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The Catlin Multitype of Sums of Squares Domains

by
Nicholas Aidoo

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Date: January, 2021
ID Number: 16334917
email: aidoon@tcd.ie

Supervisor : Prof. Andreea Nicoara
Internal Referee : Prof. Dmitri Zaitsev
External Referee : Prof. Martin Kolář

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Abstract

Given a sum of squares domain of finite D'Angelo 1-type at the origin, the model resulting from the computation of the Catlin multitype of such a domain at the origin is shown to likewise be a sum of squares domain. Based on this result, a partial normalization of the defining function of a sum of squares domain is obtained. Under the same finite type assumption, the Catlin multitype is also shown to be an invariant of the ideal of holomorphic functions defining the domain. These results are proven using Martin Kolář's algorithm for the computation of the Catlin multitype defined in [22]. For a sum of squares domain, the Kolář algorithm is restated in terms of ideals of holomorphic functions.

A commutative-algebraic way of characterizing the rank of the Levi determinant of a sum of squares domain is also presented.

In the Kolář algorithm for the computation of the Catlin multitype, polynomial transformations are required at every step to minimize the number of variables appearing in the leading polynomial. The polynomial transformations in the algorithm applied to a sum of squares domain are characterized by relating them to elementary row and column operations on the Levi matrix.

Using this characterization of the polynomial transformations and the restatement of the Kolář algorithm in terms of ideals of holomorphic functions, an algorithm that connects nicely the notion of simplifying the Jacobian module associated to a sum of squares domain with elementary row operations on the complex Jacobian matrix of the same domain is devised. By employing this algorithm, the polynomial transformations needed in the Kolář algorithm for the computation of the Catlin multitype are explicitly constructed.

Declaration

I declare that this thesis has not been submitted as an exercise for a degree at this or any other university and it is entirely my own work.

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Dedication

I dedicate this thesis to my wife Dinah Aidoo to thank her for all her support and sacrifice.

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Table of Contents

Chapter 1

Introduction

Over the years, the $\bar{\partial}$-Neumann problem has motivated a lot of work in several complex variables. Solving the central partial differential equation in several complex variables, the $\bar{\partial}$-problem, leads to a boundary value problem known as the $\bar{\partial}$-Neumann problem. While this boundary value problem is elliptic when the data is supported in the interior of the domain, its boundary conditions are non-elliptic. As a result, the study of subellipticity only needs to be done on the boundary of the domain.

Kohn in [18] and [19] solved the $\bar{\partial}$-Neumann problem in the strongly pseudoconvex case, and subsequently, Hörmander used weighted estimates in [14] to solve the problem in the more general case of a weakly pseudoconvex domain. Already in the strongly pseudoconvex case, Kohn had shown that the gain in the degree of differentiability of the solution was exactly $\frac{1}{2}$. This prompted the obvious question: When is subellipticity of the $\bar{\partial}$-Neumann problem satisfied at all? In his Acta paper [21], J.J. Kohn tackled the question of subellipticity by introducing the notion of subelliptic multipliers and as well as an algorithm to generate these multipliers. He also established that for a pseudoconvex domain with real-analytic boundary, there is an equivalence between the subellipticity of the $\overline{\partial}$-Neumann problem for (p, q) forms at a point of the domain and the property that the maximum order of contact of q-dimensional complex-analytic varieties with the boundary of the domain at the same point is finite. He proved this equivalence via the termination of his algorithm, which amounts to the statement that the constant unity function is a subelliptic multiplier.

Two important developments followed. The first was D'Angelo's work on the order of contact of complex-analytic varieties with the boundary of the domain, which is now commonly known as the D'Angelo type and is a boundary invariant. The second development was Catlin's work in the 80's in [5], [6], and [7] showing that the equivalence of the subellipticity of $\bar{\partial}$-Neumann problem with finite D'Angelo type holds for any pseudoconvex domain with smooth boundary and not just in the real-analytic case as Kohn had shown. He showed that the subelliptic gain ε in the $\bar{\partial}$-Neumann problem on (p, q) forms must satisfy $\varepsilon \leq \frac{1}{\Delta_q(M,p)}$, where $\Delta_q(M, p)$ is the D'Angelo type. He achieved this without going through the Kohn algorithm. In the process of proving this equivalence, Catlin introduced another boundary invariant called the multitype in [6]. The multitype gives a refined measure of the vanishing order of the defining function of the domain by assigning a weight to each coordinate direction. The entries of the multitype m_{n-q+1} are always bounded above by the D'Angelo q-type, and so for a pseudoconvex domain of finite D'Angelo q-type, there is always a finite number of level sets of the multitype in some neighborhood.

The multitype is difficult to compute in general, but it has several interesting properties. In order to compute it and also establish its properties, Catlin introduced another weight known as the commutator multitype. In the pseudoconvex case, the commutator multitype and the multitype equal each other. In [6] Catlin proved this equality by considering polynomial models of a domain consisting of just the terms from the Taylor expansion of the defining function that have weight 1 with respect to the multitype.

An essential ingredient of Kohn's argument in [21] that the subellipticity of the $\overline{\partial}$-Neumann problem for (p, q) forms is equivalent to finite D'Angelo q-type for real-analytic pseudoconvex domains is the Diederich-Fornæss Theorem. As proven by Diederich and Fornæss in [12], if there is a real-analytic variety of holomorphic dimension q in the boundary of a real-analytic pseudoconvex domain, then in any neighborhood of a point on that variety, there exists a complex variety of dimension q lying in the boundary of the domain. This result was generalized to smooth pseudoconvex domains by Bedford and Fornæss in [2] in 1981. Therefore, it was known even before Catlin's work in the 80's that the existence of submanifolds of holomorphic dimension q in the boundary of the domain was an obstruction to subellipticity of the $\overline{\partial}$-Neumann problem. In [6] Catlin sought to stratify the boundary of the domain in such a way as to be able to rule out the existence of such submanifolds of holomorphic dimension q. The level sets of his invariant, the multitype, precisely give this stratification as Catlin proved that each level set of the multitype sits in a submanifold of the boundary of the domain of holomorphic dimension at most $q - 1$.

In [24], A. Nicoara used this stratification in order to give a constructive proof for the termination of the Kohn algorithm in the real-analytic pseudoconvex case as opposed to Kohn's indirect proof in [21]. This prompted a question posed by D'Angelo to A. Nicoara: In the simplest possible case of a domain given by sum of squares of holomorphic functions, how does the stratification look like? We seek to answer D'Angelo's question by finding out how the entries of the multitype relate to the algebraic-geometric behavior of the ideal of holomorphic functions in the sum of squares.

The goal of this thesis is to introduce some important preparatory tools and techniques necessary for answering D'Angelo's question on the multitype level set stratification of sums of squares domains. We will focus here on the multitype computations for such domains. Our main tool is an algorithm devised by M. Kolář in [22] for the computation of the Catlin multitype when it has finite entries. In order to ensure this condition is satisfied, we will assume finite D'Angelo 1-type throughout since the latter bounds from above the last entry of the multitype; see [6].

Domains defined by sums of squares of holomorphic functions constitute a very important class in the field of several complex variables as they connect in a very natural way complex analysis with algebraic geometry. This class of domains was introduced by J.J. Kohn in his Acta paper [21] under the term *special domains*. In [10] and [11], D'Angelo studied the local geometry of real hypersurfaces by assigning to every point on the hypersurface an associated family of ideals of holomorphic functions and exploring various invariants in commutative algebra and algebraic geometry. He established a close connection between the geometry of sums of squares domains and complex algebraic geometry. Further work by Y.-T. Siu in [26] and [27] on sums of squares domains introduced new approaches for generating multipliers for general systems of partial differential equations. Owing to his initial work on sums of squares domains, Y.-T. Siu gave an extension of the special domain approach to real analytic and smooth cases. S.-Y. Kim and D. Zaitsev in [17] proposed a new class of geometric invariants called *the jet vanishing orders* and used them to establish a new selection algorithm in the Kohn's construction of subelliptic multipliers of sums of squares domains in dimension 3. Also, in a recent paper by the same authors [16], they provide a solution to the effectiveness problem in Kohn's algorithm for generating multipliers for domains including those defined by sums of squares of holomorphic functions in all dimensions. Other important results pertaining to sums of squares domains can be found in [8], [9], [13], [15], and [23].

Owing to the connection between the associated family of ideals of holomorphic functions and the geometry of sums of squares domains, a natural question is whether or not the multitype could be computed from the corresponding ideals of holomorphic functions. An answer to this question is provided in this thesis. More specifically, we show that the multitype of a sum of squares domain can be computed from the related ideal of holomorphic functions by restating the Kolář algorithm at the level of ideals. Besides the fact that working with ideals aligns better with complex algebraic geometry, this restatement also reduces significantly the amount of work involved in computing the Catlin multitype for sums of squares domains.

A *sum of squares domain* $\Omega \subset \mathbb{C}^{n+1}$ is one whose boundary defining function $r(z)$ is given by

$$r(z) = 2\mathrm{Re}(z_{n+1}) + \sum_{j=1}^{N} |f_j(z_1, \ldots, z_{n+1})|^2, \tag{1.1}$$

where $f_j(z_1, \ldots, z_{n+1})$ for all j, $1 \leq j \leq N$, are holomorphic functions vanishing at the origin in $\mathbb{C}^{n+1}$. We shall denote by $\mathcal{M} \subset \mathbb{C}^{n+1}$ the hypersurface defined by $\{z \in \mathbb{C}^{n+1} \mid r(z) = 0\}$.

The *model hypersurface* associated to $\mathcal{M}$ at the origin is given by

$$\mathcal{M}_H = \{z \in \mathbb{C}^{n+1} \mid \mathrm{H}(z, \bar{z}) = 0\}, \tag{1.2}$$

the zero locus of the homogeneous polynomial $\mathrm{H}(z, \bar{z})$ consisting of all monomials from the Taylor expansion of the defining function that have weight 1 with respect to the multitype weight. We refer to $\mathrm{H}(z, \bar{z})$ as the *model polynomial.*

$$\mathrm{H}(z, \bar{z}) = 2\mathrm{Re}(z_{n+1}) + \sum_{j=1}^{N} |h_j(z_1, \ldots, z_n)|^2, \tag{1.3}$$

where h_j is a polynomial consisting of all terms from the Taylor expansion of f_j of weight 1/2 with respect to the multitype at the origin, $j = 1, \ldots, N$. Note that the model hypersurface $\mathcal{M}_H = \{z \in \mathbb{C}^{n+1} \mid \mathrm{H}(z, \bar{z}) = 0\}$ is a decoupled sum of squares domain since variable z_{n+1} has weight 1, so no h_j can depend on it. By Catlin's results in [6], $\mathcal{M}_H$ has the same multitype at the origin as the original domain. Therefore, with respect to all multitype computations, sums of squares domains behave as if they were decoupled.

In [22] Kolář characterizes by weight the polynomial transformations that do not modify the multitype and devises an approximation algorithm yielding a weight and a partial model polynomial at each step of the algorithm. He calls the partial model polynomial the leading polynomial. At the conclusion of each step, a polynomial transformation that does not modify the multitype is supposed to be applied so that the partial model polynomial depends on the minimum number of variables. When all entries of the multitype are finite, this approximation algorithm terminates at the multitype itself. The Kolář algorithm could fail to terminate when there is at least one infinite entry in the multitype. An example of this type will be provided in the thesis.

Even though the polynomial transformations that do not modify the multitype in the Kolář algorithm play a significant role in the computation process, there is no technique in [22] to construct them. We go further by providing an explicit approach that characterizes and constructs these polynomial transformations in the Kolář algorithm for a sum of squares domain. We establish a correspondence between such a polynomial transformation and some defined sequence of elementary row and column operations on the Levi matrix of a sum of squares domain. We refer to this sequence as the *row reduction algorithm*. The termination of the row reduction algorithm at each step of the Kolář algorithm indicates that the partial model polynomial produced depends on the minimum number of variables. The algorithm naturally relates the notion of Jacobian module of a sum of squares domain to elementary row operations performed on the complex Jacobian matrix of the same domain.

The thesis is structured in the following manner: Chapter 2 defines the Catlin multitype and provides a thorough description of the Kolář algorithm as introduced in [22] for the computation of the multitype at the origin. We give an example to demonstrate that the Kolář algorithm does not always terminate if there exists at least one infinite entry of the multitype. Chapter 3 characterizes the rank of the Levi determinant of a sum of squares domain by describing it in a commutative-algebraic way.

Chapter 4 presents a key lemma for the characterization of the multitype entries of the sum of squares domain. Specifically, we establish the fact that each multitype entry can be realized by the modulus square of some monomial. Using the characterization of such monomials, we show that the model of a sum of squares domain is likewise a sum of squares domain. As an application, we produce a partial normalization of the defining function of a sum of squares domain of finite D'Angelo 1-type at the origin when the rank of its Levi matrix is nonzero. We also show in this chapter that the multitype of a sum of squares domain is an invariant of the ideal of holomorphic functions defining the domain under the assumption of finite D'Angelo 1-type at the origin. This answers positively a question posed to the author by D. Zaitsev. In the same chapter, a modified version of the Kolář algorithm in terms of ideals of holomorphic polynomials is provided.

The polynomial transformations in the Kolář algorithm for the computation of the multitype of sums of squares domains are characterized in chapter 5. This characterization is achieved through row and column operations performed on the Levi matrix of the sum of squares domain. Using the restatement of the Kolář algorithm in terms of ideals of holomorphic polynomials in chapter 4, we translate the row-column operations on the Levi matrix of a sum of squares domain into row operations on the complex Jacobian matrix of the same domain. We then explicitly construct the allowable polynomial transformations in the Kolář algorithm via a much simpler algorithm that relies on the notions of gradient ideal and Jacobian module.

Finally, in chapter 6 we summarize the entire thesis and outline further work that the author hopes to do subsequently.

Chapter 2

Catlin Multitype and the Kolář Algorithm

In this chapter, we describe the Catlin multitype and the Kolář algorithm for the computation of the multitype at the origin in [22]. The notion of weights, distinguished weights, and the multitype were introduced by D. Catlin in [6].

2.1 Preliminaries

In this section, we give some definitions and also state without proofs some theorems pertinent to our discussions in subsequent chapters. The definition of subellipticity of the $\bar{\partial}$-Neumann problem is only provided here for completeness given the topics covered in the introduction.

Definition 2.1.1. Let Ω be a domain in $\mathbb{C}^n$. Let p be a point on the boundary $\partial\Omega$ of Ω. We say that the $\bar{\partial}$-Neumann problem satisfies the subelliptic estimates on (p, q) forms at the point p if there exists a neighborhood U of p and the constants $\varepsilon > 0$ and $C > 0$ such that

$$|||\varphi|||^2_{\varepsilon,U} \leq C\big(||\bar{\partial}\varphi||^2 + ||\bar{\partial}^*\varphi||^2 + ||\varphi||^2\big) \tag{2.1}$$

for all $\varphi \in D^{(p,q)}(U)$, where $D^{(p,q)}(U)$ is the space of (p, q) forms $\varphi \in Dom(\bar{\partial}^*)$ such that $\varphi_{IJ} \in C_0^\infty(U \cap \bar{\Omega})$ for all components φ_{IJ} of φ and $|||\varphi|||_{\varepsilon,U}$ is the local Sobolev norm of order ε on U. The constant ε is referred to as the order of the subelliptic estimate.

Before we give the next definition, we recall the notion of a parameterized holomorphic curve as well as its order. A nonconstant holomorphic mapping

$$\psi : U \to \mathbb{C}^n, \tag{2.2}$$

where U is an open set of $\mathbb{C}$ is called a parameterized holomorphic curve. We now let

$$\psi : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$$

be the germ of a parameterized holomorphic curve. We define the order of ψ at 0 to be the greatest integer k for which all derivatives of order strictly less than k vanish at 0. Denoted by $v(\psi)$, the order of the parameterized holomorphic curve is sometimes referred to as the multiplicity. The multiplicity of $\psi = (\psi_1, \dots, \psi_n)$ will then be given precisely by $v(\psi) = \min_{1\leq j\leq n}\{v(\psi_j)\}$. The definition below comes from [11]:

Definition 2.1.2. Let $(\mathcal{M}, p)$ be the germ at p of a smooth real hypersurface in $\mathbb{C}^n$. Let r be a local defining function for $\mathcal{M}$ near p. The maximum order of contact of 1-dimensional complex-analytic varieties with $\mathcal{M}$ at p is given by

$$\Delta_1(\mathcal{M}, p) \;=\; \sup_{\psi} \frac{v(\psi^* r)}{v(\psi)}. \tag{2.3}$$

The D'Angelo 1-type, denoted by $\Delta_1(\mathcal{M}, p)$, is said to be finite if the quantity in (2.3) is finite.

Definition 2.1.3. The maximum order of contact of q-dimensional complex analytic varieties with a smooth real hypersurface $\mathcal{M}$ at a point p is given by

$$\Delta_q(\mathcal{M}, p) \;=\; \inf_{P} \{\Delta_1(\mathcal{M} \cap P, p)\}, \tag{2.4}$$

where P is a complex affine subspace of dimension $n - q + 1$ passing through p. Thus, the infimum is taken over all choices of P. The D'Angelo q-type $\Delta_q(\mathcal{M}, p)$, is finite if (2.4) is finite.

We state Theorems 2.1.1 and 2.1.2 from [10] without proofs. These theorems give the results that the D'Angelo type is finitely determined and that the set of points of finite type is an open subset of the boundary of the domain. Let $\mathcal{M}_k$ be the hypersurface defined by the Taylor polynomial of the defining function r of $\mathcal{M}$ to the k-th order at a point p. We denote by $\Delta_q(\mathcal{M}_k, p)$, the D'Angelo type of the hypersurface $\mathcal{M}_k$ defined by this Taylor polynomial.

Theorem 2.1.1. *Let $\mathcal{M}$ be a real hypersurface of $\mathbb{C}^n$ and let $p \in \mathcal{M}$ be a point. The following are equivalent:*

i. $\Delta_q(\mathcal{M}, p)$ *is finite.*

ii. There is an integer k_0, such that if $k \geq k_0$ then $\Delta_q(\mathcal{M}_k, p) = \Delta_q(\mathcal{M}, p)$ is finite.

iii. There is an integer k_0, such that for $k \geq k_0$, we have $\Delta_q(\mathcal{M}_k, p) \leq k$.

Remark 2.1.1. *The integer k_0 can be taken to be the ceiling of $\Delta_q(\mathcal{M}, p)$, which we will denote by $\lceil \Delta_q(\mathcal{M}, p) \rceil$.*

Theorem 2.1.2. *Let $\mathcal{M} \subset \mathbb{C}^n$ be a smooth real hypersurface. Let p_0 be a point of finite type. Then there is a neighborhood U_{p_0} of p_0 such that if p lies in U_{p_0}*

$$\Delta_q(\mathcal{M}, p) \leq 2(\Delta_q(\mathcal{M}, p_0))^{n-q}.$$

In particular, the set of points of finite type is an open subset of $\mathcal{M}$.

J. J. Kohn introduced the notion of type of a point on a pseudoconvex hypersurface in $\mathbb{C}^2$ in [20]. In [4] Thomas Bloom and Ian Graham generalized Kohn's notion to $\mathbb{C}^n$ and gave a geometric characterization of type of points on real hypersurfaces in $\mathbb{C}^n$. Here is the definition given by Bloom and Graham: Let $\mathcal{N}$ be a real C^∞ hypersurface defined in an open subset $U \subset \mathbb{C}^n$ with defining function r. Let $\mathscr{L}_k$ for $k \geq 0$ an integer, be the module, over $C^\infty(U)$, of vector fields generated by the tangential holomorphic vector fields to $\mathcal{N}$, their conjugates, and commutators of order less than or equal to k of such vector fields.

Definition 2.1.4. A point $p \in \mathcal{N}$ is of type m if $\langle \partial r(p), F(p) \rangle = 0$ for all $F \in \mathscr{L}_{m-1}$ while $\langle \partial r(p), F(p) \rangle \neq 0$ for some $F \in \mathscr{L}_m$.

Here we denote the contractions between a cotangent vector and a tangent vector by $\langle , \rangle$. We shall refer to the type at a point p as defined above as the *Bloom-Graham type.*

2.2 Computation of the Catlin Multitype

As stated in the introduction, D. Catlin in [6] devised another boundary invariant called the commutator multitype to compute the multitype on the boundary of a pseudoconvex domain since the latter cannot be computed directly from its definition. Subsequently, M. Kolář in [22] also devised an algorithm for the computation of the Catlin multitype on a general smooth hypersurface (not necessarily pseudoconvex) when all its entries are finite.

We now present some definitions we use in the thesis following the set-up of Kolář in [22] and then describe some of the tools that M. Kolář introduced. Let $\mathcal{M}$ be a hypersurface in $\mathbb{C}^{n+1}$ and $p \in \mathcal{M}$ be a Levi degenerate point. We will assume that p is a point of finite D'Angelo 1-type. Let (z, w) be local holomorphic coordinates centered at the point p, where $w = u + iv$ is the complex non-tangential variable and the complex tangential variables are in the n-tuple $z = (z_1, \dots, z_n)$ with $z_k = x_k + iy_k$. Throughout this thesis, we will compute and define weights by considering only the complex tangential variables $z_1, \dots, z_n$ as in [22].

Definition 2.2.1. A weight $\Lambda = (\mu_1, \dots, \mu_n)$ is an n-tuple of rational numbers with $0 \leq \mu_j \leq \frac{1}{2}$ satisfying:

i. $\mu_j \geq \mu_{j+1}$ for $1 \leq j \leq n-1$;

ii. For each t, either $\mu_t = 0$ or there exists a sequence of nonnegative integers $a_1, \dots, a_t$ satisfying $a_t > 0$ such that
$$\sum_{j=1}^{t} a_j \mu_j = 1.$$

Let Λ be a weight. If $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex, then we define the weighted length of α by
$$|\alpha|_\Lambda = \sum_{j=1}^{n} \alpha_j \mu_j.$$
Also if $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\hat{\alpha} = (\hat{\alpha}_1, \dots, \hat{\alpha}_n)$ are multiindices then the weighted length of the pair $(\alpha, \hat{\alpha})$ is defined by
$$|(\alpha, \hat{\alpha})|_\Lambda = \sum_{j=1}^{n} (\alpha_j + \hat{\alpha}_j) \mu_j.$$

Definition 2.2.2. A monomial $A_{\alpha\hat{\alpha}l} z^\alpha z^{\hat{\alpha}} u^l$ is said to be of weighted degree κ if
$$\kappa := l + |(\alpha, \hat{\alpha})|_\Lambda.$$

Similarly, we define the weighted order of the differential operator $D^\alpha \bar{D}^{\hat{\alpha}} D^l$ to equal to $\kappa := l + |(\alpha, \hat{\alpha})|_\Lambda$, where

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}, \quad \bar{D}^{\hat{\alpha}} = \frac{\partial^{|\hat{\alpha}|}}{\partial \bar{z}_1^{\hat{\alpha}_1} \cdots \partial \bar{z}_n^{\hat{\alpha}_n}}, \quad \text{and} \quad D^l = \frac{\partial^l}{\partial u^l}.$$

A polynomial $P(z, \bar{z}, u)$ is said to be Λ-homogeneous of weighted degree κ if it is a sum of monomials of weighted degree κ.

We shall set the variable w as well as the variables u and v to have a weight of one.

Definition 2.2.3. A weight $\Lambda = (\mu_1, \ldots, \mu_n)$ is said to be *distinguished* if there exist local holomorphic coordinates (z, w) mapping p to the origin such that the boundary-defining equation for $\mathcal{M}$ in the new coordinates is of the form

$$v = P(z, \bar{z}) + o_\Lambda(1), \tag{2.5}$$

where $P(z, \bar{z})$ is a Λ-homogeneous polynomial of weighted degree 1 without pluriharmonic terms and $o_\Lambda(1)$ denotes a smooth function whose derivatives of weighted order less than or equal to 1 vanish at zero.

