widetilde{\mathcal{T}_x}\,)^{1/2}. \tag{23}$$

But

$$\mathrm{vol}\,(G) \;=\; \int_G Dg \;=\; \int_{G/_{\mathrm{H}_x}} D[g]\;\mathrm{vol}\,(\mathrm{H}_x) \;=\; \mathrm{vol}\,(\mathrm{H}_x)\;\mathrm{vol}\left(G/_{\mathrm{H}_x}\right). \tag{24}$$

Take $\mathrm{vol}\,(G)$ to be normalized to 1.

Then the volume of the orbit of the group is:

$$\lambda(x) \;=\; \frac{1}{\mathrm{vol}\,(\mathrm{H}_x)}\;\det\left(\widetilde{\mathcal{T}_x}^{\dagger}\widetilde{\mathcal{T}_x}\right)^{1/2}. \tag{25}$$

We now assume that all stabilizers are conjugate and have the same volume $\mathrm{vol}(\mathrm{H})$. Then:

$$\int_{\mathrm{M}} f(x)\,d\mu \;=\; \frac{1}{\mathrm{vol}(\mathrm{H})}\int_{\mathrm{M}/_G} f(x)\det\left(\widetilde{\mathcal{T}_x}^{\dagger}\widetilde{\mathcal{T}_x}\right)^{1/2} dv. \tag{26}$$

With this formula we have restricted the gauge freedom by picking up only one representative from each orbit. Alternatively, we could have imposed a gauge–fixing condition and inserted it in the action together with the Faddeev–Popov determinant (this is the original Faddeev–Popov method). This would bring a delta–function of the gauge–fixing condition into the integrand and therefore would define a subspace in M. If this gauge–fixing condition is appropriate, this subspace would intersect each orbit exactly once and therefore the integration would pick up one representative of each orbit. However, the method of invariant integration is able, as we shall see later, to spot the ghost–for–ghost determinants (so far they are hidden in the volume of the stabilizer).

6 Schwarz's Resolvent Method

We now turn to Schwarz's method for evaluation of the partition function of a theory with a degenerate quadratic (in the fields) action functional by extracting the zero mode sector [1]. (Non-degenerate quadratic action functionals $\langle f, Sf \rangle$ are those, for which $Sf = 0$ if, and only if, $f = 0$.) This method is based on formal manipulations with ill-defined quantities (infinite volumes of vector spaces), ending up with an expression in terms of determinants. It is therefore necessary to give meaning to the determinant of an operator acting in an infinite dimensional space (as an additional problem, this operator might as well have negative eigenvalues which require special separate treatment). This is the topic of our next section — zeta–regularization of determinants —, we now start with the resolvent method.

Consider the following partition function:

$$Z(\lambda) = \frac{1}{N} \int_{\Gamma} \mathcal{D}f \; e^{-i\lambda S(f)}. \tag{27}$$

Here N is a normalization factor, Γ is a real vector space, λ is a positive coupling constant and $S(f)$ is a real-valued degenerate quadratic action functional:

$$S(f) = \langle f, \, \mathrm{T} f \rangle, \tag{28}$$

where T is a self-adjoint operator mapping Γ into itslef.

The inner-product $\langle \cdot \, , \, \cdot \rangle$ in Γ defines an orthogonal decomposition:

$$\Gamma = \ker \mathrm{T} \oplus (\ker \mathrm{T})^{\perp}. \tag{29}$$

Therefore the partition function is given by:

$$\begin{aligned} Z(\lambda) &= \frac{1}{N} \int_{\ker \mathrm{T} \,\oplus\, (\ker \mathrm{T})^{\perp}} \mathcal{D}f \; e^{-i\lambda S(f)} \\ &= \frac{1}{N} \, \mathrm{vol}(\ker \mathrm{T}) \int_{(\ker \mathrm{T})^{\perp}} \mathcal{D}f \; e^{-i\lambda S(f)} \\ &= \frac{1}{N} \, \mathrm{vol}(\ker S) \; \det{}' \Big(\frac{i\lambda}{\pi} \mathrm{T} \Big)^{-1/2}. \end{aligned} \tag{30}$$

In this section we will evaluate $\mathrm{vol}(\ker S)$. *Resolvent* $R(S)$ for the action functional $S(f)$ is the following chain of linear maps:

$$0 \longrightarrow \Gamma_n \xrightarrow{\mathrm{T}_n} \ldots \longrightarrow \Gamma_2 \xrightarrow{\mathrm{T}_2} \Gamma_1 \xrightarrow{\mathrm{T}_1} \ker S = \ker \mathrm{T} \longrightarrow 0. \qquad (31)$$

The maps T_i are linear and invertible and this sequence is exact, i.e. the image of each map is equal to the kernel of the following one (next we will deal with the case when the cohomology groups $\mathrm{H}^k\big(R(S)\big) = {}^{\ker \mathrm{T}_k}/_{\mathrm{Im}\,\mathrm{T}_{k+1}}$ are not trivial).

This means that $\ker \mathrm{T}_k = \mathrm{Im}\,\mathrm{T}_{k+1}$. In particular: $\mathrm{vol}(\ker \mathrm{T}_k) = \mathrm{vol}(\mathrm{Im}\,\mathrm{T}_{k+1})$ and $\mathrm{vol}(\ker \mathrm{S}) = \mathrm{vol}(\ker \mathrm{T}) = \mathrm{vol}(\mathrm{Im}\,\mathrm{T}_1)$. We therefore need to evaluate $\mathrm{vol}(\mathrm{Im}\,\mathrm{T}_1)$. Let T_k' be the restriction of the operator T_k over the space of $(\ker \mathrm{T}_k)^{\perp}$, that is, T_k': $(\ker \mathrm{T}_k)^{\perp} \longrightarrow \mathrm{Im}\,\mathrm{T}_k$. It follows that $\mathrm{vol}(\mathrm{Im}\,\mathrm{T}_k) = |\det \mathrm{T}_k'|\ \mathrm{vol}(\ker \mathrm{T}_k)^{\perp} = |\det' \mathrm{T}_k|\ \mathrm{vol}(\ker \mathrm{T}_k)^{\perp}$. The orthogonal decompositions $\Gamma_k = \ker \mathrm{T}_k \oplus (\ker \mathrm{T}_k)^{\perp}$ imply that $\mathrm{vol}(\ker \mathrm{T}_k)^{\perp} = {}^{\mathrm{vol}(\Gamma_k)}/_{\mathrm{vol}(\ker \mathrm{T}_k)}$. Therefore

$$\mathrm{vol}(\mathrm{Im}\,\mathrm{T}_k) = |\det' \mathrm{T}_k|\ \mathrm{vol}(\Gamma_k)\ \mathrm{vol}(\ker \mathrm{T}_k)^{-1}. \qquad (32)$$

We thus get the recursion formula:

$$\mathrm{vol}(\ker \mathrm{T}_{k-1}) = \det{}' (\mathrm{T}_k^{\dagger}\mathrm{T}_k)^{1/2}\ \mathrm{vol}(\Gamma_k)\ \mathrm{vol}(\ker \mathrm{T}_k)^{-1}. \qquad (33)$$

The partition function is then given by:

$$Z(\lambda) = \frac{1}{N} \prod_{k=1}^{n} [\mathrm{vol}(\Gamma_k)]^{(-1)^{k-1}} \prod_{k=1}^{n} \det{}' (\mathrm{T}_k^{\dagger}\mathrm{T}_k)^{\frac{1}{2}(-1)^{k-1}} \det{}' \Big(\frac{i\lambda}{\pi}\mathrm{T}\Big)^{-1/2}. \qquad (34)$$

If we now choose the normalization N as $\prod\limits_{k=1}^{n} [\mathrm{vol}\,(\Gamma_k)]^{(-1)^{k+1}}$, then we will end up with:

$$Z(\lambda) \;=\; \Big[\prod_{k=1}^{n} \det{}'\,(\mathrm{T}_k^{\dagger}\mathrm{T}_k)^{\frac{1}{2}(-1)^{k-1}}\Big]\,\det{}'\Big(\frac{i\lambda}{\pi}\,\mathrm{T}\Big)^{-1/2}. \tag{35}$$

In the next section we will not only give meaning to the determinant of an infinite–dimensional operator, but we will also show how to extract numerical factors out of the determinant.

Now we would like to point out the structure of the product of determinants with alternating powers entering (35). Clearly, the last determinant in the product is nothing else, but the Faddeev–Popov ghost determinant discussed earlier. It enters with a positive power (opposite to the power of the determinant of the operator of the theory). The second last determinant in the product is the ghost–for–ghost determinant. It has power opposite to the ghost determinant, as the ghost–for–ghost fields serve the same cause as the ghosts themselves — to restrict residual gauge freedom — and enter with “statistics”, opposite to the “fields before them”. The third last determinant in the product is the ghost–for–ghost–for–ghost determinant and so on. Physical theories involve the differential operator and thus each map T_k will also involve a differential operator (exterior derivative). It means that some of the spaces Γ_k will be the spaces of different p-forms and so the number n will be limited by the dimension of the manifold. We will illustrate how this method works in the last section of this part.

Consider now the case when the cohomology of the resolvent is not trivial. In this case Schwarz's method fails — it is for exact sequences only. We will describe the generalization of Schwarz's method for manifolds with non-trivial homology [14]. We have:

$$\ker \mathrm{T}_{k} = \mathrm{Im}\, \mathrm{T}_{k+1} \oplus \mathcal{H}^k\big(R(S)\big) \tag{36}$$

and

$$\mathrm{vol}\,(\ker \mathrm{T}_{k}) \;=\; \mathrm{vol}\,(\mathrm{Im}\, \mathrm{T}_{k+1})\; \mathrm{vol}\Big(\mathcal{H}^k\big(R(S)\big)\Big), \tag{37}$$

where $\mathcal{H}^k\big(R(S)\big)$ is the space of "harmonic" k-forms, associated with the resolvent, that is, these elements ω of Γ_{k}, which are "closed" ($\mathrm{T}_{k}\,\omega = 0$) and "co-closed" ($\mathrm{T}_{k}^{\dagger}\,\omega = 0$). In the last section of this part the exact sequence will be the de Rham complex, the operators T_{k} will be the exterior derivatives and the inverted commas will disappear. Equation (37) implies that we have to evaluate an additional factor: $\mathrm{vol}\Big(\mathcal{H}^k\big(R(S)\big)\Big)$, which will appear in the recursion formula (33). Everything else will be the same.

The projection map $\ker \mathrm{T}_{k} \longrightarrow \mathrm{H}^k_{\mathrm{dR}}\big(R(S)\big) = {}^{\ker \mathrm{T}_{k}}/_{\mathrm{Im}\, \mathrm{T}_{k+1}}$ induces the isomorphism:

$$\phi_{k} : \mathcal{H}^k\big(R(S)\big) \longrightarrow \mathrm{H}^k_{\mathrm{dR}}\big(R(S)\big). \tag{38}$$

Therefore:

$$\mathrm{vol}\Big(\mathcal{H}^k\big(R(S)\big)\Big) = |\det\phi_q|^{-1}\,\mathrm{vol}\Big(\mathrm{H}^k_{\mathrm{dR}}\big(R(S)\big)\Big). \tag{39}$$

Thus the recursion relation (33) gets modified:

$$\mathrm{vol}\,(\ker \mathrm{T}_k) \;=\; \frac{\det'\,(\mathrm{T}^\dagger_{k+1}\mathrm{T}_{k+1})^{1/2}}{\det\,(\phi^\dagger_k\phi_k)^{1/2}}\;\frac{\mathrm{vol}\,(\Gamma_{k+1})}{\mathrm{vol}\,(\ker \mathrm{T}_{k+1})}\;\mathrm{vol}\Big(\mathrm{H}^k_{\mathrm{dR}}\big(R(S)\big)\Big). \tag{40}$$

Finally, the partition function will be given by:

$$Z(\lambda) \;=\; \prod_{k=1}^{n}\Big[\det'\,(\mathrm{T}^\dagger_k\mathrm{T}_k)\det\,(\phi^\dagger_k\phi_k)\Big]^{\frac{1}{2}(-1)^{k-1}}\det'\Big(\frac{i\lambda}{\pi}\mathrm{T}\Big)^{-1/2}. \tag{41}$$

Here we have already normalized by:

$$N \;=\; \prod_{k=1}^{n}\left[\frac{\mathrm{vol}\,(\Gamma_k)}{\mathrm{vol}\Big(\mathrm{H}^k\big(R(S)\big)\Big)}\right]^{(-1)^k}. \tag{42}$$

7 Zeta–regularization of Determinants

Assume that we have a non-negative operator $\big(\langle x,\, Ax\rangle \geq 0,\ \forall x\big)$, acting in some infinite–dimensional space. Assume also that this operator is self-adjoint with a discrete spectrum. The determinant of this operator is given by the product of its eigenvalues λ_k $(k = 1,\ldots,\infty;\ \ \lambda_k \geq 0,\ \ \forall k)$:

$$\det A = \prod_{k=1}^{\infty}\lambda_k. \tag{43}$$

Therefore

$$\ln\det A = \sum_{k=1}^{\infty}\ln\lambda_k. \tag{44}$$

Recall that:

$$\left(\frac{d}{ds}\sum_{k=1}^{\infty}\lambda_k^{-s}\right)_{s=0}=\left(\frac{d}{ds}\sum_{k=1}^{\infty}e^{-s\ln\lambda_k}\right)_{s=0}=-\sum_{k=1}^{\infty}\ln\lambda_k. \tag{45}$$

Thus:

$$\det A=\exp\left[-\left(\frac{d}{ds}\sum_{k=1}^{\infty}\lambda_k^{-s}\right)_{s=0}\right]. \tag{46}$$

The function

$$\zeta(s,A)=\sum_{k=1}^{\infty}\lambda_k^{-s} \tag{47}$$

is the Riemann zeta–function for the operator A. λ_k in (47) are the eigenvalues — *all of which strictly positive* — of the operator A. It is obvious, that the zero modes of the operator should be promptly discarded at first. It is another matter what happens if some of the eigenvalues are negative (we will deal with this case in a while).

So, the determinant of the elliptic operator A is given by:

$$\det A \;=\; e^{-\zeta'(0,A)}. \tag{48}$$

Note that if A is elliptic operator of order n on m-dimensional compact manifold, the ζ-function of A converges for values of s greater than m/n. We can analytically continue $\zeta(s,A)$ [15] into a meromorphic function of s at $s=0$ (the only singularity of the ζ-function is a simple pole at $s=1$). Thus we will end up with a finite expression for the infinite (formally divergent) product of the eigenvalues of the operator A. This technique is called zeta–regularization of determinants. From now on we will assume that the determinants of all elliptic operators are regularized in this way.

An alternative definition of the Riemannian ζ-function is:

$$\zeta(s) \;=\; \frac{1}{\Gamma(s)} \sum_{k=1}^{\infty} \int_0^{\infty} t^{s-1}\, e^{-\lambda_k t}\, dt \;=\; \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}\, \mathrm{Tr}\, e^{-At}\, dt, \tag{49}$$

where $\Gamma(s)$ is the Gamma–function:

$$\Gamma(s) = \int_0^{\infty} t^{s-1}\, e^{-t}\, dt. \tag{50}$$

This alternative definition is based on the formula:

$$\lambda_k^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}\, e^{-\lambda_k t}\, dt. \tag{51}$$

Witten [12] first showed how to take the imaginary unit i out of the determinant. Later, in [16], this precedure was expanded for arbitrary complex numbers and the case when some of the eigenvalues of the infinite–dimensional operator A are negative is also addressed (from now on we will assume that all zero modes of all operators are discarded somehow *before* we write down the expression for a ζ-regularized determinant). Let us briefly outline this procedure.

As the order of the eigenvalues of the operator A is of no importance when we have to multiply them all to calculate the determinant, we can always assume that first come the positive ones, then the negative ones. It means that for our purposes we can write any operator A in the form:

$$A = \begin{pmatrix} A_+ & \\ & A_- \end{pmatrix}, \tag{52}$$

where $A_\pm \,:\, \Gamma_\pm \longrightarrow \Gamma_\pm$ and $\Gamma_\pm$ is the space spanned by eigenvectors of A corresponding to positive (negative) eigenvalues.

The operator $|A|$, formed by A, has positive eigenvalues only:

$$|A| = \begin{pmatrix} A_+ & \\ & -A_- \end{pmatrix}. \tag{53}$$

The ζ-regularization technique makes sense for the operator $|A|$ only:

$$\det |A| \;=\; e^{-\zeta'\left(0,\; |A|\right)}, \tag{54}$$

where

$$\zeta\left(s,\, |A|\right) \;=\; \zeta(s,\, A_+) \;+\; \zeta(s,\, -A_-). \tag{55}$$

The eta–function of the operator A is [16]:

$$\eta(s,\, A) \;=\; \sum_{k=1}^{\infty} \frac{\operatorname{sign} \lambda_k}{\lambda_k^s} \;=\; \zeta(s,\, A_+) \;-\; \zeta(s,\, -A_-). \tag{56}$$

The η-function can be analytically continued in a similar way so that $\eta(0)$ is well-defined. This number formally represents the number of positive eigenvalues less the number of negative ones.

