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The associative filtration of the dendriform operad

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Summary

Operads encode different types of algebraic structures in the same way as groups encode different type of symmetries, associative algebras encode different types of operators acting on vector spaces, etc. Similarly how one can talk about normal forms in groups and in associative algebras, it is possible to define normal forms in operads, and to develop effective methods for computing normal forms; that was done by Dotsenko and Khoroshkin in the paper [6].

Among different types of operads, the simplest one is given by "non-symmetric" operads. Non-symmetric operads are those where we can ignore the symmetries and assume that arguments of operations are always in order. (An example of such algebraic structure is noncommutative associative algebras.) In case of non-symmetric operads, the theory of Dotsenko and Khoroshkin admits an extension allowing to use constants (operations with no arguments), as demonstrated by Dotsenko and Vallette [7]. An overall introduction to effective methods of operad theory is given in the monograph of Bremner and Dotsenko [3].

The goal of this manuscript is to present of instance of using these methods to study specific questions about operads. We shall mainly study one famous type of algebras called dendriform algebras. Those were defined by Loday in [10]. For his purposes, the operad of dendriform algebras appeared as the Koszul dual of the operad of dialgebras. In a more direct way, a dendriform algebra is a vector space V with two binary operations denoted $<$ and $>$ that satisfy the following three

algebraic properties for all elements a_1, a_2, a_3 of V:

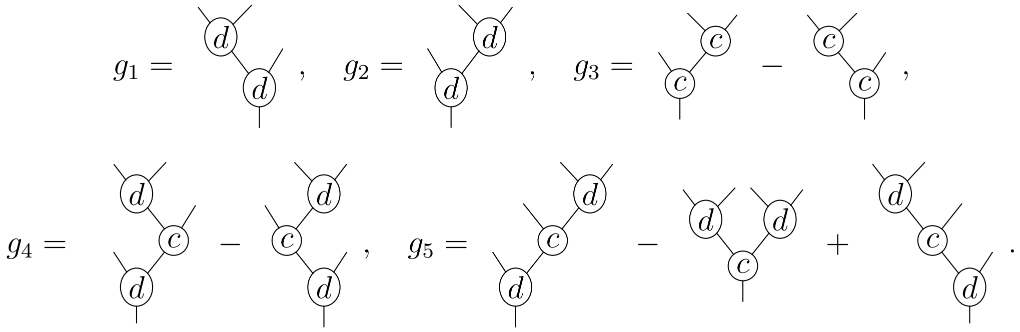
$$(a_1 > a_2) < a_3 = a_1 > (a_2 < a_3),$$
$$(a_1 < a_2) < a_3 = a_1 < (a_2 < a_3 + a_2 > a_3),$$
$$(a_1 < a_2 + a_1 > a_2) > a_3 = a_1 > (a_2 > a_3).$$

Note that all arguments in these identities stay in the same order, so dendriform algebras can be described using a non-symmetric operad $\mathbf{Dend}$.

It is well known that the sum of the two operations in any dendriform algebra, the operation $a_1 \star a_2 = a_1 < a_2 + a_1 > a_2$, is always associative. In a way, the dendriform operad might be viewed as a non-symmetric analogue of a pre-Lie algebra [4], which is a Lie algebra where the Lie bracket splits in a certain way. In fact, in each dendriform algebra, the operation $a_1 \triangleleft a_2 = a_1 < a_2 - a_2 > a_1$ satisfies the pre-Lie identity, and the corresponding embedding of the pre-Lie operad into the dendriform operad can be used to prove some nontrivial results, see for example [1].

For pre-Lie algebras, a recent result of Dotsenko [5] describes the associated graded operad for the Lie filtration on the pre-Lie operad. We shall consider a non-symmetric analogue of this question, that is the associative filtration of the dendriform operad. Using Gröbner bases, we shall give a complete description of the associated graded operad. The main result of this manuscript can be stated as follows.

Main Theorem. *Consider the associated graded operad of* $\mathbf{Dend}$ *with respect to its associative filtration. This operad is isomorphic to the quotient of the free non-symmetric operad generated by two binary generators* c *and* d *by the* *ideal I generated by the following five elements:*



Similarly to the pre-Lie case, the associated graded operad of the dendriform operad is presented by quadratic and cubic relations. However, the cubic relations have more complicated structure than the ones of [5], so methods of that paper are not applicable. However, we were able to make more use of operadic Gröbner bases than it is possible in the pre-Lie case.

Acknowledgements

I would like to thank Allah for constant support in everything. Then my thanks go to my supervisor Dr. Vladimir Dotsenko for his time and encouragement all these years. I have been extremely lucky to have a supervisor who cared so much about my work. His editorial support contributed enormously to the production of this thesis. He has provided wonderful guidance and insight without which I would never have got this far. I express my gratitude to my husband and my parents for help and advice on all matters. I also thank Kingdom of Saudi Arabia for providing funding and support me during my study and my stay in Dublin.

Contents

1 Algebraic background

In this part of the manuscript, we survey the algebraic and combinatorial background needed for stating and solving our main problem. Our main reference in studying these topics was the book of Bremner and Dotsenko [3]. Below, $\mathbb{F}$ denotes an arbitrary field. Vector spaces we work with are usually finite-dimensional or at least are direct sums of finite-dimensional components.

1.1 Ordered bases and normal forms

1.1.1 Orders of sets

A (*partial*) *order* on a set M is a binary relation $\Xi \subset M \times M$ which is:

- *irreflexive*: $(m, m) \notin \Xi$ for all $m \in M$;
- *asymmetric*: for any $m_1, m_2 \in M$, if $(m_1, m_2) \in \Xi$ then $(m_2, m_1) \notin \Xi$;
- *transitive*: for any $m_1, m_2, m_3 \in M$, if $(m_1, m_2) \in \Xi$ and $(m_2, m_3) \in \Xi$, then $(m_1, m_3) \in \Xi$.

Instead of writing $(m_1, m_2) \in \Xi$, we shall write $m_1 \prec_\Xi m_2$, or even $m_1 \prec m_2$, if Ξ is clear from the context. We shall also write $m_1 \succ m_2$ iff $m_2 \prec m_1$. We shall describe the relation $m_1 \prec m_2$ by saying that m_1 is *less than* m_2, and that m_2 is *greater than* m_1.

An order Ξ is a *total order* if for all $m_1 \neq m_2 \in M$, we have either $m_1 \prec_\Xi m_2$ or $m_1 \succ_\Xi m_2$.

A total order on a set M is said to be a *well-order*, or a *Noetherian order*, or a *well-founded order*, if each (nonempty) subset S of M has a (unique) minimal element with respect to that order.

1.1.2 Monomials and polynomials

Suppose that V is a vector space over $\mathbb{F}$ with a well-ordered basis $\{e_i\}_{i\in \mathsf{I}}$. While we do not assume V to possess any specific algebraic structure, in practice we shall be only dealing with the cases where some algebraic structure is present, and for that reason we introduce the following terminology:

- each basis element $e_i \in V$ is called a *monomial*, and each vector $v \in V$ is called a *polynomial*;
- for each polynomial

$$f = \sum_{i\in \mathsf{I}} c_i e_i \in V,$$

we call the set $\{e_i \colon c_i \neq 0\}$ the *support* of the polynomial f, and denote it by $\operatorname{supp}(f)$;

- for each nonzero polynomial f,
  - we call the maximal element of $\operatorname{supp}(f)$ the *leading monomial* of f, and denote it by $\text{LM}(f)$,
  - we call the coefficient of $\text{LM}(f)$ in f the *leading coefficient* of f, and denote it by $\text{LC}(f)$,
  - we call the corresponding term $\text{LC}(f)\,\text{LM}(f)$ of f the *leading term* of f, and denote it by $\text{LT}(f)$;
- we call a polynomial $f \in V$ with $\text{LC}(f) = 1$ *monic*.

1.1.3 Normal forms of vectors

Let S be a subset of V. We shall consider the vector space

$$\text{LT}(S) := \operatorname{span}(\text{LM}(f) \colon f \neq 0 \in S),$$

which we call the *space of leading terms of S*.

Note that the elements of $\text{LT}(S)$ are all possible linear combinations of leading terms, and not just leading terms alone.

Let S be a subset of V. A monomial e_i is said to be *linearly reduced with respect to S* if $e_i \notin \text{LM}(S)$; in other words, if e_i is not a leading monomial of an element of S. More generally, an element $f \in V$ is said to be *linearly reduced with respect to S*, if its support consists of basis monomials that are linearly reduced with respect to S.

A subset $S \subset V$ is said to be *linearly self-reduced* if each element $s \in S$ is monic and linearly reduced with respect to $S \setminus \{s\}$.

Lemma 1 ([3, Lemma 1.2.1.3])**.** *Let S be a subspace of V. Cosets of the monomials that are linearly reduced with respect to S form a basis of the quotient V/S.*

Let S be a subspace of V. We call monomials that are linearly reduced with respect to S *normal modulo* S, and linear combinations of normal monomials *normal forms*. For each f in V, we call the unique element in the coset $f + S$ that is reduced with respect to S the *normal form of* f *modulo* S. If we know a self-reduced basis B of a subspace S, the normal forms are precisely elements that are linearly reduced with respect to B, and that is the smallest set of conditions one has to check.

Proposition 1 ([3, Prop. 1.2.1.6])**.** *Every subspace $S \subset V$ has a linearly self-reduced basis B.*

1.2 Nonsymmetric operads

1.2.1 Nonsymmetric collections

A model example of an associative algebra is given by all endomorphisms of a vector space V with composition as the product. Operads arise similarly if we consider multilinear maps with any number of arguments. Instead of vector spaces, we will use a bigger category of the so-called nonsymmetric collections.

A *nonsymmetric collection* is a sequence $\mathcal{V} = \{\mathcal{V}(n)\}_{n\geq 0}$ of vector spaces. A *morphism* between two nonsymmetric collections $\mathcal{V}$ and $\mathcal{W}$ is a collection of linear maps

$$\phi_n \colon \mathcal{V}(n) \to \mathcal{W}(n), \quad n \geq 0.$$

If each ϕ_n is an embedding of a subspace, we call the collection of their images a *subcollection* of $\mathcal{W}$, and write $\mathcal{V} \subset \mathcal{W}$.

An important example of a nonsymmetric collection is the endomorphism operad of a vector space.

Example 1. The *endomorphism operad* of a vector space V is the nonsymmetric collection End_V with $\mathrm{End}_V(n) := \mathrm{Hom}(V^{\otimes n}, V)$, $n \geq 0$. In particular,

$$\mathrm{End}_V(0) = \mathrm{Hom}(\mathbb{F}, V) \cong V,$$
$$\mathrm{End}_V(1) = \mathrm{Hom}(V, V) = \mathrm{End}(V).$$

The collection End_V has a rich algebraic structure given by compositions of maps. Suppose that $f \in \mathrm{End}_V(r)$, $g_1 \in \mathrm{End}_V(n_1)$, $\ldots$, $g_n \in \mathrm{End}_V(n_r)$. The *nonsymmetric composition* $\gamma(f; g_1, \ldots, g_r)$, or $f \circ (g_1, \ldots, g_r)$ is an element of $\mathrm{End}_V(n_1 + \cdots + n_r)$ defined by the formula

$$f \circ (g_1, \ldots, g_r) \colon x_1, \ldots, x_{n_1+\cdots+n_r} \mapsto$$
$$f(g_1(x_{k_1+1}, \ldots, x_{k_1+n_1}), g_2(x_{k_2+1}, \ldots, x_{k_2+n_2}) \ldots, g_r(x_{k_r+1}, \ldots, x_{k_r+n_r})),$$

where $k_i = n_1 + \cdots + n_{i-1}$ (in particular, $k_1 = 0$).

1.2.2 Combinatorics of trees

It is very useful to depict nonsymmetric compositions by tree-shaped diagrams, so we shall fix a language for working with trees, and then use that language to define elements of operads and work with them.

Definition 1. A *rooted tree* τ consists of:

- a finite set of *vertices* $\mathrm{Vert}(\tau)$ represented as a disjoint union
$$\mathrm{Vert}(\tau) = \mathrm{Int}(\tau) \sqcup \mathrm{Leaves}(\tau) \sqcup \{r\},$$
where elements of the (possibly empty) set $\mathrm{Int}(\tau)$ are called *internal vertices*, elements of the (possibly empty) set $\mathrm{Leaves}(\tau)$ are called *leaves*, and the element r is called the *root* of τ, and denoted $\mathrm{Root}(\tau)$, and

- a *parent function*
$$\mathrm{Parent}_\tau \colon \mathrm{Vert}(\tau) \setminus \{r\} \to \mathrm{Vert}(\tau),$$
for which
$$|\,\mathrm{Parent}_\tau^{-1}(r)| = 1,$$
$$\mathrm{Parent}_\tau^{-1}(l) = \varnothing \text{ for each } l \in \mathrm{Leaves}(\tau).$$
The only requirement imposed on this function is *connectivity*: for each vertex $v \in \mathrm{Vert}(\tau) \setminus \{r\}$ there is a (unique) positive integer l and vertices $v_0 = v$, v_1, ..., $v_l = r$, such that $v_i = \mathrm{Parent}_\tau(v_{i-1})$ for all $i = 1, \ldots, l$. This number l is called the *depth* of the vertex v, and the sequence v_l, v_{l-1}, ..., v_0 is referred to as the *path from root to* v.

An *endpoint* of a tree τ is a vertex $v \in \mathrm{Vert}(\tau)$ for which $\mathrm{Parent}_\tau^{-1}(v) = \varnothing$; from the above conditions we see that each leaf of τ is an endpoint, but there may be endpoints that are not leaves.

The only tree τ for which $\mathrm{Int}(\tau)$ is empty is called the *trivial tree*.

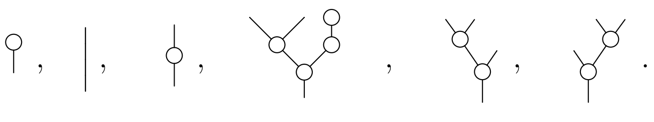
All trees we shall work with are planar rooted trees. A *planar rooted tree* τ is a rooted tree together with a *planar structure*, i.e., a total order on the preimage $\mathrm{Parent}_\tau^{-1}(v)$ for each $v \in \mathrm{Vert}(\tau)$.

Two rooted trees are said to be *isomorphic* if there is a bijection between their sets of vertices that respects all the rooted tree data and their respective planar structures.

A planar structure of a tree τ induces a total order on the set of its endpoints as follows. Let e and e' be two different endpoints of τ, and consider the paths from the root to e and e'. Suppose that the first k vertices of those paths coincide, and the $k+1$-st vertices, say v_{k+1} and v'_{k+1}, are different. Under this assumption, $\mathrm{Parent}_\tau(v_{k+1}) = \mathrm{Parent}_\tau(v'_{k+1})$, and hence the planar structure allows to compare v_{k+1} and v'_{k+1}. We say that $e \prec e'$ if $v_{k+1} \prec v'_{k+1}$.