We order the weights lexicographically. This means that for the pair of weights $\Lambda_1 = (\mu_1, \ldots, \mu_n)$ and $\Lambda_2 = (\mu_1', \ldots, \mu_n')$, $\Lambda_1 > \Lambda_2$ if for some t, $\mu_j = \mu_j'$ for $j < t$ and $\mu_t > \mu_t'$.

Definition 2.2.4. Let $\mathcal{M}$ be a hypersurface in $\mathbb{C}^{n+1}$, and let $p \in \mathcal{M}$. Let $\Lambda^* = (\mu_1, \ldots, \mu_n)$ be the greatest lower bound with respect to the lexicographic ordering of all the distinguished weights at p. The *multitype* $\mathscr{M}$ at p is defined to be the n-tuple $(m_1, \ldots, m_n)$, where $m_j = \infty$ if $\mu_j = 0$ and $m_j = \frac{1}{\mu_j}$ if $\mu_j \neq 0$. We call the multitype $\mathscr{M}$ at p finite, if the last entry $m_n < \infty$.

The next theorem from [6] clarifies the relationship between the multitype and the D'Angelo type for a pseudoconvex domain:

Theorem 2.2.1. (Catlin). *Let $\Omega \subset \mathbb{C}^{n+1}$ be a pseudoconvex domain smooth boundary. Let $p_0 \in b\Omega$ be a boundary point. If $\mathscr{M}(p_0) = (m_1, \ldots, m_n)$ is the multitype at p_0, then for each $q = 1, \ldots, n$, $m_{n+1-q} \leq \Delta_q(b\Omega, p_0)$, where $\Delta_q(b\Omega, p_0)$ is the D'Angelo q-type at p_0.*

For the purposes of this thesis, we shall assume finite D'Angelo 1-type at every point $p \in \mathcal{M}$, since by Theorem 2.2.1, this assumption ensures that all the entries of the multitype are finite, which is the exact setting in which the Kolář algorithm works.

For a weight Λ, we say the local coordinates on $\mathcal{M}$ at p are Λ-adapted if $\mathcal{M}$ is described locally to have the form in (2.5), where P is Λ-homogeneous. We shall refer to Λ^*-adapted coordinates as the multitype coordinates given such that P is Λ^*-homogeneous.

Let γ_j, $j = 1, \ldots, c$, be the length of the j-th constant piece of the multitype weight given such that c is the number of distinct entries in the multitype. Let $\sum_{i=1}^{j} \gamma_i = k_j$, then we have

$$\mu_1 = \cdots = \mu_{k_1} > \mu_{k_1+1} = \cdots = \mu_{k_2} > \cdots = \mu_{k_{c-1}} > \mu_{k_{c-1}+1} = \cdots = \mu_n,$$

where $n = k_c$. We define a monotone sequence of weights $\Lambda_1, \ldots, \Lambda_c$ which are ordered lexicographically as follows. Λ_1 is a constant n-tuple $(\mu_1, \ldots, \mu_1)$ and $\Lambda_c = \Lambda^*$ is the multitype weight. We then define the weight $\Lambda_j = (\lambda_1^j, \ldots, \lambda_n^j)$ for $1 < j < c$, by $\lambda_i^j = \mu_i$ for $i \leq k_{j-1}$ and $\lambda_i^j = \mu_{k_{j-1}+1}$ for $i > k_{j-1}$. Note that this construction yields a finite sequence of weights even if Λ^* has some infinite entries.

Definition 2.2.5. Let $\Lambda = (\lambda_1, \ldots, \lambda_n)$ be a weight and

$$\tilde{w} = w + g(z_1, \ldots, z_n, w) \text{ and } \tilde{z}_j = z_j + f_j(z_1, \ldots, z_n, w),$$

for $1 \leq j \leq n$, be a holomorphic change of variables. We say that this transformation is:

i. Λ-homogeneous if f_j is a Λ-homogeneous polynomial of weighted degree λ_j and g is a Λ-homogeneous polynomial of weighted degree 1,

ii. Λ-superhomogeneous if f_j has a Taylor expansion consisting of monomials that have weighted degree $\geq \lambda_j$ and g consists of terms of weighted degree ≥ 1,

iii. Λ-subhomogeneous if the Taylor expansion of f_j consists of terms of weighted degree $\leq \lambda_j$ and g consists of weighted degree ≤ 1.

Definition 2.2.6. Fix Λ^*-adapted local coordinates. The *leading polynomial* P is defined as

$$P(z, \bar{z}) = \sum_{|(\alpha,\hat{\alpha})|_{\Lambda^*}=1} C_{\alpha,\hat{\alpha}} z^{\alpha} \bar{z}^{\hat{\alpha}}. \tag{2.6}$$

The polynomial defined in (2.6) is exactly the polynomial that only retains the terms of weight 1. Put differently, it is a Λ^*-homogeneous polynomial of weighted degree 1 with no pluriharmonic terms, where Λ^* is the multitype weight. Following Kolář in [22], we will also denote by *leading polynomial* the polynomial consisting of all terms of weight 1 with respect to each intermediate weight Λ_j in the Kolář algorithm.

Theorem 2.2.2. (Kolář). *A biholomorphic transformation takes Λ^*-adapted coordinates into Λ^*-adapted coordinates if and only if this transformation is Λ^*-superhomogeneous.*

We will apply this theorem below in order to give a thorough explanation of the Kolář algorithm for the computation of the multitype under the assumption that all its entries are finite.

2.2.1 The Kolář Algorithm

The algorithm consists of a finite number of steps that terminate at the multitype weight. In other words, it is an approximation algorithm that generates a partial model polynomial and an intermediate weight at every step. Most importantly, the polynomial transformations that do not affect the multitype are characterized by weight at every step of the Kolář algorithm.

The algorithm starts by considering local holomorphic coordinates in which the leading polynomial in the variables z and $\bar{z}$ contains no pluriharmonic terms. The degree of the lowest order monomial in this polynomial is then equal to the Bloom-Graham type of $\mathcal{M}$ at p as defined in [4]. This gives the first multitype component m_1; see [6]. By our assumption $1 < m_1 < +\infty$. Let $m_1 = \frac{1}{\mu_1}$, and set $\Lambda_1 = (\mu_1, \dots, \mu_1)$. We then consider all Λ_1-homogeneous transformations and choose one that will make the leading polynomial P_1 to be independent of the largest number of variables. We denote this number by d_1. For such coordinates, we get the defining function of $\mathcal{M}$ to be of the form

$$v = P_1(z_1, \dots, z_{n-d_1}, \bar{z}_1, \dots, \bar{z}_{n-d_1}) + Q_1(z, \bar{z}) + o(u),$$

where P_1 is Λ_1-homogeneous of weighted degree 1 and Q_1 is $o_{\Lambda_1}(1)$. Note that due to the equal weights in Λ_1, all Λ_1-homogeneous transformations are linear. We use the result that for any weight Λ which is smaller than Λ_1 with respect to the lexicographic ordering Λ-adapted coordinates are also Λ_1-adapted coordinates. We thus have that $\mu_1 = \dots = \mu_{n-d_1}$ and $\mu_{n-d_1+1} < \mu_1$. We define the following important tools:

Let

$$Q_1(z, \bar{z}) = \sum_{|(\alpha,\hat{\alpha})|_{\Lambda_1} > 1} C^1_{\alpha,\hat{\alpha}} z^\alpha \bar{z}^{\hat{\alpha}}.$$

We define

$$\Theta_1 = \left\{ (\alpha, \hat{\alpha}) \;\middle|\; C^1_{\alpha,\hat{\alpha}} \neq 0 \text{ and } \sum_{i=1}^{n-d_1} (\alpha_i + \hat{\alpha}_i)\mu_i < 1 \right\}.$$

For every $(\gamma, \hat{\gamma}) \in \Theta_1$,

$$W_1(\gamma, \hat{\gamma}) = \frac{1 - \sum_{i=1}^{n-d_1} (\gamma_i + \hat{\gamma}_i)\mu_i}{\sum_{i=n-d_1+1}^{n} (\gamma_i + \hat{\gamma}_i)}. \tag{2.7}$$

We define the next weight Λ_2 by letting

$$\lambda_j^2 = \max_{(\alpha,\hat{\alpha}) \in \Theta_1} \mathrm{W}_1(\alpha, \hat{\alpha})$$

for $j > n - d_1$, and $\lambda_j^2 = \mu_1$ for $j \leq n - d_1$. We then complete the second step by letting P_2 be the new leading polynomial corresponding to the weight Λ_2. P_2 depends on more than $n - d_1$ variables.

We proceed by induction. At the j-th step, for $j > 2$, using coordinates from the previous step, we consider all Λ_{j-1}-homogeneous transformations and choose one that makes the leading polynomial P_{j-1} to be independent of the largest number of variables. We fix such coordinates, and let d_{j-1} be the largest number of variables, which do not show up in P_{j-1} after this change of variables. By Theorem 2.2.2, the transformations taking Λ_{j-1}-adapted coordinates into Λ_{j-1}-adapted coordinates are always Λ_{j-1}-superhomogeneous. The number of multitype entries that are added at each step of the computation depends on the difference $(d_{j-2} - d_{j-1})$. Hence we consider two cases at this step:

CASE 1: Assume that $d_{j-2} > d_{j-1}$. Also recall that for any weight Λ that is smaller than Λ_{j-1} with respect to the lexicographic ordering, Λ-adapted coordinates are also Λ_{j-1}-adapted. This implies that we get $(d_{j-2} - d_{j-1})$ multitype entries

$$\mu_{n-d_{j-2}+1} = \dots = \mu_{n-d_{j-1}} = \lambda^{j-1}_{n-d_{j-2}+1}$$

and let $\lambda_i^j = \mu_i$ for $i \leq n - d_{j-2}$. To obtain λ_i^j for $j > n - d_{j-1}$, we consider

$$v = P_{j-1}(z_1, \dots, z_{n-d_{j-1}}, \bar{z}_1, \dots, \bar{z}_{n-d_{j-1}}) + Q_{j-1}(z, \bar{z}) + o(u),$$

where Q_{j-1} is $o_{\Lambda_{j-1}}(1)$ and P_{j-1} is Λ_{j-1}-homogeneous of weighted degree 1. We define Q_{j-1}, Θ_{j-1}, and W_{j-1} in a similar way as in step two. Thus,

$$Q_{j-1}(z, \bar{z}) = \sum_{|(\alpha,\hat{\alpha})|_{\Lambda_{j-1}} > 1} C_{\alpha,\hat{\alpha}}^{j-1} z^\alpha \bar{z}^{\hat{\alpha}},$$

and also

$$\Theta_{j-1} = \left\{ (\alpha, \hat{\alpha}) |\ C_{\alpha,\hat{\alpha}}^{j-1} \neq 0 \ \text{ and } \ \sum_{i=1}^{n-d_{j-1}} (\alpha_i + \hat{\alpha}_i)\mu_i < 1 \right\}.$$

For every $(\gamma, \hat{\gamma}) \in \Theta_{j-1}$,

$$W_{j-1}(\gamma, \hat{\gamma}) = \frac{1 - \sum_{i=1}^{n-d_{j-1}} (\gamma_i + \hat{\gamma}_i)\mu_i}{\sum_{i=n-d_{j-1}+1}^{n} (\gamma_i + \hat{\gamma}_i)}. \tag{2.8}$$

So for the remaining multitype entries of Λ_j we let

$$\lambda_i^j = \max_{(\alpha,\hat{\alpha}) \in \Theta_{j-1}} W_{j-1}(\alpha, \hat{\alpha}),$$

for $i > n - d_{j-1}$.

CASE 2: Assume that $d_{j-1} = d_{j-2}$. There are zero multitype entries computed in this case and so we only determine λ_i^j for $j > n - d_{j-1}$ using (2.8). This completes the j-th step of the computation.

The process terminates after a finite number of steps to give all the entries of the multitype weight Λ^*. It is clear that case 1 advances the process. We just need to show that the number of times case 2 occurs where no multitype entries are determined can only happen finitely many times. We claim case 2 can take place at most $\lceil \frac{1}{\mu_n} \rceil^{n-d_{j-1}+1}$ times, where $\lceil \frac{1}{\mu_n} \rceil$ is the ceiling for the rational number $\frac{1}{\mu_n}$. Indeed, it comes down to the number of different values that (2.8) can have. The upper bound for the numerator is given by $\lceil \frac{1}{\mu_n} \rceil^{n-d_{j-1}}$ as the μ_i entries are decreasing, whereas the upper bound for the denominator is given by $\lceil \frac{1}{\mu_n} \rceil$.

Example 1. Let the defining function of a smooth real hypersurface $\mathcal{M} \subset \mathbb{C}^5$ near a point 0 be given by

$$r = 2\,\mathrm{Re}(z_4) + |z_1 - z_2 + z_3^2|^2 + |z_1^2 - z_2^2|^2 + |z_2^4|^2.$$

Using the Kolář algorithm, we proceed as follows: The Bloom-Graham type is 2, which implies that $\mu_1 = \frac{1}{2}$ and $\Lambda_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Thus, $P_1 = |z_1|^2 + |z_2|^2 - 2\mathrm{Re}(z_1\bar{z}_2) = |z_1 - z_2|^2$. We consider all Λ_1-homogeneous transformation and choose

$$\tilde{z}_1 = z_1 - z_2, \quad \text{and} \quad \tilde{z}_j = z_j,$$

for $j = 2, 3, 4$, to obtain a leading polynomial P_1 independent of the variables z_2 and z_3. We write r in the new variables and ignore $\sim$ when no confusion arises.

$$r = 2\mathrm{Re}(z_4) + |z_1 + z_3^2|^2 + |z_1^2 + 2z_1z_2|^2 + |z_2^4|^2 \quad \text{and} \quad P_1 = |z_1|^2 \quad \text{with} \quad d_1 = 2.$$

We then get Q_1 to be the sum

$$Q_1 = |z_3^2|^2 + 2\mathrm{Re}(z_1\bar{z}_3^2) + |z_1^2|^2 + 4\mathrm{Re}(z_1^2\bar{z}_1\bar{z}_2) + 4|z_1z_2|^2 + |z_2^4|^2.$$

We compute W_1 and find the maximum number is given by $\max(W_1) = \frac{1}{4}$. Hence $\Lambda_2 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ and $P_2 = |z_1 + z_3^2|^2$. We consider all Λ_2-homogeneous transformations and see that

$$\tilde{z}_1 = z_1 + z_3^2 \quad \text{and} \quad \tilde{z}_j = z_j$$

for $j = 2, 3, 4$ makes the leading polynomial independent of the largest number of variables, namely z_2 and z_3. So in the new variables, we get that

$$r = 2\mathrm{Re}(z_4) + |z_1|^2 + |z_1^2 + 2z_1z_3^2 + z_3^4 + 2z_1z_2 - 2z_2z_3^2|^2 + |z_2^4|^2 \text{ and } P_2 = |z_1|^2$$

with $d_2 = 2$.

$$\begin{aligned} Q_2 =& |z_1^2|^2 + 4|z_1z_3^2|^2 + 4\mathrm{Re}(z_1^2\bar{z}_1\bar{z}_3^2) + |z_3^4|^2 + 2\mathrm{Re}(z_1^2\bar{z}_3^4) + 4\mathrm{Re}(z_1z_3^2\bar{z}_3^4) + 4|z_1z_2|^2 + 4\mathrm{Re}(z_1^2\bar{z}_1\bar{z}_2) \\ &+ 8\mathrm{Re}(z_1z_3^2\bar{z}_1\bar{z}_2) + 4\mathrm{Re}(z_3^4\bar{z}_1\bar{z}_2) + 4|z_2z_3^2|^2 - 4\mathrm{Re}(z_1^2\bar{z}_2\bar{z}_3^2) - 8\mathrm{Re}(z_1z_3^2\bar{z}_2\bar{z}_3^2) \\ &- 4\mathrm{Re}(z_3^4\bar{z}_2\bar{z}_3^2) - 8\mathrm{Re}(z_1z_2\bar{z}_2\bar{z}_3^2) + |z_2^4|^2. \end{aligned}$$

We compute W_2 and select the maximum, which is $\frac{1}{6}$, and so $\Lambda_3 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6})$. Here, no Λ_3-homogeneous transformation can make P_3 to be independent of any variables. Thus,

$$P_3 = |z_1|^2 + 4|z_2z_3^2|^2 \quad \text{with} \quad d_3 = 0.$$

Example 2. Let the defining function of a smooth real hypersurface $\mathcal{M} \subset \mathbb{C}^5$ near a point 0 be given by

$$r = 2\,\mathrm{Re}(z_5) + |z_1 + z_2 + z_1z_3^2 + z_3^3 + z_4^6|^2.$$

Clearly, there exists at least one variety lying in $\mathcal{M}$. For instance, the varieties $\varphi(t) = (t, -t, 0, 0, 0)$ and $\varphi(t) = (0, 0, -t^2, t, 0)$ both lie in $\mathcal{M}$, and so the D'Angelo 1-type is infinite. Now, using the Kolář algorithm, we proceed as follows:

$$\begin{aligned} r =& 2\mathrm{Re}(z_5) + |z_1 + z_2 + z_1z_3^2 + z_3^3 + z_4^6|^2 \\ =& 2\mathrm{Re}(z_5) + |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1\bar{z}_2) + |z_1z_3^2|^2 + 2\mathrm{Re}(z_1\bar{z}_1\bar{z}_3^2) + 2\mathrm{Re}(z_2\bar{z}_1\bar{z}_3^2) + |z_3^3|^2 \\ &+ 2\mathrm{Re}(z_1\bar{z}_3^3) + 2\mathrm{Re}(z_2\bar{z}_3^3) + 2\mathrm{Re}(z_1z_3^2\bar{z}_3^3) + |z_4^6|^2 + 2\mathrm{Re}(z_1\bar{z}_4^6) + 2\mathrm{Re}(z_2\bar{z}_4^6) \\ &+ 2\mathrm{Re}(z_1z_3^2\bar{z}_4^6) + 2\mathrm{Re}(z_3^3\bar{z}_4^6). \end{aligned}$$

The Bloom-Graham type is 2, which implies that $\mu_1 = \frac{1}{2}$ and $\Lambda_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Hence $P_1 = |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1\bar{z}_2) = |z_1 + z_2|^2$. We consider all Λ_1-homogeneous transformation and choose

$$\tilde{z}_1 = z_1 + z_2, \quad \text{and} \quad \tilde{z}_j = z_j,$$

for all $j = 2, 3, 4, 5$ to obtain a leading polynomial independent of the variables z_2, z_3, and z_4. So we write r in the new variables and ignore $\sim$ when no confusion arises.

$$r = 2\mathrm{Re}(z_5) + |z_1 + z_1 z_3^2 - z_2 z_3^2 + z_3^3 + z_4^6|^2 \quad \text{and} \quad P_1 = |z_1|^2 \quad \text{with} \quad d_1 = 3.$$

$$\begin{aligned} Q_1 = {} & |z_1 z_3^2|^2 + 2\mathrm{Re}(z_1 \bar{z}_1 \bar{z}_3^2) + |z_2 z_3^2|^2 - 2\mathrm{Re}(z_1 \bar{z}_2 \bar{z}_3^2) - 2\mathrm{Re}(z_1 z_3^2 \bar{z}_2 \bar{z}_3^2) + |z_3^3|^2 \\ & + 2\mathrm{Re}(z_1 \bar{z}_3^3) + 2\mathrm{Re}(z_1 \bar{z}_3^2 \bar{z}_3^3) - 2\mathrm{Re}(z_2 z_3^2 \bar{z}_3^3) + |z_4^6|^2 + 2\mathrm{Re}(z_1 \bar{z}_4^6) + 2\mathrm{Re}(z_1 z_3^2 \bar{z}_4^6) \\ & - 2\mathrm{Re}(z_2 z_3^2 \bar{z}_4^6) + 2\mathrm{Re}(z_3^3 \bar{z}_4^6). \end{aligned}$$

We compute W_1 and find the maximum number: $\max(W_1) = \max\{\frac{1}{6}, \frac{1}{9}, \frac{1}{10}, \frac{1}{12}, \frac{1}{16}\} = \frac{1}{6}$. Hence $\Lambda_2 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. We consider all Λ_2-homogeneous transformations and show that

$$\tilde{z}_1 = z_1 - z_2 z_3^2 + z_3^3 \quad \text{and} \quad \tilde{z}_j = z_j \text{ for } j = 2, 3, 4, 5$$

makes the leading polynomial independent of the largest number of variables, namely z_2, z_3, and z_4. So in the new variables, we get that

$$r = 2\mathrm{Re}(z_5) + |z_1 + z_1 z_3^2 + z_2 z_3^4 - z_3^5 + z_4^6|^2 \quad \text{and} \quad P_2 = |z_1|^2 \quad \text{with} \quad d_2 = 3.$$

$$\begin{aligned} Q_2 = {} & |z_1 z_3^2|^2 + 2\mathrm{Re}(z_1 \bar{z}_1 \bar{z}_3^2) + |z_2 z_3^4|^2 + 2\mathrm{Re}(z_1 \bar{z}_2 \bar{z}_3^4) + 2\mathrm{Re}(z_1 z_3^2 \bar{z}_2 \bar{z}_3^4) + |z_3^5|^2 - 2\mathrm{Re}(z_1 \bar{z}_3^5) \\ & - 2\mathrm{Re}(z_1 \bar{z}_3^2 \bar{z}_3^5) - 2\mathrm{Re}(z_2 z_3^4 \bar{z}_3^5) + |z_4^6|^2 + 2\mathrm{Re}(z_1 \bar{z}_4^6) + 2\mathrm{Re}(z_1 z_3^2 \bar{z}_4^6\} + 2\mathrm{Re}(z_2 z_3^4 \bar{z}_4^6) \\ & - 2\mathrm{Re}(z_3^5 \bar{z}_4^6). \end{aligned}$$

By computing W_2, we get that the maximum number: $\max(W_2) = \max\{\frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{14}, \frac{1}{16}\} = \frac{1}{10}$. Hence $\Lambda_3 = (\frac{1}{2}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10})$. Considering all Λ_3-homogeneous transformations, we show that

$$\tilde{z}_1 = z_1 + z_2 z_3^4 - z_3^5 \text{ and } \tilde{z}_j = z_j \text{ for } j = 2, 3, 4, 5$$

makes the leading polynomial independent of the largest number of variables, namely z_2, z_3, and z_4. So in the new variables, we get that

$$r = 2\mathrm{Re}(z_5) + |z_1 + z_1 z_3^2 - z_2 z_3^6 + z_3^7 + z_4^6|^2 \quad \text{and} \quad P_3 = |z_1|^2 \quad \text{with} \quad d_3 = 3.$$

$$\begin{aligned} Q_3 = {} & |z_1 z_3^2|^2 + 2\mathrm{Re}(z_1 \bar{z}_1 \bar{z}_3^2) + |z_2 z_3^6|^2 - 2\mathrm{Re}(z_1 \bar{z}_2 \bar{z}_3^6) - 2\mathrm{Re}(z_1 z_3^2 \bar{z}_2 \bar{z}_3^6) + |z_3^7|^2 \\ & + 2\mathrm{Re}(z_1 \bar{z}_3^7) + 2\mathrm{Re}(z_1 \bar{z}_3^2 \bar{z}_3^7) - 2\mathrm{Re}(z_2 z_3^6 \bar{z}_3^7) + |z_4^6|^2 + 2\mathrm{Re}(z_1 \bar{z}_4^6) + 2\mathrm{Re}(z_1 z_3^2 \bar{z}_4^6) \\ & - 2\mathrm{Re}(z_2 z_3^6 \bar{z}_4^6) + 2\mathrm{Re}(z_3^7 \bar{z}_4^6). \end{aligned}$$

We compute W_3 and find that the maximum number is $\max(W_3) = \frac{1}{12}$. Thus, $\Lambda_4 = (\frac{1}{2}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12})$. We consider all Λ_4-homogeneous transformations and choose

$$\tilde{z}_1 = z_1 + z_4^6 \quad \text{and} \quad \tilde{z}_j = z_j,$$

for $j = 2, 3, 4, 5$, which makes the leading polynomial independent of the largest number of variables z_2, z_3, and z_4. So in the new variables, we get that

$$r = 2\mathrm{Re}(z_5) + |z_1 + z_1 z_3^2 - z_3^2 z_4^6 - z_2 z_3^6 + z_3^7|^2 \quad \text{and} \quad P_4 = |z_1|^2 \quad \text{with} \quad d_4 = 3.$$

By further computations, we get the following:

$\max(W_4) = \frac{1}{14}$ and so $\Lambda_5 = (\frac{1}{2}, \frac{1}{14}, \frac{1}{14}, \frac{1}{14})$. By considering all Λ_5-homogeneous transformations, we choose

$$\tilde{z}_1 = z_1 - z_2 z_3^6 + z_3^7 \quad \text{and} \quad \tilde{z}_j = z_j,$$

for $j = 2, 3, 4, 5$ since it makes the leading polynomial independent of the largest number of variables, namely z_2, z_3, and z_4. We now list the following without providing the details:

i. We now have $r = 2\text{Re}(z_5) + |z_1 + z_1 z_3^2 + z_2 z_3^8 - z_3^9 - z_3^2 z_4^6|^2$, hence $P_5 = |z_1|^2$, $\max(W_5) = \frac{1}{16}$ and $\Lambda_6 = (\frac{1}{2}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16})$.

ii. Among the Λ_6-homogeneous transformations, we choose: $\tilde{z}_1 = z_1 - z_3^2 z_4^6$ and $\tilde{z}_j = z_j$ for $j = 2, 3, 4, 5$ and write $r = 2\text{Re}(z_5) + |z_1 + z_1 z_3^2 + z_3^4 z_4^6 + z_2 z_3^8 - z_3^9|^2$, $P_6 = |z_1|^2$, $\max(W_6) = \frac{1}{18}$ and $\Lambda_7 = (\frac{1}{2}, \frac{1}{18}, \frac{1}{18}, \frac{1}{18})$.

iii. Among the Λ_7-homogeneous transformations, we choose: $\tilde{z}_1 = z_1 + z_2 z_3^8 - z_3^9$ and $\tilde{z}_j = z_j$ for $j = 2, 3, 4, 5$ and write $r = 2\text{Re}(z_5) + |z_1 + z_1 z_3^2 - z_2 z_3^{10} + z_3^{11} - z_3^4 z_4^6|^2$, $\mathrm{P}_7 = |z_1|^2$, $\max(\mathrm{W}_7) = \frac{1}{20}$ and $\Lambda_8 = (\frac{1}{2}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20})$.