For any real positive number α we have [16]:

$$\det\,(\alpha A) = \alpha^{\,\zeta(0,\, A)}\; e^{-\zeta'(0,\, A)}. \tag{57}$$

For any complex number $\beta = |\beta| e^{i\theta}$ we have [16]:

$$\begin{aligned} \det(\beta A) &= \det(\beta A_+)\, \det\big((-\beta)(-A_-)\big) \\ &= e^{\frac{i\pi}{2}\left[\left(\frac{2\theta}{\pi} \mp 1\right)\zeta(0,|A|)\ \pm\ \eta(0,A)\right]}\, |\beta|^{\zeta(0,\,A)}\, \det|A|. \end{aligned} \tag{58}$$

Equipped more or less with everything we need, we now proceed to

8 Abelian Duality

We will first calculate the partition function of free Maxwell theory:

$$\begin{aligned} Z_1(\lambda_1) &= \int\limits_{\Omega^1(M)} \mathcal{D}A\ e^{-\lambda_1 \int d^3x\, \sqrt{g}\ F_{\mu\nu}F^{\mu\nu}} = \int\limits_{\Omega^1(M)} \mathcal{D}A\ e^{-\lambda_1 \int d_1 A \wedge * d_1 A} \\ &= \int\limits_{\Omega^1(M)} \mathcal{D}A\ e^{-\lambda_1 \langle d_1 A,\, d_1 A\rangle} = \int\limits_{\Omega^1(M)} \mathcal{D}A\ e^{-\lambda_1 \langle A,\, d_1^\dagger d_1 A\rangle}. \end{aligned} \tag{59}$$

The integral is over the space of all one-forms Ω^1(M). We can decompose the space of all one-forms as a direct sum of the kernel of the operator entering the partition function and its orthogonal complement:

$$\Omega^1(\mathrm{M}) = \ker d_1 \oplus (\ker d_1)^\perp. \tag{60}$$

Therefore

$$\begin{aligned} Z_1(\lambda_1) &= \mathrm{vol}(\ker d_1) \int\limits_{(\ker d_1)^\perp} \mathcal{D}A\ e^{-\lambda_1 \langle A,\, d_1^\dagger d_1 A\rangle} \\ &= \mathrm{vol}(\ker d_1)\ \det{}'\Big(\frac{\lambda_1}{\pi}\, d_1^\dagger d_1\Big)^{-1/2}. \end{aligned} \tag{61}$$

The partition function is thus an ill-defined quantity — the determinant and the volume factor are infinite. Our calculations will be formal. With a zeta–regularization technique we can make the determinant finite. We can also asssume that an appropriate normalization is chosen in such way that the divergency of the volume factor is absorbed. This will make the partition function finite.

Note that vol(kerd_1) is nothing else but the Faddeev–Popov ghost determinant times the ghost–for–ghost determinant. To see that, let us calculate the same partition function using the method of invariant integration: we will exploit the gauge symmetry of the theory to restrict the integration over Ω^1(M) to integration over a lower-dimensional space — the space of the orbits of the group of gauge transformation.

The stabilizer of the group of gauge transformations $A \longrightarrow A + d_0\Omega^0(\mathrm{M})$ consists of those elements of Ω^0(M) for which $d_0\Omega^0(\mathrm{M}) = 0$, that is, the constant functions. In order to pick one representative of each equivalence class $[A]$, we impose a gauge condition, that is, we intersect the space of the orbits of the group of gauge transformations in the space of all one-forms by a hyperplane defined by those A's, for which $\partial_\mu A^\mu = 0$, i.e. $d_0^\dagger A = 0$. The integration is then performed over this hyperplane. We thus get:

$$Z_1(\lambda_1) \;=\; \frac{1}{\mathrm{vol}(\ker d_0)} \int_{\Omega^1(M)/d_0} \mathcal{D}[A]\; e^{-\lambda_1 \langle A,\, d_1^\dagger d_1 A\rangle}\, \det{}'\,(d_0^\dagger d_0)^{1/2}. \qquad (62)$$

The stabilizer of the group of gauge transformations consists of the constant functions, that is, the stabilizer is the real line. The real line can be canonically identified with the zeroth de Rham cohomology group $\mathrm{H}^0_{\mathrm{dR}}(\mathrm{M})$. The projection map $\ker d_q \longrightarrow \mathrm{H}^q_{\mathrm{dR}}(\mathrm{M})$ induces the isomorphism [14]:

$$\phi_q : \mathcal{H}^q(\mathrm{M}) \longrightarrow \mathrm{H}^q_{\mathrm{dR}}(\mathrm{M}) \tag{63}$$

where $\mathcal{H}^q(\mathrm{M})$ is the space of harmonic q-forms. Therefore:

$$\mathrm{vol}\Big(\mathcal{H}^q(\mathrm{M})\Big) = |\det \phi_q|^{-1} \ \mathrm{vol}\Big(\mathrm{H}^q_{\mathrm{dR}}(\mathrm{M})\Big). \tag{64}$$

So the volume of the stabilizer is:

$$\mathrm{vol}(\ker d_0) = \det(\phi_0^\dagger \phi_0)^{1/2} \ \mathrm{vol}\Big(\mathcal{H}^0(\mathrm{M})\Big). \tag{65}$$

The volume of the orbit of the group is proportional to the ghost–for–ghost determinant $\det(\phi_0^\dagger \phi_0)$ that extracts the zero modes from the Faddeev–Popov ghost determinant $\det(d_1^\dagger d_1)$. The ghost–for–ghost determinant is equal to the inverse of the volume of the manifold [14]:

$$\det(\phi_0^\dagger \phi_0)^{-1} = \mathrm{vol}(\mathrm{M}). \tag{66}$$

Now we will extract the scaling factor $\frac{\lambda_1}{\pi}$ from the functional determinant. Following [16] we can write:

$$\det{}'\Big(\frac{\lambda_1}{\pi}\, d_1^\dagger d_1\Big)^{-1/2} \ = \ \Big(\frac{\lambda_1}{\pi}\Big)^{-\frac{1}{2}\zeta(0,\, d_1^\dagger d_1)} \ \det{}'(d_1^\dagger d_1)^{-1/2}\,. \tag{67}$$

Thus the partition function of Maxwell theory is given by:

$$Z_1(\lambda_1) \;=\; \left(\frac{\lambda_1}{\pi}\right)^{-\frac{1}{2}\zeta(0,\, d_1^\dagger d_1)} \frac{\mathrm{vol}(\mathrm{M})^{1/2}}{\mathrm{vol}\big(\mathcal{H}^0(\mathrm{M})\big)} \frac{\det{}'(d_0^\dagger d_0)^{1/2}}{\det{}'(d_1^\dagger d_1)^{1/2}}. \tag{68}$$

We now use the fact that on odd-dimensional and two-dimensional manifolds there are no poles in the ζ–function near $s = 0$. This can be seen using Seeley's formula [15] for the ζ-function of some Laplace-type operator L on a d-dimensional manifold without a boundary:

$$\zeta(s, L) = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{A_n}{s + n - \frac{d}{2}} + \frac{J(s)}{\Gamma(s)}, \tag{69}$$

where A_n are the heat–kernel co-efficients and $J(s)$ is analytic. Then $\zeta(0,\, \Delta_p) = -\dim \mathrm{H}^p_{\mathrm{dR}}(\mathrm{M})$. Using the formula [16]:

$$\zeta(s,\, d_p^\dagger d_p) \;=\; (-1)^p \sum_{q=0}^{p} (-1)^q\, \zeta(s,\, \Delta_q), \tag{70}$$

we finally get:

$$Z_1(\lambda_1) \;=\; \lambda_1^{-\frac{1}{2}\dim \mathrm{H}^0_{\mathrm{dR}}(\mathrm{M})} \frac{\mathrm{vol}(\mathrm{M})^{1/2}}{\mathrm{vol}\big(\mathcal{H}^0(\mathrm{M})\big)} \det{}'(d_0^\dagger d_0)^{1/2}\; \det{}'(d_1^\dagger d_1)^{-1/2}. \tag{71}$$

Consider now the partition function of free scalar theory:

$$\begin{aligned} Z_0(\lambda_0) \;&=\; \int_{\Omega^0(M)} \mathcal{D}\varphi\; e^{-\lambda_0 \int d^3x\, \sqrt{g}\; \partial_\mu\varphi\, \partial^\mu\varphi} \;=\; \int_{\Omega^0(M)} \mathcal{D}\varphi\; e^{-\lambda_0 \int d_0\varphi \wedge * d_0\varphi} \\ &=\; \int_{\Omega^0(M)} \mathcal{D}\varphi\; e^{-\lambda_0 \langle d_0\varphi,\, d_0\varphi\rangle} \;=\; \int_{\Omega^0(M)} \mathcal{D}\varphi\; e^{-\lambda_0 \langle \varphi,\, d_0^\dagger d_0 \varphi\rangle}. \end{aligned} \tag{72}$$

We now decompose the space of all zero-forms $\Omega^0(M)$ in a similar way:

$$\Omega^0(M) = \ker d_0 \oplus (\ker d_0)^{\perp}. \tag{73}$$

With this decomposition the partition function becomes:

$$\begin{aligned}
Z_0(\lambda_0) &= \mathrm{vol}(\ker d_0) \int\limits_{(\ker d_0)^{\perp}} \mathcal{D}\varphi \; e^{-\lambda_0 \langle \varphi,\, d_0^{\dagger} d_0 \varphi \rangle} \\
&= \left(\frac{\lambda_0}{\pi}\right)^{\frac{1}{2}\dim \mathrm{H}^0_{\mathrm{dR}}(\mathrm{M})} \mathrm{vol}(\ker d_0)\ \det{}'(d_0^{\dagger} d_0)^{-1/2} \\
&= \left(\frac{\lambda_0}{\pi}\right)^{\frac{1}{2}\dim \mathrm{H}^0_{\mathrm{dR}}(\mathrm{M})} \frac{\mathrm{vol}\big(\mathcal{H}^0(\mathrm{M})\big)}{\mathrm{vol}(\mathrm{M})^{1/2}}\ \det{}'(d_0^{\dagger} d_0)^{-1/2}.
\end{aligned} \tag{74}$$

Therefore:

$$Z_0(\lambda_0) = \lambda_0^{\frac{1}{2}\dim \mathrm{H}^0_{\mathrm{dR}}(\mathrm{M})} \frac{\mathrm{vol}\big(\mathcal{H}^0(\mathrm{M})\big)}{\mathrm{vol}(\mathrm{M})^{1/2}}\ \det{}'(d_0^{\dagger} d_0)^{-1/2}. \tag{75}$$

The product of the partition functions of the theories is:

$$Z_0(\lambda_0) Z_1(\lambda_1) = \left(\frac{\lambda_0}{\lambda_1}\right)^{\frac{1}{2}\dim \mathrm{H}^0_{\mathrm{dR}}(\mathrm{M})} \det{}'(d_1^{\dagger} d_1)^{-1/2}. \tag{76}$$

On the other hand we have:

$$Z_1(\lambda_1) = \lambda_1^{-\frac{1}{2}\dim \mathrm{H}^0_{\mathrm{dR}}(\mathrm{M})} \mathrm{vol}(\ker d_1)\ \det{}'(d_1^{\dagger} d_1)^{-1/4}\ \det{}'(d_1^{\dagger} d_1)^{-1/4}. \tag{77}$$

The Hodge star operator is invertible and on three–dimensional manifolds we have: $\det{}'(d_1^{\dagger} d_1)^{1/2} = \det{}'(*d_1)$. (For these operators the multiplicative anomaly vanishes.) Thus (modulo a phase factor):

$$Z_1(\lambda_1) = \lambda_1^{-\frac{1}{2}\dim \mathrm{H}^0_{\mathrm{dR}}(\mathrm{M})} \det{}'(d_1^{\dagger} d_1)^{-1/4}\ \mathrm{vol}\big(\ker(*d_1)\big)\ \det{}'(*d_1)^{-1/2}. \tag{78}$$

The last two factors in this formula are exactly the partition function of Chern–Simons theory Z_{CS}. The partition function of Chern–Simons theory is a topological invariant (modulo a phase factor [16]), given by the Ray–Singer torsion of the manifold [3]:

$$Z_{CS}(\lambda_{CS}) \;=\lambda_{CS}^{-\frac{1}{2}\dim \mathrm{H}^0_{\mathrm{dR}}(\mathrm{M})}\;\tau_{\mathrm{RS}}^{1/2}(\mathrm{M}). \tag{79}$$

Therefore:

$$\left(\frac{Z_1(\lambda_1)}{Z_{CS}(\lambda_{CS})}\right)^2 \;=\; \lambda_1^{-\dim \mathrm{H}^0_{\mathrm{dR}}(\mathrm{M})}\;\det{}'(d_1^{\dagger}d_1)^{-1/2}\,. \tag{80}$$

Dividing (80) by (76) we get:

$$\frac{Z_1(\lambda_1)}{Z_0(\lambda_0)} \;=\; Z_{CS}^2\left(\sqrt{\lambda_0\lambda_1}\,\right) \;=\; (\lambda_0\lambda_1)^{-\frac{1}{2}\dim \mathrm{H}^0_{\mathrm{dR}}(\mathrm{M})}\;\tau_{\mathrm{RS}}(\mathrm{M}). \tag{81}$$

R $\longleftrightarrow$ 1/R duality means that if the coupling constants (overall scaling factors) are related as $\lambda_0 = \lambda_1^{-1}$ then both partition functions will depend on the coupling constants in the same way (one has to be careful, because the coupling constants are not dimensionless). The ratio of the partition functions is a topological invariant — the Ray–Singer torsion of the manifold. Therefore the two theories are equivalent. For manifolds for which the Ray–Singer torsion is one ($\mathbf{S}^3$ for instance), the partition functions are equal. Note that both scalar field theory and Maxwell theory are non-topological in three dimensions.

Abelian Duality

$$\frac{Z_1(\lambda_1)}{Z_0(\lambda_0)} = Z_{CS}^2\left(\sqrt{\lambda_0\lambda_1}\,\right) \tag{82}$$

is stronger than $\mathrm{R} \longleftrightarrow 1/\mathrm{R}$ duality

$$\frac{Z_1(\lambda_1)}{Z_0\left(\frac{1}{\lambda_1}\right)} = \tau_{\mathrm{RS}}(\mathrm{M}) \tag{83}$$

in the sense that if the coupling constants are not related as $\mathrm{R} \longleftrightarrow 1/\mathrm{R}$, there is still a relation — the ratio of the partition functions is given by the square of the partition function of Chern–Simons theory with coupling constant $\lambda_{CS} = \sqrt{\lambda_1\lambda_0}$, that is, by the partition function of $\mathrm{U}(1) \times \mathrm{U}(1)$ Chern–Simons theory with purely off-diagonal coupling (BF theory).

We can show the Abelian Duality by considering the following resolvent generated by the de Rham complex:

$$0 \longrightarrow \mathbb{R} \xrightarrow{\phi_0^{-1}} \Omega^0(\mathrm{M}) \xrightarrow{d_0} \mathrm{Im}\, d_0 = \ker d_1 = \ker S_1 \longrightarrow 0. \tag{84}$$

Here, for simplicity only, we have assumed that the first cohomology group of the mainfold is trivial and thus we have:

$$\mathrm{vol}(\ker d_1) = \mathrm{vol}(\mathrm{Im} d_0). \tag{85}$$

If we denote by $\tilde{d}_0$ the restriction of d_0 over $(\ker d_0)^{\perp}$, then the map

$$\tilde{d}_0 : (\ker d_0)^{\perp} \longrightarrow \mathrm{Im}\, d_0 \tag{86}$$

implies:

$$\mathrm{vol}\,(\mathrm{Im}\,d_0) \;=\; \det{}'\,(d_0^{\dagger}d_0)^{1/2}\,\mathrm{vol}\Big((\ker d_0)^{\perp}\Big) \;=\; \det{}'\,(d_0^{\dagger}d_0)^{1/2}\,\frac{\mathrm{vol}\Big(\Omega^0(\mathrm{M})\Big)}{\mathrm{vol}\,(\ker d_0)}. \quad (87)$$

We thus get:

$$\mathrm{vol}\,(\ker d_1) \;=\; \mathrm{vol}\Big(\Omega^0(\mathrm{M})\Big)\,\det{}'\,(d_0^{\dagger}d_0)^{1/2}\,\frac{1}{\mathrm{vol}\,(\ker d_0)}. \quad (88)$$

We have already seen that

$$\mathrm{vol}\,(\ker d_0) \;=\; \det\,(\phi_0^{\dagger}\phi_0)^{1/2}\;\mathrm{vol}\Big(\mathcal{H}^0(\mathrm{M})\Big). \quad (89)$$

Therefore the partition function of Maxwell theory is given by the same expression as (71).

To show that the Abelian Duality is a property of three dimensions only, consider again the de Rham complex. Due to Hodge duality, $d^{\dagger}_{m-p-1} = * \, d_p \, *$. Therefore $\det{}'\,(d^{\dagger}_{m-p-1}d_{m-p-1}) = \det{}'\,(d_p^{\dagger}d_p)$ and for even–dimensional manifolds all determinants involving the differential operator cancel each other. In higher odd dimensions, it is possible to find a relation between scalar field theory and Maxwell theory, but there will be more determinants coming in from the de Rham complex, thus non-physical theories should also be involved.