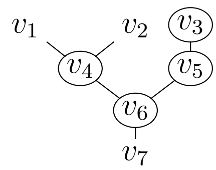
Rooted trees are conventionally depicted by diagrams made of points, little circles, and edges, that is, straight lines connecting points and circles. Each point represents a leaf or the root, each little circle represents an internal vertex, and each edge between v and v' directed downward from v to v' represents the relation $v' = \mathrm{Parent}_\tau(v)$. In particular, the root is always at the bottom of the diagram. We shall always draw planar rooted trees in the plane in a way that the planar order on $\mathrm{Parent}_\tau^{-1}(v)$ is determined by ordering the corresponding edges left-to-right.

Example 2. The following diagrams represent planar rooted trees:



The second tree is the trivial tree. The last two trees are non-isomorphic because of the different planar left-to-right structures.

The fourth diagram which we now represent with labels that give names to all the vertices



represents a rooted tree τ for which

$$r = v_7,$$

$$\mathrm{Leaves}(\tau) = \{v_1, v_2\},$$

$$\mathrm{Int}(\tau) = \{v_3, v_4, v_5, v_6\},$$

$$\mathrm{Parent}_\tau(v_1) = \mathrm{Parent}_\tau(v_2) = v_4,$$

$$\mathrm{Parent}_\tau(v_3) = v_5,$$

$$\mathrm{Parent}_\tau(v_4) = \mathrm{Parent}_\tau(v_5) = v_6,$$

$$\mathrm{Parent}_\tau(v_6) = v_7.$$

1.2.3 Two definitions of an operad

In the endomorphism operad, the compositions satisfy the "two-level associativity": a composition where each substituted operation is itself a composition $g_i \circ (h_1, \ldots, h_{p_i})$ can be computed in two different ways, either computing each $g_i \circ (h_1^{(i)}, \ldots, h_{p_i}^{(i)})$ individually or first computing $f \circ (g_1, \ldots, g_r)$, and then computing the composition of that with all the elements $h_j^{(i)}$. This leads to one classical definition of a nonsymmetric operad.

Definition 2. A *nonsymmetric operad* is a nonsymmetric collection of vector spaces $\mathcal{P} = \{\mathcal{P}(n)\}_{n \geq 0}$ equipped with an element $\mathrm{id} \in \mathcal{P}(1)$ and maps

$$\gamma^{(r)}_{n_1,\ldots,n_r} \colon \mathcal{P}(r) \otimes \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_r) \to \mathcal{P}(n_1 + \cdots + n_r)$$

(for which the shorthand notation

$$f \circ (g_1, \ldots, g_r) := \gamma^{(r)}_{n_1,\ldots,n_r}(f \otimes g_1 \otimes \cdots \otimes g_r)$$

is commonly used), which satisfy the following properties:

- *associativity*:

$$f \circ (g_1 \circ (h_1^{(1)}, \ldots, h_{q_1}^{(1)}), \ldots, g_r \circ (h_1^{(r)}, \ldots, h_{q_r}^{(r)})) = \\ (f \circ (g_1, \ldots, g_r)) \circ (h_1^{(1)}, \ldots, h_{q_1}^{(1)}, \ldots, h_1^{(r)}, \ldots, h_{q_r}^{(r)}).$$

- *unit axiom*:

$$\gamma^{(1)}_n(\mathrm{id};\alpha) = \alpha, \quad \gamma^{(r)}_{1,1,\dots,1}(\alpha;\mathrm{id},\dots,\mathrm{id}) = \alpha.$$

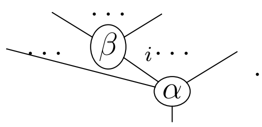
Ideals in nonsymmetric operads are defined similarly to ideals in rings. Suppose that $\mathcal{P}$ is a nonsymmetric operad. An *ideal* $\mathcal{I}$ of $\mathcal{P}$ is a subcollection $\mathcal{I} \subset \mathcal{P}$ for which the element $f \circ (g_1,\dots,g_n)$ belongs to $\mathcal{I}$ if at least one of the elements $f, g_1,\dots,g_n$ belongs to $\mathcal{I}$.

Let $f \in \mathrm{End}_V(n)$, $g \in \mathrm{End}_V(m)$, and $1 \le i \le n$. The *partial composition of f and g at the i-th slot* is the operation

$$f \circ_i g := f \circ (\mathrm{id},\dots,\mathrm{id}, g, \mathrm{id},\dots,\mathrm{id}),$$

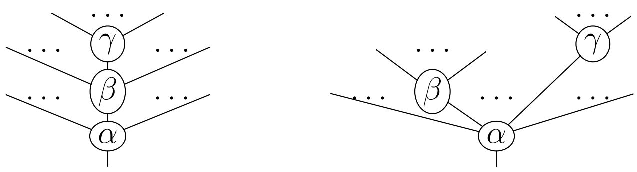
where g is at the i-th argument of f.

Formulas like that can be represented by pictures. For example, the partial composition $\alpha \circ_i \beta$ is represented by the tree



whose internal vertices are labelled by α and β.

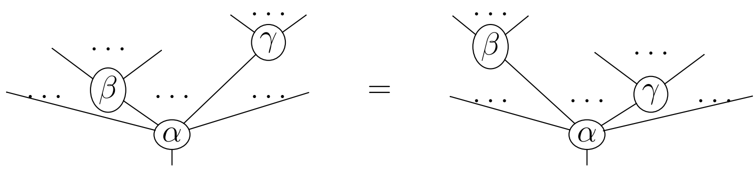
Suppose that we use partial compositions to create a single element out of three elements $\alpha \in \mathrm{End}_V(n)$, $\beta \in \mathrm{End}_V(m)$, $\gamma \in \mathrm{End}_V(r)$. This can be done in two essentially different ways represented by the following three-vertex trees:



For the first of those trees, we compose the three operations in a sequence, and this kind of composition satisfies a property that generalizes the associativity of composition of linear transformations. Basically, the corresponding composition can be computed in two different ways, and those ways must give the same result. On the level of formulas, this gives

$$(\alpha \circ_i \beta) \circ_j \gamma = \alpha \circ_i (\beta \circ_{j-i+1} \gamma) \quad \text{for } i \le j \le i+m-1.$$

For the second of those trees, we compose two operations in parallel, and this kind of composition satisfies a property that is somewhat closer to commutativity; that property is not visible on the level of associative algebras. More precisely, here two operations are composed in parallel, and there are two different ways to compute that composition depending on a choice of levels of vertices in trees. These ways must produce the same result:



On the level of formulas, this gives

$$(\alpha \circ_i \beta) \circ_j \gamma = \begin{cases} (\alpha \circ_{j-m+1} \gamma) \circ_i \beta, & i+m \leq j \leq n+m-1, \\ (\alpha \circ_j \gamma) \circ_{i+r-1} \beta, & 1 \leq j \leq i-1 \end{cases}$$

(there are two formulas, the first one corresponds to the picture above, and the second one corresponds to its mirror reflection).

This suggests the following definition.

Definition 3. A *nonsymmetric operad* is a nonsymmetric collection of vector spaces $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$ equipped with an element $\mathrm{id} \in \mathcal{P}(1)$ and maps

$$\circ_i \colon \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(n+m-1), \quad \alpha \otimes \beta \mapsto \alpha \circ_i \beta$$

which satisfy the following properties for all $\alpha \in \mathcal{P}(n)$, $\beta \in \mathcal{P}(m)$, $\gamma \in \mathcal{P}(r)$:

- *sequential axiom*:

$$(\alpha \circ_i \beta) \circ_j \gamma = \alpha \circ_i (\beta \circ_{j-i+1} \gamma) \quad \text{for } i \leq j \leq i+m-1;$$

- *parallel axiom*:

$$(\alpha \circ_i \beta) \circ_j \gamma = \begin{cases} (\alpha \circ_{j-m+1} \gamma) \circ_i \beta, & i+m \leq j \leq n+m-1, \\ (\alpha \circ_j \gamma) \circ_{i+r-1} \beta, & 1 \leq j \leq i-1; \end{cases}$$

- *unit axiom*:

$$\mathrm{id} \circ_1 \alpha = \alpha, \quad \alpha \circ_i \mathrm{id} = \alpha \quad \text{for } 1 \leq i \leq n.$$

The following result is well known.

Proposition 2. *The classical and the partial definition of a nonsymmetric operad are equivalent to each other.*

1.3 Free nonsymmetric operads

We shall now present an explicit construction of the free nonsymmetric operad with a given set of generators. Free nonsymmetric operads are spanned by decorated trees, which are often viewed as "tree-shaped tensors" whose internal vertices represent multilinear operations; they should be decorated accordingly.

An *operation alphabet* is a collection $\mathcal{X} = \{\mathcal{X}(n)\}_{n\geq 0}$ of finite sets $\mathcal{X}(n)$ indexed by nonnegative integers n. The number n is referred to as *arity* of an element $x \in \mathcal{X}(n)$. Throughout this chapter, unless otherwise specified, $\mathcal{X}$ denotes an arbitrary operation alphabet.

A *nonsymmetric tree monomial* in $\mathcal{X}$ is a pair $T = (\tau, \mathsf{x})$, where τ is a planar rooted tree and x is a labelling of all internal vertices of τ by elements of $\mathcal{X}$; each vertex v must have a label $\mathsf{x}_v \in \mathcal{X}(|\operatorname{Parent}^{-1}(v)|)$.

The tree monomial for which the underlying tree τ is the trivial tree is called the *trivial tree monomial*, or the *empty tree monomial*.

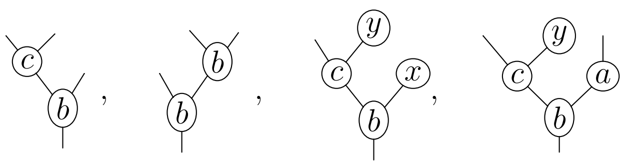
The *arity* of a tree monomial T, denoted $\operatorname{ar}(T)$, is the number of leaves of τ, and its *weight*, denoted $\operatorname{wt}(T)$, is the number of internal vertices of τ.

The set of all tree monomials in $\mathcal{X}$ of arity n is denoted $\operatorname{Tree}_{\mathcal{X}}(n)$. The collection of all these sets for all $n \geq 0$ is denoted $\operatorname{Tree}_{\mathcal{X}}$.

Example 3. Suppose that

$$\mathcal{X}(0) = \{x, y\}, \quad \mathcal{X}(1) = \{a\}, \quad \mathcal{X}(2) = \{b, c\}.$$

The following are examples of tree monomials in $\operatorname{Tree}_{\mathcal{X}}$:



The first two of them have arity 3 and weight 2, the third one has arity 1 and weight 4, and the last one has arity 2 and weight 4.

A *nonsymmetric tree polynomial* in $\mathcal{X}$ with coefficients in $\mathbb{F}$ is a linear combination of nonsymmetric tree monomials of the same arity.

We denote the vector space of all nonsymmetric tree polynomials of arity n by $\mathcal{T}(\mathcal{X})(n)$; of course we have $\mathcal{T}(\mathcal{X})(n) = \mathbb{F}\,\mathrm{Tree}_{\mathcal{X}}(n)$.

In the case of associative algebras, we use the intuitive notion of concatenation of words to define products. For operations, the notion that serves a similar purpose is that of grafting of trees. We will discuss two different types of graftings, full and partial; these correspond to compositions from the classical definition of a nonsymmetric operad and partial compositions, respectively.

Full grafting of planar trees τ_1, ..., τ_r to a planar tree τ_0 corresponds, in terms of our graphical representation, to joining the open edges corresponding to the roots of τ_1, ..., τ_r with the open edges corresponding to leaves of τ_0, in the total planar order.

Let $\tau_0, \tau_1, \ldots, \tau_r$ be planar rooted trees, and suppose that τ_0 has r leaves. We define a planar rooted tree $\tau_0 \circ (\tau_1, \ldots, \tau_r)$, called the result of *full grafting* of $\tau_1, \ldots, \tau_r$ to τ_0, to be the planar rooted tree τ for which:

$$\begin{aligned}
\mathrm{Root}(\tau) &= \mathrm{Root}(\tau_0),\\
\mathrm{Int}(\tau) &= \bigsqcup_{i=0}^{r} \mathrm{Int}(\tau_i),\\
\mathrm{Leaves}(\tau) &= \bigsqcup_{i=1}^{r} \mathrm{Leaves}(\tau_i).
\end{aligned}$$

The parent function and the planar structure on the thus defined set of vertices are induced by the respective parent functions and planar structures of τ_i, $0 \le i \le r$, with the following exceptions. For each $j = 1, \ldots, r$, for the only vertex v_j in $\mathrm{Parent}_{\tau_j}^{-1}(\mathrm{Root}(\tau_j))$, we define

$$\mathrm{Parent}_\tau(v_j) := \mathrm{Parent}_{\tau_0}(\ell_j),$$

where ℓ_j is the j-th leaf of τ_0 in the total planar order on leaves induced from the total planar order of endpoints of τ_0. This means that

$$\mathrm{Parent}_\tau^{-1}(\mathrm{Parent}_{\tau_0}(\ell_j)) = \{v_j\} \sqcup \mathrm{Parent}_{\tau_0}^{-1}(\mathrm{Parent}_{\tau_0}(\ell_j)) \setminus \{\ell_j\};$$

the total order needed by the planar structure puts v_j in the place of ℓ_j.

If all the grafted trees τ_i except for one are trivial, we end up with the definition of partial grafting. Let τ_1 and τ_2 be two rooted planar trees. Let $\ell \in \mathrm{Leaves}(\tau_1)$.

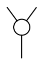
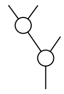
We define a planar rooted tree $\tau_1 \circ_\ell \tau_2$, called the result of *partial grafting* of τ_2 to τ_1 at ℓ, as follows. We put

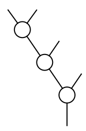
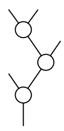
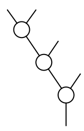
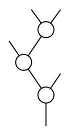
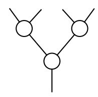
$$\mathrm{Root}(\tau_1 \circ_\ell \tau_2) = \mathrm{Root}(\tau_1),$$

$$\mathrm{Int}(\tau_1 \circ_\ell \tau_2) = \mathrm{Int}(\tau_1) \sqcup \mathrm{Int}(\tau_2),$$

$$\mathrm{Leaves}(\tau_1 \circ_\ell \tau_2) = \mathrm{Leaves}(\tau_1) \sqcup \mathrm{Leaves}(\tau_2) \setminus \{\ell\}.$$

The parent function and the planar structure on the thus defined set of vertices are induced by the respective parent functions and planar structures of τ_1 and τ_2 with two small exceptions. For the only vertex v in $\mathrm{Parent}_{\tau_2}^{-1}(\mathrm{Root}(\tau_2))$, we define $\mathrm{Parent}_{\tau_1 \circ_\ell \tau_2}(v) = \mathrm{Parent}_{\tau_1}(\ell)$. This means that $\mathrm{Parent}_{\tau_1 \circ_\ell \tau_2}^{-1}(\mathrm{Parent}_{\tau_1}(\ell)) = \{v\} \sqcup \mathrm{Parent}_{\tau_1}^{-1}(\mathrm{Parent}_{\tau_1}(\ell)) \setminus \{\ell\}$; the total order needed by the planar structure puts v in the place of ℓ.