It is obvious from the above calculations that the procedure fails to terminate. Continuing the above process yields the result that at the ν-th step,

$$\max(W_{\nu-1}) = \frac{1}{2(\nu+2)} \quad \text{and} \quad \Lambda_\nu = \left(\frac{1}{2}, \frac{1}{2(\nu+2)}, \frac{1}{2(\nu+2)}, \frac{1}{2(\nu+2)}\right),$$

where $\nu \geq 3$. From the above computation, we see that there are no multitype entries produced by the algorithm after the first step and that $d_j = 3$ for all $j \geq 1$. This implies that we have an infinite sequence of weights $\{\Lambda_j\}_{j\geq 2}$ converging to the multitype weight $\Lambda^* = (1/2, 0, 0, 0)$. We should also note for this example that the leading polynomial P_j is a sum of squares for each Λ_j-homogeneous transformation chosen with $j \geq 1$.

Chapter 3

Characterizing the Rank of the Levi Determinant

In this chapter, we shall give a commutative-algebraic way of characterizing the rank of the Levi form, which should be very helpful in understanding the behavior of domains given by sums of squares of holomorphic functions.

We shall give the following elementary definitions in order to aid the reader to understand the concepts presented in this section. The reader is directed to [1] for additional details.

Definition 3.0.1. A ring R is called a *local ring* if it has a unique maximal ideal $\mathfrak{m}$.

Definition 3.0.2. Let $\mathfrak{I}_j$ be ideals of R for $j \geq 1$. A ring R is called a *Noetherian ring* if it satisfies the following equivalent statements:

i. Every ideal in R is finitely generated;

ii. Every non empty set of ideals in R has a maximal element;

iii. For every increasing chain of ideals $\mathfrak{I}_1 \subseteq \mathfrak{I}_2 \subseteq \cdots$ there exists an integer m such that
$$\mathfrak{I}_m = \mathfrak{I}_j \text{ for all } j \geq m+1,$$
namely all increasing chains of ideals in a Noetherian ring R stabilize.

Definitions 3.0.3, 3.0.4, and 3.0.5 below are given as in [1].

Definition 3.0.3. Let $\mathfrak{p}_j$ for $j \geq 0$ be prime ideals in a ring R. We define a chain of prime ideals of R to be a finite strictly increasing sequence $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_l$; the length of the chain here is l.

We define the *dimension* of a Noetherian ring R to be the supremum of the lengths of all chains of prime ideals in R.

Definition 3.0.4. Let R be a Noetherian local ring of dimension d, and let $\mathfrak{m}$ be its maximal ideal, where $k = R/\mathfrak{m}$ is its corresponding residue field. We call R a *regular local ring* if it satisfies the following equivalent statements:

i. $\mathfrak{m}$ can be generated by d elements;

ii. $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = d$.

From the above definition, it is clear that local rings of non-singular points of a variety are regular local rings. This implies that geometrically a regular local ring corresponds to a regular point of a variety.

Definition 3.0.5. Let R be a local Noetherian ring with dimension d. A *system of parameters* is a system $\{x_1, \dots, x_d\}$ which generates an ideal that is primary to the maximal ideal $\mathfrak{m}$. We call the system $\{x_1, \dots, x_d\}$ a *regular system of parameters* if it generates the maximal ideal.

We will now define the notion of a germ. Let $p \in \mathbb{C}^n$ be given. Let $f : U \to \mathbb{C}$ and $g : V \to \mathbb{C}$ be two holomorphic functions defined on open sets U and V satisfying that $p \in U$ and $p \in V$. We say that f and g are *equivalent at a point* p if there exists some neighborhood $W \subseteq U \cap V$ of p such that $f|_W = g|_W$. Clearly, the relation defined here is an equivalence relation. We refer to the equivalence class as the germ at p of a holomorphic function in $\mathbb{C}^n$.

Definition 3.0.6. The ring of germs at p of holomorphic functions in $\mathbb{C}^n$, denoted by ${}_n\mathcal{O}_p$, is the set of germs of holomorphic functions at p equipped with the structure of a ring.

The ring ${}_n\mathcal{O}_p$ can be denoted by ${}_n\mathcal{O}$ or simply $\mathcal{O}$ if the point p is the origin. The ring ${}_n\mathcal{O}_p$ is a regular local ring of dimension n.

The following lemma constitutes our first step towards finding a commutative-algebraic way of understanding the rank of the Levi form for a sum of squares domain.

Lemma 3.0.1. *Let $\mathcal{M} \subset \mathbb{C}^{n+1}$ be a sum of squares domain whose defining function is given as*

$$r = 2Re(z_{n+1}) + \sum_{j=1}^{N} |f_j(z_1, \dots, z_n)|^2,$$

where $f_1, \dots, f_N$ are holomorphic functions near the origin. Assume that each of the generators f_j has vanishing order at least 1. Without loss of generality, we order these generators by vanishing order at the origin, that is, $v_0(f_1) \leq \cdots \leq v_0(f_N)$. Then

$$rk(\lambda_{i\bar{j}}) = rk\,(J(f)) \;\leq\; \#(f),$$

where $\#(f)$ is the number of generators with vanishing order 1, $rk(\lambda_{i\bar{j}})$ is the rank of the Levi from at 0*, and $rk\,(J(f))$ is the rank of the complex Jacobian matrix $(J(f))$. In particular, the rank of the Levi from at* 0 *equals the number of generators with linearly independent linear parts.*

Proof. Suppose all the generators have vanishing order at least 2 or simply $v_0(f_1) \geq 2$. Then $rk(J)$ is zero, which clearly equals the number of $f_i's$ with rank of order 1. The interesting case is when at least one of the holomorphic functions has vanishing order one. Let q for $1 \leq q \leq N$ be the greatest integer such that $v_0(f_q) = 1$, then $\#(f) = q$. This means that we now have a set $\mathfrak{A}$ of N generators, where the first q generators have non-zero linear parts. We now construct a new set $\mathfrak{B}$ of ordered generators as follows: The last $N - q$ elements of set $\mathfrak{A}$ become the last $N - q$ generators of set $\mathfrak{B}$.

The first generator in set $\mathfrak{A}$ becomes the first element of set $\mathfrak{B}$. Pick the second element of set $\mathfrak{A}$ and check whether its linear part and the linear part of the first generator are linearly independent. If the two linear parts are linearly independent then the second generator of set $\mathfrak{A}$ becomes the second element of set $\mathfrak{B}$; otherwise, it becomes the q-th element in set $\mathfrak{B}$. Consider the j-th element in set $\mathfrak{A}$ for $2 < j \leq q$. If its linear part and the linear parts of the elements in the first q positions in set $\mathfrak{B}$ are linearly independent, then we place it in the first empty position counting from the left of set $\mathfrak{B}$; otherwise, we place it in the first empty position counting from the right of set $\mathfrak{B}$. The process will eventually terminate since there are only q number of steps in this procedure.

In the end, we will get k generators from the sum of squares whose linear parts are linearly independent where $k = q$ or $k < q$. Since the Levi form on M near the origin is given by the expression

$$(\lambda_{i\bar{j}}) = \left(\sum_{k=1}^{N} \frac{\partial f_k}{\partial z_i} \frac{\partial f_k}{\partial \bar{z}_j} \right),$$

it follows that the ranks of $(\lambda_{i\bar{j}})$ and $rk\,(\mathrm{J}(f))$ are equal and equal to k, which is the rank of the complex Jacobian.

□

Remark 3.0.1. *By our construction, the linear parts from the first k holomorphic functions will always be linearly independent.*

Proposition 3.0.2. *Let $f_1, \ldots, f_k$ for $k \leq N$ be the generators from the sum of squares whose linear parts are linearly independent. Let $\mathfrak{I} = \langle f_1, \ldots, f_k \rangle$ be the ideal generated by $f_1, \cdots, f_k$, and let $\langle f \rangle = \langle f_1, \ldots, f_N \rangle$ be the ideal generated by the generators $f_1, \ldots, f_N$. There exists a holomorphic change of coordinates such that $f_j = z_j$ for $j = 1, \ldots, k$ and*

$$\mathcal{V}(\langle f \rangle) \subset \mathcal{V}(\mathfrak{I}).$$

Proof. We begin with a list $\{f_1, \ldots, f_k\}$ as in the hypothesis and express each of the k linear parts as $\sum_{j=1}^{n} a_{ij} z_j$ for $i = 1, \ldots, q$, and $a_{ij} \in \mathbb{C}$. Diagonalizing these k linear parts we get each f_i to be of the form $a_i z_i + C_i$, where C_i is a function in n variables such that $v_0(C_i) > 1$ and $a_i \neq 0$ for $1 \leq i \leq k$. Now let $\tilde{f}_i = a_i z_i + C_i$ and choose new holomorphic coordinates $(\tilde{z}_1, \ldots, \tilde{z}_n)$ about the origin such that

$$\tilde{f}_i = a_i z_i + C_i = \tilde{z}_i, \quad \text{for } 1 \leq i \leq k.$$

We define an $(n-k)$-dimensional linear subvariety $\mathcal{V}(\mathfrak{I}) = \{\tilde{z} \mid \tilde{z}_1 = \cdots = \tilde{z}_k = 0\}$, where $\mathfrak{I} = \left\langle \tilde{f}_1, \ldots, \tilde{f}_k \right\rangle = \langle \tilde{z}_1, \ldots, \tilde{z}_k \rangle$. The rank of the Jacobian matrix $\left(\frac{\partial \tilde{f}_i}{\partial \tilde{z}_j}(0)\right)_{i,j}$ is exactly equal to k. □

The geometric interpretation of the above proposition is that it constructs the smallest dimensional coordinate hyperplane containing the zero set of the generators of the sum of squares. Algebraically, $z_1, z_2, \ldots, z_k$ form part of a regular system of parameters at the origin with k being maximal in the sense that no holomorphic change of variables can produce a larger number.

Lemma 3.0.3. *Let $\mathcal{M} \subset \mathbb{C}^{n+1}$ be a sum of squares domain whose defining function is given as*

$$r = 2Re(z_{n+1}) + \sum_{j=1}^{N} |f_j(z_1, \ldots, z_n)|^2,$$

where f_j, for $1 \leq j \leq N$, are holomorphic functions in the neighborhood of the origin. Assume that we apply the Kolář algorithm in order to compute the multitype at 0 *and that there exists at least one non-zero Levi eigenvalue of the Levi form on $\mathcal{M}$ at* 0*. Then there exists a Λ_1-homogeneous transformation such that the number of variables in the first leading polynomial P_1 is always the same as the rank of the Levi form on $\mathcal{M}$.*

Proof. We start by ordering the generators from the sum of squares domain $\mathcal{M}$ by vanishing order. Assume that the rank of the Levi form at 0 is k. By Lemma 3.0.1 and Proposition 3.0.2, we conclude that there exist exactly k generators of vanishing order 1 for which the set of their linear parts is linearly independent. Suppose that these k generators are the first k holomorphic functions from the sum of squares domain $\mathcal{M}$.

Applying the Kolář algorithm to the sum of squares of the N holomorphic functions implies that the Bloom-Graham type will be precisely 2 since there are at least $k \geq 1$ generators with vanishing order 1. Let's assume that there are q generators with vanishing order 1 for $k \leq q \leq n$. Since the Bloom-Graham type is 2, the weight $\Lambda_1 = (1/2, \ldots, 1/2)$, and the first leading polynomial is given by $P_1 = |l_1|^2 + \cdots + |l_q|^2$, where l_j is the linear part of the generator f_j for $1 \leq j \leq q$. Next, we need to consider all Λ_1-homogeneous transformations and choose one which makes P_1 to be independent of the largest number of variables. We know that Λ_1-homogeneous transformations are always linear. Since we have k linearly independent linear parts, it suffices to show that there exists at least one linear transformation which makes the total number of variables in $l_1, \ldots, l_q$ to be exactly k, and no other linear transformation can make the total number of variables in $l_1, \ldots, l_q$ to be less than k.

By our setting, evaluating the complex Jacobian matrix at the origin will give precisely an N by n matrix. This matrix is subdivided into a q by n submatrix of coefficients of the linear parts, and an $N - q$ by n submatrix with only zero entries which are obtained as a result of the generators $f_{q+1}, \ldots, f_N$, each having a vanishing order greater than 1 and hence complex partial derivatives that vanish at the origin.

By applying Gaussian's elimination to the N by n matrix, we get after a finite number of linear transformations a k by k identity submatrix because there are exactly k linearly independent linear parts. If we consider the composition of all of these finitely many linear transformations giving the k by k identity submatrix, then we can always choose that as the linear transformation for which the number of variables in $l_1, \ldots, l_k$ is always k.

Hence if the rank of the Levi form at 0 is k for $k \geq 1$, then there is always a Λ_1-homogeneous transformation such that the first leading polynomial P_1 depends on exactly k variables. □

From the previous lemma, we see that if we consider a sum of squares domain such that k of the generators have linearly independent linear parts, then the Kolář algorithm always gives a first leading polynomial, after considering all possible Λ_1-homogeneous transformations, that is independent of $n - k$ variables. Here $n - k$ is precisely the dimension of the zero locus of $\mathfrak{I} = \langle z_1, \ldots, z_k \rangle$, whose generators are exactly the k variables on which the first leading polynomial depends. In fact, the proof of Lemma 3.0.3 constructs the linear transformation that brings the linear parts of the first k generators to the simplest form, which is $z_1, \ldots, z_k$.

Chapter 4

An Ideal Restatement of the Kolář Algorithm

A major preparatory tool to aid our quest of understanding the stratification by multitype level sets of the boundary of a domain given by a sum of squares of holomorphic functions is introduced in this chapter. As previously stated, we rely on Kolář's computation of the multitype in [22] to derive this tool. We seek to effectively interpret our results both geometrically and algebraically, and so we resort to reducing the problem to the study of the ideal of the holomorphic functions in the sum. For the transition to be effective, we need to establish that a natural modification of the Kolář algorithm still holds at the level of ideals of holomorphic functions.

4.1 Squares of Monomials and the Multitype Entries

We will show in this section that the set of monomials that give the maximum W-value at each step of the Kolář algorithm always consists of both squares of moduli of monomials and cross terms. Since the entries in the multitype depend on the maximum W-values, this will suffice to establish that for each entry of the multitype, there is always a square that gives the corresponding multitype entry.

The ensuing lemma gives the foundational result we need in order to transition from the sums of squares case to the case of ideals of holomorphic functions. We will use parts of this lemma to prove Propositions 4.1.2 and 4.2.1, namely that the model of a sum of squares is a sum of squares and that the multitype is an invariant of the ideal of holomorphic functions defining the sum of squares.

Lemma 4.1.1. *Let f and g be monomials with non-zero coefficients from the Taylor expansion of h, where h is a generator from a sum of squares domain in $\mathbb{C}^{n+1}$. Let P_t for $t \geq 1$ be the leading polynomial at step t of the Kolář algorithm, and let W_t be the quantity defined in* (2.8) *computed at the* $(t+1)$*-th step.*

A. If $W_t(|f|^2) = W_t(|g|^2)$, then $W_t(f\bar{g}) = W_t(|f|^2) = W_t(|g|^2)$.

B. If $W_t(|f|^2) < W_t(|g|^2)$, then $W_t(|f|^2) < W_t(f\bar{g}) < W_t(|g|^2)$.

C. If $W_t(|f|^2)$ cannot be computed, then *i.* $W_t(f\bar{g}) \leq W_t(|g|^2)$ *for any monomial* g *for which both* $W_t(|g|^2)$ *and* $W_t(f\bar{g})$ *can be computed.*

ii. $W_t(f\bar{g})$ *cannot be computed for any monomial* g *for which* $W_t(|g|^2)$ *cannot be computed.*

D. If f *is such that* $|f|^2$ *is in the leading polynomial* P_t*, then*

i. For any monomial g *for which* $W_t(|g|^2)$ *can be computed,* $W_t(f\bar{g}) = W_t(|g|^2)$.

ii. For any monomial g *for which* $W_t(|g|^2)$ *cannot be computed,* $W_t(f\bar{g})$ *cannot be computed as well.*

Here $W_t(f) := W_t(\alpha, \hat{\alpha})$ *where* $(\alpha, \hat{\alpha})$ *is the pair of multiindices corresponding to the monomial* f.

Remark 4.1.1. *We shall say the quantity* $W_t(f)$ *"cannot be computed" if the pair of multiindices* $(\alpha, \hat{\alpha})$ *corresponding to the monomial* f *is not an element of* Θ_t*. In other words,* $W_t(f)$ *cannot be computed if the numerator of the fraction giving* $W_t(f)$ *is not positive; see* (4.1) *below.*

Remark 4.1.2. *We note here that* $W_t(f\bar{g})$ *and* $W_t(\bar{f}g)$ *corresponding to the cross terms* $f\bar{g}$ *and* $\bar{f}g$ *respectively are equal.*

Proof. Let $z_1, \ldots, z_c$ be the variables in the leading polynomial P_t, and let the weight $\Lambda_t = (\mu_1, \ldots, \mu_c, \mu_{c+1}, \ldots, \mu_n)$. We begin by recalling that

$$W_t(\alpha, \hat{\alpha}) = \frac{1 - \sum_{i=1}^{c}(\alpha_i + \hat{\alpha}_i)\mu_i}{\sum_{i=c+1}^{n}(\alpha_i + \hat{\alpha}_i)}, \tag{4.1}$$

where $(\alpha, \hat{\alpha}) = (\alpha_1, \ldots, \alpha_n, \hat{\alpha}_1, \ldots, \hat{\alpha}_n)$ is the multiindex of the monomial whose W_t is being computed.

Let Γ_1 be the set of all non-zero monomials that consist of only variables not in P_t, let Γ_2 be the set of all non-zero monomials which consist of variables both in P_t as well as variables not in P_t, and let Γ_3 be the set of all non-zero monomials which consist of only variables in P_t. If $W_t(|f|^2)$ can be computed, then $f \in \Gamma_1$ or $f \in \Gamma_2$ only. We will now prove f cannot belong to Γ_3. We assume the opposite, namely that $f \in \Gamma_3$ and that $W_t(|f|^2)$ can be computed. The monomial $|f|^2$ has weight ≥ 1 with respect to Λ_t, and since $f \in \Gamma_3$, it follows that the numerator of $W_t(|f|^2)$ is ≤ 0. Therefore, $W_t(|f|^2)$ cannot be computed, which is a contradiction. Without loss of generality, we now specify to $f \in \Gamma_1$ or $f \in \Gamma_2$ for all parts of the lemma pertaining to the case when $W_t(|f|^2)$ can be computed (and similarly for g).

Since $f \in \Gamma_1$ or $f \in \Gamma_2$, we can write $f = f_1 f_2$ where f_1 and f_2 are monomials satisfying $f_1 \in \Gamma_3$ and $f_2 \in \Gamma_1$. Let $f_1 = z_1^{\alpha_1} \cdots z_c^{\alpha_c}$, where $z_1, \ldots, z_c$ are the variables in the leading polynomial P_t, and $f_2 = C_f z_{c+1}^{\beta_{c+1}} \cdots z_n^{\beta_n}$ for $C_f \in \mathbb{C}$. If $W_t(|f|^2)$ can be computed, by our definition of f, the multiindices α and β corresponding to monomials f_1 and f_2 respectively must satisfy $|\alpha| \geq 0$ and $|\beta| > 0$. If $|\alpha| = 0$, then $f \in \Gamma_1$, and if $|\alpha| > 0$, then $f \in \Gamma_2$. Here $|\alpha| = \alpha_1 + \cdots + \alpha_c$ as the rest of the entries are zero, and $|\beta| = \beta_{c+1} + \cdots + \beta_n$ for the same reason. Now, $|f|^2 = f_1\bar{f}_1 f_2\bar{f}_2$ with $|\alpha| = |\hat{\alpha}|$ and $|\beta| = |\hat{\beta}|$. Hence

$$W_t(|f|^2) = \frac{1 - \sum_{i=1}^{c}(\alpha_i + \hat{\alpha}_i)\mu_i}{|\beta| + |\hat{\beta}|} = \frac{\frac{1}{2} - \sum_{i=1}^{c}\alpha_i\mu_i}{|\beta|}. \tag{4.2}$$

Similarly, let $g = g_1 g_2$, where $g_1 \in \Gamma_3$ and $g_2 \in \Gamma_1$. Let γ and τ be the multiindices corresponding to monomials g_1 and g_2 respectively. Here $g_1 = z_1^{\gamma_1} \cdots z_c^{\gamma_c}$ and $g_2 = C_g z_{c+1}^{\tau_{c+1}} \cdots z_n^{\tau_n}$ for $C_g \in \mathbb{C}$. By a similar computation as carried out above for f,

$$W_t(|g|^2) = \frac{\frac{1}{2} - \sum_{i=1}^{c} \gamma_i \mu_i}{|\tau|}. \tag{4.3}$$

A. Suppose that $W_t(|f|^2) = W_t(|g|^2)$ and consider

$$\begin{aligned} W_t(f\bar{g}) &= \frac{1 - \sum_{i=1}^{c} \alpha_i \mu_i - \sum_{i=1}^{c} \hat{\gamma}_i \mu_i}{|\beta| + |\tau|} \\ &= \frac{\frac{1}{2} - \sum_{i=1}^{c} \alpha_i \mu_i + \frac{1}{2} - \sum_{i=1}^{c} \gamma_i \mu_i}{|\beta| + |\tau|} \\ &= \frac{|\beta| W_t(|f|^2) + |\tau| W_t(|g|^2)}{|\beta| + |\tau|} \qquad \text{from (4.2) and (4.3) by cross multiplication.} \\ &= W_t(|f|^2) = W_t(|g|^2) \qquad \text{by our hypothesis.} \end{aligned} \tag{4.4}$$

B. Suppose that $W_t(|f|^2) < W_t(|g|^2)$. Then from (A.) we get that

$$\begin{aligned} W_t(f\bar{g}) &= \frac{\frac{1}{2} - \sum_{i=1}^{c} \alpha_i \mu_i + \frac{1}{2} - \sum_{i=1}^{c} \gamma_i \mu_i}{|\beta| + |\tau|} \\ &= \frac{|\beta| W_t(|f|^2) + |\tau| W_t(|g|^2)}{|\beta| + |\tau|} \qquad \text{from (4.2) and (4.3).} \\ &< \frac{|\beta| W_t(|g|^2) + |\tau| W_t(|g|^2)}{|\beta| + |\tau|} \qquad \text{since } \; W_t(|f|^2) < W_t(|g|^2) \\ &= W_t(|g|^2) \end{aligned} \tag{4.5}$$

Again

$$\begin{aligned} W_t(f\bar{g}) &= \frac{|\beta| W_t(|f|^2) + |\tau| W_t(|g|^2)}{|\beta| + |\tau|} \\ &> W_t(|f|^2) \end{aligned} \tag{4.6}$$

since $W_t(|f|^2) < W_t(|g|^2)$. Thus from (4.5) and (4.6) we obtain

$$W_t(|f|^2) < W_t(f\bar{g}) < W_t(|g|^2).$$

C. i. Suppose that $W_t(|f|^2)$ cannot be computed. Then clearly $f \notin \Gamma_1$. We know that $f = f_1 f_2$ and that $|\alpha| > 0$ and $|\beta| \geq 0$. If $W_t(|f|^2)$ cannot be computed, then we get that $\sum_{i=1}^{c} (\alpha_i + \hat{\alpha}_i) \mu_i \geq 1$. Thus $\sum_{i=1}^{c} \alpha_i \mu_i \geq 1/2$ since $\alpha_i = \hat{\alpha}_i$ for all $i,\ 1 \leq i \leq c$.