R $\longleftrightarrow$ 1/R duality can be shown in a different manner — by a duality transformation. We will illustrate this by considering the following partition function:

$$Z = \int\limits_{\ker d_1} \mathcal{D}A \; e^{R\int A\wedge *A} = \int\limits_{\ker d_1} \mathcal{D}A \; e^{R\langle A,\, A\rangle}. \qquad (90)$$

Over the space of the kernel of the operator d_1 we can locally write $A = d_0\Phi$. Therefore Z becomes the partition function of free scalar field theory:

$$Z = \int\limits_{\Omega^0(M)} \mathcal{D}\Phi \; e^{R\int d_0\Phi\wedge * d_0\Phi}. \qquad (91)$$

Alternatively, we can replace the integral over the kernel of the operator d_1 by an integral over $\Omega^1(M)$ and include a Lagrange multiplier B $\big(B \in \Omega^1(M)\big)$ to keep track of the fact that A is flat:

$$Z = \int\limits_{\ker d_1} \mathcal{D}A \; e^{R\int A\wedge *A} = \int\limits_{\Omega^1(M)} \mathcal{D}A\,\mathcal{D}B \; e^{R\int A\wedge *A + \int B\wedge d_1 A}. \qquad (92)$$

If we integrate over A and absorb the resulting determinant det (R$\mathbb{I}$) in the normalization, we end up with the partition function of Maxwell theory with coupling constant $1/R$:

$$Z = \int\limits_{\Omega^1(M)} \mathcal{D}B \; e^{\frac{1}{R}\int d_1 B\wedge * d_1 B}. \qquad (93)$$

The same can be seen if we make a change in the variables in (92) — *dualization* — $A \longrightarrow A' = A + \frac{1}{R} * d_1 B$.

With this dualization the partition function becomes:

$$Z = \int\limits_{\Omega^1(M)} \mathcal{D}A \; e^{R\int A\wedge *A} \int\limits_{\Omega^1(M)} \mathcal{D}B \; e^{\frac{1}{R}\int d_1 B\wedge * d_1 B}. \qquad (94)$$

The integral over A is Gaussian and can be absorbed in the normalization factor. The remaining integral is the partition function of Maxwell theory.

Let us now include external currents J_μ in Maxwell theory and j in scalar field theory:

$$
\begin{aligned}
Z_1(J) &= \int\limits_{\Omega^1(M)} \mathcal{D}A \; e^{-\int d^3x \sqrt{g}\,(F_{\mu\nu}F^{\mu\nu} - qJ_\mu A^\mu)} = \int\limits_{\Omega^1(M)} \mathcal{D}A \; e^{-\langle A,\, d_1^\dagger d_1 A\rangle + q\,\langle J,\, A\rangle}, \\
Z_0(j) &= \int\limits_{\Omega^0(M)} \mathcal{D}\varphi \; e^{-\int d^3x \sqrt{g}\,(\partial_\mu\varphi\,\partial^\mu\varphi - ej\varphi)} = \int\limits_{\Omega^0(M)} \mathcal{D}\varphi \; e^{-\langle \varphi,\, d_0^\dagger d_0 \varphi\rangle + e\,\langle j,\, \varphi\rangle},
\end{aligned} \tag{95}
$$

where q and e are some charges.

Now extract perfect squares and perform the Gaussian integration to end up with:

$$
Z_0(j) = Z_0(J)\; \tau_{\rm RS}(\mathrm{M})\; \exp\Big(e\,\langle j,\, \frac{1}{d_0^\dagger d_0}\, j\rangle\Big)\; \exp\Big(q\,\langle J,\, \frac{1}{d_1^\dagger d_1}\, J\rangle\Big). \tag{96}
$$

If the charges and the currents are related as:

$$
j = -\sqrt{\frac{q}{e}} * d_2\,(d_1^\dagger)^{-1}\, J, \tag{97}
$$

then the Abelian duality will go through on the level of interactions with external currents.

Alternatively, if the condition for the cancelation of the terms, involving the currents, is not satisfied, then the ratio of the partition functions of scalar field theory, interacting with external current j, and Maxwell theory, interacting with external current J, would be given by the partition function of U(1) × U(1) Chern–Simons theory interacting with some current k, specified by j, J and the corresponding propagators of these two theories.

Using the definition of correlation function (as a functional derivative of the partition function with respect to the external current), we can easily relate the correlation functions of scalar field theory, Maxwell theory and Chern–Simons theory.

We have recently shown [17] (see part II of the thesis) that the partition functions of Maxwell–Chern–Simons theory and the self-dual model differ by the partition function of Chern–Simons theory (thus the two theories being equivalent). Therefore, the ratio of the partition functions of scalar field theory and Maxwell theory is equal (modulo phase ambiguities) to the square of the ratio of the partition functions of Maxwell–Chern–Simons theory and the self-dual model. We can relate the correlation functions of these five models as well.

Finally we would like to mention that Chern–Simons theory can be dynamically generated from the parity–breaking part of a theory with massive fermions [18] — as gauge–invariant regularization of the massless fermionic determinant introduces parity anomaly given by the Chern–Simons theory (see part IV of the thesis). In this sense, our result (81) implies that a theory with massive fermions (interacting with external currents) together with massless scalar fields (possibly iteracting with external currents) add up to Maxwell theory (with possible interaction with external currents). Thus we have a form of bosonization in three dimensions.

After this work was completed, our attention was kindly drawn by A. Schwarz to [19] where the ratio Z_{k-1}/Z_{m-k-1} (where m is the dimension of the manifold) is expressed as the Ray–Singer torsion. The difference between our work and [19] is in the following. In [19], the initial considerations are for the case when there are no zero modes of the Laplace operators Δ_k (acting on k-forms). When these zero modes are absent, it is rather obvious that the quotient Z_{k-1}/Z_{m-k-1} is the Ray–Singer torsion of the manifold. The case of interest appears when these zero modes are no longer neglected. In [19] a very deep analysis is given for this case: the theory of the measure of the path integrals involved is developed and certain general results are given in this direction. In our paper we have kept these zero modes all along and we have shown that even with them the quotient (Z_1/Z_0 in our case) is still given by the Ray–Singer torsion. In addition we have studied the scaling dependence of the models and we have shown the relation to R $\longleftrightarrow$ 1/R duality. We have also given treatment on the physically relevant case — interaction with external currents and correlation functions.

II. Equivalence of the Self–Dual Model and Maxwell–Chern–Simons Theory on Arbitrary Manifolds

9 Introduction

In three dimensions it is possible to add a gauge-invariant Chern–Simons term to the Maxwell gauge field action [13], [20], [21]. The resulting Maxwell–Chern–Simons theory has been analyzed completely and in [13] the entire subject of topologically massive three–dimensional gauge theories has been set up. Further Maxwell–Chern–Simons theory has been used as an effective theory for different models, such as fractional Hall effect and high-temperature superconductivity [22], [23].

The Self–Dual Model was first studied in detail by Deser et al. [13] and it was shown in [24] that the Self–Dual Model is equivalent, modulo global differences, to the Maxwell–Chern–Simons theory.

Subsequently, this equivalence has been studied by many authors using variety of techniques: in the context of bosonisation and at the quantum level (using Legendre transformation) in the abelian and non-abelian case in [25], [26]; by constraint analysis in [27] and [28]; by means of Batalin–Fradkin–Tyutin formalism [29]; in the context of duality [30], [31], [32] and many others.

We address the equivalence with differential geometric tools. It allows us to reveal global features of these models which, so far, have been overlooked. We pay particular attention to the zero modes present in the problem. These zero modes contain topological information regarding the manifold. By neglecting them, i.e. absorbing the divergence due to the zero modes in the normalization constant, this information is lost. Schwarz's method of invariant integration [3], allows us to formally consider a key part of the zero mode sector from the divergent term. This is enough, as we show, to get topological information regarding the manifold.

We show that, subject to choice of appropriate normalizations, the ratio of the partition functions of the two theories in the presence of currents is given, modulo a phase factor, by the partition function of abelian Chern–Simons theory with currents. This phase factor captures the geometrical properties of the manifold. The partition function of Chern–Simons theory contains a phase factor which captures the topological properties of the currents (their linking number) and modulo this phase factor it is a topological invariant (the Ray–Singer torsion of the manifold). Therefore the Self–Dual Model and Maxwell–Chern–Simons theory are equivalent to within a phase factor which contains geometrical information about the manifold and another phase factor which contains information about the topological properties of the currents.

10 Self–Dual Model

The Self–Dual Model was introduced in [33] as a "square root" of the Proca equation for a massive antisymmetric tensor field. Proca equation is:

$$\partial^\mu F_{\mu\nu} - m^2 A_\nu = 0, \tag{98}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This equation implies the Lorentz condition

$$\partial^\mu A_\mu = 0, \tag{99}$$

from which only two of the three components of A survive.

In [33] a "square root" is taken from Proca's equation in order to find a model in which not two, but only one mode is propagated. A self–duality condition is introduced:

$$A_\mu \;=\; \frac{1}{2m}\,\epsilon_{\mu\nu\rho}\,F^{\nu\rho}. \tag{100}$$

This condition implies Proca equation together with the Lorentz condition and is generated as an equation of motion by the following Lagrangian:

$$\mathcal{L} \;=\; -\frac{m^2}{2}\,A_\mu A^\mu \;+\; \frac{m}{2}\,\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho. \tag{101}$$

The term "self–duality" is related only to the equations of motion of the model.

11 Maxwell–Chern–Simons Theory

In three dimensions we can add a metric–independent scalar, the Chern–Simons term, to Maxwell theory [13]. The Chern–Simons term, violating $\mathcal{P}$ and $\mathcal{T}$ discrete symmetries, serves as a *topological* mass term for the U(1) gauge field A. In result the massless spinless Maxwell excitation acquires mass and spin 1. It is the topological non-triviality of the Chern–Simons term (invariant under small gauge transformations and changed by a discrete quantity — the winding number of the transformation — under large gauge transformations) that generates masses for the gauge fields. The Maxwell–Chern–Simons Lagrangian

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{k}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \tag{102}$$

implies equations of motion

$$\partial_\mu F^{\mu\nu} + \frac{ke^2}{2} \epsilon^{\nu\alpha\beta} F_{\alpha\beta} = 0, \tag{103}$$

which describe the propagation of a single degree of freedom with mass ke^2. The fact that here we also have a single degree of freedom might serve as a first naive indication of possible equivalence with the self–dual model.

That is an alternative way of providing massless gauge fields with masses and the theories, describing this phenomenon, are called topologically massive gauge theories. This mechanism has nothing in common with the standard

Higgs mechanism. We can even consider Maxwell–Chern–Simons theory in the framework of the spontaneous symmetry breaking mechanism. This will result in Maxwell–Chern–Simons–Higgs theory with two independently generated masses for the gauge fields (see [34] and the references therein):

$$m_{\pm} = \frac{m_{\mathrm{MCS}}}{2}\left[\sqrt{1+\frac{4m_{\mathrm{H}}^2}{m_{\mathrm{MCS}}^2}} \pm 1\right], \tag{104}$$

where $m_{\mathrm{MCS}} = ke^2$ (as stated above) is the *topological* mass and $m_{\mathrm{H}}^2 = 2e^2v^2$ is the square of the Higgs mass (v is the non-zero vacuum expectation value of the Higgs field). Both $m_{\pm}$ are physical mass poles of the propagator of Maxwell–Chern–Simons–Higgs theory. In the broken phase we have one real massive scalar degree of freedom (the Higgs field) and two massive gauge degrees of freedom [34].

12 Multiplicative Anomaly

Basic formulae from the linear algebra fail when we have operators acting in infinite–dimensional spaces. Of course, the most difficult question is what the determinant of such operator is. We hope that we already have a satisfactory answer — the ζ-regularised expression. The trace of such operators also causes problems, as it is an infinite sum. Let us accept the following regularization dependent convention:

$$\operatorname{tr} A \;=\; \sum_{k=1}^{\infty} \lambda_k \;=\; \zeta(-1,\, A). \tag{105}$$

However, for such type of operators two basic formulae from the finite–dimensional case *in general* no longer hold:

$$\operatorname{tr}(A+B) \;\neq\; \operatorname{tr} A + \operatorname{tr} B, \tag{106}$$

$$\det(AB) \;\neq\; (\det A)(\det B). \tag{107}$$

In the expression for the determinants the operators are with discarded zero modes.

The property $\operatorname{tr}\ln A = \ln \operatorname{tr} A$ continues to hold in the infinite–dimensional case.

Let us now define the *mulltiplicative anomaly*:

$$\begin{aligned} F(A,B) \;&=\; \ln\det(AB) \;-\; \ln\det A \;-\; \ln\det B \\ &=\; \zeta'(0,\, A) + \zeta'(0,\, B) - \zeta'(0,\, AB). \end{aligned} \tag{108}$$

The quantity $F(A,B)$ need not be zero for infinite–dimensional operators. We will illustrate the confusion caused by the multiplicative anomaly in a very simple model of calculating the functional determinant of some infinite–dimensional operator. We will show that there are two recipes, both of which seem absolutely acceptable, but leading to different answers. Dowker [35] poses the following problem. Consider the classical action:

$$S_a = \frac{1}{2}\int (\phi_1 A_1 \phi_1 + \phi_2 A_2 \phi_2)\, dx, \tag{109}$$

where $A_i = \nabla^2 + m_i^2$ and

$$S_b = \frac{1}{2}\int \tilde{\mathbf{\Phi}} A \mathbf{\Phi}\, dx, \tag{110}$$

where $\mathbf{\Phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ and $A = \begin{pmatrix} \nabla^2 + m_1^2 & 0 \\ 0 & \nabla^2 + m_2^2 \end{pmatrix}$.

Even though $S_a = S_b$, there are two different answers:

$$Z_a = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2\, e^{-S_a} = (\det A_1)(\det A_2) \tag{111}$$

$$Z_b = \int \mathcal{D}\mathbf{\Phi}\, e^{-S_b} = \det(A_1 A_2). \tag{112}$$

The answers are different because of the multiplicative anomaly.

According to Dowker [35], the natural and usual way is to consider Z_a. In a reply to this choice, Elizalde *et al.* [36] give substantial arguments in favour of Z_b, that is, to take the algebraic determinant first and then the functional determinant. We note that the multiplicative anomaly has been tested only in terms of a ζ-regularization set up. It is obvious that all resluts depend on the regularization scheme. If we consider the infinite–dimensional block matrix $A = \begin{pmatrix} M & N \\ P & Q \end{pmatrix}$ it is true that in general $\det A \neq (\det M)$ (det $Q) - (\det N)$ $(\det P)$. The ζ-regularization technique justifies the formula

$\det\begin{pmatrix} M & 0 \\ 0 & Q \end{pmatrix} = (\det M)\ (\det Q)$. However, a different regularization will not necesarilly give the same answer. In this sense, taking algebraic determinant and taking functional determinant are not necessarily commutative actions. Thus Z_a seems to be the more natural choice. However, even though the resluts are different, the physics is most likely the same.

13 Equivalence of the Self–Dual Model and Maxwell—Chern–Simons Theory on Arbitrary Manifolds

We will consider a general Riemannian manifold.

The Self-Dual Model is given by the action:

$$S_{SD} = \int_M (f_\mu f^\mu + \epsilon_{\mu\nu\lambda} f^\mu \partial^\nu f^\lambda) d^3x \;=\; \big\langle f, (\mathbb{I} + * d_1) f \big\rangle, \tag{113}$$

where d_p is the map from the space of all p-forms to the space of $(p+1)$-forms, i.e. $d_p : \Omega^p(\mathrm{M}) \longrightarrow \Omega^{p+1}(\mathrm{M})$, and $*$ is the Hodge star operator: $* : \Omega^p(\mathrm{M}) \longrightarrow \Omega^{m-p}(\mathrm{M})$ ($m = \dim \mathrm{M} = 3$). The Hodge star operator explicitly depends on the metric of the manifold M.

The partition function of the model is:

$$Z_{SD} \;=\; \int_{\Omega^1(\mathrm{M})} \mathcal{D}f \; e^{-i \langle f,\, (\mathbb{I} + * d_1) f \rangle} \tag{114}$$

The operator $\mathbb{1} + * d_1$ is self-adjoint.

Now we will extract the zero-mode dependence from the action functional.

To do so, decompose $\Omega^1(\mathrm{M})$:

$$\Omega^1(\mathrm{M}) = \ker(\mathbb{1} + * d_1) \oplus \ker(\mathbb{1} + * d_1)^{\perp}. \tag{115}$$

Therefore:

$$Z_{SD} \;=\; \mathrm{vol}\Big(\ker(\mathbb{1} + * d_1)\Big)\; \det{}'\Big(i(\mathbb{1} + * d_1)\Big)^{-1/2}. \tag{116}$$

Witten has shown [12] how to deal with i in $\det'(i\mathrm{T})$ for some operator T, using ζ-regularization technique. He found that i leads to a phase factor, depending on the η-function of the operator T and explicitly involving the metric of the manifold. For our case we have:

$$\det{}'\Big(i(\mathbb{1} + * d_1)\Big)^{-1/2} \;=\; e^{-\frac{i\pi}{4}\eta\left(0,\,(\mathbb{1}+* d_1)\right)}\; \det{}'(\mathbb{1} + * d_1)^{-1/2}. \tag{117}$$

Finally, the partition function of the Self–Dual Model is:

$$Z_{SD} \;=\; e^{-\frac{i\pi}{4}\eta\left(0,\,(\mathbb{1}+* d_1)\right)}\; \mathrm{vol}\Big(\ker(\mathbb{1} + * d_1)\Big)\; \det{}'(\mathbb{1} + * d_1)^{-1/2}. \tag{118}$$

The action of Maxwell–Chern–Simons Theory is:

$$S_{MCS} \;=\; \int\limits_{\mathrm{M}} (F_{\mu\nu}F^{\mu\nu} + \epsilon_{\mu\nu\lambda}A^{\mu}\partial^{\nu}A^{\lambda})\, d^3x \;=\; \langle A,\, d_1^{\dagger}d_1 A\rangle \,+\, \langle A,\, *d_1 A\rangle \tag{119}$$

where $d_k^{\dagger}:\ \Omega^{k+1}(\mathrm{M}) \longrightarrow \Omega^{k}(\mathrm{M})$.