Example 4. Let $\tau_1 =$  and $\tau_2 =$  . Various partial compositions of these trees are summarized in the following table:

$\tau_1 \circ_1 \tau_2$	$\tau_1 \circ_2 \tau_2$	$\tau_2 \circ_1 \tau_1$	$\tau_2 \circ_2 \tau_1$	$\tau_2 \circ_3 \tau_1$
				

Grafting of trees allows us to give an explicit construction of free nonsymmetric operads.

Definition 4. Suppose that we are given nonsymmetric tree monomials $T_0 = (\tau_0, \mathsf{x}_0) \in \mathrm{Tree}_{\mathcal{X}}(r)$, $T_1 = (\tau_1, \mathsf{x}_1) \in \mathrm{Tree}_{\mathcal{X}}(n_1)$, $\ldots$, $T_r = (\tau_r, \mathsf{x}_r) \in \mathrm{Tree}_{\mathcal{X}}(n_r)$. We define the *nonsymmetric composition*

$$T_0 \circ (T_1, \ldots, T_r)$$

to be the nonsymmetric tree monomial (τ, x), where

$$\tau = \tau_0 \circ (\tau_1, \ldots, \tau_r),$$

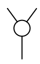
and the labelling x of $\mathrm{Int}(\tau) = \bigsqcup_{i=0}^{r} \mathrm{Int}(\tau_i)$ is given by the disjoint union of labellings x_j, $1 \le j \le r$.

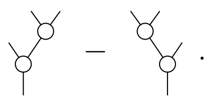
These nonsymmetric compositions may be extended by multilinearity to the collection $\mathcal{T}(\mathcal{X}) = \{\mathcal{T}(\mathcal{X})(n)\}_{n\geq 0}$ of all nonsymmetric tree polynomials of all arities, giving operations

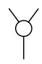
$$\gamma^{(r)}_{n_1,\dots,n_r} \colon \mathcal{T}(\mathcal{X})(r) \otimes \mathcal{T}(\mathcal{X})(n_1) \otimes \cdots \otimes \mathcal{T}(n_r) \to \mathcal{T}(\mathcal{X})(n_1 + \cdots + n_r).$$

Equipped with these operations, $\mathcal{T}(\mathcal{X})$ is the *free nonsymmetric operad generated by* $\mathcal{X}$. In addition to the notation $\mathcal{T}(\mathcal{X})$, we will use the notation $\mathcal{T}(\mathcal{M})$, where $\mathcal{M} = \{\mathcal{M}(n)\}_{n\geq 0}$ is a collection of vector spaces for which $\mathcal{M}(n) = \mathrm{span}(\mathcal{X}(n))$ for all $n \geq 0$.

Let $\mathcal{P}$ be a nonsymmetric operad, and suppose that $\mathcal{S} \subset \mathcal{P}$ is a subcollection. The *ideal of* $\mathcal{P}$ *generated by* $\mathcal{S}$, denoted by $(\mathcal{S})$, is the smallest (by inclusion) ideal of $\mathcal{P}$ containing $\mathcal{S}$. Furthemore, suppose that the nonsymmetric operad $\mathcal{P}$ is a quotient of the free operad $\mathcal{T}(\mathcal{X})$ by some ideal $\mathcal{I}$, and that the ideal $\mathcal{I}$ is generated by the collection $\mathcal{S}$. In this case, we will say that the operad $\mathcal{P}$ is *presented by generators* $\mathcal{X}$ *and relations* $\mathcal{S}$.

Example 5. One of the most famous operads presented by generators and relations is the nonsymmetric associative operad $\mathbf{As}$. It is the quotient of the free operad with one binary generator  by the ideal generated by the element



This element encodes the associator of the binary product  ; taking the quotient amounts to imposing all algebraic consequences of associativity by pre-composing and post-composing the associator with all possible operations.

1.4 Gröbner bases

Definition 5. A collection of total orders Ξ_n of $\mathrm{Tree}_{\mathcal{X}}(n)$, $n \geq 0$, is said to be a *monomial order* if the following two conditions are satisfied:

- each Ξ_n is a well-order;
- each nonsymmetric composition is a strictly increasing function in each of its arguments; that is if $T_0, T_0' \in \mathrm{Tree}_{\mathcal{X}}(r)$, $T_1, T_1' \in \mathrm{Tree}_{\mathcal{X}}(n_1)$, $\ldots$, $T_r, T_r' \in \mathrm{Tree}_{\mathcal{X}}(n_r)$, then

$$T_0 \circ (T_1, \ldots, T_r) \prec T_0' \circ (T_1, \ldots, T_r) \text{ if } T_0 \prec T_0',$$
$$T_0 \circ (T_1, \ldots, T_i, \ldots, T_r) \prec T_0 \circ (T_1, \ldots, T_i, \ldots, T_r) \text{ if } T_i \prec T_i'.$$

Unless otherwise specified, throughout this chapter, we will give definitions as well as state and prove all theoretical results for an arbitrary monomial order Ξ.

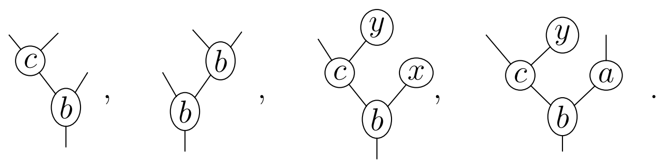
We continue with an important example of a monomial order. We denote $X := \bigsqcup_{n \geq 0} \mathcal{X}(n)$. Ler us first explain how to replace every tree monomial by a sequence of words in the alphabet X.

Definition 6. Let $T = (\tau, \mathsf{x})$ be a tree monomial. For each endpoint e of τ in the total order induced by the planar structure, we record the labels of internal vertices of the path from the root of τ to e, forming a word in the alphabet X. The sequence of these words, denoted $\mathrm{Path}(T)$, is called the *path sequence* of the tree monomial T.

Example 6. Suppose that

$$\mathcal{X}(0) = \{x, y\}, \quad \mathcal{X}(1) = \{a\}, \quad \mathcal{X}(2) = \{b, c\}.$$

Let us consider the tree monomials from Example 3



The corresponding path sequences are, respectively,

$$(bc, bc, b), \quad (b, bb, bb), \quad (bc, bcy, bx), \quad (bc, bcy, ba).$$

Note that the two path sequences (bc, bcy, bx) and (bc, bcy, ba) from the example we just considered look deceptively similar, but if we recall that the letter x corresponds to an operation of arity zero (that is, constants), while the letter a corresponds to a unary operation, we instantly see that the path sequences correspond to tree monomials whose underlying trees are combinatorially different. This observation is the key to the following result.

Lemma 2 ([3, Lemma 3.4.1.4])**.** *A tree monomial* $T = (\tau, \mathsf{x})$ *is uniquely determined by the sequence* $\mathrm{Path}(T)$*:*

$$\textit{if } \mathrm{Path}(T_1) = \mathrm{Path}(T_2) = p, \ \textit{then } T_1 = T_2.$$

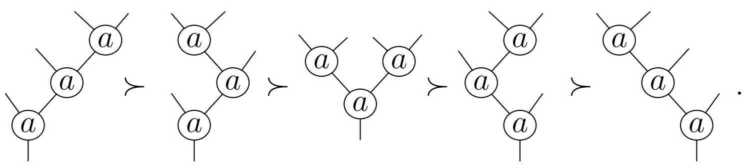
For the purpose of this manuscript, we shall use just one type of ordering of tree monomials. We refer to [3, Sec. 3.4.1] for more general definition.

Definition 7. Suppose that $\mathcal{X}(0) = \mathcal{X}(1) = \varnothing$. Let us fix some order Ξ of $X := \bigsqcup_{n\geq 2} \mathcal{X}(n)$. The *reverse path-lexicographic order* of tree monomials is defined as follows. To compare two monomials T_1 and T_2 of the same arity, we apply the following rule:

- compare the degrees of the first elements of their path sequences; if they are different, the monomial with the shorter one is bigger;
- if there is a tie, compare the first elements of their path sequences letter-by-letter;
- if there is a tie, move to the second elements of the path sequences, and proceed as above, etc.

It is possible to show that this order is always a monomial order.

Example 7. Let $\mathcal{X}(2) = \{a\}$. For the order we defined, we have



This follows from comparing the corresponding path sequences

$$(a, a^2, a^3, a^3) \succ (a, a^3, a^3, a^2) \succ (a^2, a^2, a^2, a^2) \succ (a^2, a^3, a^3, a) \succ (a^3, a^3, a^2, a).$$

The final step before defining Gröbner bases is setting up a formalism of divisibility of tree monomials, and the long division algorithm.

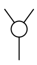
Let τ be a rooted tree. Suppose that $V' \subset \mathrm{Int}(\tau)$ is a nonempty subset satisfying the following conditions:

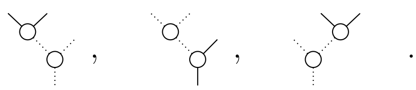
- there exists just one vertex $v' \in V'$ for which $\mathrm{Parent}_\tau(v')$ is not in V',
- for each vertex $v'' \in V'$ there is a (unique) nonnegative integer l and vertices $v_0 = v'$, v_1, …, $v_l = v''$, such that $v_i = \mathrm{Parent}_\tau(v_{i-1})$ for all $i = 1, \ldots, l$,
- for each vertex v'' in V' the preimage $\mathrm{Parent}_\tau^{-1}(v')$ is either contained in V' or is disjoint from V'.

Each such subset V' defines a planar rooted tree τ' called a *subtree* of τ. We put

$$\mathrm{Root}(\tau') = \mathrm{Parent}_\tau(v'),$$
$$\mathrm{Int}(\tau') = V',$$
$$\mathrm{Leaves}(\tau') = \left(\bigcup_{v' \in V'} \mathrm{Parent}_\tau^{-1}(V') \right) \setminus V',$$

and use the induced parent function and the induced planar structure on the thus defined set of vertices. If $\mathrm{Parent}_\tau(v') = r$, we say that the subtree τ' and the ambient tree τ *share the root*. If $\mathrm{Int}(\tau')$ is a proper subset of $\mathrm{Int}(\tau)$, we say that τ' is a *proper subtree* of τ.

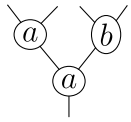
Example 8. In each of the following trees, the vertices connected by dotted lines form a subtree isomorphic to  :

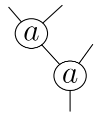
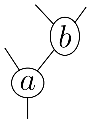


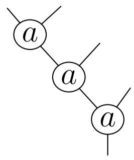
We can now give a definition of divisibility that generalises divisibility of words in free monoids.

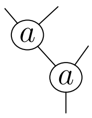
A tree monomial $T_1 = (\tau_1, \mathsf{x}_1)$ is *divisible* by a (nontrivial) tree monomial $T_2 = (\tau_2, \mathsf{x}_2)$ if the tree τ_1 contains a subtree τ_1' isomorphic to the tree τ_2, and the labels of internal vertices of that subtree in the monomial T_1 match the labels of τ_2 in the monomial T_2.

Example 9. Let $\mathcal{X} = \{a, b\}$. The monomial

 $\in \mathcal{T}(\mathcal{X})(4)$

has two different divisors of weight 2: the "left divisor"  and the "right divisor" . In comparison, the tree monomial

 $\in \mathcal{T}(\mathcal{X})(4)$

has two divisors of weight 2 both of which are occurrences of the monomial .

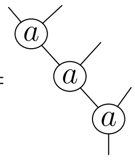
It is possible to show that this notion of divisibility of monomials is equivalent to algebraic divisibility: if $T_1 = (\tau_1, \mathsf{x}_1)$ and $T_2 = (\tau_2, \mathsf{x}_2)$ be two tree monomials, then T_1 is divisible by T_2 if and only if it can be obtained from T_2 by iterated nonsymmetric compositions with elements of $\mathcal{T}(\mathcal{X})$.

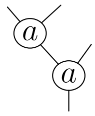
Suppose that T_1 and T_2 are tree monomials, and T_1 is divisible by T_2. In this case, there is an *insertion* operation

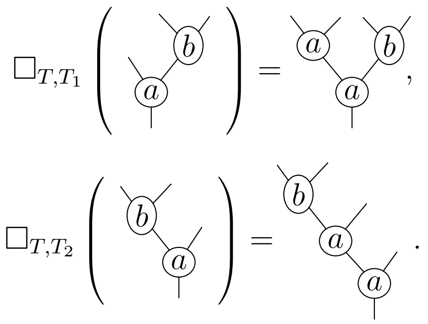
$$\square_{T_1,T_2} \colon \mathcal{T}(\mathcal{X})(\mathrm{ar}(T_2)) \to \mathcal{T}(\mathcal{X})(\mathrm{ar}(T_1)).$$

If $T = (\tau, \mathsf{x})$ is a tree monomial of the same arity as T_2, the insertion operation replaces the subtree τ_1' by τ (ensuring that each subtree of τ_1 that was grafted at a certain leaf of τ_1' gets grafted at the respective leaf of τ), and changing labels of internal vertices accordingly. Then, this operation is extended by linearity to all tree polynomials of the same arity. Our notation is not completely precise, since there may be several different divisors T_2 inside T_1. We always assume that the operation $\square_{T_1,T_2}$ inserts everything at a particular occurrence of T_2 inside T_1 which is implicit.

Example 10. Let $\mathcal{X} = \{a, b\}$. Consider the tree monomial

$T =$  $\in \mathcal{T}(\mathcal{X})(4)$

from Example 9. This monomial has two occurrences of  as a divisor; let us denote the one sharing the root with T by T_1, and the other one by T_2. We have



One very useful feature of the insertion operations is that they allow us to give an explicit description of an ideal generated by a given collection $\mathcal{S}$ in the free operad which is a suitable replacement of the description "the ideal (S) is the linear span of all elements $r_1 s r_2$ for all $r_1, r_2 \in T(X)$, $s \in S$" in the associative case.

Proposition 3 ([3, Prop. 3.4.2.10])**.** *Let $\mathcal{S} \subset \mathcal{T}(\mathcal{X})$. The ideal $(\mathcal{S})$ generated by $\mathcal{S}$ can be described explicitly as the linear span of all insertions $\square_{T_1,T_2}(f)$, where T_1 is a monomial, T_2 is a divisor of T_1, and $f \in \mathcal{S}(\mathrm{ar}(T_2))$.*

The key property of insertions is the following one.

Proposition 4 ([3, Prop. 3.4.2.12])**.** *Suppose that T_1 is a tree monomial, and T_2 is a divisor of T_1. Then for each $g \in \mathcal{T}(\mathcal{X})(\mathrm{ar}(T_2))$, we have*

$$\mathrm{LM}(\square_{T_1,T_2}(g)) = \square_{T_1,T_2}(\mathrm{LM}(g)). \tag{1}$$

Let $\mathcal{S}$ be a subset of $\mathcal{T}(\mathcal{X})$. A tree monomial T is said to be *reduced with respect to* $\mathcal{S}$ if $T \notin (\mathrm{LM}(\mathcal{S}))$; in other words, if T is not divisible by any of the leading monomials of elements of $\mathcal{S}$.