Let $g = g_1 g_2$, where $g_1 \in \Gamma_3$ and $g_2 \in \Gamma_1$. $W_t(|g|^2)$ and $W_t(f\bar{g})$ can be computed, and so g cannot belong to Γ_3, which implies the multiindex corresponding to g_2

must satisfy $|\tau| > 0$. Therefore,

$$\begin{aligned} W_t(f\bar{g}) &= \frac{\frac{1}{2} - \sum_{i=1}^{c} \alpha_i \mu_i + \frac{1}{2} - \sum_{i=1}^{c} \gamma_i \mu_i}{|\beta| + |\tau|} \\ &= \frac{\frac{1}{2} - \sum_{i=1}^{c} \alpha_i \mu_i + |\tau|\, W_t(|g|^2)}{|\beta| + |\tau|} \qquad \text{from (4.3)} \\ &\leq \frac{|\tau|}{|\beta| + |\tau|} W_t(|g|^2) \quad \text{since } \frac{1}{2} - \sum_{i=1}^{c} \alpha_i \mu_i \leq 0 \\ &\leq W_t(|g|^2) \quad \text{since } \frac{|\tau|}{|\beta| + |\tau|} \leq 1. \end{aligned} \tag{4.7}$$

ii. From (C.i.) we know that $f = f_1 f_2 \notin \Gamma_1$ and also that $\sum_{i=1}^{c} \alpha_i \mu_i \geq 1/2$ since $\alpha_i = \hat{\alpha}_i$ for all $i,\ 1 \leq i \leq c$. Similarly, for $g = g_1 g_2$ it means that $g \notin \Gamma_1$ and that $\sum_{i=1}^{c} \gamma_i \mu_i \geq 1/2$. Thus $\sum_{i=1}^{c} (\alpha_i + \gamma_i) \mu_i \geq 1$, and so the pair of multiindices corresponding to the cross term $f\bar{g}$ is not in the set Θ_t. Hence the number $W_t(f\bar{g})$ cannot be computed.

D. i. Suppose that f is such that $|f|^2$ is in the leading polynomial P_t. Then $f \in \Gamma_3$, which implies that $f = f_1 f_2$ with $|\beta| = 0$, that is, $f = C f_1$ for C a non-zero constant. Let $g = g_1 g_2$ be any monomial such that $W_t(|g|^2)$ can be computed. Clearly, $g \notin \Gamma_3$, and so $|\tau| > 0$ whereas $|\gamma| \geq 0$. We know from (4.3) that

$$W_t(|g|^2) = \frac{\frac{1}{2} - \sum_{i=1}^{c} \gamma_i \mu_i}{|\tau|}$$

Given that $|f|^2$ is a term in P_t, $\sum_{i=1}^{c} (\alpha_i + \hat{\alpha}_i) \mu_i = 1$ since P_t is a Λ_t-homogeneous polynomial of weighted degree 1. Thus $\sum_{i=1}^{c} \alpha_i \mu_i = 1/2$ since $\alpha_i = \hat{\alpha}_i$ for all $i,\ 1 \leq i \leq c$. Now

$$\begin{aligned} W_t(f\bar{g}) &= \frac{\frac{1}{2} - \sum_{i=1}^{c} \alpha_i \mu_i + \frac{1}{2} - \sum_{i=1}^{c} \gamma_i \mu_i}{|\beta| + |\tau|} \\ &= \frac{\frac{1}{2} - \sum_{i=1}^{c} \alpha_i \mu_i + \frac{1}{2} - \sum_{i=1}^{c} \gamma_i \mu_i}{|\tau|} \qquad \text{since } |\beta| = 0 \\ &= \frac{\frac{1}{2} - \sum_{i=1}^{c} \gamma_i \mu_i}{|\tau|} \quad \text{since } \sum_{i=1}^{c} \alpha_i \mu_i = \frac{1}{2} \\ &= W_t(|g|^2) \end{aligned} \tag{4.8}$$

ii. Let $g = g_1 g_2$ be any monomial such that $W_t(|g|^2)$ cannot be computed. From (D.i.) we know that $\sum_{i=1}^{c} \alpha_i \mu_i = 1/2$ and from (C.ii.) we know that $\sum_{i=1}^{c} \gamma_i \mu_i \geq 1/2$. Thus

$$\sum_{i=1}^{c} (\alpha_i + \hat{\gamma}_i) \mu_i = \sum_{i=1}^{c} (\alpha_i + \gamma_i) \mu_i = \frac{1}{2} + \sum_{i=1}^{c} \gamma_i \mu_i \geq 1.$$

Hence the pair of multiindices corresponding to the cross term $f\bar{g}$ does not belong to the set Θ_t, and so the number $W_t(f\bar{g})$ cannot be computed.

$\square$

Proposition 4.1.2. *Let $\mathcal{M} \subset \mathbb{C}^{n+1}$ be a hypersurface whose defining function is given as*

$$r(z) = 2Re(z_{n+1}) + \sum_{j=1}^{N} |f_j(z_1, \ldots, z_n)|^2,$$

where $f_1, \ldots, f_N$ are holomorphic functions near the origin and assume that the D'Angelo 1-type is finite. Then the leading polynomial $P_t(z_1, \ldots, z_{n-d_t}, \bar{z}_1, \ldots, \bar{z}_{n-d_t})$ obtained at step t of the Kolář algorithm, for $t \geq 1$, is a sum of squares of holomorphic polynomials. In particular, the final leading polynomial $P(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)$ corresponding to the multitype weight is likewise a sum of squares.

Remark 4.1.3. *The model hypersurface is given by the zero locus of the homogeneous polynomial consisting of all monomials from the Taylor expansion of the defining function that have weight 1 with respect to the multitype weight. This lemma shows that the model of a sum of squares domain is always a sum of squares domain under the assumption of finite D'Angelo 1-type.*

Proof. We order the generators f_k by vanishing order. We truncate each generator f_k up to order $\beta = \lceil \Delta_1(\mathcal{M}, 0) \rceil$, the ceiling of the D'Angelo 1-type of $\mathcal{M}$ at the origin, and denote it by f_k^β. Since a sum of squares domain is pseudoconvex, by Theorem 2.2.1, no terms of higher order than β ought to come into the computation of the multitype. We order the terms in the truncated generator f_k^β by vanishing order and also use the reverse lexicographic ordering to reorder the monomials with the same combined degree. Now let $f_{k,i} = C_{k,i} z_1^{\alpha_1^{k,i}} \cdots z_n^{\alpha_n^{k,i}}$ be the i-th monomial in the generator f_k^β after ordering by vanishing order. Let the number of distinct combined degrees in the Taylor expansion f_k^β be κ_k, and let $\eta_{k,j}$ be the number of non-zero monomials with the same combined degree $\nu_{k,j}$ in f_k^β. Thus,

$$|f_k^\beta|^2 = |f_{k,1} + \cdots + f_{k,\eta_k}|^2, \text{ where } \eta_k = \sum_{j=1}^{\kappa_k} \eta_{k,j}.$$

Here η_k is the total number of monomials with nonzero coefficients in the power series expansion of the generator f_k up to order β. In the expansion of $|f_k^\beta|^2$, we have two types of terms: squares $|f_{k,i}|^2$ and cross terms $2\text{Re}(f_{k,j}\bar{f}_{k,i})$. For simplicity sake, for each monomial $f_{k,i}$ in the generator f_k^β, we write the terms from the expansion of $|f_k^\beta|^2$ into an expression of the form

$$|f_{k,i}|^2 + \sum_{j=1}^{i-1} 2\text{Re}(f_{k,j}\bar{f}_{k,i}), \tag{4.9}$$

for $i = 1, 2, \ldots, \eta_k$.

Define $P_{t,k}$ for $t \geq 1$ to be the sum in the leading polynomial P_t at step t consisting of terms from the expansion of $|f_k^\beta|^2$. We could have $P_{t,k} = 0$ for some k, $1 \leq k \leq N$, and for some $t \geq 1$ if there are no terms from the expansion of $|f_k^\beta|^2$ in the leading polynomial P_t after the t-th step. Thus, $P_t = \sum_{k=1}^{N} P_{t,k}$. In order to show that the final leading polynomial P_t is a sum of squares, it will suffice to show that each $P_{t,k}$ obtained at each step of the Kolář algorithm is a sum of squares. Note that trivially 0 is a sum of squares.

The Bloom-Graham type is $2\nu_1$, where $\nu_1 := \min_{1\leq k\leq N}\{\nu_{k,1}\} = \nu_{1,1}$ since the monomials from the expansion of each $|f_k^\beta|^2$ as well as the generators f_k are ordered by vanishing order. Thus, the leading polynomial P_1 consists of all terms of combined degree $2\nu_1$. Clearly, $P_{1,1} \neq 0$, but $P_{1,k} = 0$ for every k such that $\nu_{k,1} > \nu_1$. Let k be such that $P_{1,k} \neq 0$, and denote by $m_k \geq 1$ the number of monomials from the Taylor expansion of f_k^β that have combined degree ν_1. By writing the terms from the expansion of $|f_k^\beta|^2$ into the form given in (4.9), it is easy to see that the first m_k squares $|f_{k,i}|^2$ for $i = 1, \dots, m_k$ as well as the $\binom{m_k}{2}$ cross terms $2\mathrm{Re}(f_{k,j}\bar{f}_{k,i})$ for $j < i$ all have combined degree $2\nu_1$. Thus

$$P_{1,k} = |f_{k,1} + \cdots + f_{k,m_k}|^2, \tag{4.10}$$

which is obviously a sum of squares. If $d_1 = 0$, then we are done and P_1 becomes the leading polynomial with multitype weight Λ_1. On the other hand, if $d_1 = n - c$ for $c < n$, then we proceed to the next step.

In the second step, we assume without loss of generality that the W_1 value of at least one square in Q_1 can be computed. Then the maximum W_1 value exists, which further implies that some terms from the expansion of $|f_k^\beta|^2$ for some k will be added to the leading polynomial P_1 in order to obtain P_2. $P_{2,k}$ might be 0 for some k if no terms from the expansion of $|f_k^\beta|^2$ end up in P_2 after this step. Obviously, the interesting case is when $P_{2,k} \neq 0$. Consider k such that $P_{2,k} \neq 0$. Suppose that u_k squares from the expansion of $|f_k^\beta|^2$ give the maximum W_1 value, and let $|f_{k,c_l}|^2$ for $l = 1, \dots, u_k$ be these squares. Here c_l for $l = 1, \dots, u_k$ is some positive integer between 1 and η_k, and $c_l < c_{l+1}$ for $l = 1, \dots, u_k - 1$. The argument now splits into two cases:

CASE 1: $P_{1,k} = 0$. By Lemma 4.1.1 part A, we get exactly $\binom{u_k}{2}$ cross terms $2\mathrm{Re}(f_{k,c_e}\bar{f}_{k,c_l})$, which give the maximal value for W_1 and $c_e < c_l$ for $1 \leq e, l \leq u_k$. Combining the u_k squares with the $\binom{u_k}{2}$ cross terms gives

$$P_{2,k} = |f_{k,c_1} + \cdots + f_{k,c_{u_k}}|^2. \tag{4.11}$$

CASE 2: $P_{1,k} \neq 0$. Then from Lemma 4.1.1 part Di, we know that for each square $|f_{k,c_l}|^2$ there are exactly m_k cross terms $2\mathrm{Re}(f_{k,j}\bar{f}_{k,c_l})$ for all $j < c_l$ and $j = 1, \dots, m_k$ as well as $l = 1, \dots, u_k$ that give the maximal value for W_1. We also know from the first case that there are $\binom{u_k}{2}$ cross terms $2\mathrm{Re}(f_{k,c_e}\bar{f}_{k,c_l})$ that give the maximal value for W_1, and so we obtain the result that

$$P_{2,k} = |f_{k,1} + \cdots + f_{k,m_k} + f_{k,c_1} + \cdots + f_{k,c_{u_k}}|^2. \tag{4.12}$$

In the $(t+1)$-th step, we begin by first assuming that the sum $P_{t,k}$ for some k is a sum of squares because if $P_{t,k} = 0$ the argument is identical to the one given in Case 1. Thus let $P_{t,k}$ be given as $P_{t,k} = |f_{k,b_1} + \cdots + f_{k,b_{v_k}}|^2$, where v_k is the total number of squares from the expansion of $|f_k^\beta|^2$ in P_t after step t with $b_j < b_{j+1}$ for $j = 1, \dots, v_k - 1$. Let's assume that the W_t value of at least one square in Q_t can be computed. This implies that the maximal value for W_t exists. Assume that s_k squares from the expansion of $|f_k^\beta|^2$ give the maximum W_t value, and let $|f_{k,a_l}|^2$ for $l = 1, \dots, s_k$ be these squares.

From Lemma 4.1.1 part Di, we know that for each square $|f_{k,a_l}|^2$ there are exactly v_k cross terms $2\text{Re}(f_{k,j}\bar{f}_{k,a_l})$ for all $j < a_l$ where $j = b_1, \dots, b_{v_k}$ and $l = 1, \dots, s_k$, which give the maximal value for W_t. By Lemma 4.1.1 part A, there are $\binom{s_k}{2}$ cross terms $2\text{Re}(f_{k,a_e}\bar{f}_{k,a_l})$ which give the maximal value for W_t, and so we obtain the result that

$$P_{t+1,k} = |f_{k,b_1} + \cdots + f_{k,b_{v_k}} + f_{k,a_1} + \cdots + f_{k,a_{s_k}}|^2. \tag{4.13}$$

Hence $P_{t+1,k}$ is a sum of squares. Since the leading polynomial P_t at each step is a sum of the $P_{t,k}$'s, the leading polynomial at each step and subsequently the final leading polynomial is a sum of squares.

Since every change of variables that is allowed in the Kolář algorithm sends each square to a square and keeps the weight of terms in the leading polynomial P_t the same, it is easy to see that the leading polynomial is still a sum of squares after any change of variables.

Our assumption of finite D'Angelo 1-type implies that the last entry of the multitype is bounded, and so all entries of the multitype weight will be finite. This means that the Kolář algorithm will definitely terminate after a finite number of steps, and so the above procedure can only occur finitely many times. We conclude therefore that the final leading polynomial corresponding to the last weight in the procedure, the multitype weight, is always a sum of squares too.

□

Lemma 4.1.3. *Assume that the D'Angelo 1-type of the hypersurface $\mathcal{M}$ of a sum of squares domain at the origin is finite. Let $\mathcal{M}_0$ be the model hypersurface of $\mathcal{M}$ given by the defining function*

$$r = 2Re(z_{n+1}) + P,$$

where P is the model polynomial. Then P cannot contain the variable z_{n+1}.

Proof. Let P be the model polynomial of a sum of squares domain $\mathcal{M}$ and assume that the D'Angelo 1-type of $\mathcal{M}$ at the origin is finite. From Proposition 4.1.2 we know that P is a sum of squares and that every monomial from its expansion is of weighted degree one with respect to the multitype weight. Now, let's assume that P depends on the variable z_{n+1}, whose weight equals 1. Since P is a sum of squares of holomorphic polynomials which vanish at the origin, every monomial from its expansion cannot be harmonic. By our assumption, at least one of these non-harmonic monomials depends on the variable z_{n+1} and so has a weighted degree strictly greater than one with respect to the multitype weight. This contradicts the fact that P is a model polynomial. Hence, P does not depend on the variable z_{n+1}. □

Armed with Proposition 4.1.2 and Lemma 4.1.3, we can strengthen Catlin's normalization result from [6] for a sum of squares domain of finite D'Angelo 1-type. Catlin proved that the model hypersurface of a pseudoconvex domain whose Levi form has rank p at the origin has a defining function of the form:

$$\begin{aligned} r_0 =& 2\text{Re}(z_{n+1}) + \sum_{k=1}^{p} |z_k|^2 + 2\text{Re}\left(\sum_{k=1}^{p} z_k h_k(z_{p+1}, \dots, z_n, \bar{z}_{p+1}, \dots, \bar{z}_n)\right) \\ &+ h_{p+1}(z_{p+1}, \dots, z_n, \bar{z}_{p+1}, \dots, \bar{z}_n) \end{aligned}$$

for $h_1, \dots, h_{p+1}$ polynomials.

Our normalization result is the following:

Proposition 4.1.4. *Let $\mathcal{M} \subset \mathbb{C}^{n+1}$ be a hypersurface whose defining function is given by*

$$r = 2Re(z_{n+1}) + \sum_{j=1}^{N} |f_j|^2,$$

where $f_1, \dots, f_N$ are holomorphic functions in the neighborhood of the origin. Assume that the D'Angelo 1-type of $\mathcal{M}$ at 0 is finite. Let

$$r_0 = 2Re(z_{n+1}) + P(z, \bar{z})$$

be the defining function of the model hypersurface $\mathcal{M}_0$ of $\mathcal{M}$, where $P(z, \bar{z})$ is a polynomial of weighted degree 1 with respect to Λ^*, the multitype weight at the origin. Assume that the rank of the Levi form of $\mathcal{M}_0$ at 0 is p. There exists a polynomial change of variables preserving Λ^* such that the new defining function r_0^* is of the form

$$r_0^* = 2Re(z_{n+1}) + \sum_{k=1}^{p} |z_k|^2 + P^*(z_{p+1}, \dots, z_n, \bar{z}_{p+1}, \dots, \bar{z}_n),$$

where $P^*(z_{p+1}, \dots, z_n, \bar{z}_{p+1}, \dots, \bar{z}_n)$ is a sum of squares of holomorphic polynomials in the variables $z_{p+1}, \dots, z_n$.

Proof. We assume $p > 0$, else there is nothing to prove. Let $\mathrm{A} = (a_{k\bar{l}})_{1 \le k,l \le n}$ be the Levi matrix of $\mathcal{M}_0$, and assume without loss of generality that the first p variables $z_1, \dots, z_p$ are the only ones that contribute to the rank of A. Note that due to homogeneity all entries in the $p \times p$ upper left principal submatrix are complex numbers. From Proposition 4.1.2 we know that $\mathrm{P}(z, \bar{z})$ is a sum of squares, and so the defining function r_0 is plurisubharmonic. Thus, $\mathcal{M}_0$ is pseudoconvex. Therefore, the Levi matrix A of r_0 is positive semi-definite, and so each principal minor of A is nonnegative. There exists then a linear transformation that transforms A into a Hermitian matrix whose $p \times p$ upper left principal submatrix is the identity matrix. In fact, this linear transformation can be taken to be the identity on variables $z_{p+1}, \dots, z_{n+1}$. As such, this linear transformation preserves Λ^* by Theorem 2.2.2. Due to Proposition 4.1.2, after our change of variables, r_0 has become $\tilde{r}_0$ given by

$$\tilde{r}_0 = 2\mathrm{Re}(z_{n+1}) + \sum_{k=1}^{p} |z_k + g_k|^2 + \tilde{P}(z_{p+1}, \dots, z_n, \bar{z}_{p+1}, \dots, \bar{z}_n),$$

where each g_k is a polynomial in the variables $z_{p+1}, \dots, z_n$, with vanishing order at least 2 and $\tilde{P}(z_{p+1}, \dots, z_n, \bar{z}_{p+1}, \dots, \bar{z}_n)$ is a sum of squares of holomorphic polynomials in the variables $z_{p+1}, \dots, z_n$ only. By homogeneity, g_k has weight 1/2 for $k = 1, \dots, p$. To finish the proof, we make the following change of variables that once again preserves Λ^* by Theorem 2.2.2: $z_k \to z_k^*$ for $k = 1, \dots, n+1$ where $z_k^* = z_k + g_k$ for $k = 1, \dots, p$ and $z_k^* = z_k$ for $k = p+1, \dots, n+1$. ☐

We shall prove another corollary to Proposition 4.1.2, but before we state it, we consider the following:

Let $\mathcal{M} \subset \mathbb{C}^{n+1}$ be a sum of squares domain given by the defining function

$$r = 2\text{Re}(z_{n+1}) + \sum_{k=1}^{N} |f_k(z_1, \ldots, z_n)|^2,$$

where $f_1, \ldots, f_N$ are holomorphic functions near the origin. Assume that the D'Angelo 1-type is finite at the origin and that β is the ceiling of the D'Angelo 1-type. We truncate each holomorphic function f_k to the order β and let $f_{k,i}$ be the i-th monomial in the power series expansion of each generator f_k after ordering by vanishing order. The ideal corresponding to the defining function r is given as $\mathcal{I} = (z_{n+1}, f_1^\beta, \ldots, f_N^\beta)$. We know that the term z_{n+1} has weight 1. Following the original algorithm of Kolář, we shall ignore the term z_{n+1} and work with the corresponding ideal $\mathcal{I} = (f_1^\beta, \ldots, f_N^\beta)$.