In this case the topological invariance is explicitly violated by the Maxwell term.

On three–dimensional manifolds we have: $d_1^\dagger = * d_1 *$. Therefore we may write the partition function as:

$$Z_{MCS} = \int_{\Omega^1(M)} \mathcal{D}A \; e^{-i\left\langle A, \; \left(*d_1 + (*d_1)^2\right) A\right\rangle}. \tag{120}$$

The operator $*d_1 + (*d_1)^2$ is self-adjoint.

We proceed to explicitly calculate the partition function. The theory has a gauge invariance under gauge transformations $A_\mu \longrightarrow A_\mu - \partial_\mu \lambda$, i.e.:

$$A \longrightarrow A \; + \; d_0 \, \Omega^0(M). \tag{121}$$

To proceed we pick up one representative of each equivalence class $[A]$, where $[A] = \{A + d_0 \, \Omega^0(M)\}$. To do this we impose the gauge condition $\partial_\mu A^\mu = 0$, that is $d_0^\dagger \, A = 0$.

This ensures that the space of orbits of the gauge group in the space of all one-forms is orthogonal to the space of those A's, for which $d_0^\dagger A = 0$ and so we will pick up only one representative of each orbit.

Then the operator d_0 plays the role of $\widetilde{\mathcal{T}_x}^\dagger$ of the Section 5, i.e. the stabilizer consists of those elements of $\Omega^0(M)$, for which $d_0 \, \Omega^0(M) = 0$ (the constant functions). Hence: H= $\mathbb{R}$.

Therefore:

$$Z_{MCS} = \frac{1}{\mathrm{vol}(H)} \int_{\Omega^1(M)/G} \mathcal{D}A \; e^{-i\left\langle A, \; \left(*d_1 + (*d_1)^2\right) A\right\rangle} \; \det{}' (d_0^\dagger d_0)^{1/2}. \tag{122}$$

The operator in the exponent has zero-modes.

Let $A \in \ker\left(*d_1 + (*d_1)^2\right)$, i.e. $*\,d_1 A + *d_1 *\, d_1 A = 0$. There are two situations to consider. We can take $*\,d_1 A = 0$, that is: $A \in \ker(*d_1)$, or $A \notin \ker(*d_1)$, i.e. $*\,d_1 A \neq 0$, but $*\,d_1 A = -(*\,d_1)^2 A$. In the second case $*\,d_1$ has inverse $(*\,d_1)^{-1}$. Therefore: $A = -*\,d_1 A$, which means that $(\mathbb{1} + *\,d_1)A = 0$, i.e. $A \in \ker(\mathbb{1} + *\,d_1)$.

By definition $\ker(*\,d_1) \bigcap \ker(\mathbb{1} + *\,d_1) = \emptyset$.

It is easy to see that $\ker(*\,d_1)$ and $\ker(\mathbb{1} + *\,d_1)$ are orthogonal:

Let $f \in \ker(\mathbb{1} + *\,d_1)$ and $g \in \ker(*\,d_1)$.

$\langle f,\, g\rangle = \langle f,\, g\rangle + \langle f,\, *\,d_1\, g\rangle = \left\langle f,\, (\mathbb{1} + *\,d_1)\, g\right\rangle = \left\langle (\mathbb{1} + *\,d_1) f,\, g\right\rangle = 0$,

since $(\mathbb{1} + *\,d_1)$ is self-adjoint and $f \in \ker(\mathbb{1} + *\,d_1)$.

So, $\ker(*\,d_1)$ is the orthogonal complement of $\ker(\mathbb{1} + *\,d_1)$.

Therefore we can write:

$$\ker\left(*d_1 + (*d_1)^2\right) = \ker(*d_1) \oplus \ker(\mathbb{1} + *\,d_1) \tag{123}$$

and

$$\mathrm{vol}\left(\ker(*d_1 + (*d_1)^2\right) = \mathrm{vol}\left(\ker(*\,d_1)\right)\, \mathrm{vol}\left(\ker(\mathbb{1} + *\,d_1)\right). \tag{124}$$

On the other hand, since $\int \mathcal{D}\omega\; e^{-\langle \omega,\, T\omega\rangle} = \mathrm{vol}(\ker \mathrm{T})\, \det{}'(\mathrm{T})^{-1/2}$, the partition function is given by:

$$Z_{MCS} = \frac{e^{-\frac{i\pi}{4}\eta\left(0,\; *d_1 + (*d_1)^2\right)}}{\mathrm{vol}(\mathrm{H})}\; \det{}'(d_0^\dagger d_0)^{1/2}\; \det{}'\left(*d_1 + (*d_1)^2\right)^{-1/2}$$

$$= e^{-\frac{i\pi}{4}\eta\left(0,\ *d_1+(*d_1)^2\right)} \ \mathrm{vol}\Big(\ker(*d_1)\Big)\ \mathrm{vol}\Big(\ker(\mathbb{I}+*\,d_1)\Big)$$

$$\det{}'\Big(*d_1+(*d_1)^2\Big)^{-1/2}. \qquad (125)$$

Let us now consider $\det{}'\Big(*d_1+(*d_1)^2\Big)^{-1/2}$.

In the infinite-dimensional case we have to take into account the multiplicative anomaly, i.e. the fact that the determinant of a product of operators is not always equal to the product of the determinants of the operators.

For our case we will show that:

$$\det{}'\Big(*d_1+(*d_1)^2\Big) \ = \ (-1)^{i\pi\psi}\ \det{}'(*d_1)\ \det{}'(\mathbb{I}+*d_1), \qquad (126)$$

where: $\psi=\zeta\Big(0,-\big(*d_1+(*d_1)^2\big)_-\Big)-\zeta\big(0,-(*d_1)_-\big)-\zeta\big(0,-(\mathbb{I}+*d_1)_-\big)$.

The meaning of $\zeta(0,\ A_-)$ will become clear from the context of the proof.

Take A to be some operator without zero-modes. We saw that we can write A in the form: $A=\begin{pmatrix} A_+ & \\ & A_- \end{pmatrix}$, $|A|=\begin{pmatrix} A_+ & \\ & -A_- \end{pmatrix}$, where $A_\pm:\Gamma_\pm\to\Gamma_\pm$ and $\Gamma_\pm$ is the space spanned by eigenvectors of A corresponding to positive (negative) eigenvalues. The operator $|A|$ has positive eigenvalues only.

Let $A=*d_1,\ B=\mathbb{I}+*\,d_1$.

For the determinants:

$$\det|A|=\prod_{n=1}^{\infty}|\lambda_n|, \qquad\qquad \det|B|=\prod_{n=1}^{\infty}|1+\lambda_n| \qquad (127)$$

we write formal expressions which are always to be ζ-regularized.

The multiplicative anomaly is:

$$
\begin{aligned}
F\big(|A|,|B|\big) &= \frac{d}{ds}\Big[\zeta\big(s,|A|\big)+\zeta\big(s,|B|\big)-\zeta\big(s,|AB|\big)\Big]_{s=0} \\
&= \frac{d}{ds}\Big[\zeta(s,A_+)+\zeta(s,-A_-)+\zeta(s,B_+)+\zeta(s,-B_-) \\
&\quad -\zeta\big(s,(AB)_+\big)-\zeta\big(s,-(AB)_-\big)\Big]_{s=0}. \qquad (128)
\end{aligned}
$$

For all operators entering this expression we can apply the analysis of [15]. This analysis holds for the case of a smooth and compact manifold. The Seeley–De Witt formula:

$$
\zeta(s,U)=\frac{1}{\Gamma(s)}\Big[\sum_{n=0}^{\infty}\frac{A_n}{s+n-\frac{D}{2}}+J(s)\Big], \qquad (129)
$$

where A_n are the heat-kernel co-efficients, D is the dimension of the manifold and $J(s)$ is some analytic function, leads to the fact that the multiplicative anomaly will vanish when $D=2$ or D is odd.

$$
\det'\big(*d_1+(*d_1)^2\big) \;=\; (-1)^{i\pi\psi}\,\det'(*d_1)\,\det'(\mathbb{I}+*d_1), \qquad (130)
$$

where: $\psi=\zeta\Big(0,-\big(*d_1+(*d_1)^2\big)_-\Big)-\zeta\big(0,-(*d_1)_-\big)-\zeta\big(0,-(\mathbb{I}+*d_1)_-\big)$.

Note that the appearance of the phase factor is not due to the multiplicative anomaly. We have used the fact that the multiplicative anomaly vanishes for the moduli of the operators. However, we are forced to include some phase ambiguity which is related to the “negative” parts of the operators — otherwise we would not be able to define a zeta-function regularized expressions for operators which have negative eigenvalues*.

It follows that the partition function of Maxwell–Chern–Simons theory can be written as:

$$
\begin{aligned}
Z_{MCS} &= \frac{e^{-\frac{i\pi}{2}\psi-\frac{i\pi}{4}\eta\left(0,\ *d_1+(*d_1)^2\right)}}{\mathrm{vol(H)}} \ \det{}'(d_0^\dagger d_0)^{1/2}\ \det{}'(*d_1)^{-1/2}\ \det{}'(\mathbb{1}+*d_1)^{-1/2} \\
&= e^{-\frac{i\pi}{2}\psi-\frac{i\pi}{4}\eta\left(0,\ *d_1+(*d_1)^2\right)}\ \mathrm{vol}\Big(\ker(*d_1)\Big)\ \mathrm{vol}\Big(\ker(\mathbb{1}+*\,d_1)\Big) \\
&\qquad \det{}'(*d_1)^{-1/2}\ \det{}'(\mathbb{1}+*d_1)^{-1/2}\,. \qquad (131)
\end{aligned}
$$

The $*$-operator is invertible, hence: $\ker(*d_1) = \ker d_1$.

As in the previous part of the thesis, the stabilizer is the same (the constant functions). Therefore:

$$
\begin{aligned}
Z_{MCS} &= \frac{e^{-\frac{i\pi}{2}\psi-\frac{i\pi}{4}\eta\left(0,\ *d_1+(*d_1)^2\right)}}{\mathrm{vol}\Big(\mathcal{H}^0(\mathrm{M})\Big)}\ \frac{\det{}'(d_0^\dagger d_0)^{1/2}}{\det(\phi_0^\dagger\phi_0)^{1/2}}\ \det{}'(*d_1)^{-1/2}\ \det{}'(\mathbb{1}+*d_1)^{-1/2} \\
&= e^{-\frac{i\pi}{2}\psi-\frac{i\pi}{4}\eta\left(0,\ *d_1+(*d_1)^2\right)+\frac{i\pi}{4}\eta(0,\ \mathbb{1}+*\,d_1)}\ \frac{\mathrm{vol}\Big(\ker(*d_1)\Big)}{\det{}'(*d_1)^{1/2}}\ Z_{SD}. \qquad (132)
\end{aligned}
$$

From this expression follows that the ratio of the partition functions of Maxwell–Chern–Simons theory and the Self–Dual Model is equal, modulo phase factor, to the partition function of pure abelian Chern–Simons theory. Thus:

$$
\frac{Z_{MCS}}{Z_{SD}} = e^{i\alpha}\ Z_{CS}\,, \qquad (133)
$$

*For other examples of phase ambiguities associated with ζ-regularised determinants see for instance [16] and [37].

where

$$\alpha = -\frac{\pi}{2}\zeta\Big(0, -\big(*d_1 + (*d_1)^2\big)_-\Big) + \frac{\pi}{2}\zeta\big(0, -(*d_1)_-\big) + \frac{\pi}{2}\zeta\big(0, -(\mathbb{1} + *d_1)_-\big)$$
$$+ \frac{\pi}{4}\eta(0,\ *d_1) + \frac{\pi}{4}\eta(0,\ \mathbb{1} + *d_1) - \frac{\pi}{4}\eta\big(0,\ *d_1 + (*d_1)^2\big). \quad (134)$$

The partition function of abelian Chern–Simons theory is equal [3], modulo a phase factor [16], to the square root of the Ray–Singer torsion [2] which is a topological invariant of the manifold given by [14]:

$$\tau_{RS}(M) = \prod_{q=0}^{3}\Big(|\det\phi_q|\ |\det' d_q|\Big)^{(-1)^q}. \quad (135)$$

This formula refers for the general case of non-trivial homology.

So the absolute value of ratio of the partition functions of Maxwell–Chern–Simons theory and the Self–Dual Model is independent of the metric of the manifold and consequently these two theories are equivalent to within a phase factor on arbitrary manifolds.

Consider now the partition function of Maxwell–Chern–Simons theory with an external source J coupled to the fields A:

$$Z_{MCS}(J) = \int_{\Omega^1(\mathrm{M})} \mathcal{D}A\ e^{-i\,\langle A,\,(*d_1 + (*d_1)^2)A\rangle\ +\ \langle J,\ A\rangle}. \quad (136)$$

For consistency we require that:

$$J \in \ker\big(*d_1 + (*d_1)^2\big)^{\perp}. \quad (137)$$

Decompose $\Omega^1(\mathrm{M})$:

$$\Omega^1(\mathrm{M}) = \ker\big(*d_1 + (*d_1)^2\big) \oplus \ker\big(*d_1 + (*d_1)^2\big)^{\perp}. \quad (138)$$

Thus:

$$Z_{MCS}(J) = \mathrm{vol}\Big(\ker(*d_1)\Big)\ \mathrm{vol}\Big(\ker(\mathbb{I} + *\,d_1)\Big)$$
$$\int\limits_{\ker\left(*d_1+(*d_1)^2\right)^{\perp}} \mathcal{D}A\ e^{-i\ \left\langle A,\ *d_1(\mathbb{I}\ +*d_1)A\right\rangle\ +\ \langle J,\ A\rangle}. \qquad (139)$$

The integral gives:

$$\int\limits_{\ker\left(*d_1+(*d_1)^2\right)^{\perp}} \mathcal{D}A\ e^{-i\ \left\langle A,\ *d_1(\mathbb{I}\ +*d_1)A\right\rangle\ +\ \langle J,\ A\rangle}$$
$$= \det{}'\Big[i\Big(*d_1 + (*d_1)^2\Big)\Big]^{-1/2}\ e^{i\,\langle J,\ \frac{1}{*d_1\ +\ (*d_1)^2}\ J\rangle}. \qquad (140)$$

We obtain:

$$Z_{MCS}(J) = e^{-\frac{i\pi}{2}\psi - \frac{i\pi}{4}\eta\left(0,\ *d_1+(*d_1)^2\right)}\ \mathrm{vol}\Big(\ker(*d_1)\Big)\ \mathrm{vol}\Big(\ker(\mathbb{I} + *\,d_1)\Big)$$
$$\det{}'(*d_1)^{-1/2}\det{}'(\mathbb{I} + *d_1)^{-1/2}\, e^{i\,\langle J,\ \frac{1}{*d_1+(*d_1)^2}\ J\rangle}. \qquad (141)$$

Here we again identify the Ray–Singer torsion. Namely, with a suitable choice of normalization N:

$$\frac{1}{N}\ \mathrm{vol}\Big(\ker(*d_1)\Big)\ \det{}'(*d_1)^{-1/2} = \tau_{RS}(\mathrm{M})^{1/2}\,. \qquad (142)$$

Hence:

$$Z_{MCS}(J) = e^{-\frac{i\pi}{2}\psi - \frac{i\pi}{4}\eta\left(0,\ *d_1+(*d_1)^2\right)}\ \det{}'(\mathbb{I}+ *d_1)^{-1/2}\,\mathrm{vol}\Big(\ker(\mathbb{I} + *\,d_1)\Big)$$
$$\tau_{RS}(M)^{1/2}\ \ e^{i\,\langle J,\ \frac{1}{*d_1+(*d_1)^2}\ J\rangle} \qquad (143)$$

The determinant entering this expression can be written as:

$$\det{}'(\mathbb{I} + *d_1)^{-1/2} = e^{\frac{i\pi}{4}\eta(0,\ \mathbb{I}+*d_1)}\ \det{}'\Big[-i\,(\mathbb{I} + *d_1)\Big]^{-1/2}$$

$$= e^{\frac{i\pi}{4}\eta(0,\, \mathbb{I}+*d_1)}\; e^{i\,\langle J,\, \frac{1}{\mathbb{I}+*d_1}\, J\rangle} \int\limits_{\ker(\mathbb{I}+*\, d_1)^{\perp}} \mathcal{D}A\; e^{-i\,\left\langle A,\, -(\mathbb{I}+*d_1)A\right\rangle\, +\, \langle J,\, A\rangle} \tag{144}$$

In the integral we now change the variables from A to iA. The Jacobian of this change of variables is $\det'(i\mathbb{I})$ which is a constant and we can absorb it in the normalization factor.