In general, a tree polynomial f is said to be *reduced with respect to $\mathcal{S}$* if it is equal to a linear combination of tree monomials which are reduced with respect to $\mathcal{S}$. A subset $\mathcal{S} \subset \mathcal{T}(\mathcal{X})$ is said to be *self-reduced* if each element $s \in \mathcal{S}$ is monic and reduced with respect to $\mathcal{S} \setminus \{s\}$.

Let $f, g \in \mathcal{T}(\mathcal{X})$ be two nonzero elements. We say that f is *reducible with respect to g* if $\mathrm{LM}(f)$ is not reduced with respect to $\{g\}$, or, in plain words, if the leading monomial of f is divisible by the leading monomial of g, that is,

$$\mathrm{LM}(f) = \square_{T_1,T_2}(\mathrm{LM}(g))$$

for some tree monomials

$$T_1 \in \mathrm{Tree}_{\mathcal{X}}(\mathrm{ar}(\mathrm{LM}(f))), \quad T_2 \in \mathrm{Tree}_{\mathcal{X}}(\mathrm{ar}(\mathrm{LM}(g))).$$

In that case, the *reduction of f with respect to g*, denoted by $r_g(f)$, is defined by the formula

$$r_g(f) = f - \frac{\mathrm{LC}(f)}{\mathrm{LC}(g)}\square_{T_1,T_2}(g).$$

It is easy to see that for all elements $f, g \in \mathcal{T}(\mathcal{X})$ such that $r_g(f)$ is defined, we have

$$r_g(f) = 0 \quad \text{or} \quad \mathrm{LM}(r_g(f)) \prec \mathrm{LM}(f).$$

One can view a reduction as one step of a version of the long division algorithm, that computes, for every $f \in \mathcal{T}(\mathcal{X})$, an element $\tilde{f}$ reduced with respect to $\mathcal{S}$, for which $\mathrm{LT}(\tilde{f}) \preceq \mathrm{LT}(f)$ and

$$f + (\mathcal{S}) = \tilde{f} + (\mathcal{S}),$$

see [3, Algorithm 3.4.2.16] for details. An important consequence of that is the following result.

Lemma 3 ([3, Lemma 3.4.2.19])**.** *Suppose that $\mathcal{I}$ is an ideal of $\mathcal{T}(\mathcal{X})$. Monomials that are reduced with respect to $\mathcal{I}$ form a basis of the quotient $\mathcal{T}(\mathcal{X})/\mathcal{I}$.*

It is also possible to use long division to find, for each finite set, a finite self-reduced set that generates the same ideal, see [3, Algorithm 3.4.2.20].

Proposition 5 ([3, Prop. 3.4.3.1])**.** *Let $\mathcal{I}$ be an ideal of $\mathcal{T}(\mathcal{X})$. The space of leading terms* $\mathrm{LT}(\mathcal{I})$ *is an ideal of* $\mathcal{T}(\mathcal{X})$*.*

We are now ready to define a Gröbner basis of an ideal.

Definition 8. Let $\mathcal{I}$ be an ideal of $\mathcal{T}(\mathcal{X})$. We say that $\mathcal{G} = \{\mathcal{G}(n) \subset \mathcal{I}(n)\}$ is a *Gröbner basis* of $\mathcal{I}$ with respect to a given monomial order Ξ if the set of leading monomials $\mathrm{LM}(\mathcal{G}) := \{\mathrm{LM}(g) \colon g \in \mathcal{G}\}$ generates the ideal of leading terms of the ideal $\mathcal{I}$:

$$\mathrm{LT}(\mathcal{I}) = (\mathrm{LM}(\mathcal{G})).$$

A Gröbner basis which is a self-reduced subset of $\mathcal{T}(\mathcal{X})$ is said to be *reduced*.

It is easy to show that a Gröbner basis of an ideal $\mathcal{I} \subset \mathcal{T}(\mathcal{X})$ generates $\mathcal{I}$.

Proposition 6 ([3, Prop. 3.4.3.4])**.** *Let $\mathcal{I}$ be an ideal of $\mathcal{T}(\mathcal{X})$. Then $\mathcal{G} \subset \mathcal{I}$ is a Gröbner basis if and only if the cosets of monomials that are reduced with respect to $\mathcal{G}$ form a basis of the quotient $\mathcal{T}(\mathcal{X})/\mathcal{I}$.*

A useful way to re-phrase this proposition is the following statement.

Theorem 1. *Let $\mathcal{I}$ be an ideal of $\mathcal{T}(\mathcal{X})$. A sequence of subsets $\mathcal{G} \subset \mathcal{I}$ is a Gröbner basis if and only if the normal forms modulo $\mathcal{I}$ are precisely the elements that are reduced with respect to $\mathcal{G}$.*

It is possible to show that each ideal $\mathcal{I} \subset \mathcal{T}(\mathcal{X})$ has a unique reduced Gröbner basis.

1.5 Computing Gröbner bases

Let us outline the most famous constructive approach to computing Gröbner bases for ideals of $\mathcal{T}(\mathcal{X})$. In the ideal of main interest to us further in Chapter 2, the Gröbner basis is going to be infinite, and this computational approach is not going to be our main tool. Still, it was useful to us to guess the infinite Gröbner basis which we then studied by other methods.

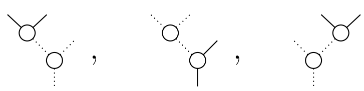
To define S-polynomials for trees, we need to make precise what we mean by overlaps of trees.

Definition 9. An *overlap* of two planar trees τ_1 and τ_2 is the data of a nontrivial rooted tree τ and isomorphisms $f_i\colon \tau \to \tau_i'$ where τ_i' is a subtree of τ_i, $i = 1, 2$, satisfying the following properties:

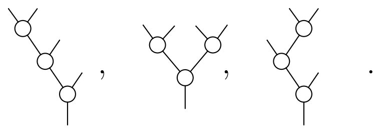
- at least one of τ_i' shares the root with τ_i,
- $f_1^{-1}(\mathrm{Int}(\tau_1)) \cup f_2^{-1}(\mathrm{Int}(\tau_2)) = \mathrm{Int}(\tau)$, but $f_1^{-1}(\mathrm{Int}(\tau_1)) \neq \mathrm{Int}(\tau)$ and $f_2^{-1}(\mathrm{Int}(\tau_2)) \neq \mathrm{Int}(\tau)$, and also $f_1^{-1}(\mathrm{Int}(\tau_1)) \cap f_2^{-1}(\mathrm{Int}(\tau_2)) \neq \varnothing$ (each internal vertex of τ is an internal vertex in at least one of τ_i, not all internal vertices are internal vertices of just one of them, and at least one of internal vertices is an internal vertex of both, so the overlap is nontrivial),
- for each $\ell \in \mathrm{Leaves}(\tau)$, at least one of the $f_1(\ell)$ and $f_2(\ell)$ is a leaf in τ_i,
- at least one of τ_i' is a proper subtree of τ_i.

Two planar rooted trees that have an overlap can be *merged* along it by identifying the vertices of τ_1' with the corresponding vertices of τ_2', and consider the naturally induced parent function and planar structure. The overlap conditions guarantee that the result of that identification is a planar rooted tree again.

Example 11. Let us consider the three trees



from Example 8. Each two of those trees form an overlap (the dashed edges mark the common parts). Merging the first and the second one (respectively, the first and the third one, the second and the third one) along their overlap, we obtain, respectively, the trees



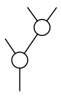
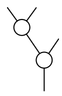
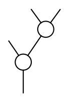
Definition 10. Let $g_1, g_2 \in \mathcal{T}(\mathcal{X})$ be two monic polynomials. We say that the leading monomials $\mathrm{LM}(g_1)$ and $\mathrm{LM}(g_2)$ form an *overlap* if they have a *small common multiple*, a tree monomial T and its two proper divisors T_1 and T_2 for which

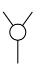
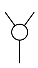
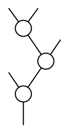
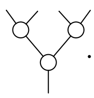
$$\mathrm{LM}(g_1) = T_1, \quad \mathrm{LM}(g_2) = T_2,$$

and the underlying tree of T is the result of merging of the underlying trees of T_1 and T_2 along an overlap. We call the element

$$S_T(g_1, g_2) := \Box_{T,T_1}(g_1) - \Box_{T,T_2}(g_2)$$

an *S-polynomial* of g_1 and g_2; the common term cancels, since both g_1 and g_2 are monic.

Example 12. Let $g_1 = g_2 =$  $-$  , and suppose we are using the reverse path-lexicographic order. Then  is the leading monomial; as we know from Example 11, it has an overlap with itself. The corresponding S-polynomial is equal to

$\Big(g_1 \circ_3$  $\Big) - \Big($  $\circ_2 g_2\Big) =$  $-$  .

The following theorem leads to a constructive way of checking that some subset of an ideal is a Gröbner basis.

Definition 11. Let $\mathcal{I} = (\mathcal{G})$ be an ideal of $\mathcal{T}(\mathcal{X})$. Consider the representation of an element $f \in \mathcal{I}$ as a combination of insertions of $g_1, \ldots, g_N \in \mathcal{G}$:

$$f = \sum_{i=1}^{N} c_i \Box_{\widetilde{T}_i, T_i}(g_i), \tag{2}$$

where $T_i = \mathrm{LM}(g_i)$. We call $\max(\widetilde{T}_i)$ the *parameter* of this linear combination.

If $f = S_T(g_1, g_2)$ is the S-polynomial of $g_1, g_2 \in \mathcal{G}$ (with all the notation as above in Definition 10), then it has an obvious representation

$$f = \square_{T,T_1}(g_1) - \square_{T,T_2}(g_2),$$

with parameter T. We call a representation of that S-polynomial *nontrivial* if its parameter is smaller than T.

Theorem 2 (Diamond lemma, [3, Th. 3.5.1.6]). *Let $\mathcal{G} \subset \mathcal{T}(\mathcal{X})$ be self-reduced, and let $\mathcal{I} = (\mathcal{G})$. The following statements are equivalent:*

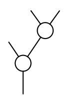
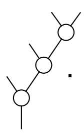
(i) *$\mathcal{G}$ is a Gröbner basis of $\mathcal{I}$.*

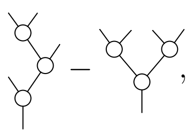
(ii) *Every S-polynomial $S_T(g_1, g_2)$ has reduced form 0 with respect to $\mathcal{G}$.*

(iii) *Every S-polynomial $S_T(g_1, g_2)$ admits a nontrivial representation of the form* (2)*.*

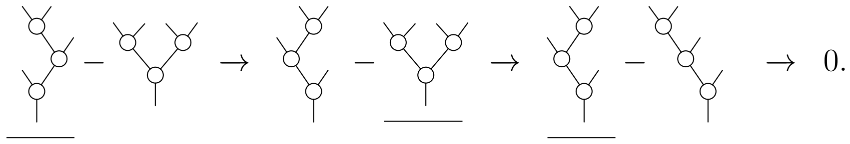
(iv) *Every element $f \in \mathcal{I}$ admits a representation of the form* (2) *with parameter* $\mathrm{LM}(f)$*.*

Theorem 2 leads naturally to a recipe for computing reduced Gröbner bases: given a set of generators of an ideal, one has to compute all pairwise S-polynomials, adjoin all reduced forms of those to the set of generators, and repeat the same. If this procedure terminates then its output is the reduced Gröbner basis of $\mathcal{I}$.

Example 13. Let us consider the associative operad, and apply the method we just outlined, using the reverse path-lexicographic order of tree monomials. The leading term of the associativity relation is , and the only small common multiple of that element with itself is the tree monomial . The corresponding S-polynomial is computed in Example 12; it is equal to



and can be reduced to zero by the following chain of reductions:



We conclude that the defining relation of $\mathbf{As}$ forms a Gröbner basis, and that the normal forms are given by "left combs" (left-growing trees, left-normed products).

1.6 Gröbner bases for commutative algebras

An important technique we shall use is the historically first instance of Gröbner bases, that is Gröbner bases for commutative algebras. Let us give a quick summary of the results, specifically emphasizing similarity with the case of nonsymmetric operads; we refer to [3, Chapter 7] for more information and further references.

Let $X = \{x_1, \dots, x_n\}$. A *monomial order* on $\mathbb{F}[X]$ is a total well-order on the set of all monomials in $x_1, \dots, x_n$ which is *multiplicative*: for all monomials m, m', m'', if $m' \prec m''$ then $mm' \prec mm''$. In this manuscript, we shall use just the *lexicographic* order `lex`, which is not the fastest for computation but the best for elimination and solving equations. To compare two monomials $v = x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$ and $w = x_1^{f_1} x_2^{f_2} \cdots x_n^{f_n}$, we search for the least index i for which $e_i \neq f_i$. Then $v \prec_{\texttt{lex}} w$ if and only if $e_i < f_i$.

Definition 12. Let $X = \{x_1, \dots, x_n\}$, and let I be an ideal of $\mathbb{F}[X]$. We say that $G \subset I$ is a *Gröbner basis* of I with respect to a given monomial order Ξ if the set of leading monomials $\mathrm{LM}(G) := \{\mathrm{LM}(g) \colon g \in G\}$ generates the ideal of leading terms of the ideal I:

$$\mathrm{LT}(I) = (\mathrm{LM}(G)).$$

A Gröbner basis which is a self-reduced subset of $\mathbb{F}[X]$ is said to be *reduced*.

It is easy to show that a Gröbner basis of an ideal $I \subset \mathbb{F}[X]$ generates I, and that each ideal $\mathcal{I} \subset \mathcal{T}(\mathcal{X})$ has a unique reduced Gröbner basis.

The *multivariate long division* of a polynomial g by a set $f_1, \dots, f_\ell \in \mathbb{F}[X]$ is a repetition of simple *reduction steps*. For each of those steps, we search for a term cm in $\mathrm{supp}(g)$ (where $c \in \mathbb{F}$ and m is a monomial), where m is a multiple $m = m'm''$ of the leading monomial $m' = \mathrm{LM}(f_i)$ for some $i = 1, \dots, \ell$, and then replace g by $g' = g - c(m/m')f_i$. We repeatedly perform these steps, and the algorithm terminates when there are no more such terms in g. The final value of g is the *remainder* of the original value of g modulo $f_1, \dots, f_\ell$. (It might depend on the order of elementary steps we chose.)