From Proposition 4.1.2, we know that all leading polynomials produced are sums of squares. Therefore, any leading polynomial P_j can be written in the form

$$P_j = \sum_{k=1}^{N} \Big| \sum_{i=1}^{v_k} f_{k,a_i} \Big|^2, \tag{4.14}$$

where the f_{k,a_i}'s are the monomials from the generator f_k^β of weighted degree $\frac{1}{2}$ with respect to Λ_j. We will associate to every leading polynomial P_j the ideal $\mathcal{I}_{P_j}$ given by

$$\mathcal{I}_{P_j} = \left(\sum_{i=1}^{v_1} f_{1,a_i}, \ldots, \sum_{i=1}^{v_N} f_{N,a_i} \right). \tag{4.15}$$

It is convenient to introduce notation for each square in P_j. Let $P_{j,k} = \Big| \sum_{i=1}^{v_k} f_{k,a_i} \Big|^2$. Then its associated ideal $\mathcal{I}_{P_{j,k}}$ can be expressed as

$$\mathcal{I}_{P_{j,k}} = \left(\sum_{i=1}^{v_k} f_{k,a_i} \right) = (f_{k,a_1} + \cdots + f_{k,a_{v_k}}) \tag{4.16}$$

Clearly,

$$\mathcal{I}_{P_j} = \sum_{k=1}^{N} \mathcal{I}_{P_{j,k}}. \tag{4.17}$$

Recall also that each monomial in every leading polynomial is of weighted degree one with respect to the corresponding weight. As a result, the weighted degree of any monomial f_{k,a_i} is exactly one half with respect to the corresponding weight.

Thus, given $\mathcal{I}_{P_{j,k}} = (f_{k,a_1} + \cdots + f_{k,a_{v_k}})$, the f_{k,a_i}'s are exactly the monomials from the generator f_k^β of weighted degree $\frac{1}{2}$ with respect to Λ_j.

Set the ideal $\mathcal{I} = \mathcal{I}_0$. For $j \geq 1$, the ideals $\mathcal{I}_{P_j}$, $\mathcal{I}_j$, and $\mathcal{I}_{P_{j,k}}$ can be described as follows:

1. $\mathcal{I}_{P_j}$ is the ideal whose generators are precisely the terms from the generators of the ideal I_{j-1} having weighted order exactly $\frac{1}{2}$ with respect to the weight Λ_j. We refer to the ideal $\mathcal{I}_{P_j}$ as the *leading polynomial ideal.*

2. The ideal $\mathcal{I}_j$ is the ideal obtained after applying the chosen Λ_j-homogeneous transformation, which makes the generators of $\mathcal{I}_{P_j}$ to be independent of the largest number of variables, to $\mathcal{I}_{j-1}$. Simply put, I_j is the ideal obtained after changing variables in the ideal I_{j-1}.

3. $\mathcal{I}_{P_{j,k}}$ is the principal ideal whose generator is the sum of monomials in the generator f_k^β with weighted degree exactly $\frac{1}{2}$ with respect to the weight Λ_j. The ideal $\mathcal{I}_{P_{j,k}}$ is the zero ideal if no monomial in the generator f_k^β has weighted degree $\frac{1}{2}$ with respect to the weight Λ_j.

4.2 The Kolář Algorithm (Ideal Version)

Set the ideal $\mathcal{I} = \mathcal{I}_0$, and compute the vanishing order at the origin of $\mathcal{I}_0$, which is the same as the degree ν_1 of the lowest order monomial in $\mathcal{I}_0$. We define the Bloom-Graham type as twice the vanishing order of $\mathcal{I}_0$. This gives the first entry of the multitype m_1 and so let $m_1 = 1/\mu_1$, where $\mu_1 = 2\nu_1$. Set the first weight to be $\Lambda_1 = (\mu_1, \dots, \mu_1)$.

In the second step, consider all Λ_1-homogeneous transformations, and choose one that will make the set of all generators of the leading polynomial ideal $\mathcal{I}_{P_1}$ to be independent of the largest number of variables. Denote this number by d_1. In the local coordinates after such a Λ_1-homogeneous transformation, we obtain that $\mathcal{I}_{P_1}$ is the ideal whose generators consist of those monomials in the variables $z_1, \dots, z_{n-d_1}$, which are of weighted degree $1/2$ with respect to Λ_1. Apply the chosen Λ_1-homogeneous transformation to the ideal $\mathcal{I}_0$ to obtain the ideal $\mathcal{I}_1$. The rest of the terms from the generators in $\mathcal{I}_1$, which are not in $\mathcal{I}_{P_1}$, have weighted degrees strictly greater than $\frac{1}{2}$ with respect to Λ_1.

We shall now give a slightly modified version of Kolář's Θ_1 and W_1. If $\alpha^{k,j} = (\alpha_1^{k,j}, \dots, \alpha_n^{k,j})$ is the multiindex of a monomial $f_{k,j}$ from any generator of $\mathcal{I}_1$, which is not in $\mathcal{I}_{P_1}$, then $f_{k,j}$ is of the form

$$f_{k,j} = C_{k,j}^1 z^{\alpha^{k,j}} \quad \text{and} \quad |\alpha^{k,j}|_{\Lambda_1} > \frac{1}{2}.$$

Let

$$\Theta_1 = \left\{ \alpha^{k,j} \;\middle|\; C_{k,j}^1 \neq 0 \text{ and } \sum_{i=1}^{n-d_1} \alpha_i^{k,j} \mu_i < \frac{1}{2} \right\}.$$

For every $\alpha^{k,j} \in \Theta_1$,

$$W_1(\alpha^{k,j}) = \frac{\frac{1}{2} - \sum_{i=1}^{n-d_1} \alpha_i^{k,j} \mu_i}{\sum_{i=n-d_1+1}^{n} \alpha_i^{k,j}}. \tag{4.18}$$

The next weight Λ_2 is defined by letting

$$\lambda_i^2 = \max_{\alpha^{k,j} \in \Theta_1} W_1(\alpha^{k,j})$$

for $i > n - d_1$, and $\lambda_i^2 = \mu_1$ for $i \le n - d_1$. To complete the second step, we let $\mathcal{I}_{P_2}$ be the second leading polynomial ideal corresponding to the weight Λ_2. The generators of $\mathcal{I}_{P_2}$ depend on more than $n - d_1$ variables.

We proceed by induction. At the step t, for $t > 2$, we consider all Λ_{t-1}-homogeneous transformations and choose one that makes the generators of the leading polynomial ideal $\mathcal{I}_{P_{t-1}}$ to be independent of the largest number of variables. Denote this number by d_{t-1}. Apply this Λ_{t-1}-homogeneous transformation to the previous ideal $\mathcal{I}_{t-2}$ in the $(t-1)$-th step to obtain the ideal $\mathcal{I}_{t-1}$. We know from the Kolář algorithm that the number of multitype entries that are added at each step of the computation depends on the difference $(d_{t-2} - d_{t-1})$. We consider two cases:

CASE 1: Assume that $d_{t-2} > d_{t-1}$. Again recall that for any weight Λ that is smaller than Λ_{t-1} with respect to the lexicographic ordering, Λ-adapted coordinates are also Λ_{t-1}-adapted. This implies that we get $(d_{t-2} - d_{t-1})$ multitype entries

$$\mu_{n-d_{t-2}+1} = \cdots = \mu_{n-d_{t-1}} = \lambda^{t-1}_{n-d_{t-2}+1}$$

and let $\lambda_i^t = \mu_i$ for $i \leq n - d_{t-2}$. Here, $\mathcal{I}_{P_{t-1}}$ is the ideal whose generators are sums of monomials in the variables $z_1, \ldots, z_{n-d_{t-1}}$ that are Λ_{t-1}-homogeneous of weighted degree $\frac{1}{2}$. To obtain λ_i^t for $i > n - d_{t-1}$, we consider the rest of the monomials from the generators in $\mathcal{I}_{t-1}$ that are not in $\mathcal{I}_{P_{t-1}}$. Using these monomials that have weighted degree strictly greater than $\frac{1}{2}$ with respect to Λ_{t-1}, we define Θ_{t-1} and compute W_{t-1} in a similar way as in step two. Such monomials are of the form $f_{k,j} = C^{t-1}_{k,j} z^{\alpha^{k,j}}$ for multiindex $\alpha^{k,j} = (\alpha_1^{k,j}, \ldots, \alpha_n^{k,j})$ satisfying $|\alpha^{k,j}|_{\Lambda_{t-1}} > \frac{1}{2}$. Thus,

$$\Theta_{t-1} = \left\{ \alpha^{k,j} \;\middle|\; C^{t-1}_{k,j} \neq 0 \text{ and } \sum_{i=1}^{n-d_{t-1}} \alpha_i^{k,j} \mu_i < \frac{1}{2} \right\}.$$

For every $\alpha^{k,j} \in \Theta_{t-1}$,

$$W_{t-1}(\alpha^{k,j}) = \frac{\frac{1}{2} - \sum_{i=1}^{n-d_{t-1}} \alpha_i^{k,j} \mu_i}{\sum_{i=n-d_{t-1}+1}^{n} \alpha_i^{k,j}}. \tag{4.19}$$

So for the remaining multitype entries of Λ_j, we let

$$\lambda_i^t = \max_{\alpha^{k,j} \in \Theta_{t-1}} W_{t-1}(\alpha^{k,j}),$$

for $i > n - d_{t-1}$.

CASE 2: Assume that $d_{t-1} = d_{t-2}$. There are zero multitype entries computed in this case, and so we only determine λ_i^t for $t > n - d_{t-1}$ using (4.19). This completes the step t of the algorithm.

We can thus establish a one-to-one correspondence between the leading polynomial P_t for $t \geq 1$ and the intermediate ideal $\mathcal{I}_{P_t}$ introduced above. Since working with ideals of holomorphic functions is often easier than with real-valued polynomials, the restatement of the Kolář algorithm simplifies multitype computations for a sum of squares domain. We work with considerably fewer terms in the case of the ideals as compared to sums of squares. In particular, for each modulus square of a generator consisting of m monomials, Kolář's original algorithm involves working with m squares plus $\binom{m}{2}$ cross terms, whereas this restatement in terms of ideals involves computations for only m monomials.

We give a corollary to Proposition 4.1.2:

Corollary 4.2.0.1. *Let $\mathcal{M} \subset \mathbb{C}^{n+1}$ be a hypersurface whose defining function is given as*

$$r = 2Re(z_{n+1}) + \sum_{j=1}^{N} |f_j(z_1, \ldots, z_n)|^2,$$

where $f_1, \ldots, f_N$ are holomorphic functions near the origin. Assume that the D'Angelo 1-type is finite. Then for $\ell \geq 1$, each monomial from every generator of the leading polynomial ideal $\mathcal{I}_{P_\ell}$ obtained at the ℓ-th step of the Kolář algorithm has weighted degree $\frac{1}{2}$ with respect to the weight Λ_ℓ.

Proof. Let f_k^β be the Taylor expansion of the holomorphic function f_k to the order β, where β is the ceiling of the D'Angelo 1-type. We order the generators by vanishing order and let $\mathcal{I} = (f_1^\beta, \ldots, f_N^\beta)$. Now assume that the vanishing order of the ideal $\mathcal{I}$ is $\nu > 0$. Then the Bloom-Graham type is precisely 2ν and the weight $\mu_1 = \frac{1}{2\nu}$ with $\Lambda_1 = (\frac{1}{2\nu}, \ldots, \frac{1}{2\nu})$. Thus, $\mathcal{I}_{P_1}$ is not the zero ideal.

For every k such that $\mathcal{I}_{P_{1,k}}$ is not the zero ideal,

$$\mathcal{I}_{P_{1,k}} = (f_{k,1}, \ldots, f_{k,m_k}),$$

where each monomial $f_{k,i}$, for $1 \leq i \leq m_k$, has weighted degree $\frac{1}{2}$ with respect to the weight Λ_1. Next, assume that in the second step, the ideal $\mathcal{I}_{P_{1,k}}$ is the same as the ideal $\mathcal{I}_{P_{2,k}}$. We know that the entries corresponding to each variable in the monomial $f_{k,i}$, for $1 \leq i \leq m_k$, are the same in both weights Λ_1 and Λ_2. Therefore, each monomial from the generator $\sum_{i=1}^{m_k} f_{k,i}$ has weighted degree $\frac{1}{2}$ with respect to Λ_2 as well. Assume that the principal ideal $\mathcal{I}_{P_{2,k}}$ is generated by the sum $\sum_{i=1}^{m_k} f_{k,i} + \sum_{j=1}^{\gamma_k} f_{k,b_j}$. Since the new sum $\sum_{j=1}^{\gamma_k} f_{k,b_j}$ corresponds to the new weight Λ_2, each monomial f_{k,b_j} has weighted degree $\frac{1}{2}$ with respect to Λ_2. Therefore, every monomial in the generator of the ideal $\mathcal{I}_{P_{2,k}}$ has weighted degree $\frac{1}{2}$ with respect to Λ_2.

Next, we assume that for $\ell \geq 2$

$$\mathcal{I}_{P_{\ell,k}} = (f_{k,a_1} + \cdots + f_{k,a_{v_k}}),$$

where each monomial f_{k,a_i}, for $1 \leq i \leq v_k$, has weighted degree exactly equal to $\frac{1}{2}$ with respect to the weight Λ_ℓ.

Now, assume that at step $\ell + 1$, the ideal $\mathcal{I}_{P_{\ell+1,k}}$ is the same as the ideal $\mathcal{I}_{P_{\ell,k}}$. Every monomial in the generator of $\mathcal{I}_{P_{\ell+1,k}}$ also has weighted degree equal to $\frac{1}{2}$ with respect to the weight $\Lambda_{\ell+1}$ because even though Λ_ℓ is not the same as $\Lambda_{\ell+1}$, the weight corresponding to each variable in $f_{k,a_{v_k}}$ is the same in both weights Λ_ℓ and $\Lambda_{\ell+1}$.

Next, assume that at step $\ell+1$ the sum $\sum_{i=1}^{u_k} f_{k,b_{u_k}}$ is added to the sum $\sum_{i=1}^{v_k} f_{k,a_{v_k}}$ to obtain the generator of the ideal

$$\mathcal{I}_{P_{\ell+1,k}} = (f_{k,a_1} + \cdots + f_{k,a_{v_k}} + f_{k,b_1} + \cdots + f_{k,b_{u_k}}).$$

This implies that each monomial $f_{k,b_{u_k}}$ has weighted degree $\frac{1}{2}$ with respect to the weight $\Lambda_{\ell+1}$. Again, each monomial $f_{k,a_{v_k}}$ is of weighted degree $\frac{1}{2}$ with respect to the weight $\Lambda_{\ell+1}$ since the weight corresponding to each variable in $f_{k,a_{v_k}}$ is the same in both weights Λ_ℓ and $\Lambda_{\ell+1}$. Thus, every monomial from the generator of the ideal $\mathcal{I}_{P_{\ell+1,k}}$ has weighted degree $\frac{1}{2}$ with respect to the weight $\Lambda_{\ell+1}$. $\square$

The example that follows is the ideal restatement of the Kolář algorithm applied to the defining function given in Example 1.

Example 3. Let $\mathcal{M} \subset \mathbb{C}^4$ be a hypersurface whose defining function is given by

$$r = 2\mathrm{Re}(z_4) + |z_1 - z_2 + z_3^2|^2 + |z_1^2 - z_2^2|^2 + |z_2^4|^2.$$

The associated ideal then becomes

$$\mathcal{I} = (z_1 - z_2 + z_3^2, z_1^2 - z_2^2, z_2^4) = \mathcal{I}_0.$$

The vanishing order here equals one, and so the Bloom-Graham type is 2, which implies that $\mu_1 = \frac{1}{2}$ and $\Lambda_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Hence

$$\mathcal{I}_{P_1} = (z_1 - z_2) = \mathcal{I}_{P_{1,1}}.$$

So $\mathcal{I}_{P_{1,2}}$ is the zero ideal. We choose a Λ_1-homogeneous transformation which makes $\mathcal{I}_{P_1}$ to be independent of the largest number of variables. Let $\tilde{z}_1 = z_1 - z_2$ and $\tilde{z}_j = z_j$, $j = 2, 3, 4$. We shall ignore $\sim$ where no confusion arises. Thus we get $d_1 = 2$

$$\mathcal{I}_{P_1} = (z_1) = \mathcal{I}_{P_{1,1}}.$$

Applying these variable changes to $\mathcal{I}_0$ gives

$$\mathcal{I}_1 = (z_1 + z_3^2, z_1^2 + 2z_1z_2, z_2^4).$$

From Example 1, we know that the next multitype entry is $\frac{1}{4}$, and so $\Lambda_2 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$.

$$\mathcal{I}_{P_2} = (z_1 + z_3^2),$$

where $\mathcal{I}_{P_{2,1}} = \mathcal{I}_{P_2}$ and $\mathcal{I}_{P_{2,2}}$ is the zero ideal. Again, we choose a Λ_2-homogeneous transformation which makes $\mathcal{I}_{P_2}$ to be dependent on only the variable z_1. Let $\tilde{z}_1 = z_1 + z_3^2$ and $\tilde{z}_j = z_j$, $j = 2, 3, 4$. Again, we ignore the sign $\sim$. Here $d_2 = 2$ and

$$\mathcal{I}_{P_2} = (z_1),$$

where $\mathcal{I}_{P_{2,1}} = \mathcal{I}_{P_2}$ and $\mathcal{I}_{P_{2,2}}$ is the zero ideal. We apply the new coordinates to $\mathcal{I}_1$ to get

$$\mathcal{I}_2 = (z_1, z_1^2 + 2z_1z_3^2 + z_3^4 + 2z_1z_2 - 2z_2z_3^2, z_2^4).$$

The multitype entry at this step is $\frac{1}{6}$, and $\Lambda_3 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6})$. We get that

$$\mathcal{I}_{P_3} = (z_1, -2z_2z_3^2) = (z_1, 2z_2z_3^2).$$

Here $\mathcal{I}_{P_{3,1}} = (z_1)$, and $\mathcal{I}_{P_{3,2}} = (2z_2z_3^2)$.

No Λ_3-homogeneous transformation can make $\mathcal{I}_{P_3}$ to be independent of any variables and so $d_3 = 0$. Thus, the multitype weight $\Lambda^* = \Lambda_3$ and the final leading polynomial ideal is

$$\mathcal{I}_{P_3} = (z_1, z_2z_3^2).$$

Proposition 4.2.1. *Let $0 \in \mathcal{M} \subset \mathbb{C}^{n+1}$ be a hypersurface whose defining function is given by*

$$r(z) = 2Re(z_{n+1}) + \sum_{j=1}^{N} |f_j(z_1, \ldots, z_n)|^2,$$

where $f_1, \ldots, f_N$ are holomorphic functions near the origin. Let $\mathcal{M}' \subset \mathbb{C}^{n+1}$ be another hypersurface whose defining function is given as

$$u(z) = 2Re(z_{n+1}) + \sum_{j=1}^{l-1} |f_j|^2 + |h_l f_l - \sum_{c \neq l} h_c f_c|^2 + \sum_{j=l+1}^{N} |f_j|^2,$$

for some fixed l, where h_j is a holomorphic function near the origin for every $j = 1, \ldots, N$. Assume that the D'Angelo 1-type of $\mathcal{M}$ is finite at the origin. Then the multitype obtained by applying Kolář algorithm to both $r(z)$ and $u(z)$ is the same provided that $\langle f_1, \cdots, f_N \rangle$ and

$$\left\langle f_1, \cdots, f_{l-1}, h_l f_l - \sum_{c \neq l} h_c f_c, f_{l+1}, \cdots f_N \right\rangle$$

represent the same ideal in the ring $\mathcal{O}$.

Remark 4.2.1. *Modifying only one generator at a time makes the bookkeeping in the computation of the multitype easier to follow.*

Remark 4.2.2. *Assume that there exist some l such that $1 \leq l \leq N$ and holomorphic functions near the origin $h_1, \ldots, h_{l-1}, h_{l+1}, \ldots, h_N$ such that $f_l = \sum_{c \neq l} h_c f_c$. It is clear that $\langle f_1, \cdots, f_N \rangle$ and $\langle f_1, \cdots, f_{l-1}, f_{l+1}, \cdots f_N \rangle$ represent the same ideal in $\mathcal{O}$. Therefore, applying Proposition 4.2.1 with $h_l \equiv 1$ shows that adding in the square $|f_l|^2$ or taking it away makes absolutely no difference as far as the multitype computation goes. This observation will be crucial in the corollary that follows.*

Proof. We will show that the multitype obtained by applying the Kolář algorithm to both $r(z)$ and $u(z)$ is the same. Let $\beta = \lceil \Delta_1(\mathcal{M}, 0) \rceil$ be the ceiling of the D'Angelo 1-type of $\mathcal{M}$ at the origin. We truncate each generator f_k as well as each holomorphic function h_k at the order β and denote them by f_k^β and h_k^β respectively. Denote by $f_{k,i}$ the i-th monomial from the Taylor expansion of f_k^β after ordering by vanishing order and reverse lexicographic order for the monomials with the same vanishing order.

By Lemma 4.1.1, we know that each entry of the multitype is realized by a square. If no square from the expansion of $|f_l^\beta|^2$ contributes to the entries of the multitype, then the multitype entries for both defining functions $r(z)$ and $u(z)$ are the same, and there is nothing to prove. We thus assume that there exists at least one square from the expansion of $|f_l^\beta|^2$ that contributes to the entries of the multitype and that no nonzero multiple of that square exists in any of the expansions of $|f_1^\beta|^2, \ldots, |f_{l-1}^\beta|^2, |f_{l+1}^\beta|^2, \ldots, |f_N^\beta|^2$.

Next, we claim that if $h_l(0) = 0$, then no square from the expansion of $|h_l^\beta f_l^\beta|^2$ can contribute to the entries of the multitype. Indeed, let $h_{l,i}$ be the i-th monomial from the Taylor expansion of h_l^β after ordering by vanishing order and reverse lexicographic order for the monomials with the same vanishing order. For every monomial $f_{l,j}$ in f_l^β, the monomial $h_{l,i} f_{l,s}$ in $h_l^\beta f_l^\beta$ has greater combined degree than that of $f_{l,j}$ for every

$i \geq 1$. As a result, $|h_{l,i}f_{l,j}|^2$ cannot give the Bloom-Graham type since the combined degree of $|f_{l,j}|^2$ is strictly less than the combined degree of $|h_{l,i}f_{l,j}|^2$. Furthermore, $W_t(|h_{l,i}f_{l,j}|^2)$ cannot be computed if $W_t(|f_{l,j}|^2)$ gives the maximum W_t-value. By the same argument, if $f_l = \sum_{c\neq l} g_c f_c$, for g_c with $c = 1, \ldots, l-1, l+1, \ldots, N$ holomorphic functions near the origin, then no square from the expansion of $|f_l^\beta|^2$ can contribute to the entries of the multitype unless it is the square of the nonzero constant term of some g_c multiplied by a monomial of f_c that in itself gives that same multitype entry. Since we assumed the contrary, f_l cannot be written in terms of the other generators. By our hypothesis, however, $\langle f_1, \cdots, f_N\rangle$ and $\left\langle f_1, \cdots, f_{l-1}, h_l f_l - \sum_{c\neq l} f_c h_c, f_{l+1}, \cdots f_N \right\rangle$ represent the same ideal in the ring $\mathcal{O}$. Putting these two facts together along with our assumption that there exists at least one square from the expansion of $|f_l^\beta|^2$ that contributes to the entries of the multitype and that no nonzero multiple of that square exists in any of the expansions of $|f_1^\beta|^2, \ldots, |f_{l-1}^\beta|^2, |f_{l+1}^\beta|^2, \ldots, |f_N^\beta|^2$, we conclude that $h_l(0) \neq 0$. Without loss of generality, assume $h_l \equiv 1$. We shall show that modifying the function f_l^β in the sum of squares by the sum $\sum_{c\neq l} f_c^\beta h_c^\beta$ does not alter the multitype. We further assume that the sum $\sum_{c\neq l} f_c^\beta h_c^\beta \neq 0$. By Lemma 4.1.1, it suffices to focus on how the omission of some squares of monomials from the term $|f_l^\beta - \sum_{c\neq l} h_c^\beta f_c^\beta|^2$ in the defining function $u(z)$ affects our results. We now break our argument into two cases:

CASE 1: Assume that there exists a monomial m in f_l^β whose square $|m|^2$ from the expansion of $|f_l^\beta|^2$ gives the Bloom-Graham type. We consider two subcases here:

i. Assume that m is in the expression $f_l^\beta - \sum_{c\neq l} h_c^\beta f_c^\beta$. Then no monomial in the sum $\sum_{c\neq l} h_c^\beta f_c^\beta$ cancels out m. Clearly, the weights obtained at the first step of the Kolář algorithm are the same for both $r(z)$ and $u(z)$ since $|m|^2$ belongs to both defining functions.

ii. Assume that m is not in the expression $f_l^\beta - \sum_{c\neq l} h_c^\beta f_c^\beta$. Hence, m gets cancelled out in the expression $f_l^\beta - \sum_{c\neq l} h_c^\beta f_c^\beta$ and so does not appear in $u(z)$. Let ψ be the monomial in the sum $\sum_{c\neq l} h_c^\beta f_c^\beta$ that cancels out m, and write $\psi = h_{c,i}f_{c,j}$ for some $c \neq l$, where $h_{c,i}$ is some monomial in h_c^β and $f_{c,j}$ is some monomial in f_c^β. By our assumption, ψ equals m, and its square $|\psi|^2$ gives the Bloom-Graham type as well. The monomial $h_{c,i}$ cannot have vanishing order 1 or higher; otherwise, $f_{c,j}$ must have combined degree less than that of ψ, which contradicts the fact that $|\psi|^2$ gives the Bloom-Graham type. Thus, $h_{c,i} = h_{c,1} \in \mathbb{C}$ and $m = h_{c,1}f_{c,j}$. Hence, the square $|f_{c,j}|^2$ gives the Bloom-Graham type as well. Even though there is the cancellation in $u(z)$, the weight obtained at the first step having applied the Kolář algorithm to $r(z)$ and $u(z)$ is the same. More specifically, the squares $|f_{c,j}|^2$ and $|m|^2$ appear in the expansions of $|f_c^\beta|^2$ and $|f_l^\beta|^2$ respectively.