The product of this determinant with the volume element gives (modulo normalization factor) the partition function of the Self–Dual Model with current $\hat{J} = -iJ$. Therefore:

$$Z_{MCS}(J) \;=\; e^{i\alpha - \frac{i\pi}{4}\eta(0,\, *d_1)}\; \tau_{RS}(\mathrm{M})^{1/2}\; e^{i\,\langle J,\, \frac{1}{*d_1}\, J\rangle}\, Z_{SD}(\hat{J}). \tag{145}$$

The first exponent contains the geometrical information of the manifold via the η-function, while the second one yields the linking number of the currents. The partition function of pure abelian Chern–Simons theory in the presence of a current J is:

$$Z_{CS}(J) \;=\; e^{-\frac{i\pi}{4}\,\eta(0,\, *d_1)}\; \tau_{RS}(\mathrm{M})^{1/2}\; e^{i\,\langle J,\, \frac{1}{*d_1}\, J\rangle}. \tag{146}$$

Therefore, at the level of currents, the ratio of the partition functions of Maxwell–Chern–Simons theory and Self–Dual Model is a topological invariant to within a phase factor:

$$\frac{Z_{MCS}(J)}{Z_{SD}(\hat{J})} \;=\; e^{i\alpha}\; \tau_{RS}(\mathrm{M})^{1/2}\; e^{i\,\langle J,\, \frac{1}{*d_1}\, J\rangle} \;=\; e^{i\alpha} Z_{CS}(J), \tag{147}$$

where:

$$\alpha = -\frac{\pi}{2}\zeta\Big(0, -\big(*d_1 + (*d_1)^2\big)_-\Big) + \frac{\pi}{2}\zeta\big(0, -(*d_1)_-\big) + \frac{\pi}{2}\zeta\big(0, -(\mathbb{I} + *d_1)_-\big)$$

$$-\frac{\pi}{4}\eta\big(0,\ *d_1 + (*d_1)^2\big) + \frac{\pi}{4}\eta(0,\ *d_1) + \frac{\pi}{4}\eta(0,\ \mathbb{I} + *\, d_1). \qquad (148)$$

The factor α contains the geometrical information of the manifold.

Note that the correlation functions can be calculated in the usual way by functionally differentiating the partition functions with respect to the external current. Equation (147) allows us to relate the correlation functions of the models.

As an example, let us take the manifold to be S^3 (hence the Ray–Singer torsion is 1 [2]) and let us suppose that the currents do not link. Then we get $Z_{CS}(J) = 1$ and therefore Maxwell–Chern–Simons theory is equivalent to the Self–Dual Model to within a phase factor which captures the geometrical properties of the manifold. If the currents link then the partition functions differ by an additional phase which captures the topological features of the currents.

The main differences between our results and those of earlier authors can be summarized as follows. We consider arbitrary manifolds and show, for the complete theories, the surprising result that the ratio of these two theories is itself a complete topological field theory (i.e. Chern–Simons theory). We also note that when the manifold is $\mathbb{R}^3(S^3)$ and, as considered by earlier authors, with no topological entanglement of currents, then the partition function of the Chern–Simons theory is 1. This result is in exact agreement with the results obtained by earlier authors.

After completion of this work our attention was drawn by P. J. Arias and J. Stephany to [27] and [32]. The relationship between Maxwell–Chern–Simons theory and the Self–Dual Model, without the phase factor, was established in these works by different analyses.

III. A Remark on Schwarz's Topological Field Theory

14 Introduction

In this section, for the sake of completeness, we will review, following [38] Schwarz's method of evaluation of the partition function of Schwarz's Topological Field Theory [3], [38] in terms of the Ray–Singer torsion [2]. As this pattern is of paramount importance for the following section, we will have to repeat some things already discussed in this thesis.

Schwarz's result has turned out to be a very important issue in Topological Quantum Field Theory; for example it is used to evaluate the semi-classical approximation for the Chern–Simons partition function [12], [39], which gives a QFT–predicted formula for an asymptotic limit of the Witten–Reshetikhin–Turaev 3-manifold invariant [40] since this invariant arises as the partition function of the Chern–Simons gauge theory on the 3-manifold [12] (see also [41] for a review of Schwarz's Topological Quantum Field Theory in a general context, and [42] for some explicit results in the case of hyperbolic 3-manifolds.)

The partition function Z of Scwarz's Topological Field theory is a priori a formal, mathematically ill-defined quantity and its evaluation [3], [38] is by formal manipulations which in the end lead to a mathematically meaningful result — the Ray–Singer torsion of the background manifold.

In this part of the thesis we will show that there is an alternative formal evaluation of the partition function which results in the trivial answer $Z = 1$. This result amounts to a novel perspective on analytic torsion: we find that it can be formally written as a certain ratio of volumes of spaces of differential forms which is formally equal to 1 by Hodge duality.

We begin by recalling the evaluation of the partition function

$$Z = \frac{1}{V} \int \mathcal{D}\omega \; e^{-S(\omega)}. \tag{149}$$

of Schwarz's Topological Field Theory [3], [38]. Here V is a normalisation factor to be specified below. The background manifold ("spacetime") M is closed, oriented, riemannian, and has odd dimension $n = 2m+1$. For simplicity we assume m is odd; then the following variant of Schwarz's topological field theory can be considered [38]: the field $\omega \in \Omega^m(\mathrm{M}, \mathrm{E})$ is an m-form on M with values in some flat $O(N)$ vectorbundle E over M. The action functional is

$$S(\omega) = \int_{\mathrm{M}} \omega \wedge d_m \omega = \langle \omega \, , * d_m \omega \rangle. \tag{150}$$

Here $d_p : \; \Omega^p \to \Omega^{p+1} \; \big(\Omega^p \equiv \Omega^p(\mathrm{M}, \mathrm{E})\big)$ is the exterior derivative twisted by a flat connection on E (which we surpress in the notation) and a sum over vector indices is implied in the expression for the action*. A choice of metric on M determines an inner product in each Ω^p, given in terms of the Hodge operator $*$ by

$$\langle \omega,\ \omega' \rangle \;=\; \int_{\mathrm{M}} \omega \wedge * \omega'. \tag{151}$$

Let $\ker S$ denote the radical of the quadratic functional S and $\ker d_p$ the nullspace of d_p. Then $\ker S = \ker d_m$, and after decomposing the integration space as $\Omega^m = \ker S \oplus (\ker S)^{\perp}$ the partition function can be formally evaluated to get

$$Z \;=\; \frac{\mathrm{vol}\,(\ker S)}{V}\ \det{}'\left((*\, d_m)^2\right)^{-1/4} \;=\; \frac{\mathrm{vol}\,(\ker S)}{V}\ \det{}'\,(d_m^{\dagger} d_m)^{-1/4}. \tag{152}$$

(we are ignoring certain phase and scaling factors — see [16] for these). Here $\mathrm{vol}\,(\ker S)$ denotes the formal volume of $\ker S$. The obvious normalisation choice, $V = \mathrm{vol}\,(\ker S)$, does not preserve a certain symmetry property which the partition function has when S is non-degenerate [38]; therefore we do not use this but instead proceed, following Schwarz, by introducing a resolvent for S. For simplicity we assume that the cohomology of d vanishes, i.e. $\mathrm{Im}\, d_p = \ker d_{p+1}$ for all p. Then S has the resolvent

$$0 \longrightarrow \Omega^0 \xrightarrow{d_0} \Omega^1 \xrightarrow{d_1} \dots \longrightarrow \Omega^{m-1} \xrightarrow{d_{m-1}} \ker S \longrightarrow 0, \tag{153}$$

*Note that the action vanishes if m is even.

which we use in the following to formally rewrite $\mathrm{vol}(\ker S)$. As before, the orthogonal decompositions

$$\Omega^p = \ker d_p \oplus (\ker d_p)^\perp \tag{154}$$

give the formal relations

$$\mathrm{vol}(\ker \Omega^p) = \mathrm{vol}(\ker d_p)\ \mathrm{vol}\Big((\ker d_p)^\perp\Big). \tag{155}$$

As we saw before, the maps d_p restrict to isomorphisms $d_p : \ker d_p^\perp \xrightarrow{\cong} \ker d_{p+1}$, giving the formal relations

$$\mathrm{vol}(\ker d_{p+1}) = |\det{}' d_p|\ \mathrm{vol}\Big((\ker d_p)^\perp\Big). \tag{156}$$

Combining what we have so far we get a recursion relation, similar to the one before:

$$\mathrm{vol}(\ker d_{p+1}) = \det{}'(d_p^\dagger d_p)^{1/2}\ \mathrm{vol}(\Omega^p)\ \mathrm{vol}(\ker d_p)^{-1}. \tag{157}$$

Now with a simple induction argument and starting with $\mathrm{vol}(\ker S) = \mathrm{vol}(\ker d_m)$ gives the formal relation:

$$\mathrm{vol}(\ker S) = \prod_{p=0}^{m-1}\Big[\det{}'(d_p^\dagger d_p)^{1/2}\,\mathrm{vol}(\Omega^p)\Big]^{(-1)^p}. \tag{158}$$

A natural choice of normalisation is now*

$$V = \prod_{p=0}^{m-1} V(\Omega^p)^{(-1)^p}. \tag{159}$$

*This choice can be motivated by the fact that, in an analogous finite-dimensional setting, the partition function then continues to exhibit a certain symmetry property which it has when S is non-degenerate [38].

Thus we finally get:

$$Z = \left[\prod_{p=0}^{m-1} \det{}'(d_p^\dagger d_p)^{\frac{1}{2}(-1)^p} \right] \det{}'(d_m^\dagger d_m)^{-1/4}. \tag{160}$$

These determinants can be given well-defined meaning via zeta–regularisation [2], resulting in a mathematically meaningful expression for the partition function. As a simple consequence of Hodge duality, we have $\det{}'(d_p^\dagger d_p) = \det{}'(d_{n-p-1}^\dagger d_{n-p-1})$ (with $n = \dim \mathrm{M}$), which allows us to re-write the partition function as:

$$Z = \tau_{RS}(\mathrm{M})^{1/2}. \tag{161}$$

where

$$\tau_{RS}(\mathrm{M}) = \prod_{p=0}^{n-1} \det{}'(d_p^\dagger d_p)^{\frac{1}{2}(-1)^p} \tag{162}$$

is the Ray–Singer analytic torsion [2]. It is independent of the metric — it depends only on M and d. This variant of Schwarz's result has the advantage that the resolvent is relatively simple. The cases where m need not be odd, and the cohomology of d need not vanish, are covered in [3] (see also [38] for the latter case). Everything we do in the following has a straightforward extension to these more general settings, but for the sake of simplicity and brevity we have omitted this.

15 The Partition Function of Schwarz's Topological Field Theory and the Ray – Singer Torsion as a Volume Ratio Anomaly

We now proceed to derive a different answer for Z to the one above. Our starting point is (155) and (156) which we consider as a formal expression for Z, i.e. we do not carry out the zeta regularisation of the determinants. Instead, we formally write

$$\det{}'(d_p^\dagger d_p)^{1/2} \;=\; \frac{\mathrm{vol}(\ker d_{p+1})}{\mathrm{vol}\Big((\ker d_{p+1})^{\perp}\Big)}. \tag{163}$$

We will also use the whole de Rham complex instead of the resolvent (which is a de Rham complex, "truncated" at a suitable point for calculating the volume of the kernel of the action). Thus we will get information, related to the manifold in general and to the partition function of Schwarz's topological field theory in particular (as it can be expressed in terms of the Ray–Singer torsion).

Substituting (163) in (162) and using (157) we find*:

$$\tau_{RS}(\mathrm{M}) \;=\; \frac{\mathrm{vol}(\Omega^1)\ \mathrm{vol}(\Omega^3)\ \ldots\ \mathrm{vol}(\Omega^n)}{\mathrm{vol}(\Omega^0)\ \mathrm{vol}(\Omega^2)\ \ldots\ \mathrm{vol}(\Omega^{n-1})}. \tag{164}$$

*This relation is obtained without any restriction on m, i.e. for arbitrary odd n.

Formally, the ratio of volumes on the right hand side equals 1, due to

$$\mathrm{vol}\,(\Omega^p) \;=\; \mathrm{vol}\,(\Omega^{n-p}). \tag{165}$$

This is a formal consequence of the Hodge star operator being an orthogonal isomorphism from Ω^p to Ω^{n-p}. (Recall that $\langle *\omega,\, *\omega' \rangle \;=\; \langle \omega,\, \omega' \rangle$ for all $\omega,\, \omega' \in \Omega^p$.) This implies $Z = 1$ due to (161), (164) and (165).

The formal relation (164) shows that analytic torsion can be considered as a "volume ratio anomaly": The ratio of the volumes on the right hand side of (164) is formally equal to 1, but when $\tau_{RS}(\mathrm{M})$ is given well-defined meaning via zeta–regularisation of the determinants, a non-trivial value results in general.

It is also interesting to consider the case where n is even — in this case, using the same arguments, we get in place of (164) the formal relation

$$\frac{\mathrm{vol}\,(\Omega^0)\;\mathrm{vol}\,(\Omega^2)\;\ldots\;\mathrm{vol}\,(\Omega^n)}{\mathrm{vol}\,(\Omega^1)\;\mathrm{vol}\,(\Omega^3)\;\ldots\;\mathrm{vol}\,(\Omega^{n-1})} \;=\; \prod_{p=0}^{n-1} \det{}'(d_p^\dagger d_p)^{\frac{1}{2}(-1)^p} \;=\; 1. \tag{166}$$

The last equality is an easy consequence of Hodge duality and continues to hold after the determinants are given well-defined meaning via zeta–regularisation. On the other hand, the ratio of volumes on the left hand side is no longer formally equal to 1 by Hodge duality.

As a conclusion at this stage of the thesis, we would like to state the following. It is well known that the Ray–Singer invariants on even–dimensional manifolds are all trivial. We give an alternative form of these invariants and show that they are trivial for odd–dimensional manifolds as well, unless a well–defined meaning is given via ζ-regularization. In even dimensions using the fact that the Ray–Singer invariants (expressed in this alternative form) are trivial, we end up with new trivial invariants, given by volume ratios. From here we can formally express volumes of spaces of different p-forms as functions of each other and use that to formally calculate determinants of Laplacians.

16 The Discrete Analogue

It is interesting to see how these resluts extend in a discrete set up. Given a simplicial complex K triangulating M, a discrete version of Schwarz's topological field theory can be constructed which captures the topological quantities of the continuum theory [14], [43]. The discrete theory uses $\widehat{\mathrm{K}}$, the cell decomposition dual to K, as well as K itself. This necessitates a field doubling in the continuum theory prior to discretisation. An additional field ω' is introduced and the original action $S(\omega) = \langle \omega \,, * \, d_m \omega \rangle$ is replaced by the doubled action

$$\widetilde{S}(\omega, \omega') \;=\; \left\langle \begin{pmatrix} \omega \\ \omega' \end{pmatrix}, \begin{pmatrix} 0 & * \, d_m \\ * \, d_m & 0 \end{pmatrix} \begin{pmatrix} \omega \\ \omega' \end{pmatrix} \right\rangle \;=\; 2 \int_{\mathrm{M}} \omega' \wedge d_m \omega. \qquad (167)$$

This theory (known as the abelian BF theory [41]) has the same topological content as the original one; in particular its partition function, $\widetilde{Z}$, can be evaluated in an analogous way to get $\widetilde{Z} = Z^2 = \tau_{RS}(\mathrm{M})$. The discretisation prescription is [14], [43]:

$$(\omega,\, \omega') \;\rightarrow\; (\alpha,\, \alpha') \;\in\; C^m(\mathrm{K}) \times C^m(\widehat{\mathrm{K}}) \tag{168}$$

$$\widetilde{S}(\omega,\, \omega') \;\rightarrow\; \widetilde{S}(\alpha,\, \alpha') \;=\; \Big\langle \begin{pmatrix} \alpha \\ \alpha' \end{pmatrix},\, \begin{pmatrix} 0 & *^{\widehat{\mathrm{K}}}\, d^{\widehat{\mathrm{K}}}_m \\ *^{\mathrm{K}}\, d^{\mathrm{K}}_m & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \alpha' \end{pmatrix} \Big\rangle. \tag{169}$$