Definition 13. Let G be a finite set of monic polynomials generating the ideal $I = (G)$. For $f_1, f_2 \in G$ the monomials d, m_1, m_2 are uniquely determined by the following conditions:

$$d = \gcd(\,\mathrm{LM}(f_1),\, \mathrm{LM}(f_2)\,), \qquad \mathrm{LM}(f_1) = dm_1, \qquad \mathrm{LM}(f_2) = dm_2.$$

Clearly, we have

$$m_1\,\mathrm{LM}(f_2) = m_2\,\mathrm{LM}(f_1).$$

The polynomial

$$g = m_1 f_2 - m_2 f_1.$$

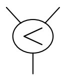
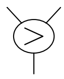
is called the *S-polynomial* of the polynomials f_1 and f_2.

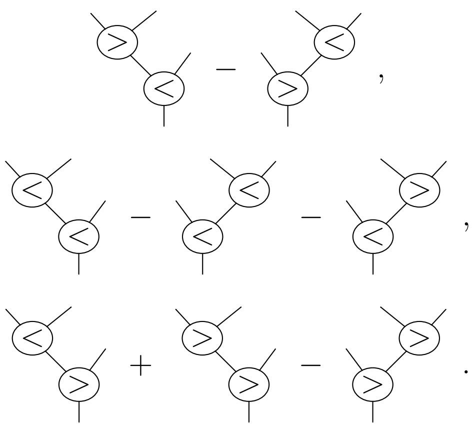
S-polynomials are used for Buchberger's criterion, which leads directly to an algorithm for computing a Gröbner basis of an ideal from an arbitrary set of generators for the ideal.

Theorem 3 (Buchberger's criterion)**.** *Let $G = \{f_1, \ldots, f_\ell\}$ be a subset of $\mathbb{F}[x_1, \ldots, x_n]$ generating the ideal $I = (G)$. Then G is a Gröbner basis for I if and only if every S-polynomial of two elements $f_i, f_j \in G$ has remainder 0 after multivariate long division by G.*

The instance below where we use computer algebra for computing a commutative Gröbner basis is essentially an application of the algorithm arising from this criterion.

2 The associative filtration of the dendriform operad

This part of the manuscript is dedicated to the main results we obtained. They concern the operad of dendriform algebras. Recall that dendriform algebras can be encoded by a non-symmetric operad $\mathbf{Dend}$; it is the quotient of the free operad with two generators  and  by the ideal generated by the elements



Some computations of Gröbner bases for this operad for various monomial orders exist in the literature, see, for example [3, Sec. 3.6.2] and [11]. We are, however, not going to specifically focus on finding further Gröbner bases; rather we consider a different basis of generators of this operad, and study some constructions arising from this choice using methods related to operadic Gröbner bases.

2.1 Filtrations of operads

We start this section with a definition of a filtration of a non-symmetric operad; this is an analogue of the corresponding definition for associative algebras [8].

Definition 14. Let $\mathcal{O}$ be a non-symmetric operad. A filtration of $\mathcal{O}$, denoted $F^\bullet\mathcal{O}$, is a sequence

$$\mathcal{O} =: F^0\mathcal{O} \supseteq F^1\mathcal{O} \supseteq F^2\mathcal{O} \supseteq \cdots,$$

(so that for each n, we have $\mathcal{O}(n) \supseteq F^1\mathcal{O}(n) \supseteq F^2\mathcal{O}(n) \supseteq \cdots$) which is compatible with the operad structure: for each partial composition

$$\circ_i \colon \mathcal{O}(n) \otimes \mathcal{O}(m) \to \mathcal{O}(n+m-1),$$

its restriction to the subspace

$$F^k\mathcal{O}(n) \otimes F^l\mathcal{O}(n) \subseteq \mathcal{O}(n) \otimes \mathcal{O}(m)$$

has the image in $F^{k+l}\mathcal{O}(n+m-1)$.

The main reason filtrations are used is to form associated graded objects that are usually "simpler". In the case of associative algebras, a filtered algebra may be noncommutative but its associated graded algebra might turn out to be commutative, etc. For operads, the associated graded object of an operad $\mathcal{O}$ with a filtration $F^\bullet\mathcal{O}$ is denoted $\mathbf{gr}_F\,\mathcal{O}$, and is defined as follows. The underlying object of $\mathbf{gr}_F\,\mathcal{O}$ is, in each arity n,

$$\mathbf{gr}_F\,\mathcal{O}(n) := \bigoplus_{i\geq 0} F^i\mathcal{O}(n)/F^{i+1}\mathcal{O}(n).$$

We shall introduce notation for the individual summands of this formula,

$$\mathbf{gr}^i_F\,\mathcal{O}(n) := F^i\mathcal{O}(n)/F^{i+1}\mathcal{O}(n).$$

The operad structure on $\mathbf{gr}_F\,\mathcal{O}$ is induced by the operad structure of $\mathcal{O}$. It is enough to define operadic compositions on the individual summands $\mathbf{gr}^i_F\,\mathcal{O}$. Let us take two elements $\alpha \in \mathbf{gr}^i_F\,\mathcal{O}(n)$ and $\beta \in \mathbf{gr}^j_F\,\mathcal{O}(m)$, and lift them to elements $\tilde{\alpha} \in F^i\mathcal{O}(n)$ and $\tilde{\beta} \in F^j\mathcal{O}(m)$. By definition of a filtration, we have $\tilde{\alpha} \circ_p \tilde{\beta} \in F^{i+j}\mathcal{O}(n+m-1)$. We define $\alpha \circ_p \beta$ to be the coset

$$\tilde{\alpha} \circ_p \tilde{\beta} + F^{i+j+1}\mathcal{O}(n+m-1) \in F^{i+j}\mathcal{O}(n+m-1)/F^{i+j+1}\mathcal{O}(n+m-1).$$

Let us show that we obtained a well defined operation on $\mathbf{gr}_F\,\mathcal{O}$. Indeed, suppose that we choose a different lifting $\tilde{\alpha}$. This amounts to adding an element from $F^{i+1}\mathcal{O}(n)$, changing $\tilde{\alpha} \circ_p \tilde{\beta}$ by an element of $F^{i+j+1}\mathcal{O}(n+m-1)$, so the coset

$$\tilde{\alpha} \circ_p \tilde{\beta} + F^{i+j+1}\mathcal{O}(n+m-1)$$

does not change. The same argument applies to $\tilde{\beta}$.

One important example of a filtration is a filtration defined by powers of an ideal. If $I \subseteq \mathcal{O}$ is an operad ideal,then we may define $F^k\mathcal{O} := I^k$, the span of elements obtained by operad compositions involving at least k elements of I. By direct inspection, this is a filtration. In this work, we only consider this kind of filtration.

We shall now define one of the key notions of this manuscript. Recall that the sum of the two operations $<$ and $>$ in any dendriform algebra is associative.

Definition 15. Let us consider the operad ideal $A \subset \mathbf{Dend}$ generated by the associative operation $\star$. Concretely, this is the linear span of all operad compositions involving the operation $\star$ at least once. As above, this ideal leads to a filtration

$$\mathbf{Dend} = A^0 \supset A^1 \supset A^2 \supset \cdots,$$

where A^k is the linear span of all operad compositions involving the operation $\star$ at least k times. We shall call this filtration the *associative filtration* of the dendriform operad.

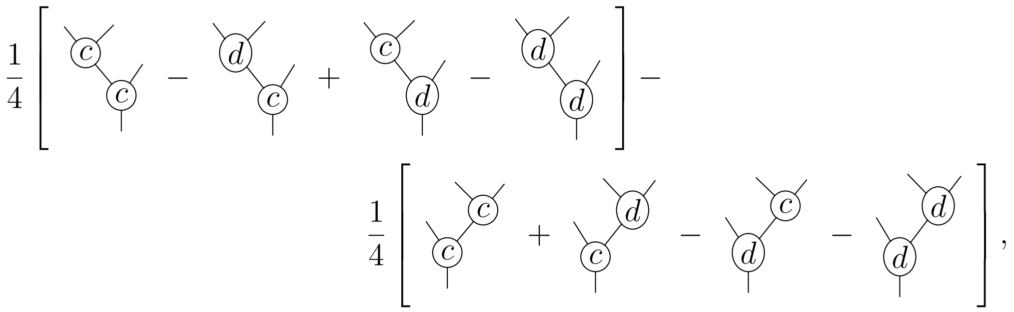
The main result of this thesis is a complete description of the associated graded operad with respect to this filtration, which we denote $\mathbf{gr}\,\mathbf{Dend}$. For that, we shall use a new system of generators of $\mathbf{Dend}$,

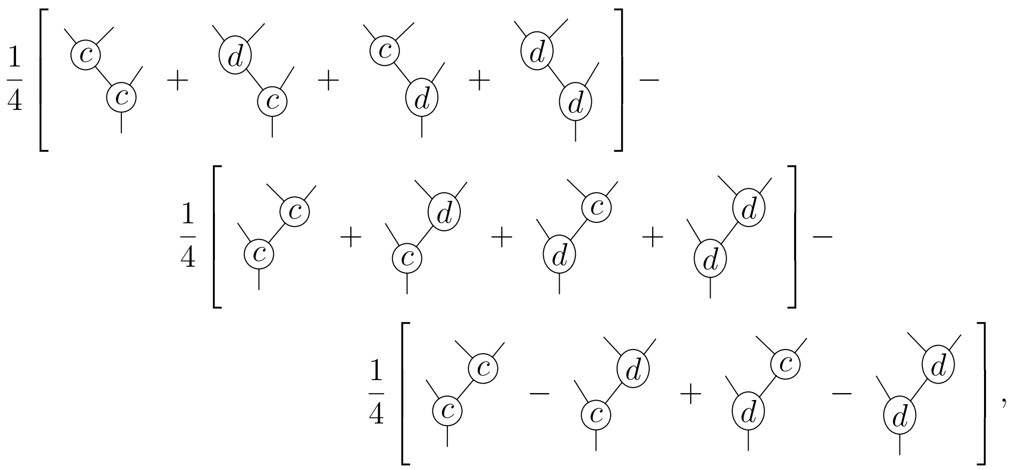
$$\star = < + > , \qquad \diamond = < - > .$$

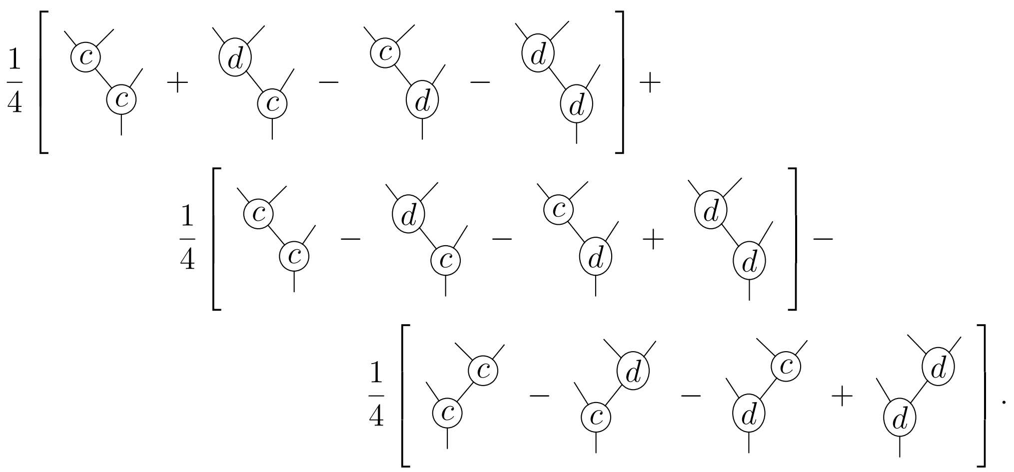
Since the operad $\mathbf{Dend}$ is generated by the operations $\diamond$ and $\star$, the operad $\mathbf{gr}\,\mathbf{Dend}$ is generated by the cosets of those operations, $\diamond + A^1 \in A^0/A^1$ and $\star + A^2 \in A^1/A^2$.

2.2 Defining relations of the new presentation of **Dend**

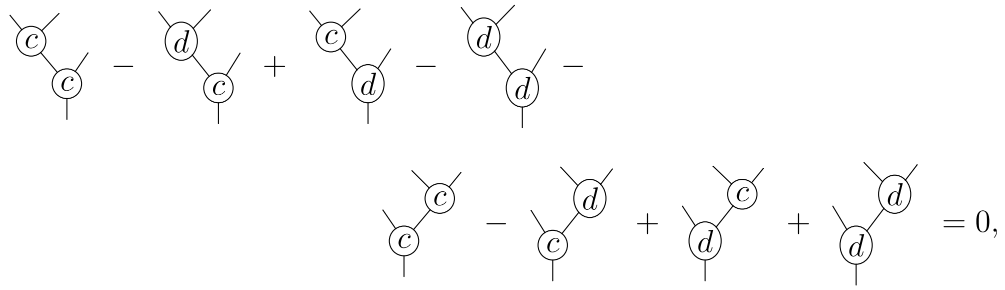
Our first step is to compute the defining relations between the new generators of the operad **Dend**. In the following computations, we shall denote all binary operations by Latin letters, using a for $>$, b for $<$, c for $\star$ and d for $\diamond$. Since $c = a + b$ and $d = b - a$, we have $a = \frac{c-d}{2}$, $b = \frac{c+d}{2}$. Substituting these into the defining relations of the dendriform operad, we obtain the following relations:

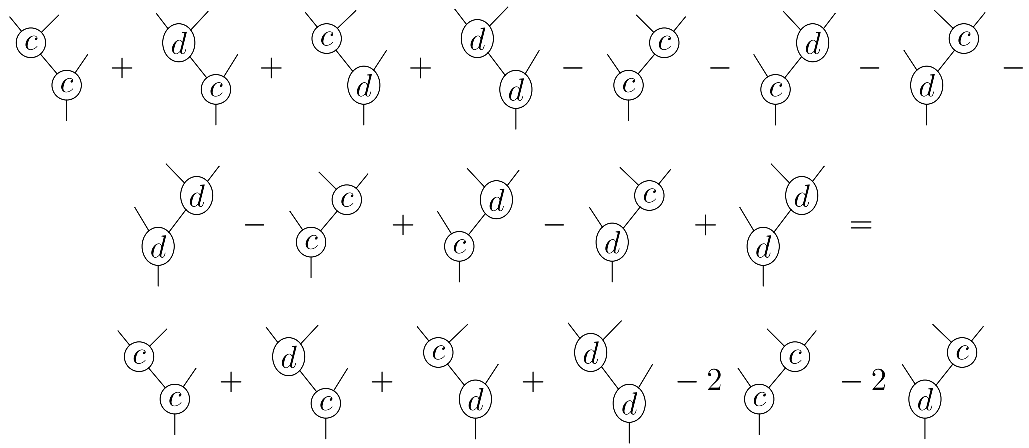


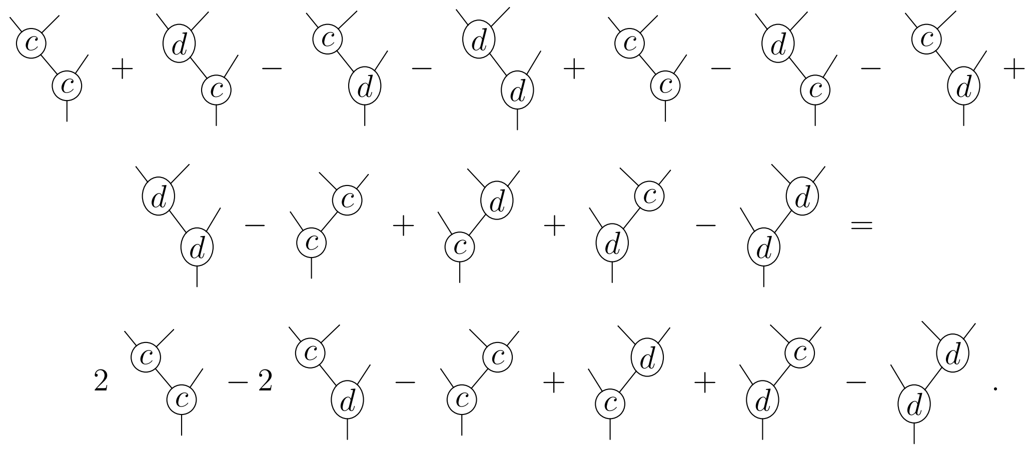


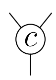
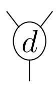