CASE 2: Assume that there exists a monomial m in f_l^β whose square $|m|^2$ from the expansion of $|f_l^\beta|^2$ gives the maximum W_t-value at the $(t+1)$-th step for $t \geq 1$.

i. Assume that m is in the expression $f_l^\beta - \sum_{c\neq l} h_c^\beta f_c^\beta$. Then no monomial in the sum $\sum_{c\neq l} h_c^\beta f_c^\beta$ cancels out m, and so the weights obtained at the $(t+1)$-th step are the same for both $r(z)$ and $u(z)$ since $|m|^2$ belongs to both defining functions.

ii. Assume that m is not in the expression $f_l^\beta - \sum_{c\neq l} h_c^\beta f_c^\beta$. Then m gets cancelled out by some monomial ψ in the sum $\sum_{c\neq l} f_c^\beta h_c^\beta$. Let $\psi = h_{c,s} f_{c,j}$ for some s and j, where $h_{c,s}$ is some monomial in h_c^β and $f_{c,j}$ is some monomial in f_c^β. This implies that $m = \psi$ and $|\psi|^2$ gives the maximal W_t-value at step $t+1$ as well. Now let's assume that $h_{c,s} \notin \mathbb{C}$. Then the combined degree of $f_{c,j}$ is less than that of ψ.

If $W_t(|f_{c,j}|^2)$ cannot be computed, then $W_t(|\psi|^2)$ cannot be computed, which gives a contradiction. Therefore, we assume that $W_t(|f_{c,j}|^2)$ can be computed.

At this point let us recall the definition of Γ_1, Γ_2, and Γ_3 as given in the proof of Lemma 4.1.1. Let Γ_1 be the set of all non-zero monomials that consist of only variables not in P_t, let Γ_2 be the set of all non-zero monomials which consist of variables both in P_t as well as variables not in P_t, and let Γ_3 be the set of all non-zero monomials which consist of only variables in P_t. Also, recall that for any monomial f if $W_t(|f|^2)$ can be computed, then $f \in \Gamma_1$ or $f \in \Gamma_2$ only. Again, we shall write any monomial f in the form $f = \gamma_1 \gamma_2$ where γ_1 and γ_2 are monomials satisfying $\gamma_1 \in \Gamma_3$ and $\gamma_2 \in \Gamma_1$. Recall that

$$W_t(|f|^2) = \frac{1 - \sum_{i=1}^{\kappa} (\alpha_i + \hat{\alpha}_i)\mu_i}{\sum_{i=\kappa+1}^{n} (\alpha_i + \hat{\alpha}_i)},$$

where $(\alpha_1, \ldots, \alpha_n, \hat{\alpha}_1, \ldots, \hat{\alpha}_n)$ is the multiindex of the monomial $|f|^2$ whose W_t is being computed, κ is the number of variables in the leading polynomial P_t, and $W_t(|f|^2)$ is the W_t-value of the term $|f|^2$.

We shall now consider the number $W_t(|\psi|^2)$ given that $W_t(|f_{c,j}|^2)$ can be computed. From Lemma 4.1.1, $f_{c,j} \in \Gamma_1$ or Γ_2. Since the monomial $h_{c,s}$ can belong to Γ_1, Γ_2, or Γ_3, we shall consider three subcases below and assume that $f_{c,j} \in \Gamma_1$ or Γ_2 in each case:

a. Assume that $h_{c,s} \in \Gamma_1$. Clearly, $W_t(|f_{c,j}|^2)$ and $W_t(|\psi|^2)$ both have the same numerator and the denominator of $W_t(|\psi|^2)$ is greater than that of $W_t(|f_{c,j}|^2)$ because $h_{c,s} \in \Gamma_1$. Thus, $W_t(|f_{c,j}|^2)$ is greater than $W_t(|\psi|^2)$, which is a contradiction to our hypothesis that $W_t(|\psi|^2)$ is maximal at step $t+1$.

b. Assume that $h_{c,s} \in \Gamma_2$. Here, the numerator of $W_t(|\psi|^2)$ is smaller than the numerator of $W_t(|f_{c,j}|^2)$ since $h_{c,s}$ contains a monomial in Γ_3. Also, the denominator of $W_t(|\psi|^2)$ is greater than the denominator of $W_t(|f_{c,j}|^2)$ because $h_{c,s}$ contains a monomial in Γ_1. Thus, $W_t(|f_{c,j}|^2)$ is greater than $W_t(|\psi|^2)$, which is again a contradiction.

c. Assume that $h_{c,s} \in \Gamma_3$. Then $W_t(|f_{c,j}|^2)$ is always greater than $W_t(|\psi|^2)$ for $f_{c,j} \in \Gamma_1$ or Γ_2 since the denominators of both numbers are equal and the numerator of $W_t(|\psi|^2)$ is less than that of $W_t(|f_{c,j}|^2)$. This gives a contradiction since $W_t(|\psi|^2)$ is maximal at step $t+1$.

From cases (a), (b), and (c) we can see that if $h_{c,s} \notin \mathbb{C}$, then $W_t(|\psi|^2)$ cannot be the maximum at step $t+1$, and so we have a contradiction to our hypothesis in all three cases. Hence $h_{c,s} \in \mathbb{C}$ and so $m = h_{c,s} f_{c,j}$. Clearly, $W_t(|f_{c,j}|^2)$ gives the maximal value at step $t+1$, too. This implies that if we apply the Kolář algorithm to both $r(z)$ and $u(z)$, then the multitype entry at the $(t+1)$-th step will be the same for both defining functions. The squares $|f_{c,j}|^2$ and $|m|^2$ appear in the expansions of $|f_c^\beta|^2$ and $|f_l^\beta|^2$ respectively. We see that regardless of the cancellation in $u(z)$, the weight obtained at the $(t+1)$-th step remains unchanged.

Clearly, the case when $h_l(0) \neq 0$ combines the analysis for the cases when $h_l \equiv 1$ and $h_l(0) = 0$. $\square$

Corollary 4.2.1.1. *Let $0 \in \mathcal{M} \subset \mathbb{C}^{n+1}$ be a hypersurface whose defining function is given by*

$$r(z) = 2Re(z_{n+1}) + \sum_{j=1}^{N} |f_j(z_1, \dots, z_n)|^2,$$

where $f_1, \dots, f_N$ are holomorphic functions near the origin. Let $\mathcal{M}' \subset \mathbb{C}^{n+1}$ be another hypersurface whose defining function is given as

$$u(z) = 2Re(z_{n+1}) + \sum_{j=1}^{S} |g_j(z_1, \dots, z_n)|^2.$$

Assume that the D'Angelo 1-type of $\mathcal{M}$ is finite at the origin. Then the multitype obtained by applying the Kolář algorithm to both $r(z)$ and $u(z)$ is the same provided that $\langle f_1, \cdots, f_N \rangle$ and $\langle g_1, \cdots, g_S \rangle$ represent the same ideal in the ring $\mathcal{O}$. In other words, the multitype is an invariant of the ideal $\langle f_1, \dots, f_N \rangle$ of generators.

Proof. Let the ideals associated to the hypersurfaces $\mathcal{M}$ and $\mathcal{M}'$ be given by $\langle f \rangle = \langle f_1, \dots, f_N \rangle$ and $\langle g \rangle = \langle g_1, \dots, g_S \rangle$ respectively, and suppose that $\langle f \rangle = \langle g \rangle$. By Remark 4.2.2 following the statement of Proposition 4.2.1, we know that adding in the square of any element of the ideal $\langle f_1, \dots, f_N \rangle$ does not modify the multitype because that element can be written in terms of the generators $f_1, \dots, f_N$ with coefficients in $\mathcal{O}$. Since $\langle f_1, \dots, f_N \rangle = \langle g_1, \dots, g_S \rangle$, each g_j is an element of $\langle f_1, \dots, f_N \rangle$ and can be written in terms of $f_1, \dots, f_N$ with coefficients in $\mathcal{O}$. Therefore,

$$r(z) = 2\text{Re}(z_{n+1}) + \sum_{j=1}^{N} |f_j(z_1, \dots, z_n)|^2$$

has the same multitype at the origin as

$$r_1(z) = 2\text{Re}(z_{n+1}) + \sum_{j=1}^{N} |f_j(z_1, \dots, z_n)|^2 + |g_1(z_1, \dots, z_n)|^2,$$

and inductively, the same multitype at the origin as

$$r_S(z) = 2\text{Re}(z_{n+1}) + \sum_{j=1}^{N} |f_j(z_1, \dots, z_n)|^2 + \sum_{k=1}^{S} |g_k(z_1, \dots, z_n)|^2.$$

Now, we apply the argument in reverse. Since $\langle g_1, \dots, g_S \rangle = \langle f_1, \dots, f_N \rangle$, each f_j is an element of $\langle g_1, \dots, g_S \rangle$ and can be written in terms of $g_1, \dots, g_S$ with coefficients in $\mathcal{O}$. Therefore, by Remark 4.2.2,

$$u(z) = 2\text{Re}(z_{n+1}) + \sum_{k=1}^{S} |g_k(z_1, \dots, z_n)|^2$$

has the same multitype at the origin as

$$u_1(z) = 2\text{Re}(z_{n+1}) + \sum_{k=1}^{S} |g_k(z_1, \dots, z_n)|^2 + |f_1(z_1, \dots, z_n)|^2,$$

and inductively, as

$$r_S(z) = 2\mathrm{Re}(z_{n+1}) + \sum_{k=1}^{S} |g_k(z_1, \ldots, z_n)|^2 + \sum_{j=1}^{N} |f_j(z_1, \ldots, z_n)|^2.$$

We conclude that $r(z)$ and $u(z)$ have the same multitype at the origin, namely that the multitype is an invariant of the ideal of generators. $\square$

Chapter 5

Polynomial Transformations in the Kolář Algorithm

We give an explicit construction of the polynomial transformations that are performed in the Kolář algorithm in this chapter. To achieve that, we relate polynomial transformations to pairs of row-column operations on the Levi matrix of a sum of squares domain. We then associate a polynomial transformation to a certain sequence of such row-column operations.

The following elementary lemma is included for completeness:

Lemma 5.0.1. *The composition of* Λ*-homogeneous transformations is* Λ*-homogeneous.*

Proof. Let Λ be a weight, and let λ_i be the entry corresponding to the variable z_i in Λ. Denote by $\mathcal{S}_1$ and $\mathcal{S}_2$ the Λ-homogeneous transformations given by

$$\tilde{z}_i^1 = p_i^1(z_1, \dots, z_n) \quad \text{and} \quad \tilde{z}_i^2 = p_i^2(\tilde{z}_1^1, \dots, \tilde{z}_n^1)$$

respectively for $1 \leq i \leq n$, where each polynomial p_i^k for $k = 1, 2$ is of weighted degree λ_i with respect to the weight Λ.

Now consider the transformation $\mathcal{S}_1 \circ \mathcal{S}_2$ given by

$$\tilde{z}_i^2 = p_i^2\big(p_1^1(z_1, \dots, z_n), \dots, p_n^1(z_1, \dots, z_n)\big).$$

From the statements above, we can deduce that each monomial in p_i^k is of weighted degree λ_i with respect to the weight Λ. Hence $\mathcal{S}_1 \circ \mathcal{S}_2$ is Λ-homogeneous as well.

□

5.1 Operations on the Levi Matrix

Let $z = (z_1, \dots, z_n)$ and $\mathbb{C}[z, \bar{z}]$ be a polynomial ring in the variables z and $\bar{z}$ over $\mathbb{C}$ the field of complex numbers.

Definitions 5.1.1 and 5.1.2 below are slightly modified versions of those given in [3] since we are working on the polynomial ring $\mathbb{C}[z, \bar{z}]$ and strictly with the Levi matrix, which is Hermitian.

Definition 5.1.1. Let A be an $n \times n$ Levi matrix of a sum of squares domain $\Omega \subset \mathbb{C}^n$. We say that the following types of operations on the rows (columns) are called *elementary row (column) operations.*

i. Interchanging two rows (columns). Denote by $R_i \leftrightarrow R_j$ ($C_i \leftrightarrow C_j$) the operation of interchanging the i-th and j-th rows (columns).

ii. Multiplying the elements of one row (column) by a nonzero α ($\bar{\alpha}$) $\in \mathbb{C}$. Denote by $\alpha \mathrm{R}_i$ ($\bar{\alpha}\mathrm{C}_i$) the operation of multiplying the i-th row (column) by a nonzero α ($\bar{\alpha}$) $\in \mathbb{C}$.

iii. Adding to the elements of one row (column) h ($\bar{h}$) times the corresponding elements of a different row (column), where $h \in \mathbb{C}[z]$. Denote by $\mathrm{R}_j + h\mathrm{R}_i$ ($\mathrm{C}_j + \bar{h}\mathrm{C}_i$) the operation of adding to the elements of the j-th row (column) h ($\bar{h}$) times the corresponding elements of the i-th row (column).

Definition 5.1.2. An elementary matrix is a matrix obtained by performing a single elementary row or column operation on an identity matrix. Thus, we give the following definitions:

i. E_{ij} the matrix obtained by interchanging the i-th and j-th rows of the identity matrix and denote by $E_{\bar{\imath}\bar{\jmath}}$ the matrix obtained by interchanging the i-th and j-th column of the identity matrix. We note that $E_{ij} = E_{\bar{\imath}\bar{\jmath}}$.

ii. $D_i(\alpha)$ is the matrix obtained by multiplying the i-th row of the identity matrix by a nonzero $\alpha \in \mathbb{C}$ and $D_{\bar{\imath}}(\bar{\alpha})$ is the matrix obtained by multiplying the i-th column of the identity matrix by a nonzero $\bar{\alpha}$.

iii. Let $h \in \mathbb{C}[z]$. Then $L_{ij}(h)$ is the matrix obtained from the identity matrix by adding to the elements of the j-th row h times the corresponding elements of the i-th row and $L_{\bar{\imath}\bar{\jmath}}(\bar{h})$ is the matrix obtained from the identity matrix by adding to the elements of the j-th column $\bar{h}$ times the corresponding elements of the i-th column.

The matrices $E_{ij}, E_{\bar{\imath}\bar{\jmath}}, D_i(\alpha), D_{\bar{\imath}}(\bar{\alpha}), L_{ij}(h)$, and $L_{\bar{\imath}\bar{\jmath}}(\bar{h})$ are known as *elementary matrices.* We shall refer to the matrices $E_{ij}, D_i(\alpha)$, and $L_{ij}(h)$ as *elementary row matrices* and the matrices $E_{\bar{\imath}\bar{\jmath}}, D_{\bar{\imath}}(\bar{\alpha})$, and $L_{\bar{\imath}\bar{\jmath}}(\bar{h})$ as *elementary column matrices.*

By multiplying the Levi matrix A on the left by row elementary matrices, we obtain the row operations given in definition 5.1.1, and by multiplying on right of the Levi matrix A by column elementary matrices, we obtain the column operations given in definition 5.1.1.

For the subsequent lemmas, we shall assume the following:

Let $\mathcal{M} \subset \mathbb{C}^{n+1}$ be the boundary of a sum of squares domain defined by $\{r < 0\}$, where

$$r = 2\mathrm{Re}(z_{n+1}) + \sum_{j=1}^{N} |f_j(z_1, \ldots, z_n)|^2,$$

and $f_1, \ldots, f_N$ are holomorphic functions in the neighborhood of the origin. Let

$$r_0 = 2\mathrm{Re}(z_{n+1}) + \mathrm{P}(z, \bar{z})$$

be the defining function of the model hypersurface $\mathcal{M}_0$ of $\mathcal{M}$, where $P(z, \bar{z})$ is a polynomial of weighted degree 1 with respect to the multitype weight at the origin Λ^* of $\mathcal{M}$. Let A be the $n \times n$ Levi matrix of the model $\mathcal{M}_0 \subset \mathbb{C}^{n+1}$, where we ignore the contribution of the $(n+1)^{st}$ coordinate as the holomorphic functions in the sum of squares do not depend on it.

Lemma 5.1.1. *Assume that the D'Angelo 1-type of the hypersurface $\mathcal{M}$ at 0 is finite. Let $i \in \{1, \ldots, n\}$ be fixed, and let $h \in \mathbb{C}[z]$ for $z = (z_1, \ldots, z_n)$ be a nonzero monomial independent of z_i. Let h_ℓ denote the derivative of h with respect to the variable z_ℓ, which is $\partial_{z_\ell} h$ with $l \in \{1, \ldots, n\} \setminus \{i\}$. Furthermore, let $h_\ell(\tau)$ denote h_ℓ with every factor of z_l replaced by a factor of τ. Performing the elementary row and column operations $R_\ell - h_\ell R_i \to R_\ell$ and $C_\ell - \bar{h}_\ell C_i \to C_\ell$ on the Levi matrix A of r_0 for all variables z_ℓ in h corresponds to the polynomial transformation*

$$\tilde{z}_i = z_i + \int_0^{z_\ell} h_\ell(\tau)\, d\tau = z_i + h; \qquad \tilde{z}_\omega = z_\omega \text{ for } \omega \neq i$$

in the sense that the new matrix $\tilde{A}$ obtained after these elementary operations is Hermitian and is the Levi matrix of the new defining function of the sum of squares domain after the change of variables $z_\omega \to \tilde{z}_\omega$ for $\omega = 1, \ldots, n+1$ has taken place.

Remark 5.1.1. *The reader should note that while only variables $z_1, \ldots, z_n$ play a role in the behavior of the Levi matrix, $\mathbb{C}^{n+1}$ is the underlying space, so all changes of variables described in this chapter will take place in $\mathbb{C}^{n+1}$ and leave z_{n+1} unchanged.*

Proof. Suppose that the defining function r_0 of the model hypersurface $\mathcal{M}_0$ is of the form

$$r_0 = 2\mathrm{Re}(z_{n+1}) + \sum_{t=1}^{N} |g_t|^2,$$

where $P(z, \bar{z}) = \sum_{t=1}^{N} |g_t|^2$ and $g_t = \sum_{l=1}^{b_t} m_{t,l}$ is a polynomial consisting of monomials $m_{t,l}$. From Lemma 4.1.3, it is clear that $P(z, \bar{z})$ cannot depend on the variable z_{n+1}. Let $m_{t,l} = C_{t,l} \prod_{\delta=1}^{n} z_\delta^{\alpha_\delta^{t,l}}$ with $C_{t,l} \in \mathbb{C}$. For each t and for $l_1, l_2 \in \{1, \ldots, b_t\}$, every monomial from the expansion of $|g_t|^2$ can be written as

$$m_{t,l_1} \overline{m}_{t,l_2} = C_{t,l_1} \overline{C}_{t,l_2} \prod_{\delta=1}^{n} z_\delta^{\alpha_\delta^{t,l_1}} \bar{z}_\delta^{\hat{\alpha}_\delta^{t,l_2}}.$$

By writing each term $m_{t,l_1} \overline{m}_{t,l_2}$ for all t in the new coordinates, we obtain $P(z, \bar{z})$ in the new coordinates. Hence it suffices to show that applying the specified elementary row and column operations to the Levi matrix of the monomial $m_{t,l_1} \overline{m}_{t,l_2}$ corresponds to the polynomial transformation $\tilde{z}_i = z_i + h$; $\tilde{z}_\omega = z_\omega$ for $\omega \neq i$.

Denote by D the (i, j, k, u) submatrix of the Levi matrix of the monomial $m_{t,l_1} \overline{m}_{t,l_2}$, and let $\mathrm{D} = (d_{e\bar{\kappa}})_{e,\kappa=i,j,k,u}$, where $d_{e\bar{\kappa}}$ is given by

$$d_{e\bar{\kappa}} = C_{t,l_1} \overline{C}_{t,l_2} \alpha_e^{t,l_1} \hat{\alpha}_\kappa^{t,l_2} z_e^{\alpha_e^{t,l_1}-1} \bar{z}_e^{\hat{\alpha}_e^{t,l_2}} z_\kappa^{\alpha_\kappa^{t,l_1}} \bar{z}_\kappa^{\hat{\alpha}_\kappa^{t,l_2}-1} \prod_{\substack{\delta=1 \\ \delta \neq e,\kappa}}^{n} z_\delta^{\alpha_\delta^{t,l_1}} \bar{z}_\delta^{\hat{\alpha}_\delta^{t,l_2}}.$$

Let $h = C z_{a_1}^{\beta_{a_1}}, \ldots, z_{a_s}^{\beta_{a_s}}$, where $C \in \mathbb{C}$, $\beta = (\beta_1, \ldots, \beta_n)$ is a multiindex, and $a_1, \ldots, a_s \in \{1, \ldots, n\} \setminus \{i\}$. Now, assume that $j, k \in \{a_1, \ldots, a_s\}$ with $j \neq k$ and $u \notin \{a_1, \ldots, a_s\}$. Perform the elementary operations $\mathrm{R}_\ell - h_\ell \mathrm{R}_i \to \mathrm{R}_\ell$ and $\mathrm{C}_\ell - \bar{h}_\ell \mathrm{C}_i \to \mathrm{C}_\ell$ for all variables

z_ℓ in h on D to get

$$\begin{pmatrix} d_{i\bar{i}} & d_{i\bar{j}} - \bar{h}_j d_{i\bar{i}} & d_{i\bar{k}} - \bar{h}_k d_{i\bar{i}} & d_{i\bar{u}} \\ d_{j\bar{i}} - h_j d_{i\bar{i}} & d_{j\bar{j}} - h_j d_{i\bar{j}} - \bar{h}_j d_{j\bar{i}} + |h_j|^2 d_{i\bar{i}} & d_{j\bar{k}} - h_j d_{i\bar{k}} - \bar{h}_k d_{j\bar{i}} + h_j \bar{h}_k d_{i\bar{i}} & d_{j\bar{u}} - h_j d_{i\bar{u}} \\ d_{k\bar{i}} - h_k d_{i\bar{i}} & d_{k\bar{j}} - \bar{h}_j d_{k\bar{i}} - h_k d_{i\bar{j}} + \bar{h}_j h_k d_{i\bar{i}} & d_{k\bar{k}} - h_k d_{i\bar{k}} - \bar{h}_k d_{k\bar{i}} + |h_k|^2 d_{i\bar{i}} & d_{k\bar{u}} - h_k d_{i\bar{u}} \\ d_{u\bar{i}} & d_{u\bar{j}} - \bar{h}_j d_{u\bar{i}} & d_{u\bar{k}} - \bar{h}_k d_{u\bar{i}} & d_{u\bar{u}} \end{pmatrix}.$$

The matrix above is the Levi matrix for the monomial in the new coordinates

$$\tilde{m}_{l_1}\overline{\tilde{m}}_{l_2} = C_{t,l_1}\overline{C}_{t,l_2}(\tilde{z}_i - h)^{\alpha_e^{t,l_1}}(\bar{\tilde{z}}_i - \bar{h})^{\hat{\alpha}_e^{t,l_2}} \prod_{\delta \neq i} z_\delta^{\alpha_\delta^{t,l_1}} \bar{z}_\delta^{\hat{\alpha}_\delta^{t,l_2}},$$

which is obtained after applying the polynomial transformation $\tilde{z}_i = z_i + h;\ \ \tilde{z}_\omega = z_\omega$ for $\omega \neq i$ to $m_{t,l_1}\overline{m}_{t,l_2}$.