Here $C^p(\mathrm{K}) = C^p(\mathrm{K},\, \mathrm{E})$ is the space of p-cochains on K with values in the flat $O(N)$ vectorbundle E; $d^{\mathrm{K}}_p : C^p(\mathrm{K}) \rightarrow C^{p+1}(\mathrm{K})$ is the coboundary operator twisted by a flat connection on E; $C^q(\widehat{\mathrm{K}})$ and $d^{\widehat{\mathrm{K}}}_q$ are the corresponding objects for $\widehat{\mathrm{K}}$; $*^{\mathrm{K}} : C^p(\mathrm{K}) \rightarrow C^{n-p}(\widehat{\mathrm{K}})$ and $*^{\widehat{\mathrm{K}}} : C^q(\widehat{\mathrm{K}}) \rightarrow C^{n-q}(\mathrm{K})$ are the duality operators, induced by the duality between p-cells of K and $(n$–$p)$-cells of $\widehat{\mathrm{K}}$. The p-cells of K and $\widehat{\mathrm{K}}$ determine canonical inner products in $C^p(\mathrm{K})$ and $C^p(\widehat{\mathrm{K}})$ for each p, and with respect to these the duality operators are orthogonal maps. (The definitions and background can be found in [44]; see also [2] and [14].) As before, we are assuming that m is odd and that the cohomology of the flat connection on E vanishes: $\mathrm{H}^*(\mathrm{M},\, \mathrm{E}) = 0$. Then the partition function of the discrete theory, denoted by $\widetilde{Z}_{\mathrm{K}}$, can be evaluated by formal manipulations analogous to those in before (see [14], [43]) and the resulting expression can be written as either

$$\widetilde{Z}_{\mathrm{K}} \;=\; \tau(\mathrm{K}, d^{\mathrm{K}}) \qquad \text{or} \qquad \widetilde{Z}_{\mathrm{K}} \;=\; \tau(\widehat{\mathrm{K}}, d^{\widehat{\mathrm{K}}}), \tag{170}$$

where

$$\tau(\mathrm{K}, d^{\mathrm{K}}) = \prod_{p=0}^{n-1} \det{}' (\partial^{\mathrm{K}}_{p+1} d^{\mathrm{K}}_p)^{\frac{1}{2}(-1)^p} \tag{171}$$

and $\tau(\widehat{\mathrm{K}}, d^{\widehat{\mathrm{K}}})$ is defined analogously. Here $\partial^{\mathrm{K}}_{p+1}$ denotes the adjoint of d^{K}_p (it can be identified with the boundary operator on the (p+1)-chains of K). The quantities $\tau(\mathrm{K}, d^{\mathrm{K}})$ and $\tau(\widehat{\mathrm{K}}, d^{\widehat{\mathrm{K}}})$ coincide; in fact (171) is the Reidemeister combinatorial torsion (also called the R-torsion) of M determined by the given flat connection on E, and is the same for all cell decompositions K of M [2], [45]. (This is analogous to the metric-independence of analytic torsion.) Moreover, the analytic and combinatorial torsions coincide [46], so the discrete partition function in fact reproduces the continuum one:

$$\widetilde{Z}_{\mathrm{K}} = \widetilde{Z}. \tag{172}$$

We now present an analogue of the formal argument which led to $Z = 1$ earlier. Consider

$$\tau(\mathrm{K}, d^{\mathrm{K}})\ \tau(\widehat{\mathrm{K}}, d^{\widehat{\mathrm{K}}}) = \prod_{p=0}^{n-1} \det{}' (\partial^{\mathrm{K}}_{p+1} d^{\mathrm{K}}_p)^{\frac{1}{2}(-1)^p} \det{}' (\partial^{\widehat{\mathrm{K}}}_{p+1} d^{\widehat{\mathrm{K}}}_p)^{\frac{1}{2}(-1)^p}. \tag{173}$$

Using the analogues of (155) and (163) in the present setting,

$$\mathrm{vol}(C^p(\mathrm{K})) = \mathrm{vol}(\ker d^{\mathrm{K}}_p)\, \mathrm{vol}\big((\ker d^{\mathrm{K}}_p)^{\perp}\big) \tag{174}$$

and

$$\det{}' (d^{\dagger}_p d_p)^{1/2} = \frac{\mathrm{vol}(\ker d^{\mathrm{K}}_{p+1})}{\mathrm{vol}\big((\ker d^{\mathrm{K}}_p)^{\perp}\big)}, \tag{175}$$

and the corresponding $\widehat{\mathrm{K}}$ relations, we find an analogue of the formal relation (164):

$$\tau(\mathrm{K}, d^{\mathrm{K}})\ \tau(\widehat{\mathrm{K}}, d^{\widehat{\mathrm{K}}}) \;=\; \frac{\mathrm{vol}\big(C^1(\mathrm{K})\big)\,\mathrm{vol}\big(C^3(\mathrm{K})\big)\,\mathrm{vol}\big(C^n(\mathrm{K})\big)}{\mathrm{vol}\big(C^0(\mathrm{K})\big)\,\mathrm{vol}\big(C^2(\mathrm{K})\big)\,\mathrm{vol}\big(C^{n-1}(\mathrm{K})\big)} \times \frac{\mathrm{vol}\big(C^1(\widehat{\mathrm{K}})\big)\,\mathrm{vol}\big(C^3(\widehat{\mathrm{K}})\big)\,\mathrm{vol}\big(C^n(\widehat{\mathrm{K}})\big)}{\mathrm{vol}\big(C^0(\widehat{\mathrm{K}})\big)\,\mathrm{vol}\big(C^2(\widehat{\mathrm{K}})\big)\,\mathrm{vol}\big(C^{n-1}(\widehat{\mathrm{K}})\big)}. \tag{176}$$

Formally, the right hand side equals 1 due to

$$\mathrm{vol}\big(C^p(\mathrm{K})\big) \;=\; \mathrm{vol}\big(C^{n-p}(\widehat{\mathrm{K}})\big). \tag{177}$$

This is a formal consequence of the duality operator being an orthogonal isomorphism from $C^p(\mathrm{K})$ to $C^{n-p}(\widehat{\mathrm{K}})$ $\big($i.e. $\langle *^{\mathrm{K}}\alpha,\, *\alpha'\rangle = \langle \alpha,\, \alpha'\rangle$ for all $\alpha,\, \alpha' \in C^p(\mathrm{K})\big)$. This implies that, formally,

$$\widetilde{Z}_{\mathrm{K}} \;=\; \Big[\tau(\mathrm{K}, d^{\mathrm{K}})\ \tau(\widehat{\mathrm{K}}, d^{\widehat{\mathrm{K}}})\Big]^{1/2} \;=\; 1. \tag{178}$$

Thus we see that combinatorial torsion can also be considered as a "volume ratio anomaly" in an analogous way to analytic torsion.

Finally, in the n even case it is straightforward to find a combinatorial analogue of the formal relation (166) — we leave this to the reader.

IV. Absence of Cross–Confinement for Dynamically Generated Multi–Chern–Simons Theories

17 Introduction

In a recent paper Cornalba et al. [47] have proposed a novel topological way of confining charged particles. The method uses the special properties of U(1) × U(1) Chern–Simons gauge theory, interacting with external sources in two spatial dimensions, with a scalar Higgs field providing condensates. The idea of the approach is to note that, when charge/flux constraints of a certain type are not satisfied, the fall off of the Higgs fields at infinity will not be fast enough and will lead to configurations with infinite energy; hence, such configurations are confined. The analysis is based on number–theoretic properties of the couplings and charges and shows the intriguing possibility for confinement even for integral charge particles. The confinement mechanism is topological in origin.

A Chern–Simons term of the form considered in [47] can be dynamically generated as the parity–breaking part of the low–momentum region of the effective action of a three–dimensional U(1) × ... × U(1) Maxwell gauge filed theory with fermions, after integrating out the fermionic degrees of freedom [18].

Indeed, we carry out this procedure for the system at non-zero temperature. The effective action, obtained by us, following the approach of [18] has the correct temperature dependence for the multiple U(1) Chern–Simons term and yields in its zero-temperature limit a multiple Chern–Simons term of the form considered in [47], [48], [49].

Such multiple U(1) gauge theories have been considered before, for example: in the study of spontaneously broken abelian Chern–Simons theories [48], [49]; in the study of two–dimensional superconductivity without parity violation [50].

Our original motivation was to investigate if the mechanism for cross–confinement, proposed in [47], continues to hold for the system with a temperature slightly deviated from zero and if confinement is lost for high temperature with the system still in the Higgs phase.

Surprisingly, with this dynamically generated parity–breaking term, the arguments of Cornalba et al. [47] do not hold, namely, the proposed scheme of confinement is not possible. This result is valid, as we show, for zero and non–zero temperatures. In this model it is not possible to eliminate the screening of the long–range Coulomb interactions. We claim that, if confinement occurs, it happens when the broken $U(1) \times \ldots \times U(1)$ gauge symmetry is restored in at least one of the directions of the gauge group.

By the standard Higgs mechanism, the gauge group is spontaneously broken down to a product of the cyclic groups $Z_1 \times \ldots \times Z_N$. This residual symmetry represents the non-trivial holonomy of the Goldstone boson. The photon fields $A_\mu^{(i)}$, $i = 1, ..., N$ now acquire masses by their coupling to the Goldstone bosons. In the broken Higgs phase the Higgs currents are proportional in magnitude to the massive vector fields and screen the Coulomb interaction and we are left with purely quantum Aharonov–Bohm interactions [48], [49]. In this phase, at temperature well below the critical, all conserved charges can reside in the zero–momentum mode due to the bosonic character of the particles. When the temperature increases, some of the charges get excited out of the condensate and at sufficiently high temperature the condensate becomes thermally disordered and the symmetry is restored. When this happens the charges introduced by the matter currents will not be screened and the energy of the Coulomb field will logarithmically diverge with distance (in two spatial dimensions) and this will lead to confinement.

18 Spontaneous Symmetry Breaking

In this section we will briefly describe, following [51], the physics of a phenomenon known as Spontaneous Symmetry Breaking. Consider a system, which has a certain stable symmetric configuration. Say, the temperature (this choice is motivated by our further analysis) is such a parameter in this theory, that in a certain low temperature range this symmetric configuration persists. Above some critical value of the temperature, this symmetric configuration becomes unstable and the new ground state is no longer symmetric (as it was for lower values of the temperature). In classical field theory this corresponds to a global symmetry of the Lagrangian (it is invariant under a symmetry group G), with ground state not obeying this symetry (it is invariant under a subgroup H of the group G). This leads to the appearance of massless particles (called Goldstone particles). These particles are not necessarily observable. The bigger the subroup H (i.e. the less degenerate the vacuum), the smaller the number of the Goldstone particles. Actually, the number of the Goldstone particles is equal to the dimension of the coset $G/_H$ and does not depend on the representation of G and on the form of the potentail term. The Goldstone particles could be bosons or fermions (in certain supersymmetric theories). As the number of the degrees of freedom in the theory is preserved, the appearance of Goldstone particles leads to disappearance of some of the original particles. If we start off with scalar fields (massless or massive — each have one degree of freedom), then the number of the surviving scalar fields together with the Goldstone particles equals the original number of scalar fields.

Upon quantization, there are some subtleties, but in general, the ideas extend easily from the classical case.

It is very interesting to consider the case when the symmetry is local (gauge symmetry). This phenomenon is known as Higgs mechanism. When talking about scalar fields and local symmetries, gauge fields A_μ should be introduced in order to guarantee the invariance of the Lagrangian under this gauge symmetry. These gauge fields enter via the covariant derivatives of the scalar fields. A gauge–invariant kinetic term $F_{\mu\nu}F^{\mu\nu}$ does no harm to the gauge–invariance of the Lagrangian and can also be added. The photons, introduced by A_μ, have two degrees of freedom (each) and are *á priori* massless. As a result of the Spontaneous Symmetry Breaking, the Goldstone bosons "eat" the massless photon and this amounts to the appearance of *massive* photon fields and disappearance of some of the original scalar fields, necessary to produce Goldstone bosons, the Goldstone bosons themselves and some of the massless photons. What counts in this mechanism is not the preservation of the number of particles, but the number of the degrees of freedom (a massive photon has three degrees of freedom).

As we have mentioned before, the Spontaneous Symmetry Breaking mechanism is another way of endowing particles with masses (cf. topologically massive gauge theories) and these two known mechanisms can work together.

19 Finite Temperature Field Theory

The partition function of a statistical system is:

$$Z(\beta) \;=\; \mathrm{Tr}\, e^{-\beta H}, \tag{179}$$

where β is the inverse of the equilibrium temperature $\left(\beta = \frac{1}{kT}\right)$ and H is the ensemble Hamiltonian. The trace is taken over some complete basis. Generally, this partition function cannot be evaluated exactly. Matsubara formalism [52] in Temperature Field Theory shows how to perturbatively calculate this partition function. We would like to give, following [53], a short introduction to one of the reincarnations of Matsubara formalism — the path integral approach. In 2+1 dimensions the action is:

$$S(\varphi) \;=\; \int_{t_1}^{t_2} dt \int d^2x\, \mathcal{L}(\varphi), \tag{180}$$

where $\mathcal{L}(\varphi)$ is the Lagrangian density and φ is the quantum field. Let us identify

$$t_1 - t_2 \;=\; i\beta, \tag{181}$$

that is, let us limit the time integration to integration over a finite interval. The time dependence is called temperature dependence. The partition function is then given by:

$$Z(\beta) \;=\; \mathrm{Tr}\, e^{-\beta H} \;=\; \int \mathcal{D}\varphi\, e^{-S_E}, \tag{182}$$

where S_E is the Euclidean action. To make the propagator dependent non-trivially on the tempreature, one should introduce (anti) periodic boundary conditions for the fundamental fields of the theory. Since the fields are defined only within a finite time interval, we can Fourier expand them over a set of *discrete*, or Matsubara, frequencies $\omega_n = \frac{n\pi}{\beta}$, where $n = 0, \pm1, \pm2, \ldots$. For boson fields one should take only the even frequencies and for fermion fileds — only the odd ones. Since the remaining spatial coordinates are still continuous, it is now easy to calculate the partition function using diagramatic methods, similar to those in zero temperature quantum field theory (see [53] and the references therein). The only difference would be in the fact that the propagator has all the temperature dependence in it.

20 Induced Parity–Breaking Term at Finite Temperature

Unless the co-efficient of the Chern–Simons term is quantized, the Chern–Simons term is not invariant under large gauge transformations, that is, transformations with non-zero winding number. At finite temperature, when the euclidean time (temperature) is compactified, we have non-trivial S^1 geometry and the quantization law of the Chern–Simons co-efficient plays an important role. Gauge field theories with charged fermions can be made gauge–invariant under both small and large gauge transformations at any temperatures (by ζ-regularization of the fermionic determinant). The price for that is the appearance of parity anomaly, in the form of the Chern–Simons term, after integrating out the fermionic degrees of freedom [18]. At zero temperature the Chern–Simons term enters with a quantized co-efficient, so there are no problems with large gauge invariance. At non-zero temperature it was firstly believed that the Chern–Simons co-efficient remains unchanged (with this resulting in preservation of the gauge invariance). However, this co-efficient turns out to be a smooth function of the temperature and thus one might expect a gauge anomaly appearing (as this part of the action will no longer be invariant under large gauge transformations). In [18] it is shown that, for both the abelian and non-abelian cases, although the perturbative expansion leads to a non-quantized temperature-dependent Chern–Simons co-efficient, the whole action is still invariant under large gauge transformations — there is no clash between temperature dependence and gauge invariance — the violation of the gauge invariance by the Chern–Simons term is compensated by non-local higher order terms in the perturbative expansion. In the next section we will briefly describe a confinement mechanism, proposed by Cornalba *et al.* [47], which involves a φ^4 scalar field theory coupled to a Chern–Simons term, and in the section following this one we will incorporate massive fermions instead of this Chern–Simons term in order to investigate the possibilities for confinement at any temperature, after dynamically generating a Chern–Simons term (when integrating out the fermionic degrees of freeedom).

21 Cross–Confinement in Multi–Chern–Simons Theories

Cornalba *et al.* [47] proposed a classical mechanism of confinement in 2+1 dimensions based on number–theoretic properties of the charges and the currents involved in the model. This possibility for confinement in two spatial dimensions is the following [47]: the Coulomb field of a charged particle decays as $1/r$ and the field energy diverges logarithmically at large distances. The proposed mechanism in its simplest form goes as follows [47]: a $\mathrm{U}(1)_A \times \mathrm{U}(1)_B$ gauge theory with off-diagonal Chern–Simons term is considered. As a result, an electric charge with respect to one gauge group induces a magnetic flux with respect to the other. If we have a condensed scalar field, charged with respect to gauge group $\mathrm{U}(1)_A$, it will quantize the $\mathrm{U}(1)_A$ flux. If the values of this quantized flux are not in accordance to the value, "desired" by the electric charge, then this electric charge will be confined. Otherwise, the Coulomb interaction will be screened and the electric charge will not be confined. The general case of Chern–Siomns coupling (as 2×2 and 3×3 matrices with arbitrary entries) is also considered and the conditions for confinement are given after similar (slightly more complicated) considerations. In the following section we will show that this mechanism does not work if the Chern–Simons term is dynamically generated in the sense discussed above.