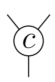
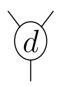
Expanding these and collecting similar terms, we obtain the following three relations:



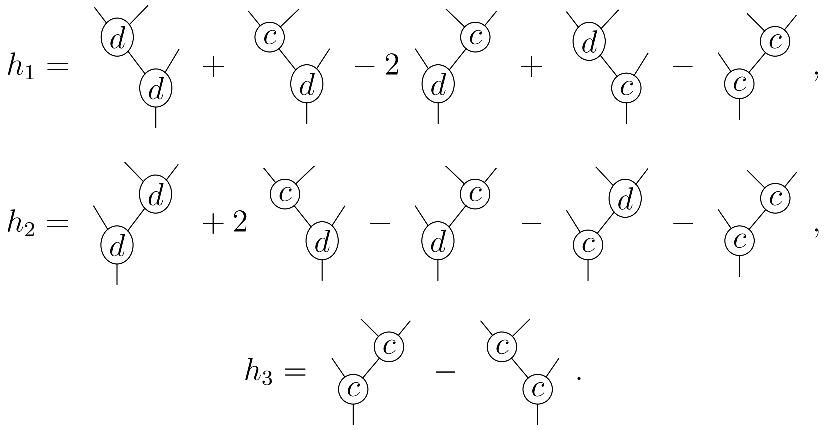




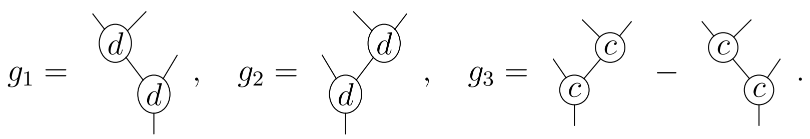
To simplify these relations, we impose a particular ordering of tree monomials in the free nonsymmetric operad generated by  and  ; this ordering will be the main ordering used in our work.

Definition 16. The *associative filtration ordering* of the free nonsymmetric operad generated by  and  is defined as follows. To compare two tree monomials, we first compare the number of their vertices labelled d; if there is no tie, the tree with the greater number of such vertices is greater. Otherwise, we compare them using the reverse path-lexicographic ordering corresponding to the order $d > c$.

A direct computation shows that, for the order we just imposed, the linearly self-reduced set of elements spanning the space of ternary relations of the dendriform operad consists of the following three elements:



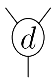
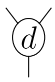
Since the filtration of **Dend** arises from counting the number of vertices labelled c in a tree, the associated graded relations in $\mathbf{gr}\,\mathbf{Dend}$ of h_1, h_2 and h_3 will be, respectively,

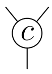


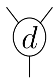
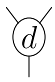
Our main goal is to describe all relations of the operad $\mathbf{gr}\,\mathbf{Dend}$. Our original approach was to compute the Gröbner basis of the operad **Dend** for a suitable ordering, and look at what happens when we go to its associated graded one. However, the formulas that are obtained this way are not very nice. However, we did enough computations to find some extra relations and have a plausible conjecture about all relations. In the next sections, we present arguments leading to this conjecture and its proof, leaving the heaviest computations "behind the scenes".

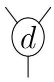
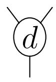
2.3 Some new relations of the operad $\mathbf{gr\,Dend}$

Our next step is to use the new presentation of the operad $\mathbf{Dend}$ that we just got, and find some relations in the operad $\mathbf{gr\,Dend}$. Let us remark that the relations found in the previous section are not all the defining relations of the operad $\mathbf{gr\,Dend}$; for example, in the expression

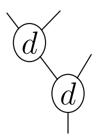
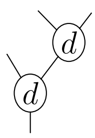
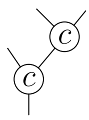
$h_1 \circ_1$  $-$  $\circ_1 h_1$

the terms without  cancel, but this expression is different from zero in $\mathbf{Dend}$, and so has a non-zero projection to $\mathbf{gr\,Dend}$. However, in $\mathbf{gr\,Dend}$ we have

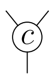
$g_1 \circ_1$  $-$  $\circ_1 g_1 = 0,$

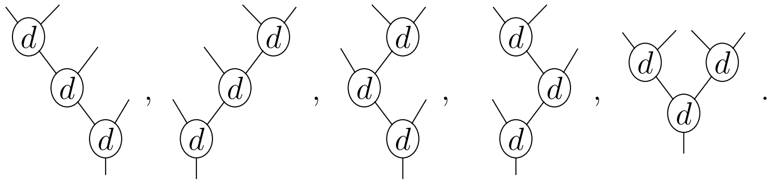
so this means that the associated graded of the element $h_1 \circ_1$  $-$  $\circ_1 h_1$ should be expected to be an additional relation in $\mathbf{gr\,Dend}$. Our approach for identifying elements like that will use Gröbner bases for non-symmetric operads for the associative filtration ordering we defined above.

For this ordering,the leading monomials of the relations of $\mathbf{Dend}$ are

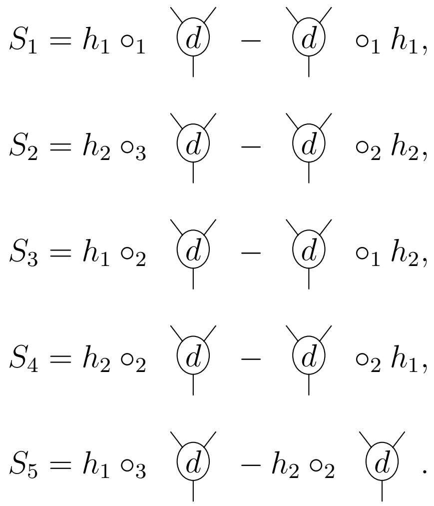
$\mathsf{LM}(h_1) =$  , $\mathsf{LM}(h_2) =$  , $\mathsf{LM}(h_3) =$  .

Let us determine the part of the Gröbner basis that consists of trees with at most three internal vertices; this means that we only look at S-polynomials between the original relations.

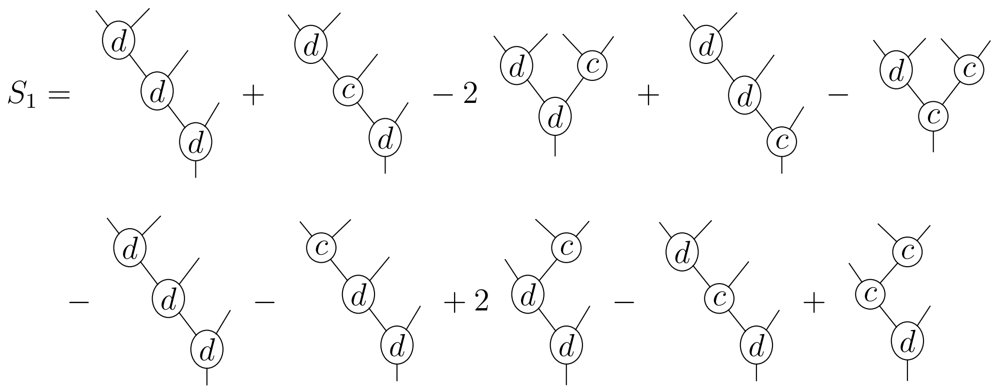
There is a small common multiple of the leading term of h_3 with itself, which does not give rise to a non-trivial S-polynomial, since the operad generated by the operation  is the associative operad, for which the S-polynomial can be reduced to zero, as we established in Example 13. There are also five small common multiples of the leading terms of h_1 and h_2 between themselves,



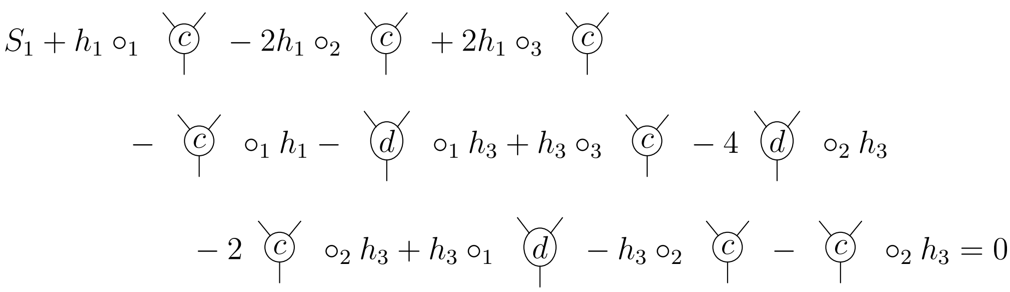
The corresponding S-polynomials are:



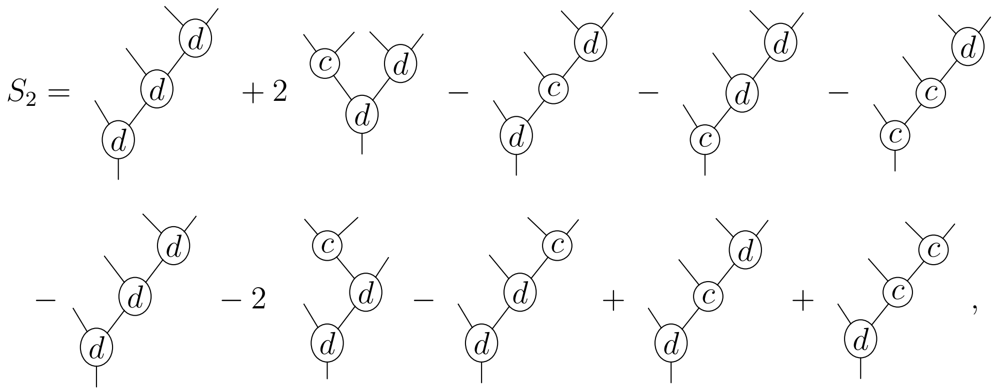
Let us first consider the S-polynomial S_1. We have



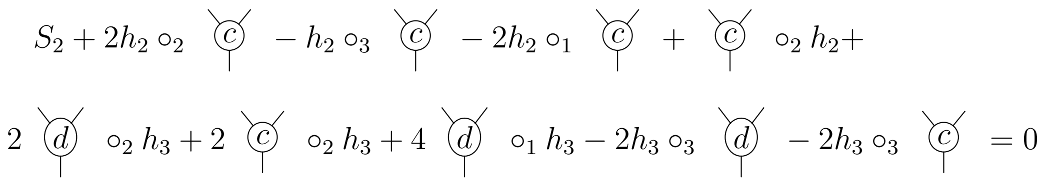
By a direct computation, we have the following sequence of reductions:



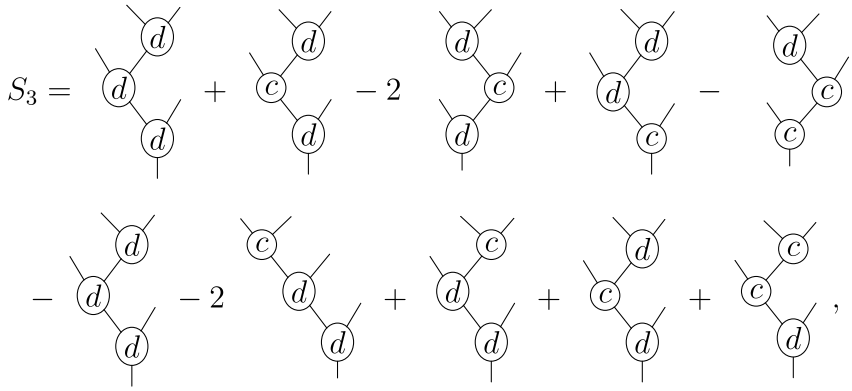
Similarly, we have



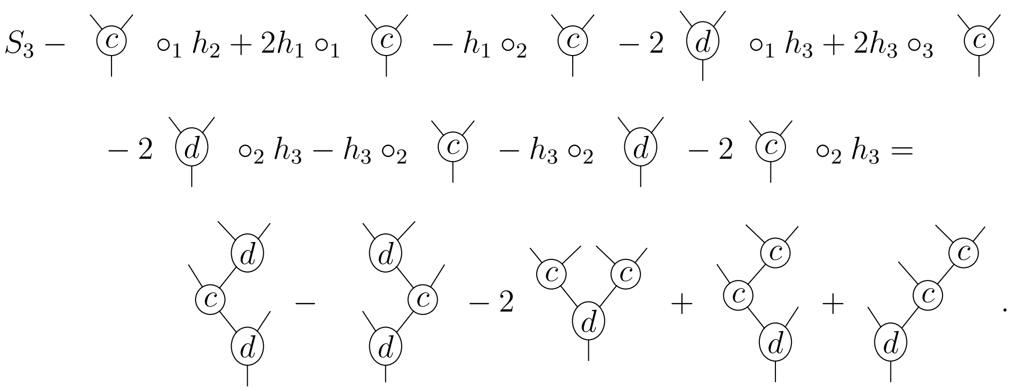
and the following sequence of reductions:



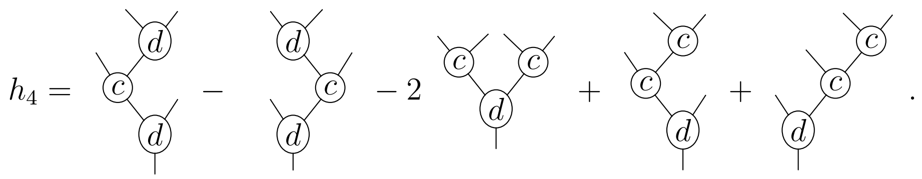
The S-polynomial S_3 does not get reduced to zero. We have