We now prove that the matrix $\tilde{\mathrm{A}}$ is Hermitian. Since the matrix A is Hermitian, we will show that applying the operation $\mathrm{R}_\ell - h_\ell \mathrm{R}_i \to \mathrm{R}_\ell$ and $\mathrm{C}_\ell - \bar{h}_\ell \mathrm{C}_i \to \mathrm{C}_\ell$ to A gives a matrix that is Hermitian as well. Let $\mathrm{A} = (a_{k\bar{l}})_{1 \leq k,l \leq n}$ be the Levi matrix. Apart from row ℓ and column ℓ, there is no change to A, which is Hermitian. Let $a_{\ell\bar{k}}$ be an entry in row ℓ. Then the entry $a_{k\bar{\ell}}$ is in column ℓ and satisfies the property that $a_{\ell\bar{k}} = \bar{a}_{k\bar{\ell}}$. Performing the elementary operations $\mathrm{R}_\ell - h_\ell \mathrm{R}_i \to \mathrm{R}_\ell$ and $\mathrm{C}_\ell - \bar{h}_\ell \mathrm{C}_i \to \mathrm{C}_\ell$ on A gives a new matrix $\tilde{\mathrm{A}}$ with $a_{\ell\bar{k}} - h_\ell a_{i\bar{k}}$ in row ℓ and $a_{k\bar{\ell}} - \bar{h}_\ell a_{k\bar{i}}$ in column ℓ. Now

$$a_{\ell\bar{k}} - h_\ell a_{i\bar{k}} = \bar{a}_{k\bar{\ell}} - \bar{\bar{h}}_\ell \bar{a}_{k\bar{i}} = \overline{a_{k\bar{\ell}} - \bar{h}_\ell a_{k\bar{i}}}.$$

Thus, the new matrix $\tilde{\mathrm{A}}$ is Hermitian as well. $\square$

Lemma 5.1.2. *Assume that the D'Angelo 1-type of the hypersurface $\mathcal{M}$ at 0 is finite. Let i and j with $1 \leq i,j \leq n$ be given. Performing both elementary operations $R_i \leftrightarrow R_j$ and $C_i \leftrightarrow C_j$ on the Levi matrix A corresponds to the polynomial transformation*

$$\tilde{z}_i = \int_0^{z_j} d\tau = z_j;\ \ \tilde{z}_j = \int_0^{z_i} d\tau = z_i; \qquad \tilde{z}_\omega = z_\omega \text{ for } \omega \neq i,j$$

in the sense that the new matrix $\tilde{A}$ obtained after these elementary operations is Hermitian and is the Levi matrix of the new defining function of the sums of squares domain after the change of variables $z_\omega \to \tilde{z}_\omega$ for $\omega = 1,\ldots,n+1$ has taken place.

Furthermore, at step e of the Kolář algorithm for all e, the weighted degree with respect to Λ_e of each monomial in $P(z,\bar{z})$ remains unchanged under the above polynomial transformation, where Λ_e is the corresponding weight at this step of the algorithm.

Proof. Suppose that the defining function r_0 of the model hypersurface $\mathcal{M}_0$ is of the form

$$r_0 = 2\mathrm{Re}(z_{n+1}) + \sum_{t=1}^{N} |g_t|^2,$$

where $P(z,\bar{z}) = \sum_{t=1}^{N} |g_t|^2$ and $g_t = \sum_{l=1}^{b_t} m_{t,l}$ is a polynomial consisting of monomials $m_{t,l}$. As in the proof of Lemma 5.1.1, let $m_{t,l} = C_{t,l}\prod_{\delta=1}^{n} z_\delta^{\alpha_\delta^{t,l}}$. We also know from the proof of Lemma 5.1.1 that to obtain our desired result, it suffices to show that applying the operations $\mathrm{R}_i \leftrightarrow \mathrm{R}_j$ and $\mathrm{C}_i \leftrightarrow \mathrm{C}_j$ on the Levi matrix of the monomial $m_{t,l_1}\overline{m}_{t,l_2}$ corresponds to the polynomial transformation $\tilde{z}_i = z_j;\;\; \tilde{z}_j = z_i;\;\; \tilde{z}_\omega = z_\omega$ for $\omega \neq i, j$ and $l_1, l_2 \in \{1, 2, \dots, b_t\}$.

Let $\mathrm{A} = (a_{i\bar{j}})_{1\le i,j\le n}$, where $a_{i\bar{j}} = \sum_{t=1}^{N}(\partial_{z_i} g_t)(\partial_{\bar{z}_j}\bar{g}_t)$. Then for each t, the term of the form $(\partial_{z_i} m_{t,l_1})(\partial_{\bar{z}_j}\overline{m}_{t,l_2})$ of the entry $a_{i\bar{j}}$ for $l_1, l_2 \in \{1, 2, \dots, b_t\}$ can be written as

$$(\partial_{z_i} m_{t,l_1})(\partial_{\bar{z}_j}\overline{m}_{t,l_2}) = C_{t,l_1}\overline{C}_{t,l_2}\alpha_i^{t,l_1}\hat{\alpha}_j^{t,l_2} z_i^{\alpha_i^{t,l_1}-1}\bar{z}_i^{\hat{\alpha}_i^{t,l_2}} z_j^{\alpha_j^{t,l_1}}\bar{z}_j^{\hat{\alpha}_j^{t,l_2}-1}\prod_{\substack{\delta=1\\ \delta\neq i,j}}^{n} z_\delta^{\alpha_\delta^{t,l_1}}\bar{z}_\delta^{\hat{\alpha}_\delta^{t,l_2}}.$$

Now, let

$$B_I^t = C_{t,l_1}\overline{C}_{t,l_2}\alpha_i^{t,l_1}\hat{\alpha}_j^{t,l_2}\prod_{\substack{\delta=1\\ \delta\neq I}}^{n} z_\delta^{\alpha_\delta^{t,l_1}}\bar{z}_\delta^{\hat{\alpha}_\delta^{t,l_2}},$$

where $I = i, j$. We shall therefore consider the i, j, k submatrix of the Levi matrix of the term $m_{t,l_1}\overline{m}_{t,l_2}$ and ignore t when no confusion arises. Thus, the corresponding i, j, k submatrix has the following entries:

$$\begin{pmatrix}
z_i^{\alpha_i^{l_1}-1}\bar{z}_i^{\hat{\alpha}_i^{l_2}-1}B_i & z_i^{\alpha_i^{l_1}-1}\bar{z}_j^{\hat{\alpha}_j^{l_2}-1}z_j^{\alpha_j^{l_1}}\bar{z}_i^{\hat{\alpha}_i^{l_2}}B_{i,j} & z_i^{\alpha_i^{l_1}-1}\bar{z}_k^{\hat{\alpha}_k^{l_2}-1}z_k^{\alpha_k^{l_1}}\bar{z}_i^{\hat{\alpha}_i^{l_2}}B_{i,k} \\
\bar{z}_i^{\hat{\alpha}_i^{l_1}-1}z_j^{\alpha_j^{l_2}-1}\bar{z}_j^{\hat{\alpha}_j^{l_1}}z_i^{\alpha_i^{l_2}}\bar{B}_{i,j} & z_j^{\alpha_j^{l_1}-1}\bar{z}_j^{\hat{\alpha}_j^{l_2}-1}B_j & \bar{z}_k^{\hat{\alpha}_k^{l_1}-1}z_j^{\alpha_j^{l_2}-1}\bar{z}_j^{\hat{\alpha}_j^{l_1}}z_k^{\alpha_k^{l_2}}\bar{B}_{j,k} \\
\bar{z}_i^{\hat{\alpha}_i^{l_1}-1}z_k^{\alpha_k^{l_2}-1}\bar{z}_k^{\hat{\alpha}_k^{l_1}}z_i^{\alpha_i^{l_2}}\bar{B}_{i,k} & z_k^{\alpha_k^{l_1}-1}\bar{z}_j^{\hat{\alpha}_j^{l_2}-1}z_j^{\alpha_j^{l_1}}\bar{z}_k^{\hat{\alpha}_k^{l_2}}B_{j,k} & z_k^{\alpha_k^{l_1}-1}\bar{z}_k^{\hat{\alpha}_k^{l_2}-1}B_k
\end{pmatrix}.$$

Now, perform the elementary operations $\mathrm{R}_i \leftrightarrow \mathrm{R}_j$ and $\mathrm{C}_i \leftrightarrow \mathrm{C}_j$ on the matrix above to obtain the matrix

$$\begin{pmatrix}
z_j^{\alpha_j^{l_1}-1}\bar{z}_j^{\hat{\alpha}_j^{l_2}-1}B_j & \bar{z}_i^{\hat{\alpha}_i^{l_1}-1}z_j^{\alpha_j^{l_2}-1}\bar{z}_j^{\hat{\alpha}_j^{l_1}}z_i^{\alpha_i^{l_2}}\bar{B}_{i,j} & \bar{z}_k^{\hat{\alpha}_k^{l_1}-1}z_j^{\alpha_j^{l_2}-1}\bar{z}_j^{\hat{\alpha}_j^{l_1}}z_k^{\alpha_k^{l_2}}\bar{B}_{j,k} \\
z_i^{\alpha_i^{l_1}-1}\bar{z}_j^{\hat{\alpha}_j^{l_2}-1}z_j^{\alpha_j^{l_1}}\bar{z}_i^{\hat{\alpha}_i^{l_2}}B_{i,j} & z_i^{\alpha_i^{l_1}-1}\bar{z}_i^{\hat{\alpha}_i^{l_2}-1}B_i & z_i^{\alpha_i^{l_1}-1}\bar{z}_k^{\hat{\alpha}_k^{l_2}-1}z_k^{\alpha_k^{l_1}}\bar{z}_i^{\hat{\alpha}_i^{l_2}}B_{i,k} \\
z_k^{\alpha_k^{l_1}-1}\bar{z}_j^{\hat{\alpha}_j^{l_2}-1}z_j^{\alpha_j^{l_1}}\bar{z}_k^{\hat{\alpha}_k^{l_2}}B_{j,k} & \bar{z}_i^{\hat{\alpha}_i^{l_1}-1}z_k^{\alpha_k^{l_2}-1}\bar{z}_k^{\hat{\alpha}_k^{l_1}}z_i^{\alpha_i^{l_2}}\bar{B}_{i,k} & z_k^{\alpha_k^{l_1}-1}\bar{z}_k^{\hat{\alpha}_k^{l_2}-1}B_k
\end{pmatrix}.$$

Clearly, the matrix obtained after the elementary row and column operations is Hermitian as well. Also, the second matrix is the Levi matrix of the term

$$\tilde{m}_{l_1}\overline{\tilde{m}}_{l_2} = \tilde{z}_i^{\alpha_j^{l_1}}\bar{\tilde{z}}_i^{\hat{\alpha}_j^{l_2}}\tilde{z}_j^{\alpha_i^{l_1}}\bar{\tilde{z}}_j^{\hat{\alpha}_i^{l_2}}B_{i,j},$$

where $\tilde{z}_i = z_j;\;\; \tilde{z}_j = z_i;\;\; \tilde{z}_\omega = z_\omega$, for $\omega \neq i, j$. Now, by including the t, which was ignored in the second matrix, the resultant matrix then becomes the Levi matrix of the term

$$\tilde{m}_{t,l_1}\overline{\tilde{m}}_{t,l_2} = \tilde{z}_i^{\alpha_j^{t,l_1}}\bar{\tilde{z}}_i^{\hat{\alpha}_j^{t,l_2}}\tilde{z}_j^{\alpha_i^{t,l_1}}\bar{\tilde{z}}_j^{\hat{\alpha}_i^{t,l_2}}B_{i,j}^t.$$

We now give a proof of the second part of the lemma, which states: At step $e \geq 1$ of the Kolář algorithm, the weighted degree with respect to Λ_e of each monomial in $P(z, \bar{z})$ remains unchanged under the polynomial transformation $\tilde{z}_i = z_j;\ \tilde{z}_j = z_i;\ \tilde{z}_\omega = z_\omega$ for $\omega \neq i, j$, where Λ_e is the corresponding weight at this step of the algorithm.

Let $\Lambda_e = (\mu_1, \ldots, \mu_n)$ be the weight at step e of the Kolář algorithm, and let μ_s for $s = 1, \ldots, n$ be the weight corresponding to the variable z_s. We know from the proof of the first part of this lemma that each monomial from the expansion of $P(z, \bar{z})$ is given by

$$m_{t,l_1}\overline{m}_{t,l_2} = C_{t,l_1}\overline{C}_{t,l_2}\prod_{\delta=1}^{n} z_\delta^{\alpha_\delta^{t,l_1}}\bar{z}_\delta^{\hat{\alpha}_\delta^{t,l_2}},$$

for $t = 1, \ldots, N$ and $l_1, l_2 \in \{1, \ldots, b_t\}$. Hence we will show that the weighted degree of the monomial $m_{t,l_1}\overline{m}_{t,l_2}$ with respect to Λ_e remains unchanged under the specified polynomial transformation. Let $\beta^t_{l_2,l_1}$ be the weighted degree of the monomial $B^t_{i,j}$. Then the weighted degree of $m_{t,l_1}\overline{m}_{t,l_2}$ is given by

$$\beta^t_{l_2,l_1} + (\alpha_i^{t,l_1} + \hat{\alpha}_i^{t,l_2})\mu_i + (\alpha_j^{t,l_1} + \hat{\alpha}_j^{t,l_2})\mu_j. \tag{5.1}$$

Clearly, the monomial in the new coordinates

$$\tilde{m}_{t,l_1}\overline{\tilde{m}}_{t,l_2} = \tilde{z}_i^{\alpha_j^{t,l_1}}\bar{\tilde{z}}_i^{\hat{\alpha}_j^{t,l_2}}\tilde{z}_j^{\alpha_i^{t,l_1}}\bar{\tilde{z}}_j^{\hat{\alpha}_i^{t,l_2}}B^t_{i,j}$$

has a weighted degree equal to that given in (5.1) since the weights μ_i and μ_j correspond to the variables $z_i = \tilde{z}_j$ and $z_j = \tilde{z}_i$ respectively and μ_ω is the weight corresponding to the variable $\tilde{z}_\omega,\ \omega \neq i, j$. □

The next lemma gives us a more convenient way to perform the elementary row and column operations when there exists at least one diagonal entry that is nonzero. This lemma transforms any such diagonal entry into the number 1.

Lemma 5.1.3. *Assume that the D'Angelo 1-type of the hypersurface $\mathcal{M}$ at 0 is finite. Let i with $1 \leq i \leq n$ be given. Assume that the $(i, \bar{\imath})$ entry of the Levi matrix A is a real number $|\alpha|^2$, $\alpha \neq 0$. Performing both elementary operations $\frac{1}{\alpha}R_i \to R_i$ and $\frac{1}{\bar{\alpha}}C_i \to C_i$ on A corresponds to the polynomial transformation*

$$\tilde{z}_i = \int_0^{z_i} \alpha\ d\tau = \alpha z_i; \qquad \tilde{z}_k = z_k \text{ for } k \neq i$$

in the sense that the new matrix $\tilde{A}$ obtained after these elementary operations is Hermitian and is the Levi matrix of the new defining function of the sums of squares domain after the change of variables $z_k \to \tilde{z}_k$ for $k = 1, \ldots, n+1$ has taken place. As a result of this change of variables, the $(i, \bar{\imath})$ entry of $\tilde{A}$ equals 1.

Proof. Using the fact that we are working with the model $\mathcal{M}_0$, we conclude that the weight corresponding to the variable z_i is $1/2$ since the $(i, \bar{\imath})$ entry is a nonzero real number. The defining function r_0 of the model will therefore contain the sum of the form $\sum_{j=1}^{q} |\gamma_j z_i + f_j|^2$, where f_j is a polynomial not depending on the variable z_i, q is a positive integer, and γ_j is a nonzero complex number.

For $k \neq i$, $1 \leq k \leq n$, the entries $(i, \bar{\imath}),\ (i, \bar{k}),$ and $(k, \bar{\imath})$ of the Levi matrix A are

$$\sum_{j=1}^{q} |\gamma_j|^2,\quad \sum_{j=1}^{q} \gamma_j \frac{\partial}{\partial \bar{z}_k}\bar{f}_j,\quad \sum_{j=1}^{q} \bar{\gamma}_j \frac{\partial}{\partial z_k} f_j \tag{5.2}$$

respectively. Let the real number $\sum_{j=1}^{q} |\gamma_j|^2 = |\alpha|^2$.

Performing the operations $\frac{1}{\alpha}\mathrm{R}_i \to \mathrm{R}_i$ and $\frac{1}{\bar{\alpha}}\mathrm{C}_i \to \mathrm{C}_i$ gives a matrix $\tilde{\mathrm{A}}$ such that for $i \neq k$ the entries $(i,\bar{\imath}),\ (i,\bar{k}),$ and $(k,\bar{\imath})$ are

$$1, \quad \sum_{j=1}^{q} \frac{\gamma_j}{\alpha} \frac{\partial}{\partial \bar{z}_k} \bar{f}_j, \quad \sum_{j=1}^{q} \frac{\bar{\gamma}_j}{\alpha} \frac{\partial}{\partial z_k} f_j \tag{5.3}$$

respectively.

Now consider the sum contained in r_0 below:

$$\begin{aligned} \sum_{j=1}^{q} |\gamma_j z_i + f_j|^2 &= \sum_{j=1}^{q} |\gamma_j|^2 |z_i|^2 + \sum_{j=1}^{q} 2\mathrm{Re}(\gamma_j z_i \bar{f}_j) + \sum_{j=1}^{q} |f_j|^2 \\ &= |\alpha|^2 |z_i|^2 + \sum_{j=1}^{q} 2\mathrm{Re}(\gamma_j z_i \bar{f}_j) + \sum_{j=1}^{q} |f_j|^2. \end{aligned} \tag{5.4}$$

Substituting the polynomial transformation $\tilde{z}_i = \alpha z_i$ into the sum in (5.4) gives the equation

$$|\tilde{z}_i|^2 + \sum_{j=1}^{q} 2\mathrm{Re}\Big(\tilde{z}_i\Big(\frac{\gamma_j}{\alpha}\bar{f}_j\Big)\Big) + \sum_{j=1}^{q} |f_j|^2 = \sum_{j=1}^{q} \Big|\frac{\gamma_j \tilde{z}_i}{\alpha} + f_j\Big|^2. \tag{5.5}$$

Thus, the new defining function after the change of variables $z_k \to \tilde{z}_k$ for $k = 1, \dots, n+1$ has taken place contains the sum in (5.5). When the Levi form of the new defining function is computed, its entries $(i,\bar{\imath}),\ (i,\bar{k}),$ and $(k,\bar{\imath})$ are precisely those in (5.3).

To prove that $\tilde{A}$ is Hermitian, let $A = (a_{i\bar{j}})_{1 \le i,j \le n}$, and let the i-th row and i-th column of the matrix A be of the form $(a_{i\bar{1}} \quad a_{i\bar{2}} \quad \cdots \quad a_{i\bar{n}})$ and $(a_{1\bar{i}} \quad a_{2\bar{i}} \quad \cdots \quad a_{n\bar{i}})^T$ respectively. Applying both elementary operations $\alpha\mathrm{R}_i \to \mathrm{R}_i$ and $\bar{\alpha}\mathrm{C}_i \to \mathrm{C}_i$ to A gives the matrix $\tilde{\mathrm{A}}$ whose i-th row and i-th column are of the form $(\alpha a_{i\bar{1}} \quad \cdots \quad \alpha a_{i\bar{n}})$ and $(\bar{\alpha} a_{1\bar{i}} \quad \cdots \quad \bar{\alpha} a_{n\bar{i}})^T$ respectively. Clearly, $\tilde{\mathrm{A}}$ is Hermitian since $\alpha a_{i\bar{j}} = \overline{\bar{\alpha} a_{j\bar{i}}}$ for $1 \le j \le n$. □

Example 4. Let the defining function of a sum of squares domain $\mathcal{M} \subset \mathbb{C}^5$ be given by

$$r = 2\mathrm{Re}(z_5) + |z_1|^2 + |z_2|^2 + |2z_1 + z_2 + z_3^3 z_4|^2 + |z_3^5|^2 + |z_4^5|^2.$$

The Levi matrix of the defining function r is given by

$$\mathrm{A} = \begin{pmatrix} 5 & 2 & 6\bar{z}_3^2\bar{z}_4 & 2\bar{z}_3^3 \\ 2 & 2 & 3\bar{z}_3^2\bar{z}_4 & \bar{z}_3^3 \\ 6z_3^2 z_4 & 3z_3^2 z_4 & 9|z_3^2 z_4|^2 + 25|z_3^4|^2 & 3z_3^2 z_4 \bar{z}_3^3 \\ 2z_3^3 & z_3^3 & 3\bar{z}_3^2\bar{z}_4 z_3^3 & |z_3^3|^2 + 25|z_4^4|^2 \end{pmatrix}.$$

Apply lemma 5.1.3 by performing the elementary operations $\frac{1}{\sqrt{5}}\mathrm{R}_1 \to \mathrm{R}_1$ and $\frac{1}{\sqrt{5}}\mathrm{C}_1 \to \mathrm{C}_1$ on the matrix above to get

$$\xrightarrow[\frac{1}{\sqrt{5}}\mathrm{C}_1\to\mathrm{C}_1]{\frac{1}{\sqrt{5}}\mathrm{R}_1\to\mathrm{R}_1}\begin{pmatrix} 1 & \frac{2}{\sqrt{5}} & \frac{6}{\sqrt{5}}\bar{z}_3^2\bar{z}_4 & \frac{2}{\sqrt{5}}\bar{z}_3^3 \\ \frac{2}{\sqrt{5}} & 2 & 3\bar{z}_3^2\bar{z}_4 & \bar{z}_3^3 \\ \frac{6}{\sqrt{5}}z_3^2z_4 & 3z_3^2z_4 & 9|z_3^2z_4|^2+25|z_3^4|^2 & 3z_3^2z_4\bar{z}_3^3 \\ \frac{2}{\sqrt{5}}z_3^3 & z_3^3 & 3\bar{z}_3^2\bar{z}_4z_3^3 & |z_3^3|^2+25|z_4^4|^2 \end{pmatrix} = \tilde{\mathrm{A}}.$$

By lemma 5.1.3, this corresponds to the polynomial transformation $\tilde{z}_1 = \sqrt{5}z_1;\ \tilde{z}_k = z_k$ for $k = 2,3,4,5$. Let's denote this transformation by $\tilde{\mathcal{S}}$. Thus, substituting the change of variable $\tilde{z}_1 = \sqrt{5}z_1;\ \tilde{z}_k = z_k$ for $k = 2,3,4,5$ into r yields $\tilde{r}$ given by

$$\begin{aligned}\tilde{r} &= 2\mathrm{Re}(\tilde{z}_5) + \left|\frac{1}{\sqrt{5}}\tilde{z}_1\right|^2 + |\tilde{z}_2|^2 + \left|\frac{2}{\sqrt{5}}\tilde{z}_1 + \tilde{z}_2 + \tilde{z}_3^3\tilde{z}_4\right|^2 + |\tilde{z}_3^5|^2 + |\tilde{z}_4^5|^2 \\ &= 2\mathrm{Re}(\tilde{z}_5) + \left|\tilde{z}_1 + \frac{2}{\sqrt{5}}\tilde{z}_2 + \frac{2}{\sqrt{5}}\tilde{z}_3^3\tilde{z}_4\right|^2 + \left|\frac{1}{\sqrt{5}}\tilde{z}_2 + \frac{1}{\sqrt{5}}\tilde{z}_3^3\tilde{z}_4\right|^2 + |\tilde{z}_2|^2 + |\tilde{z}_3^5|^2 + |\tilde{z}_4^5|^2,\end{aligned} \tag{5.6}$$

where the second line is obtained by gathering terms in the first line of (5.6). Clearly, the Levi matrix of $\tilde{r}$ is the same as the matrix $\tilde{\mathrm{A}}$.