22 Absence of Cross–Confinement for Dynamically Generated Multi–Chern–Simons Theories

We will first determine the parity–breaking part of the effective action for U(1) $\times \ldots \times$ U(1) Maxwell gauge field theory coupled to massive fermions and ϕ^4 scalar field theory in 3 dimensions at finite temperature. Contact with the multiple Chern–Simons term, considered in [47], [48], [49], is made by taking the zero-temperature limit. The effective action for the low–momentum region of the theory is:

$$e^{-\Gamma(A^{(k)},M_k)} = \int \prod_{k=1}^{N} \mathcal{D}\psi_k\, \mathcal{D}\bar{\psi}_k\, \mathcal{D}\phi \ \exp\Big\{ -\int_0^\beta d\tau \int d^2x \Big[\sum_{k=1}^{N} \Big(\bar{\psi}_k \not{D}_k^f \psi_k + j^{(k)} A^{(k)} \Big) +$$

$$+ (D_\alpha\phi)(D^\alpha\phi)^* - m^2\phi\phi^* - \lambda(\phi\phi^*)^2\Big]\Big\}, \qquad (183)$$

where $\not{D}^f_k = \not\partial + iQ_{kj}\not{A}^{(j)} + M_k$ are the fermionic covariant derivatives with Q_{kj}, $k, j = 1, \ldots, N$ being the matrix of the fermionic charges with respect to the N gauge groups, $D_\alpha = \partial_\alpha + iq_k A^{(k)}_\alpha, k = 1, \ldots, N$ are the covariant derivatives for the scalar field with q_i being the charge of the scalar field with respect to the i^{th} gauge group. In this action $\beta = \frac{1}{T}$ is the inverse temperature and Dirac matrices are in the representation $\gamma_\mu = \sigma_\mu$. We have also introduced external currents coupled to the gauge fields.

We shall consider first the parity-breaking part of the fermionic part of the action and at this stage the scalar field is only a spectator.

For this purpose we will follow the approach of Fosco *et al.* [18].

The fermionic fields obey antiperiodic boundary conditions, while the gauge fields are periodic. The considered class of configurations for the gauge fields is:

$$A^{(k)}_3 = A^{(k)}_3(\tau), \quad A^{(k)}_{1,2} = A^{(k)}_{1,2}(x), \quad k = 1, \ldots, N. \qquad (184)$$

There is a family of gauge transformation parameters, which allow us to gauge the time-components $A^{(i)}_3(\tau)$ to the constants $a^{(i)}$ [18]. This makes the Dirac operator invariant under translations in the time coordinate (as the dependence on τ comes solely from the $A^{(i)}_3$ fields) and therefore we could Fourier–expand ψ_i and $\bar\psi_i$ over the Matsubara modes. As we mentioned above, the fermion fields are expanded over the odd frequencies:

$$\psi_i(\tau, x) = \frac{1}{\beta}\sum_{n=-\infty}^{\infty} e^{i\omega_n\tau}\ \psi_n^{(i)}(x), \tag{185}$$

$$\bar{\psi}_i(\tau, x) = \frac{1}{\beta}\sum_{n=-\infty}^{\infty} e^{-i\omega_n\tau}\ \bar{\psi}_n^{(i)}(x), \tag{186}$$

where $\omega_n = (2n+1)\frac{\pi}{\beta}$. For a single U(1) theory with one fermion field and without scalars we have [18]:

$$\begin{aligned}
\det(\not{\partial} + ie\not{A} + M) &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ \exp\Big[-\int_0^\beta d\tau \int d^2x\ \bar{\psi}\,(\not{\partial} + ie\not{A} + M)\,\psi\Big] \\
&= \int \prod_{n=-\infty}^{\infty} \mathcal{D}\psi_n(x)\mathcal{D}\bar{\psi}_n(x)\ \exp\Big[\frac{1}{\beta}\sum_{n=-\infty}^{\infty}\int d^2x\,\bar{\psi}_n(x) \\
&\qquad \times\ \big(\not{d} + M + i\gamma_3(\omega + e\tilde{A}_3)\big)\psi_n(x)\Big] \\
&= \prod_{n=-\infty}^{\infty} \det\big(\not{d} + \rho_n\, e^{i\gamma_3\phi_n}\big),
\end{aligned} \tag{187}$$

where $\tilde{A}_3 = \frac{1}{\beta}\int_0^\beta A_3(\tau)\,d\tau$, $\not{d} = \gamma_j(\partial_j + ieA_j)$ is the Dirac operator in the remaining spatial coordinates, $\rho_n = \sqrt{M^2 + (\omega + e\tilde{A}_3)^2}$ and $\phi_n = \mathrm{arctg}\Big(\frac{\omega + e\tilde{A}_3}{M}\Big)$. It is shown in [18] that:

$$\begin{aligned}
\det(\not{\partial} + ie\not{A} + M) &= \prod_{n=-\infty}^{\infty} \det\Big(\not{d} + M + i\gamma_3(\omega + e\tilde{A}_3)\Big) \\
&= \prod_{n=-\infty}^{\infty} J_n(A, M)\ \det(\not{d} + \rho_n),
\end{aligned} \tag{188}$$

where $J_n(A, M)$ is the anomalous Fujikawa jacobian [54]:

$$J_n(A, M) = \exp\Big[-\frac{ie\phi_n}{2\pi}\int d^2x\,\epsilon_{jk}\partial_j A_k\Big]. \tag{189}$$

The parity–odd part of the action is then given by [18]:

$$\Gamma_{odd} = -\sum_{n=-\infty}^{\infty} \ln J_n(A, M) = \frac{ie}{2\pi}\sum_{n=-\infty}^{\infty} \phi_n \int d^2x\,\epsilon_{jk}\partial_j A_k$$

$$= \frac{ie}{2\pi} \frac{M}{|M|} \operatorname{arctg}\Big[\operatorname{th}\Big(\frac{\beta|M|}{2}\Big) \operatorname{tg}\Big(\frac{e}{2} \int_0^\beta d\tau\, A_3(\tau)\Big)\Big] \int d^2x\, \epsilon_{jk}\partial_j A_k \quad (190)$$

and

$$\lim_{T\to 0} \Gamma_{odd} = \frac{1}{2} \frac{M}{|M|} S_{CS}. \quad (191)$$

In [18] the sumation was performed in the following way:

$$\sum_{n=-\infty}^{\infty} \operatorname{arctg}\Big(\frac{\omega + e\tilde{A}_3}{M}\Big) = \frac{M}{|M|} \sum_{n=-\infty}^{\infty} \operatorname{arctg}\Big[\frac{(2n+1)\pi + x}{y}\Big], \quad (192)$$

where $x = e\beta\tilde{A}_3$ and $y = \beta|M|$ are dimensionless parameters. Then we have:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \operatorname{arctg}\Big(\frac{\omega + e\tilde{A}_3}{M}\Big) &= \frac{M}{|M|} \sum_{n=-\infty}^{\infty} \int_0^x du \frac{d}{du} \operatorname{arctg}\Big[\frac{(2n+1)\pi + u}{y}\Big] \\ &= \frac{M}{|M|} \int_0^x du \sum_{n=-\infty}^{\infty} \frac{y}{y^2 + [(2n+1)\pi + u]^2}. \end{aligned} \quad (193)$$

The summation is done [18] with the formula:

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n-x_1)(n-x_2)} = \frac{\pi[\operatorname{cotg}(\pi x_1) - \operatorname{cotg}(\pi x_2)]}{x_1 - x_2}. \quad (194)$$

Following similar steps, one finds that the parity-odd bit of the fermion part of our effective action is given by:

$$\Gamma_{odd} = \frac{i}{2\pi} \sum_{k,j=1}^{N} \sum_{n=-\infty}^{+\infty} \phi_n^{(k)} \int \epsilon_{lm} Q_{kj} \partial_l A_m^{(j)} d^2x, \quad (195)$$

where $\phi_n^{(k)} = \operatorname{arctg}\Big(\frac{\omega_n + Q_{kj}a^{(j)}}{M_k}\Big)$ and the Matsubara frequenies ω_n are given above.

Performing the summation we get that for U(1) $\times \ldots \times$ U(1) gauge group the parity-odd part of the action is:

$$e^{-\Gamma_{odd}} = \int \mathcal{D}\phi \, \exp\Big\{-\frac{i}{2\pi}\sum_{k,j,n=1}^{N} \operatorname{arctg}\Big[\operatorname{th}\Big(\frac{\beta M_k}{2}\Big)\, \operatorname{tg}\Big(\frac{1}{2}\int_0^{\beta} Q_{kj}A_3^{(j)}(\tau)d\tau\Big)\Big] \times \int \epsilon_{lm}Q_{kn}\partial_l A_m^{(n)} d^2x + \int_0^{\beta} d\tau \int \Big[\,(D_\alpha\phi)(D^\alpha\phi)^* - m^2\phi\phi^* - \lambda(\phi\phi^*)^2 + j^{(k)}A^{(k)}\Big]d^2x\Big\}. \tag{196}$$

As the temperature T approaches 0 (that is, $\beta \to \infty$) this reduces to U(1) $\times \ldots \times$ U(1) Chern–Simons gauge theory.

We will use now the effective parity–odd temperature dependent action (with the induced U(1) $\times \ldots \times$ U(1) parity breaking term) to re-examine the confinement argument of Cornalba *et al.* [47]. First of all, let us perform the integration (using Stokes' theorem) of the gauge fields over the spatial coordinates. This gives the relation with the magnetic fluxes Φ_l:

$$e^{-\Gamma_{odd}} = \int \mathcal{D}\phi \, \exp\Big\{-\frac{i}{2\pi}\sum_{k,j,n=1}^{N} \operatorname{arctg}\Big[\operatorname{th}\Big(\frac{\beta M_k}{2}\Big)\, \operatorname{tg}\Big(\frac{1}{2}\int_0^{\beta} Q_{kj}A_3^{(j)}(\tau)d\tau\Big)\Big] Q_{kn}\Phi_n + \int_0^{\beta} d\tau \int \Big[(D_\alpha\phi)(D^\alpha\phi)^* - m^2\phi\phi^* - \lambda(\phi\phi^*)^2 + j^{(k)}A^{(k)}\Big]d^2x\Big\}. \tag{197}$$

The equations of motion, obtained by varying the action with respect to the magnetic fields, are:

$$-\frac{i}{4\pi}\sum_{k,m,n=1}^{N} \frac{\operatorname{th}\left(\frac{\beta M_k}{2}\right) Q_{kl}\, Q_{kn}\, \Phi_n}{\cos^2\left(\frac{1}{2}\int_0^{\beta} Q_{km}A_3^{(m)} d\tau\right) + \operatorname{th}^2\left(\frac{\beta M_k}{2}\right)\sin^2\left(\frac{1}{2}\int_0^{\beta} Q_{km}A_3^{(m)} d\tau\right)}$$

$$- q_l \int (\phi D_3 \phi^* - \phi^* D_3 \phi) d^2x \; + \; \int \rho^{(l)} d^2x \; = \; \int \partial_j F^{(l)\, j3}\, d^2x, \tag{198}$$

where $\rho^{(l)} = j_3^{(l)}$ are the charge densities. Here we have included explicitely the contribution of the Maxwell term $F^{(i)}_{\mu\nu} F^{(i)\,\mu\nu}$. We ignore temperature dependent terms which come from $O(A^4)$ terms in the effective action. These are of higher order ($\sim Q^4$) in the fermionic chagres. The Coulomb charges on the right hand side vanish because all U(1) fields are massive.

Denote by u the integral over the third component of the conserved Nöther current: $u = \int (\phi D_3 \phi^* - \phi^* D_3 \phi) d^2x$ and by $C^{(l)} = \int \rho^{(l)} d^2x$ the total external charge. So, we have:

$$\mu \Phi = C - uq, \tag{199}$$

where $\Phi = \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_N \end{pmatrix}$, $q = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix}$, $C = \begin{pmatrix} C^{(1)} \\ \vdots \\ C^{(N)} \end{pmatrix}$, and:

$$\mu_{ln} = \frac{i}{4\pi} \sum_{k,m=1}^{N} \frac{\mathrm{th}\left(\frac{\beta M_k}{2}\right) Q_{kl}\, Q_{kn}}{\cos^2\left(\frac{1}{2}\int\limits_0^\beta Q_{km} A_3^{(m)} d\tau\right) \; + \; \mathrm{th}^2\left(\frac{\beta M_k}{2}\right) \sin^2\left(\frac{1}{2}\int\limits_0^\beta Q_{km} A_3^{(m)} d\tau\right)} . \tag{200}$$

As in [47] there is another condition which must be satisfied by the magnetic fluxes. The Higgs field ϕ should be completely condensed, i.e. $\phi(x) = ve^{i\sigma(x)}$, where $\sigma(x)$ is the Goldstone boson field (the mass and the coupling constants of the scalar field are temperature–dependent). In order that this holds we have to require that the covariant derivative of the scalar field vanishes. After integration we get:

$$2\pi l = q_1 \Phi_1 + \ldots + q_N \Phi_N = {}^t q \, \Phi, \tag{201}$$

where $2\pi l$ is the non-trivial holonomy of the Goldstone boson (reflecting a topological property of the Higgs field). Combining the two conditions (7) and (9) for the fluxes we get:

$$\begin{aligned} \mu \Phi &= C - uq, \\ 2\pi l &= {}^t q \, \Phi. \end{aligned} \tag{202}$$

Following the analysis of [47] we identify u as a continuous parameter, representing the ability of the condensate to screen the electric charge.

The matrix μ can be written as $\mu = {}^t Q \, F(\beta) \, Q$, where $F(\beta)$ is a diagonal matrix with entries:

$$F_{kj}(\beta) = -\frac{\delta_{kj}}{4\pi i} \frac{\mathrm{th}\left(\frac{\beta M_k}{2}\right)}{\cos^2\left(\frac{1}{2}\int\limits_0^\beta Q_{km} A_3^{(m)} d\tau\right) + \mathrm{th}^2\left(\frac{\beta M_k}{2}\right) \sin^2\left(\frac{1}{2}\int\limits_0^\beta Q_{km} A_3^{(m)} d\tau\right)} . \tag{203}$$

As $F(\beta)$ is diagonal we can always write μ in the form:

$$\mu = {}^t Q(T) \, Q(T), \tag{204}$$

where $Q_{mn}(T) = F^{1/2}_{mj}(\beta)\, Q_{jn}$.

Let us now try to eliminate the screening in (10) by inverting the matrix μ. We get that if the determinant of μ is not zero and if

$$ {}^t q\, \mu^{-1} q \;=\; 0, \qquad (205) $$

then the screening would be eliminated (the condition ${}^t q\ \mu^{-1} q = 0$ is the condition for confinement, proposed by Cornalba et al. [47]. According to their analysis, if the determinant of μ vanishes, then μ^{-1} should be interpreted as the transposed matrix of co-factors).

Assuming that the determinant of μ is not zero, we can re-write this as:

$$ {}^t\!\left(\tilde{Q}(T)q\right)\, \tilde{Q}(T)q \;=\; 0, \qquad (206) $$

where $\tilde{Q}(T)$ is the matrix of co-factors. This equation shows that the vector $\tilde{Q}(T)q$ is orthogonal to itself ("orthogonal" with respect to the matrix multiplication of column vectors) and, therefore, this is the null vector:

$$ \tilde{Q}(T)\, q \;=\; 0. \qquad (207) $$

This is an equation for the values of the boson field charges, which would eliminate the screening mechanism. As we see, we can have a non–trivial solution if, and only if, $\det Q = 0$, which contradicts to our initial assumption ($\det \mu \neq 0$). Therefore, we cannot eliminate the screening. Otherwise, this theory would be inconsistent with the induced parity–breaking term. This argument is valid for all values of the temperature.

Intuitively, one can expect that if condition (201) is violated, after elimination of screening, there would be currents which would not fall off faster than $1/r$ at infinity and the resulting long–range forces will lead to diverging energies. We argue that condition (201) can never be violated — this condition represents the fact that we are left with a residual symmetry after the spontaneous symmetry breakdown. If this condition does not hold, it would mean that the symmetry is restored. This, on its turn, will lead to diverging energy straight away, but not in the broken Higgs phase.

We conclude that confinement is not possible in the Higgs phase in the presence of the dynamically generated parity–breaking term (which coincides with Chern–Simons term in zero–temperature limit). If there are configurations with infinte energy, they must necessarily be outside the broken Higgs phase — where the gauge symmetry is restored.

V. Operator Formalism for Chern–Simons Theories

23 Introduction

Take a three–dimensional manifold M, which is a connected sum of two pieces M_1 and M_2. The boundaries of each of the pieces are the same, just the orientations are opposite and the corresponding Hilbert spaces are canonically dual to each other. The path integral in each of the ingredients determines vectors in the Hilbert spaces and, "according to the general ideas of quantum field theory", Witten introduces [12] the partition function of the theory as the product

$$Z(\mathrm{M}) = \langle \chi, \psi \rangle, \tag{208}$$

where χ belongs to the Hilbert space associated with the boundary of M_1 and ψ belongs to the Hilbert space associated with the boundary of M_2. The essential part is to construct the states χ and ψ.

We will show how this idea is applied to Chern–Simons theory, following the work of Labastida and Ramallo [55] and of Bos and Nair [56].

We will start with an oriented compact three–dimensional surface without boundary M and a U(1) bundle E with connection A_μ. Abelian Chern–

Simons theory is given by:

$$Z(\mathrm{M}) \;=\; \int [\mathcal{D}A_\mu]\; e^{ikS(A_\mu)}, \tag{209}$$

where

$$S(A_\mu) \;=\; \frac{1}{2\pi} \int\limits_{\mathrm{M}} A \wedge dA. \tag{210}$$

The path integral is over the gauge orbits and k is an arbitrary integer guaranteeing invariance with respect to gauge transformations.