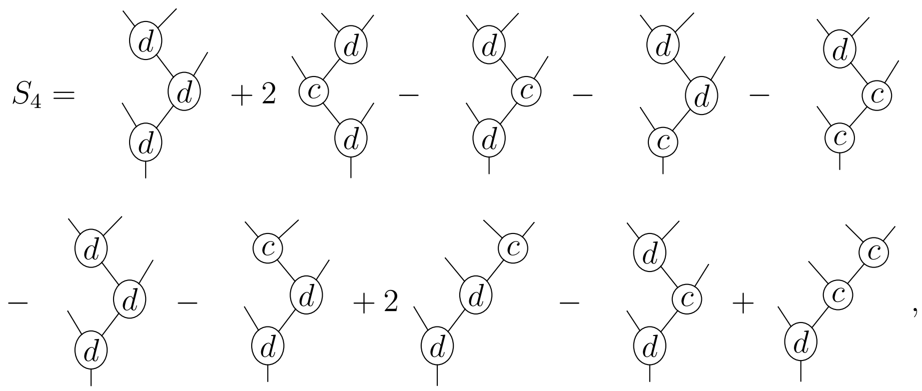
and the following sequence of reductions:



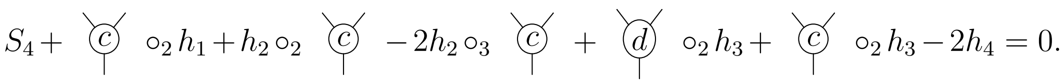
We set



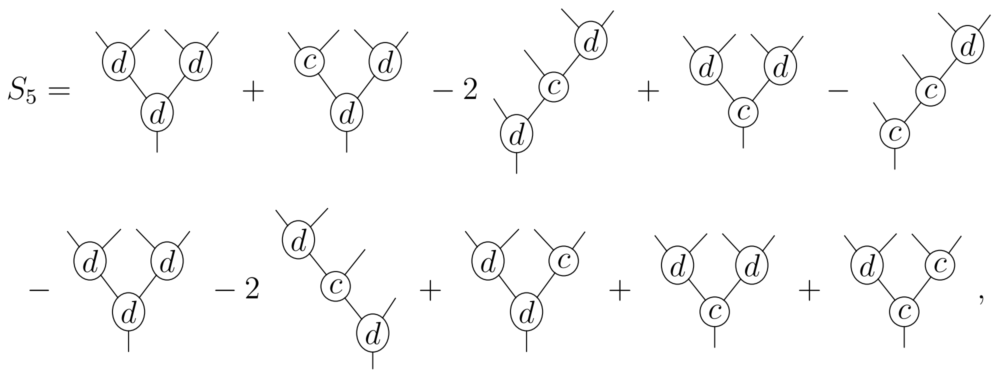
Next, we have



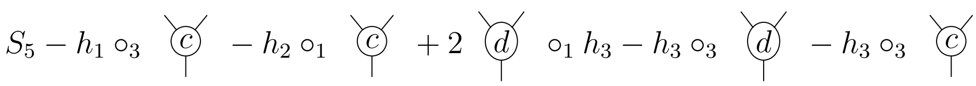
and the following sequence of reductions:



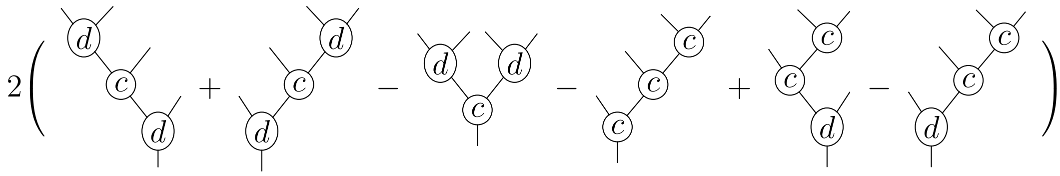
Finally, for the S-polynomial S_5, we have



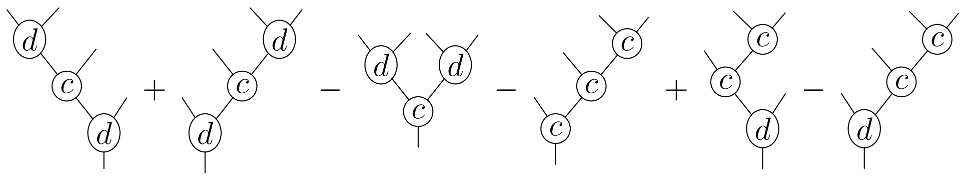
and the sequence of reductions



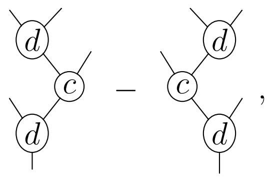
which produces the element



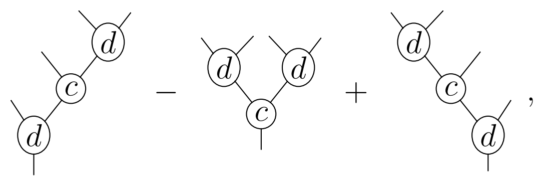
that cannot be reduced further. We set

$h_5 =$  .

In the associated graded operad $\mathbf{gr}\,\mathbf{Dend}$, the two relations h_4 and h_5 we found lead to the relations

$g_4 =$  ,

and

$g_5 =$  ,

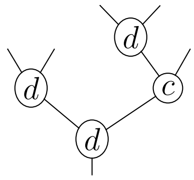
where we multiplied the first of them by -1 to make the leading coefficient equal to 1.

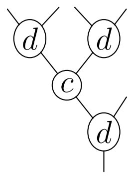
Clearly, these relations go not follow from the relations g_1, g_2, and g_3. In principle, there could be more relations. However, we shall be able to prove that all relations of $\mathbf{gr}\,\mathbf{Dend}$ follow from the relations g_1, g_2, g_3, g_4, and g_5.

Our proof of completeness of defining relations that we found will proceed as follows. First, we shall compute some particular elements of the ideal generated by the relations g_1, g_2, g_3, g_4, and g_5, which will give us information about the associated ideal of leading terms. Using those elements, we shall be able to produce a "combinatorial" upper bound on components of $\mathbf{gr}\,\mathbf{Dend}$, that is a set of monomials that span the components. Using the method of generating functions, we shall convert that combinatorial bound into a numerical bound on dimensions of those components. That bound, compared with the actual known dimensions, will allow us to conclude the completeness.

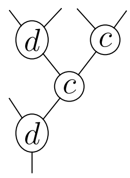
2.4 More relations in the operad $\mathrm{gr}\,\mathrm{Dend}$

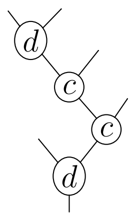
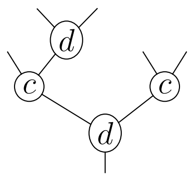
In this section, we find some relations in the operad $\mathbf{gr}\,\mathbf{Dend}$ that make nontrivial contributions to the ideal of leading terms of this operad.

We start by considering the small common multiple  of the leading terms of the relations g_1 and g_4. The corresponding S-polynomial is equal to the monomial

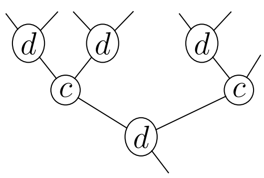
 ,

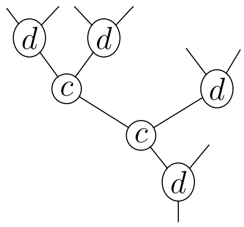
which cannot be reduced using the existing elements, and so contributes to the ideal of the leading terms. For the reasons that will be apparent shortly, we shall denote this monomial $g_{1,1}$.

Further, let us consider the small common multiple  of the leading terms of the relations g_3 and g_4. The The corresponding S-polynomial is equal to

 $-$  ,

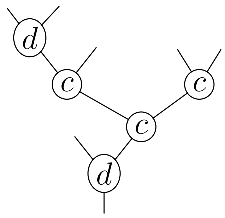
which cannot be reduced using the existing elements, and so contributes to the ideal of the leading terms. For the reasons that shall be apparent shortly, we shall denote this monomial $g_{4,2}$.

Next, we consider the small common multiple  of the leading terms of the relations $g_{1,1}$ and g_4. The corresponding S-polynomial is equal to

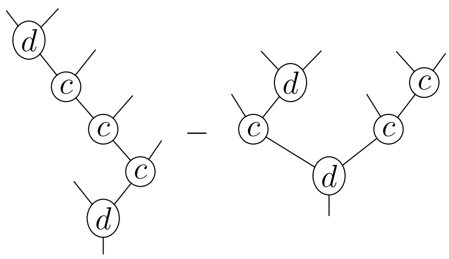

,

which cannot be reduced using the existing elements, and so contributes to the ideal of the leading terms. Its similarity with the element $g_{1,1}$ is now apparent, and we denote it $g_{1,2}$.

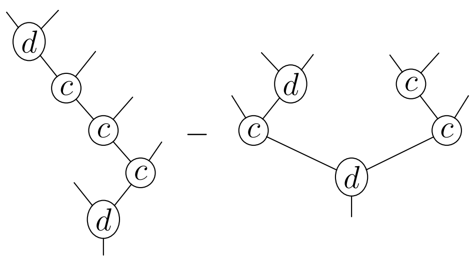
As one more step, we consider the small common multiple



of the leading terms of the relations g_3 and $g_{4,1}$. The corresponding S-polynomial is equal to


,

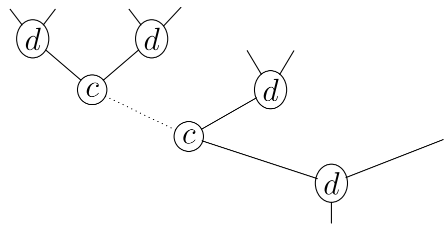
which can be easily reduced to


,

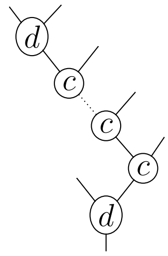
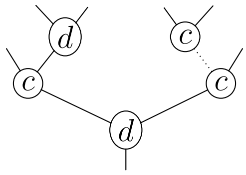
and that element cannot be reduced further using the existing elements, and so contributes to the ideal of the leading terms. Its similarity with the element $g_{4,1}$ is now apparent, and we denote it $g_{4,3}$.

We are now ready to state the key result of this section.

Proposition 7. *For each $n \geq 1$, the ideal of defining relations of $\mathbf{gr\,Dend}$ contains the elements*

$g_{1,n} =$ 

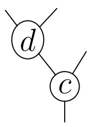
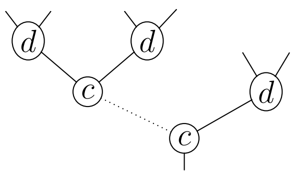
and

$g_{4,n} =$  $-$  ,

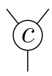
where the dotted lines denote repetitions of the same fragment, and the number of vertices labelled c is equal to n.

Let us remark that the element g_1 can also be included in the first series as $g_{1,0}$.

Proof. Let us note that $g_{1,n}$ is of arity $2n+3$ and $g_{4,n}$ is of arity $n+3$. Using the pattern observed above, this can be easily established by induction on n. Indeed, we have

$g_{1,n+1} = g_{1,n} \circ_{2n+3}$  $- \, g_4 \circ_1$  ,

and also

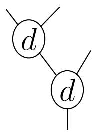
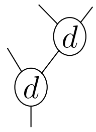
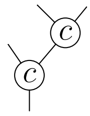
$g_{4,n} \circ_{n+3}$ 

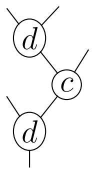
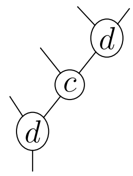
easily reduces to $g_{4,n+1}$ using the associativity relation g_3. □

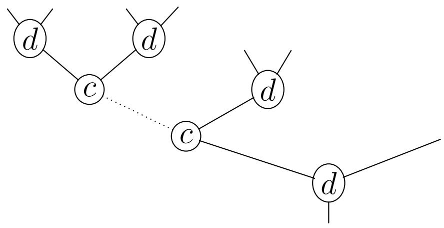
Our work shows that the ideal of leading terms of $\mathbf{gr\,Dend}$ is very likely to have an infinite Gröbner basis. Using the Buchberger's criterion for the purpose of checking that we have a Gröbner basis is, in case of infinite family of polynomials, a very hard task. So we shall bypass it by dimension counting. This will be done in the next section.

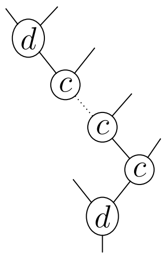
2.5 Dimension counting

In previous sections, we found many elements that contribute to the ideal of the leading term of $\mathbf{gr}\,\mathbf{Dend}$. Let us list them here:

$\mathsf{LM}(g_1) =$  , $\mathsf{LM}(g_2) =$  , $\mathsf{LM}(g_3) =$  ,

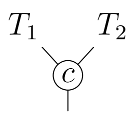
$\mathsf{LM}(g_4) =$  , $\mathsf{LM}(g_5) =$  ,

$\mathsf{LM}(g_{1,n}) =$  ,

$\mathsf{LM}(g_{4,n}) =$ 

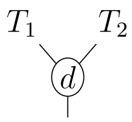
Since all normal monomials with respect to the ideal of the leading terms form a basis of an operad, the normal monomials with respect to the leading terms that we listed form a spanning set of $\mathbf{gr}\,\mathbf{Dend}$: if it happens that more monomials are needed to generate the ideal of the leading terms, there will be fewer normal monomials needed altogether. Let us describe the normal monomials with respect to these leading terms.

Observation 1: if we have a monomial with the root labelled by c, that is a monomial of the form

 ,

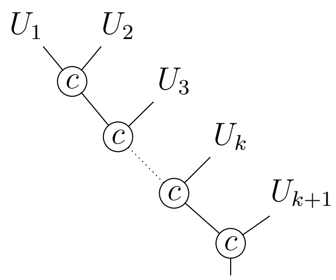
where T_1 and T_2 are normal monomials, then this monomial is normal as long as the root of T_2 is labelled by d, since the only leading term with the root labelled by c is $\mathsf{LM}(g_3)$.

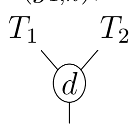
Observation 2: Let we now consider monomials whose root is labelled by d, that is monomials of the form



where T_1 and T_2 are normal monomials. For such a monomial to be normal, we need both T_1 and T_2 to *not* have d as the root label; therefore, each of them is either a leaf or a monomial with the root labelled c. We observe that for every leading term of our relations, one of the children of the root vertex is a leaf, therefore there are no conditions on normality that would involve T_1 and T_2 simultaneously, so we can consider them separately.

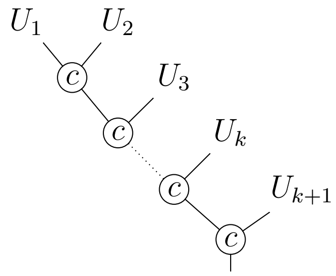
Observation 3: Let us denote by k the length of the maximal "left comb" of T_1 made of vertices labelled c starting from the root, so that T_1 has the following shape:

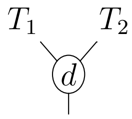


Here each of the trees U_1, ..., U_{k+1} is either a leaf or has a root labelled d (for U_1 this follows from the maximality of the left comb, and for U_2, ..., U_{k+1} from being normal with respect to $\mathsf{LM}(g_3)$). The only constraint on the left-growing trees is imposed by $\mathsf{LM}(g_{1,n})$, $n \geq 1$, so if all U_i are normal, and at least one of them is a leaf, then 

is a normal monomial.