The choice of vectors χ and ψ in [55] is as follows:

$$\Psi(A_{\bar{z}}) \;=\; \int\limits_{\mathrm{M}_1} [\mathcal{D}A_\mu]\; \exp\Big[\frac{ik}{2} S(A_\mu) - \frac{k}{2\pi} \int\limits_{\partial\mathrm{M}_1} d^2\sigma\, A_z A_{\bar{z}}\Big], \tag{211}$$

$$\Phi(A_z) \;=\; \int\limits_{\mathrm{M}_2} [\mathcal{D}A_\mu]\; \exp\Big[\frac{ik}{2} S(A_\mu) + \frac{k}{2\pi} \int\limits_{\partial\mathrm{M}_2} d^2\sigma\, A_z A_{\bar{z}}\Big]. \tag{212}$$

Here a holomorphic representation is chosen: $A_z = \frac{1}{2}(A_1 - iA_2)$ is fixed on $\partial\mathrm{M}_2$ and $A_{\bar{z}} = \frac{1}{2}(A_1 + iA_2)$ is fixed on $\partial\mathrm{M}_1$. A_0 is taken to be orthogonal to the boundary and σ are the local coordinates on the Riemann surface $\Sigma = \partial\mathrm{M}_1 = -\partial\mathrm{M}_2$.

The partition function is then [55]:

$$Z(\mathrm{M}) = \langle \Psi(A_{\bar{z}}), \Phi(A_z) \rangle = \int\limits_{\mathrm{M}} [\mathcal{D}A_z \mathcal{D}A_{\bar{z}}]\; e^{\frac{k}{\pi}\int\limits_{\Sigma} d^2\sigma\, A_z A_{\bar{z}}}\; \Psi(A_{\bar{z}})\, \Phi(A_z). \tag{213}$$

We have:

$$\Psi(A_{\bar{z}}) \;=\; \overline{\Phi(A_z)} \tag{214}$$

and [56]:

$$[A_z(\omega,\bar{\omega}),\ A_{\bar{z}}(\omega',\bar{\omega}')] \;=\; \frac{\pi}{k}\,\delta(\omega-\omega')\,\delta(\bar{\omega}-\bar{\omega}'). \tag{215}$$

Therefore

$$A_z \;=\; \frac{\pi}{k}\,\frac{\delta}{\delta A_{\bar{z}}}. \tag{216}$$

We will now determine the vectors $\Psi(A_{\bar{z}})$ and $\Phi(A_z)$ using the symmetries of the model. Under a gauge transformation (say on M_1) $A_\mu \longrightarrow A_\mu + g^{-1}\partial_\mu g$, the vector $\Psi(A_{\bar{z}})$ transforms as [55]:

$$\Psi(A_{\bar{z}}) \;\longrightarrow\; \exp\Big[-k\Big(\gamma(g) + \langle u_a,\, g\rangle\Big)\Big]\;\Psi(A_{\bar{z}}), \tag{217}$$

where

$$\gamma(g) \;=\; \frac{1}{2\pi}\int\limits_{\Sigma} d^2\sigma\; g^{-1}\partial_z g\; g^{-1}\partial_{\bar{z}}\, g \tag{218}$$

and

$$\langle u_a,\, g\rangle \;=\; \frac{1}{\pi}\int\limits_{\Sigma} d^2\sigma\; A_{\bar{z}}\,\partial_z g\, g^{-1}. \tag{219}$$

The gauge transformations are classified by the winding number around non-contractible loops in the manifold. In [55] it is assumed that M_1 is a solid ball with g handles for which a canonical set of closed contours $\{\alpha_i,\ \beta_j \mid i,j = 1,\ldots,g\}$ is chosen. Let $\omega_i = \omega_i(z)dz$ be the basis for the space of holomorphic one–forms, defined via $\int_{\alpha_i}\omega_j = \delta_{ij}$ and let $\tau_{ij} = \int_{\beta_i}\omega_j$.

We also have $\int d^2z \; \omega_i(z) \; \overline{\omega_j(z)} \;\; = \;\; -2i \;\, \mathrm{Im}\,\tau_{ij}$ (the period matrix τ is a symmetric $g \times g$ matrix with positive definite imaginary part). We can therefore parametrize the gauge fields [55]:

$$A_{\bar{z}} \;\; = \;\; (u_a \, u)^{-1} \; \partial_{\bar{z}} \, (u_a \, u), \tag{220}$$

where u is single–valued map connected to the identity map and

$$u_a \;\; = \;\; \exp\Big[\pi \int^{\bar{z}} \overline{\omega(z)} \; (\mathrm{Im}\,\tau)^{-1} \; a \;\; - \;\; \pi \, \bar{a} \, (\mathrm{Im}\,\tau)^{-1} \int^{\bar{z}} \omega(z)\Big]. \tag{221}$$

We can therefore choose [55]:

$$\Psi_p(A_{\bar{z}}) \;\; = \;\; \xi \, e^{-k\gamma(u)} \, \psi_p(a), \tag{222}$$

where ξ is some a-independent constant, $\gamma(u)$ is defined above and the set of functions $\{\psi_p(a) \mid p = 1, \ldots, k^g\}$ is given by:

$$\psi_p(a) \;\; = \;\; e^{\frac{k\pi}{2} a \,.\, (\mathrm{Im}\,\tau)^{-1} \,.\, a} \; \Theta\begin{bmatrix} p/k \\ 0 \end{bmatrix}(ka|k\tau). \tag{223}$$

Here $\Theta[{}^{p/k}_{\;0}](ka|k\tau)$ is the Jacobi theta–function with characteristics (for a deep analysis see [57]). As we see, we got not one, but a whole basis of vectors $\Psi_p(A_{\bar{z}})$ — these are the Wilson line operators around non-contractible cycles.

24 On the Chern–Simons Parameter k in

$$Z(\mathrm{M}) = \int [\mathcal{D}A] \exp\Big(ik \int_{\mathrm{M}} A \wedge d\,A\Big)$$

We will now show that the parameter k must be even.

It is enough to consider:

$$A_{\bar{z}} \;=\; u_a^{-1}\,\partial_{\bar{z}}\,u_a. \tag{224}$$

Thus, under gauge transformation, we have:

$$A_{\bar{z}} \;\longrightarrow A_{\bar{z}}^{g} \;=\; u_a^{-1}\,\partial_{\bar{z}}\,u_a \;+\; g^{-1}\,\partial_{\bar{z}}\,g. \tag{225}$$

Therefore [55]:

$$\Psi(A_{\bar{z}}) \;=\; e^{-\gamma_{2k}(u_a)}, \tag{226}$$

$$\Psi(A_{\bar{z}}^{g}) \;=\; e^{-\gamma_{2k}(u_a g)}, \tag{227}$$

where $\gamma_{2k}(u_a)$ should be such that any single–valued map g must satisfy

$$\gamma_{2k}(u_a g) \;=\; \gamma_{2k}(u_a) \;-\; k\Big(\gamma(g) \;+\; \langle u_a,\, g\rangle\Big). \tag{228}$$

Under large gauge transformations with a map which winds n_i times around β_i and m_j times around α_j [55]:

$$g \,=\, \exp\Big[-\pi(n + m\bar{\tau})(\mathrm{Im}\,\tau)^{-1}\int^{z}\omega(z) \,+\, \pi\int^{\bar{z}}\overline{\omega(z)}\,(\mathrm{Im}\,\tau)^{-1}\,(n + \tau m)\Big] \tag{229}$$

we have:

$$u_a \;\longrightarrow\; u_{a+n+\tau m} \;=\; u_a\,g \tag{230}$$

and

$$\exp\Big[-\gamma_{2k}(u_{a+n+\tau m})\Big] = \exp\Big[\frac{k\pi}{2}\Big(2n(\mathrm{Im}\,\tau)^{-1}a + n(\mathrm{Im}\,\tau)^{-1}n + n(\mathrm{Im}\,\tau)^{-1}m\tau$$
$$+ 2m\bar{\tau}(\mathrm{Im}\,\tau)^{-1}a + m\bar{\tau}(\mathrm{Im}\,\tau)^{-1}n$$
$$+ m\bar{\tau}(\mathrm{Im}\,\tau)^{-1}m\tau\Big)\Big]\exp\Big[-\gamma_{2k}(u_a)\Big]. \qquad (231)$$

As far as the behaviour of $\psi_p(a)$ under large gauge transformations, we have:

$$\exp\Big[\frac{k\pi}{2}\, a\,(\mathrm{Im}\,\tau)^{-1}a\Big]$$
$$\longrightarrow \exp\Big[\frac{k\pi}{2}\,\Big(m\tau(\mathrm{Im}\,\tau)^{-1}a + m\tau(\mathrm{Im}\,\tau)^{-1}n + m\tau(\mathrm{Im}\,\tau)^{-1}m\tau\Big)\Big] \qquad (232)$$

and [57]:

$$\Theta\Big[{q \atop 0}\Big](a+n|\tau) = \Theta\Big[{q \atop 0}\Big](a|\tau), \qquad (233)$$
$$\Theta\Big[{q \atop 0}\Big](a+m\tau|\tau) = \exp\Big(-i\pi m\tau m - 2\pi i m a\Big)\,\Theta\Big[{q \atop 0}\Big](a|\tau). \qquad (234)$$

In order to match the transformation law (231) to the transformation laws (232), (233) and (234), we have to make sure that $\exp\Big(\frac{k\pi}{2}\, imn\Big)$ matches to $\exp\Big(-\frac{k\pi}{2}\, imn\Big)$. That is, we have to make sure that for arbitrary integers m and n we have:

$$\exp\Big(k\pi imn\Big) = 1. \qquad (235)$$

Thus the Chern–Simons parameter must be even.

References

[1] A. S. Schwarz: *Quantum Field Theory and Topology*, Springer–Verlag, Berlin (1993);

[2] D. B. Ray and I. M. Singer: Adv. Math. **7**, 145 (1971);

[3] A. S. Schwarz: Lett. Math. Phys. **2**, 247 (1978);
A. S. Schwarz: Commun. Math. Phys. **67**, 1 (1979);

[4] E. Witten: Lectures given at 1996–1997 Quantum Field Theory program at IAS, Princeton, USA, in: P. Deligne, P. Etingof, D. S. Freed, L. Jeffrey, D. Kazhdan, J. Morgan, D. R. Morrison and E. Witten, eds: *Quantum Fields and Strings: A Course for Mathematicians*, American Mathematical Society, Providence (1999);

[5] S. S. Chern and J. Simons: Ann. Math. **99**, 48 (1974);

[6] L. D. Faddeev and V. N. Popov: Phys. Lett. B**25**, 29 (1967);

[7] M. Nakahara: *Geometry, Topology and Physics*, Institute of Physics Publishing Ltd., Bristol and Philadelphia (1990);

[8] J. M. F. Labastida and C. Lozano: Lectures given by J. M. F. Labastida at the CERN–La Plata–Santiago de Compostela Workshop on Trends in Theoretical Physics held at La Plata in May 1997; hep-th/9709192;

[9] S. S. Chern: in S. S. Chern, editor: *Studies in Mathematics, Volume 27 — Global Differential Geometry*, The Mathematical Association of America inc. (1989);

[10] E. Witten: Commun. Math. Phys. **117**, 353 (1988);

[11] S. K. Donaldson: J. Diff. Geom. **18**, 279 (1983);

S. K. Donaldson: Topology, **29**, 257 (1990);

[12] E. Witten: Commun. Math. Phys. **121**, 351 (1989);

[13] S. Deser, R. Jackiw and S. Templeton: Ann. Phys. **140**, 372 (1982);

S. Deser, R. Jackiw and S. Templeton: Phys. Rev. Lett. **48**, 975 (1982);

[14] D. H. Adams: hep-th/9612009;

[15] R. T. Seeley: Amer. Math. Soc. Proc. Symp. Pure Math **10**, 288 (1967);

E. Elizalde, L. Vanzo and S. Zerbini: Commun. Math. Phys. **194**, 613 (1998);

[16] D. H. Adams and S. Sen: Phys. Lett. B**353**, 495 (1995);

[17] E. M. Prodanov and S. Sen: Phys. Rev. **D**59, 065019 (1999);

[18] C. D. Fosco, G. L. Rossini and F. A. Schaposnik: Phys. Rev. D**56**, 6547 (1997);

C. D. Fosco, G. L. Rossini and F. A. Schaposnik: Phys. Rev. Lett. **79**, 1980–1983, (1997); Erratum–ibid. **79**, 4296 (1997);

S. Deser, L. Griguolo and D. Seminara: Phys. Rev. Lett. **79**, 1976 (1997);

[19] A. S. Schwarz and Yu. S. Tyupkin: Nucl. Phys. B**242**, 436 (1984);

[20] R. Jackiw and S. Templeton: Phys. Rev. D**23**, 2291 (1981);

[21] J. Schonfeld: Nucl. Phys. B**185** 157, (1981);

[22] F. Wilczek: *Fractional Statistics and Anyon Superconductivity*, World Scientific (1990);

[23] A. P. Balachandran, L. Chandar, E. Ercolessi, T. R. Govindarajan and R. Shankar: Int. J. Mod. Phys. A**8**, 3417 (1994);

[24] S. Deser and R. Jackiw: Phys. Lett. B**139**, 371 (1984);

[25] E. Fradkin and F. A. Schaposnik: Phys. Lett. B**338**, 253 (1994);

[26] N. Bralic, E. Fradkin, V. Manias and F. A. Schaposnik: Nucl. Phys. B**446**, 144 (1995);

[27] P. J. Arias and A. Restuccia: Phys. Lett. B**347**, 241 (1995);

[28] R. Banerjee, H. J. Rothe and K. D. Rothe: Phys. Rev. D**52**, 3750 (1995);

[29] R. Banerjee and H. J. Rothe: Nucl. Phys. B**447**, 183 (1995);

[30] Y. I. Kogan: Mod. Phys. Lett. A**6**, 501 (1991);

[31] J. C. Le Guillou, E. F. Moreno, C. Nunez and F. A. Schaposnik: Mod. Phys. Lett. A**12**, 2707 (1997);

[32] J. Stephany: Phys. Lett. B**390**, 128 (1997);

[33] P. K. Townsend, K. Pilch and P. van Nieuwenhuizen: Phys. Lett. B**136**, 38 (1984);

[34] G. V. Dune: Les Houches Lectures 1998, hep-th/9902115;

[35] J. S. Dowker: hep-th/9803200;

[36] E. Elizalde, A. Filippi, L. Vanzo and S. Zerbini: hep-th/9804072;

[37] S. Deser, L. Griguolo and D. Seminara: Phys. Rev. D**57**, 7444 (1998);

[38] D. H. Adams and S. Sen.: hep-th/9503095;

[39] D. Freed and R. Gompf: Commun. Math. Phys. **141**, 79 (1991);
L. Jeffrey: Commun. Math. Phys. **147**, 563 (1992);
L. Rozansky: Commun. Math. Phys. **171**, 279 (1995);
D. H. Adams: Phys. Lett. B**417**, 53 (1998);

[40] N. Reshetikhin and V. Turaev: Invent. Math. **103**, 547 (1991);

[41] D. Birmingham, M. Blau, G. Thompson and M. Rakowski: Phys. Rep. **209**, 129 (1991);

[42] A. A. Bytsenko, L. Vanzo and S. Zerbini: Nucl. Phys. B**505**, 641 (1997);
A. A. Bytsenko, A. E. Goncalves and W. da Cruz: Mod. Phys. Lett. A**13**, 2453 (1998);
A. A. Bytsenko, A. E. Goncalves, M. Simoes and F. L. Williams: hep-th/9901054;

[43] David H. Adams: Phys. Rev. Lett. **78**, 4155 (1997);

[44] B. A. Dubrovin, A. T. Fomenko and S. P. Novikov: *Modern Geometry – Methods and Applications: Part III Introduction to Homology Theory*, Springer–Verlag, New York (1990);

[45] K. Reidemeister: Hamburger Abhandl. **11**, 102 (1935);
W. Franz and J. Reine: Angew. Math. **173**, 245 (1935);
J. Milnor: Bull. A. M. S. **72**, 348 (1966);

[46] J. Cheeger: Ann. Math. **109**, 259 (1979); W. Müller: Adv. Math. **28**, 233 (1978);

[47] L. Cornalba and F. Wilczek: Phys. Rev. Lett. **78**, 4679–4681, (1997);

[48] M. de Wild Propitius: Nucl.Phys. B**489**, 297–359, (1997);

[49] M. de Wild Propitius: Phys. Lett. B**410**, 188–194, (1997);

[50] N. Dorey and N. E. Mavromatos: Nucl. Phys. B**386**, 614–682, (1992);

[51] L. Ryder: *Quantum Field Theory*, Cambridge University Press (1996);

[52] T. Matsubara: Prog. Theor. Phys. **14**, 351 (1951);

[53] A. Das: *Finite Temperature Field Theory*, World Scientific (1997);

[54] K. Fujikawa: Phys. Rev. Lett. **42**, 1195 (1979);
K. Fujikawa: Phys. Rev. D**21**, 2848 (1980);

[55] J. M. F. Labastida and A. V. Ramallo: Phys. Lett. B 27, 92 (1989);

[56] M. Bos and V. P. Nair: Phys. Lett. B 23, 61 (1989);

[57] N. Ja. Vilenkin and A. U. Klimuk: *Representation of Lie Groups and Special Functions*, Kluwer Academic publishers (1991).