Observation 4: Similarly, let us denote by k the length of the maximal "left comb" of T_2 made of vertices labelled c starting from the root, so that T_2 has the following shape:



Again, here each of the trees U_1, ..., U_{k+1} is either a leaf or has a root labelled d. The constraints on the right-growing trees are imposed by $\mathrm{LM}(g_{4,n})$, $n \geq 1$ and $\mathrm{LM}(g_5)$, so U_1 and U_{k+1} must be leaves, and then 

is a normal monomial.

We shall now convert this description into the language of generating functions. Let us introduce the following sequences:

- c_n is the number of normal monomials with $n \geq 2$ leaves with the root labelled c,
- d_n is the number of normal monomials with $n \geq 2$ leaves with the root labelled d,
- l_{n-1} is the number of normal monomials with $n \geq 2$ leaves with the root labelled d, and the right child of the root being a leaf,
- r_{n-1} is the number of normal monomials with $n \geq 2$ leaves with the root labelled d, and the left child of the root being a leaf.

We shall use the corresponding generating functions

$$C(x) := \sum_{n\geq 2} c_n x^n, \quad D(x) := \sum_{n\geq 2} d_n x^n,$$
$$L(x) := \sum_{n\geq 2} l_n x^n, \quad R(x) := \sum_{n\geq 2} r_n x^n.$$

Our Observation 1 immediately leads to the recurrence relation

$$c_n = 1 \cdot b_{n-1} + (c_2 + d_2)b_{n-2} + \cdots + (c_{n-2} + d_{n-2})d_2 + (c_{n-1} + d_{n-1}) \cdot 1,$$

where the factors 1 correspond to the cases where we have a leaf on the left or on the right; on the level of generating functions, this means

$$C(x) = (x + C(x) + D(x))(x + D(x)).$$

Our Observation 2 leads to the recurrence relation

$$d_n = l_1 r_{n-1} + l_2 r_{n-2} + \cdots + l_{n-1} r_1,$$

since the subscripts in the sequences l and r count precisely the arity of the left-growing and the right-growing subtree; on the level of generating functions, this means

$$D(x) = L(x)R(x).$$

Our observations 3 and 4 are a little bit harder to write in terms of recurrence relations but are easy to write on the level of generating functions, since the composite of series $F(G(x))$ is known [1, Sec. 5.1] to correspond to grafting all possible trees enumerated by $G(x)$ at the leaves of all possible trees enumerated by $F(x)$. With that in mind, our Observation 3 means that

$$L(x) = ((x + D(x)) + (x + D(x))^2 + \cdots + (x + D(x))^p + \cdots) - \\ (D(x) + D(x)^2 + \cdots + D(x)^p + \cdots).$$

Indeed, $x + D(x)$ enumerates normal monomials that are either of arity one or have the root labelled d, so

$$(x + D(x)) + (x + D(x))^2 + \cdots + (x + D(x))^p + \cdots$$

enumerates the results of substituting such trees into arbitrary left combs, and $D(x)$ enumerates normal monomials that have the root labelled d, so

$$D(x) + D(x)^2 + \cdots + D(x)^p + \cdots$$

enumerates the results of substituting such trees into arbitrary left combs, and this is precisely what we need to discard. Similarly, our Observation 4 means that

$$R(x) = x + x^2 + x(x + D(x))x + x(x + D(x))^2 x + \cdots + x(x + D(x))^p x + \cdots$$

Indeed, the term $x(x + D(x))^p x$ counts the left combs with $p+2$ leaves where at each of the middle p leaves one may graft a normal monomial with the root labelled d.

Let us also introduce the generating series $N(x) = x + C(x) + D(x)$; its coefficient of x^n is the number of all monomials of arity n that are normal with respect to our leading terms. Collecting together the formulas that we obtained, and using the geometric series formula, we see that our five generating series $C(x)$, $D(x)$, $L(x)$, $R(x)$, and $N(x)$, considered together with the formal variable x, satisfy the following system of equations:

$$\begin{cases} N = x + C + D, \\ C = (x + D)N, \\ D = LR, \\ L = \dfrac{x + D}{1 - x - D} - \dfrac{D}{1 - D}, \\ R = x + \dfrac{x^2}{1 - x - D}. \end{cases}$$

To deal with this system of equations, we make all equations polynomial by clearing the denominators; after that we may use commutative Gröbner bases. Namely, we shall work in the ring of polynomials in six variables C, D, L, R, N, x with rational coefficients, order the variables $C > D > L > R > N > x$, and consider the induced lexicographic order. Computing the reduced Gröbner basis for this order would allow us to eliminate the variables C, D, L, and R, and find an equation relating N to x. We use the online calculator

`http://magma.maths.usyd.edu.au/calc/`

of `Magma` [2] to perform this computation. The corresponding code is

```
Q := RationalField();
P<C,D,L,R,N,x> := PolynomialRing(Q, 6, "lex");
I := ideal<P | C+D+x-N, C-(x+D)*N, D-L*R,
L*(1-x-D)*(1-D)-(x+D)*(1-D)+D*(1-x-D), (R-x)*(1-x-D)-x^2>;
GroebnerBasis(I);
```

The output of this computation is the following list of polynomials:

$$C + Nx - N + x,$$
$$D - Nx,$$
$$LN + 1/2L - 1/2N^2 - 1/2N,$$
$$Lx + 2L - Nx - N - x,$$
$$R - Nx^2 - x^2 - x,$$
$$N^2x + 2Nx - N + x.$$

The last polynomial does indeed depend only on N and x, giving a polynomial relation

$$N^2x + N(2x - 1) + x = 0,$$

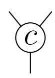
which is a quadratic equation for N. Solving it, we obtain

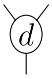
$$N = \frac{(1 - 2x) \pm \sqrt{(2x - 1)^2 - 4x^2}}{2x} = \frac{(1 - 2x) \pm \sqrt{1 - 4x}}{2x}.$$

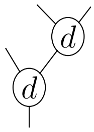
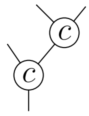
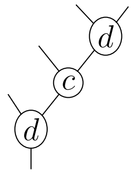
Since N does not contain negative powers of x, we must choose the minus sign, so that

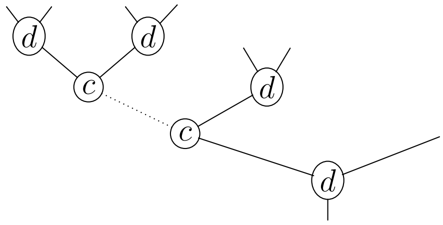
$$N = \frac{(1 - 2x) - \sqrt{1 - 4x}}{2x} = \frac{1 - \sqrt{1 - 4x}}{2x} - 1.$$

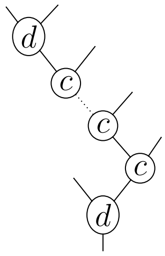
It is well known (see e.g. [12]) that the series $\frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$, $n \geq 0$. By subtracting 1 from it, we merely consider Catalan numbers for $n \geq 1$. We proved the following result.

Proposition 8. *Consider, in the free non-symmetric operad $\mathcal{T}$ generated by* 

and  *the monomial ideal* J *generated by the elements*

$m_2 =$  , $m_3 =$  , $m_5 =$  ,

$m_{1,n} =$  , $n \geq 0$,

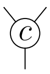
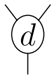
$m_{4,n} =$  , $n \geq 1$.

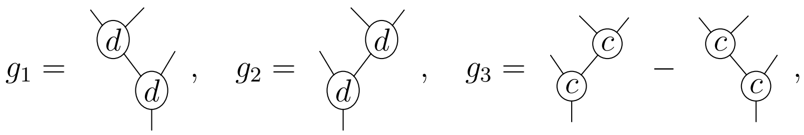
Then $\dim(\mathcal{T}/J)(n) = \frac{1}{n+1}\binom{2n}{n}$ *for all* $n \geq 1$.

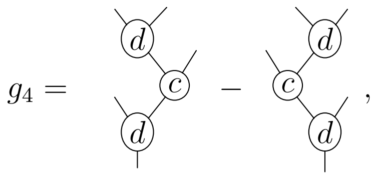
Proof. Our results above imply this result immediately: monomials that are normal with respect to these elements clearly form a basis in the quotient. $\square$

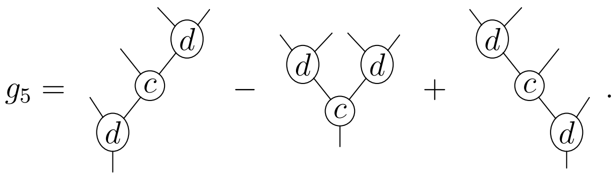
2.6 Presentation of the operad $\mathrm{gr}\,\mathbf{Dend}$

We are finally ready to state and prove the main result of this manuscript.

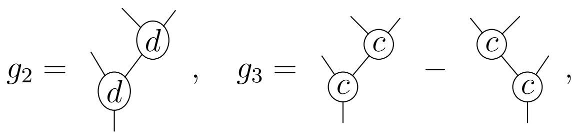
Theorem 4. *Consider the operad $\mathrm{gr}\,\mathbf{Dend}$ which is obtained as the associated graded operad of $\mathbf{Dend}$ for the associative filtration of that operad. This operad is isomorphic to the quotient of the free non-symmetric operad $\mathcal{T}$ generated by*  *and*  *by the ideal I generated by the following five elements:*

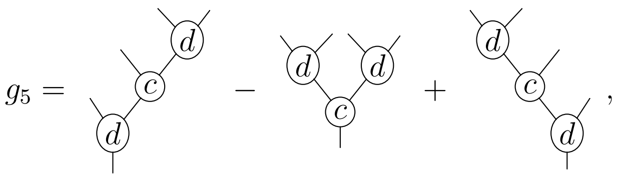


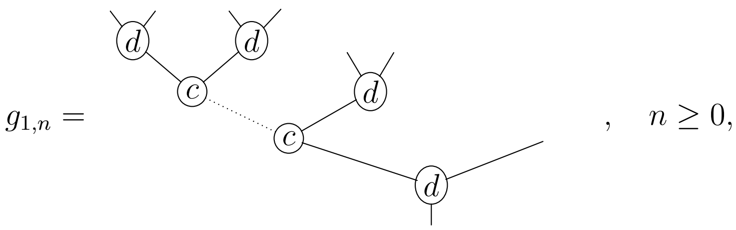




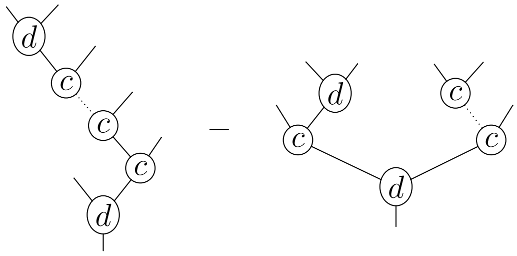
Moreover, the elements







$g_{4,n} =$



$, \quad n \geq 1$

form a Gröbner basis of this operad for the associative filtration ordering.

Proof. The proof goes in several steps. First, we note that from the computations of Section 2.3 it follows directly that the ideal of defining relations of the operad $\mathbf{gr}\,\mathbf{Dend}$ contains the ideal I. Therefore, we have

$$\dim \mathbf{gr}\,\mathbf{Dend}(n) \leq \dim(\mathcal{T}/I)(n).$$

Moreover, by Proposition 7, the generators of the ideal J from Proposition 8 belong to the ideal of leading terms of I. Therefore, we have

$$\dim(\mathcal{T}/I)(n) = \dim(\mathcal{T}/\mathsf{LT}(I))(n) \leq \dim(\mathcal{T}/J)(n) = \frac{1}{n+1}\binom{2n}{n}.$$

However, passing to the associated graded operad does not change the dimensions of components, so

$$\dim \mathbf{gr}\,\mathbf{Dend}(n) = \dim \mathbf{Dend}(n),$$

and by a result of [10, Prop. 5.7], $\dim \mathbf{Dend}(n)$ is given by the Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

Thus, we obtained a chain of inequalities with the same lower and upper bound:

$$\frac{1}{n+1}\binom{2n}{n} = \dim \mathbf{gr}\,\mathbf{Dend}(n) \leq \dim(\mathcal{T}/I)(n) = \\ \dim(\mathcal{T}/\mathsf{LT}(I))(n) \leq \dim(\mathcal{T}/J)(n) = \frac{1}{n+1}\binom{2n}{n}$$

In such a chain, all inequalities must be equalities. The inequality

$$\dim \mathbf{gr}\,\mathbf{Dend}(n) \leq \dim(\mathcal{T}/I)(n)$$

is an equality for all n if and only if $\mathbf{gr}\,\mathbf{Dend} \cong \mathcal{T}/I$; the inequality

$$\dim(\mathcal{T}/\mathsf{LT}(I))(n) \leq \dim(\mathcal{T}/J)(n)$$

is an equality if and only if $J = \mathsf{LT}(I)$, which means that the elements listed in the statement of the theorem form a Gröbner basis of I. $\square$

Bibliography

[1] Bergeron, N., Loday, J.-L.: The symmetric operation in a free pre-Lie algebra is magmatic. Proc. Amer. Math. Soc. 139:55, 1585–1597 (2011).

[2] Bosma, W., Cannon, J., Playoust, C.: The Magma algebra system. I. The user language, J. Symbolic Comput., **24**, 235–265 (1997).

[3] Bremner, M., Dotsenko, V.: Algebraic operads. An algorithmic companion. CRC Press, Boca Raton, FL, 2016. xvii+365 pp.

[4] Chapoton, F., Livernet, M.: Pre-Lie algebras and the rooted trees operad. Int. Math. Res. Notices 8, 395–408 (2001).

[5] Dotsenko, V.: Algebraic structures of F-manifolds via pre-Lie algebras, Ann. Mat. Pura Appl. (1923 –), 198:2, 517–527 (2019).

[6] Dotsenko, V., Khoroshkin, A.: Gröbner bases for operads. Duke Math. J. 153:2, 363–396 (2010).

[7] Dotsenko, V., Vallette, B.: Higher Koszul duality for associative algebras. Glasgow Math. J. 55:A, 55-74 (2014).

[8] Filtered algebra. E. B. Vinberg (originator), Encyclopedia of Mathematics. URL: `http://www.encyclopediaofmath.org/index.php?title=Filtered_algebra&oldid=12513`.

[9] Loday, J.-L., Vallette, B.: Algebraic operads. Grundlehren Math. Wiss. 346, Springer, Heidelberg, 2012. xviii+512 pp.

[10] Loday, J.-L.: Dialgebras. In: "Dialgebras and related operads", Lect. Notes Math. 1763. 7–66 (2001).

[11] Madariaga, S.: Gröbner–Shirshov bases for the non-symmetric operads of dendriform algebras and quadri-algebras. J. Symbolic Comput. 60, 1–14 (2014).

[12] Stanley, R. P.: Catalan Numbers. Cambridge University Press, Cambridge, 2015.