Lemma 5.1.4. *Assume that the D'Angelo 1-type of $\mathcal{M}$ at 0 is finite. For a given i, $1 \le i \le n$, assume that the entries in the $(i,\bar{\imath})$, $(i,\bar{\jmath})$ and $(j,\bar{\imath})$ positions of the Levi matrix A are $1, \bar{u} + \bar{g}_j$ and $u + g_j$ respectively, where u is a nonzero complex number, g is a polynomial of order at least 2 not depending on the variable z_i, and $g_j = \partial_{z_j} g$. Performing both elementary operations $R_j - uR_i \to R_j$ and $C_j - \bar{u}C_i \to C_j$ on A corresponds to the polynomial transformation*

$$\tilde{z}_i = z_i + \int_0^{z_j} u\ d\tau = z_i + uz_j; \qquad \tilde{z}_k = z_k \text{ for } k \ne i$$

in the sense that the new matrix $\tilde{A}$ obtained from A after both elementary operations $R_j - uR_i \to R_j$ and $C_j - \bar{u}C_i \to C_j$ have been performed is Hermitian and is the Levi matrix of the new defining function of the sums of squares domain after the change of variables $z_k \to \tilde{z}_k$ for $k = 1, \ldots, n+1$ has taken place. The entries $(i,\bar{\imath})$, $(i,\bar{\jmath})$ and $(j,\bar{\imath})$ of $\tilde{A}$ are $1, \bar{g}_j$ and g_j respectively.

Proof. By our assumptions regarding the form of the entries of the Levi matrix A, the defining function r_0 can be expressed as

$$r_0 = 2\mathrm{Re}(z_{n+1}) + |z_i|^2 + 2\mathrm{Re}(z_i\overline{uz_j}) + 2\mathrm{Re}(z_i\bar{g}) + \gamma,$$

where g and γ are polynomials not depending on the variable z_i. Therefore, for $j \ne k$ the i,j,k submatrix of the Levi matrix A has the following entries:

$$\begin{pmatrix} 1 & \bar{u} + \bar{g}_j & \bar{g}_k \\ u + g_j & \gamma_{j\bar{j}} & \gamma_{j\bar{k}} \\ g_k & \bar{\gamma}_{k\bar{j}} & \gamma_{k\bar{k}} \end{pmatrix}$$

with the notation $\gamma_{e\bar{\ell}} = \partial_{z_e}\partial_{\bar{z}_\ell}\gamma$.

Performing the elementary operations $\mathrm{R}_j - u\mathrm{R}_i \to \mathrm{R}_j$ and $\mathrm{C}_j - \bar{u}\mathrm{C}_i \to \mathrm{C}_j$ on A gives a new matrix $\tilde{\mathrm{A}}$ whose i, j, k submatrix has entries as follows:

$$\begin{pmatrix} 1 & \bar{g}_j & \bar{g}_k \\ g_j & \gamma_{j\bar{j}} - |u|^2 - u\bar{g}_j - \bar{u}g_j & \gamma_{j\bar{k}} - u\bar{g}_k \\ g_k & \gamma_{k\bar{j}} - \bar{u}g_k & \gamma_{k\bar{k}} \end{pmatrix}$$

After the change of variables $\tilde{z}_i = z_i + uz_j$, $\tilde{z}_k = z_k$ for $k \neq i$, the defining function r_0 has the form

$$\tilde{r}_0 = 2\mathrm{Re}(\tilde{z}_{n+1}) + |\tilde{z}_i|^2 + 2\mathrm{Re}(\tilde{z}_i\bar{g}) - 2\mathrm{Re}(u\tilde{z}_j\bar{g}) + \gamma - |u|^2|\tilde{z}_j|^2.$$

The Levi matrix of $\tilde{r}_0$ has the same entries as that of the matrix $\tilde{\mathrm{A}}$.

The fact that $\tilde{\mathrm{A}}$ is Hermitian follows in the same manner as in the proof of Lemma 5.1.1 with h_ℓ replaced by u.

□

Remark 5.1.2. *We remark here that the row and column operations performed on the matrix A commute.*

Example 5. We continue where we left off in example 4. From equation (5.6), we write the defining function $\tilde{r}$ as

$$\tilde{r} = 2\mathrm{Re}(\tilde{z}_5) + \left|\tilde{z}_1 + \frac{2}{\sqrt{5}}\tilde{z}_2 + \frac{2}{\sqrt{5}}\tilde{z}_3^3\tilde{z}_4\right|^2 + \left|\frac{1}{\sqrt{5}}\tilde{z}_2 + \frac{1}{\sqrt{5}}\tilde{z}_3^3\tilde{z}_4\right|^2 + |\tilde{z}_2|^2 + |\tilde{z}_3^5|^2 + |\tilde{z}_4^5|^2$$

and its Levi matrix $\tilde{\mathrm{A}}$ as

$$\tilde{\mathrm{A}} = \begin{pmatrix} 1 & \frac{2}{\sqrt{5}} & \frac{6}{\sqrt{5}}\bar{z}_3^2\bar{z}_4 & \frac{2}{\sqrt{5}}\bar{z}_3^3 \\ \frac{2}{\sqrt{5}} & 2 & 3\bar{z}_3^2\bar{z}_4 & \bar{z}_3^3 \\ \frac{6}{\sqrt{5}}z_3^2z_4 & 3z_3^2z_4 & 9|z_3^2z_4|^2 + 25|z_3^4|^2 & 3z_3^2z_4\bar{z}_3^3 \\ \frac{2}{\sqrt{5}}z_3^3 & z_3^3 & 3\bar{z}_3^2\bar{z}_4z_3^3 & |z_3^3|^2 + 25|z_4^4|^2 \end{pmatrix},$$

ignoring the sign $\sim$ on the variables as no confusion arises. $\tilde{\mathrm{A}}$ satisfies the hypothesis of lemma 5.1.4 and so we perform the elementary operations $\mathrm{R}_2 - \frac{2}{\sqrt{5}}\mathrm{R}_1 \to \mathrm{R}_2$ to get

$$\xrightarrow{\mathrm{R}_2 - \frac{2}{\sqrt{5}}\mathrm{R}_1 \to \mathrm{R}_2} \begin{pmatrix} 1 & \frac{2}{\sqrt{5}} & \frac{6}{\sqrt{5}}\bar{z}_3^2\bar{z}_4 & \frac{2}{\sqrt{5}}\bar{z}_3^3 \\ 0 & \frac{6}{5} & \frac{3}{5}\bar{z}_3^2\bar{z}_4 & \frac{1}{5}\bar{z}_3^3 \\ \frac{6}{\sqrt{5}}z_3^2z_4 & 3z_3^2z_4 & 9|z_3^2z_4|^2 + 25|z_3^4|^2 & 3z_3^2z_4\bar{z}_3^3 \\ \frac{2}{\sqrt{5}}z_3^3 & z_3^3 & 3\bar{z}_3^2\bar{z}_4z_3^3 & |z_3^3|^2 + 25|z_4^4|^2 \end{pmatrix} \text{ and}$$

and $\mathrm{C}_2 - \frac{2}{\sqrt{5}}\mathrm{C}_1 \to \mathrm{C}_2$ on the matrix above to get

$$\xrightarrow{C_2-\frac{2}{\sqrt{5}}C_1\to C_2}\begin{pmatrix} 1 & 0 & \frac{6}{\sqrt{5}}\bar{z}_3^2\bar{z}_4 & \frac{2}{\sqrt{5}}\bar{z}_3^3 \\ 0 & \frac{6}{5} & \frac{3}{5}\bar{z}_3^2\bar{z}_4 & \frac{1}{5}\bar{z}_3^3 \\ \frac{6}{\sqrt{5}}z_3^2z_4 & \frac{3}{5}z_3^2z_4 & 9|z_3^2z_4|^2+25|z_3^4|^2 & 3z_3^2z_4\bar{z}_3^3 \\ \frac{2}{\sqrt{5}}z_3^3 & \frac{1}{5}z_3^3 & 3\bar{z}_3^2\bar{z}_4z_3^3 & |z_3^3|^2+25|z_4^4|^2 \end{pmatrix}=\hat{\mathrm{A}}.$$

By lemma 5.1.4, these elementary operations correspond to the polynomial transformation $\hat{z}_1=\tilde{z}_1+\frac{2}{\sqrt{5}}\tilde{z}_2$; $\hat{z}_k=\tilde{z}_k$ for $k=2,3,4,5$. Denote this polynomial transformation by $\mathcal{S}_2$.

Substituting the change of variables $\hat{z}_1=\tilde{z}_1+\frac{2}{\sqrt{5}}\tilde{z}_2$; $\hat{z}_k=\tilde{z}_k$ for $k=2,3,4,5$ into $\tilde{r}$ yields $\hat{r}$ given by

$$\begin{aligned}\hat{r}&=2\mathrm{Re}(\hat{z}_5)+\left|\hat{z}_1-\frac{2}{\sqrt{5}}\hat{z}_2+\frac{2}{\sqrt{5}}\hat{z}_2+\frac{2}{\sqrt{5}}\hat{z}_3^3\hat{z}_4\right|^2+\left|\frac{1}{\sqrt{5}}\hat{z}_2+\frac{1}{\sqrt{5}}\hat{z}_3^3\hat{z}_4\right|^2+|\hat{z}_2|^2+|\hat{z}_3^5|^2+|\hat{z}_4^5|^2\\ &=2\mathrm{Re}(\hat{z}_5)+\left|\hat{z}_1+\frac{2}{\sqrt{5}}\hat{z}_3^3\hat{z}_4\right|^2+\left|\frac{1}{\sqrt{5}}\hat{z}_2+\frac{1}{\sqrt{5}}\hat{z}_3^3\hat{z}_4\right|^2+|\hat{z}_2|^2+|\hat{z}_3^5|^2+|\hat{z}_4^5|^2\end{aligned}\tag{5.7}$$

and its Levi matrix is the same as $\hat{\mathrm{A}}$.

We shall now apply lemma 5.1.3. Thus, we perform the elementary operations $\frac{\sqrt{5}}{\sqrt{6}}\mathrm{R}_2\to\mathrm{R}_2$ and $\frac{\sqrt{5}}{\sqrt{6}}\mathrm{C}_2\to\mathrm{C}_2$ on $\hat{\mathrm{A}}$ to get $\check{\mathrm{A}}$, which is given by

$$\xrightarrow[\frac{\sqrt{5}}{\sqrt{6}}C_2\to C_2]{\frac{\sqrt{5}}{\sqrt{6}}R_2\to R_2}\begin{pmatrix} 1 & 0 & \frac{6}{\sqrt{5}}\bar{z}_3^2\bar{z}_4 & \frac{2}{\sqrt{5}}\bar{z}_3^3 \\ 0 & 1 & \frac{3}{\sqrt{30}}\bar{z}_3^2\bar{z}_4 & \frac{1}{\sqrt{30}}\bar{z}_3^3 \\ \frac{6}{\sqrt{5}}z_3^2z_4 & \frac{3}{\sqrt{30}}z_3^2z_4 & 9|z_3^2z_4|^2+25|z_3^4|^2 & 3z_3^2z_4\bar{z}_3^3 \\ \frac{2}{\sqrt{5}}z_3^3 & \frac{1}{\sqrt{30}}z_3^3 & 3\bar{z}_3^2\bar{z}_4z_3^3 & |z_3^3|^2+25|z_4^4|^2 \end{pmatrix}=\check{\mathrm{A}},$$

and we ignore the sign $\vee$ on the variables. By lemma 5.1.3, the corresponding polynomial transformation is $\check{z}_2=\frac{\sqrt{6}}{\sqrt{5}}\hat{z}_2$; $\check{z}_k=\hat{z}_k$ for $k=1,3,4,5$. Denote this polynomial transformation by $\mathcal{S}_3$. Substituting this transformation into $\hat{r}$ yields $\check{r}$ given by

$$\check{r}=2\mathrm{Re}(\check{z}_5)+\left|\check{z}_1+\frac{2}{\sqrt{5}}\check{z}_3^3\check{z}_4\right|^2+\left|\check{z}_2+\frac{1}{\sqrt{30}}\check{z}_3^3\check{z}_4\right|^2+\frac{1}{6}|\check{z}_3^3\check{z}_4|^2+|\check{z}_3^5|^2+|\check{z}_4^5|^2.$$

Before we state and prove lemma 5.1.5 below, we shall see how the Kolář algorithm directly relates to the concept of elementary operations discussed so far by considering the following: We apply the Kolář algorithm to the defining function given in example 4 by

$$r=2\mathrm{Re}(z_5)+|z_1|^2+|z_2|^2+|2z_1+z_2+z_3^3z_4|^2+|z_3^5|^2+|z_4^5|^2.$$

The Bloom-Graham type is 2 and the weight $\Lambda_1=(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Also,

$P_1=|2z_1+z_2|^2+|z_1|^2+|z_2|^2$ and $Q_1=2\mathrm{Re}(2z_1\bar{z}_3^3\bar{z}_4)+2\mathrm{Re}(z_2\bar{z}_3^3\bar{z}_4)+|z_3^3z_4|^2+|z_3^5|^2+|z_4^5|^2$.

Now we choose a Λ_1-homogeneous transformation that makes P_1 to be independent of the largest number of variables. Here we choose the composition of the all the transformations given in examples 4 and 5 above, which is

$$\mathcal{S}_3 \circ \mathcal{S}_2 \circ \mathcal{S}_1 : \quad \check{z}_1 = \sqrt{5} z_1 + \frac{2}{\sqrt{5}} z_2; \;\; \check{z}_2 = \frac{\sqrt{6}}{\sqrt{5}} z_2; \;\; \check{z}_k = z_k \text{ for } k = 3, 4, 5.$$

Applying this linear change of variables to r gives

$$P_1 = |\check{z}_1|^2 + |\check{z}_2|^2 \text{ and } Q_1 = 2\mathrm{Re}\left(\frac{2}{\sqrt{5}} z_1 \bar{\check{z}}_3^3 \bar{\check{z}}_4\right) + 2\mathrm{Re}\left(\frac{1}{\sqrt{30}} z_2 \bar{\check{z}}_3^3 \bar{\check{z}}_4\right) + |\check{z}_3^3 \check{z}_4|^2.$$

Computing W_1 for all monomials in Q_1 gives $\max W_1 = \frac{1}{8}$ and so $\Lambda_2 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8}\right)$. Thus,

$$P_2 = \left|\check{z}_1 + \frac{2}{\sqrt{5}} \check{z}_3^3 \check{z}_4\right|^2 + \left|\check{z}_2 + \frac{1}{\sqrt{30}} \check{z}_3^3 \check{z}_4\right|^2 + \frac{1}{6} |\check{z}_3^3 \check{z}_4|^2 \text{ and } Q_2 = |\check{z}_3^5|^2 + |\check{z}_4^5|^2. \tag{5.8}$$

Lemma 5.1.5. *Assume that the D'Angelo 1-type of $\mathcal{M}$ at 0 is finite. At step j of the Kolář algorithm applied to the defining function r_0 to compute the multitype at 0, let the leading polynomial P_j and leftover polynomial Q_j be of the form*

$$P_j = |z_i + m + g|^2 + \gamma \quad \text{and} \quad Q_j = \lambda$$

respectively, where m is a nonzero monomial of degree at least 2 independent of the variable z_i, g is a polynomial of degree at least 2 independent of the variable z_i, and γ as well as λ are polynomials independent of the variable z_i. Denote the derivative $\partial_{z_j} m \neq 0$ by m_j. For a given i, $1 \le i \le n$, the elementary operations $R_j - m_j R_i \to R_j$ and $C_j - \overline{m}_j C_i \to C_j$ performed on the Levi matrix A of r_0, for all variables z_j in m, correspond to the polynomial transformation

$$\tilde{z}_i = z_i + \int_0^{z_j} m_j(\tau)\, d\tau = z_i + m; \qquad \tilde{z}_k = z_k \text{ for } k \neq i,$$

for any j, where $m_j(\tau) = m_j(z_1, \ldots, z_{j-1}, \tau, z_{j+1}, \ldots, z_n)$ is a nonzero monomial.

Furthermore, the new matrix $\tilde{A}$ obtained from A after both elementary operations $R_j - m_j R_i \to R_j$ and $C_j - \overline{m}_j C_i \to C_j$ have been performed is also Hermitian and is the Levi matrix of the new defining function of the sums of squares domain after the change of variables $z_k \to \tilde{z}_k$ for $k = 1, \ldots, n+1$ has taken place.

Proof. Let $m = C z_{a_1}^{\alpha_{a_1}} \cdots z_{a_d}^{\alpha_{a_d}}$ be a nonzero monomial for some positive integer d, a nonzero complex constant C, and $a_j \in \{1, \ldots n\} \setminus \{i\}$. Also,

$$m_{a_j} = \frac{\partial m}{\partial z_{a_j}} = C \alpha_{a_j} z_{a_1}^{\alpha_{a_1}} \cdots z_{a_{j-1}}^{\alpha_{a_{j-1}}} z_{a_j}^{\alpha_{a_j}-1} z_{a_{j+1}}^{\alpha_{a_{j+1}}} \cdots z_{a_d}^{\alpha_{a_d}}.$$

By our assumptions regarding P_j and Q_j, we conclude that the defining function r_0 is of the form

$$r_0 = 2\mathrm{Re}(z_{n+1}) + |z_i + m + g|^2 + \gamma + \lambda.$$

The i, j, k submatrix of the Levi matrix A for $a_1 \le j, k \le a_d$ with $j \neq k$ is given by

$$\begin{pmatrix} 1 & \overline{m}_j + \bar{g}_j & \overline{m}_k + \bar{g}_k \\ m_j + g_j & |m_j + g_j|^2 + \gamma_{j\bar{j}} + \lambda_{j\bar{j}} & \bar{b} \\ m_k + g_k & b & |m_k + g_k|^2 + \gamma_{k\bar{k}} + \lambda_{k\bar{k}} \end{pmatrix}$$

with the notation $\gamma_{e\bar{\ell}} = \partial_{z_e}\partial_{\bar{z}_\ell}(\gamma)$, $\lambda_{e\bar{\ell}} = \partial_{z_e}\partial_{\bar{z}_\ell}(\lambda)$, $g_e = \partial_{z_e} g$, and $b = \overline{m}_j m_k + \overline{m}_j g_k + m_k \bar{g}_j + \bar{g}_j g_k + \gamma_{k\bar{j}} + \lambda_{k\bar{j}}$.

Perform both elementary operations $\mathrm{R}_\ell - m_\ell \mathrm{R}_i \to \mathrm{R}_\ell$ and $\mathrm{C}_\ell - \overline{m}_\ell \mathrm{C}_i \to \mathrm{C}_\ell$ on A for all $\ell \in \{a_1, a_2, \ldots, a_d\}$ including $\ell = j, k$. The new Levi matrix $\tilde{\mathrm{A}}$, for $a_1 \le j, k \le a_d$ with $j \neq k$, has the i, j, k submatrix given by

$$\begin{pmatrix} 1 & \bar{g}_j & \bar{g}_k \\ g_j & |g_j|^2 + \gamma_{j\bar{j}} + \lambda_{j\bar{j}} & g_j \bar{g}_k + \gamma_{j\bar{k}} + \lambda_{j\bar{k}} \\ g_k & \bar{g}_j g_k + \gamma_{k\bar{j}} + \lambda_{k\bar{j}} & |g_k|^2 + \gamma_{k\bar{k}} + \lambda_{k\bar{k}} \end{pmatrix}$$

Clearly, for any $j \in \{a_1, a_2, \ldots, a_d\}$, $\int_0^{z_j} m_j(\tau) d\tau = m$. Substituting the polynomial transformation $\tilde{z}_i = z_i + \int_0^{z_j} m_j(\tau)\, d\tau = z_i + m$; $\tilde{z}_\ell = z_\ell$ for $\ell \neq i$ into the defining function r_0 gives a new defining function $\tilde{r}_0$ of the form

$$\tilde{r}_0 = 2\mathrm{Re}(\tilde{z}_{n+1}) + |\tilde{z}_i + g| + \gamma + \lambda.$$

Also, the Levi matrix of the new defining function $\tilde{r}_0$ has the same entries as those of the matrix $\tilde{\mathrm{A}}$.

From the above analysis, we can deduce that if the polynomial g is nonzero, then we can apply this lemma a finite number of times to get the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma_{j\bar{j}} + \lambda_{j\bar{j}} & \gamma_{j\bar{k}} + \lambda_{j\bar{k}} \\ 0 & \gamma_{k\bar{j}} + \lambda_{k\bar{j}} & \gamma_{k\bar{k}} + \lambda_{k\bar{k}} \end{pmatrix}$$

and the new defining function

$$r_0^* = 2\mathrm{Re}(z_{n+1}^*) + |z_i^*| + \gamma + \lambda.$$

Again, the fact that $\tilde{\mathrm{A}}$ is Hermitian follows in the same manner as in the proof of Lemma 5.1.1 with h_ℓ replaced by m_ℓ. $\square$

Example 6. We continue where we left off in example 5. Let the leading polynomial P_1 and the leftover polynomial Q_1 obtained from the defining function r after applying the Kolář algorithm be given as in equation (5.8)

$$P_2 = |\check{z}_1 + \frac{2}{\sqrt{5}} \check{z}_3^3 \check{z}_4|^2 + |\check{z}_2 + \frac{1}{\sqrt{30}} \check{z}_3^3 \check{z}_4|^2 + \frac{1}{6} |\check{z}_3^3 \check{z}_4|^2 \ \text{ and } \ Q_2 = |\check{z}_3^5|^2 + |\check{z}_4^5|^2.$$

We shall apply lemma 5.1.5 since its hypotheses are satisfied. We thus perform the elementary operations $\mathrm{R}_3 - \frac{6}{\sqrt{5}} z_3^2 z_4 \mathrm{R}_1 \to \mathrm{R}_3$ and $\mathrm{C}_3 - \frac{6}{\sqrt{5}} \bar{z}_3^2 \bar{z}_4 \mathrm{C}_1 \to \mathrm{C}_3$ on $\check{\mathrm{A}}$ to get

$$\xrightarrow[\mathrm{C}_3 - \frac{6}{\sqrt{5}} \bar{z}_3^2 \bar{z}_4 \mathrm{C}_1 \to \mathrm{C}_3]{\mathrm{R}_3 - \frac{6}{\sqrt{5}} z_3^2 z_4 \mathrm{R}_1 \to \mathrm{R}_3} \begin{pmatrix} 1 & 0 & 0 & \frac{2}{\sqrt{5}} \bar{z}_3^3 \\ 0 & 1 & \frac{3}{\sqrt{30}} \bar{z}_3^2 \bar{z}_4 & \frac{1}{\sqrt{30}} \bar{z}_3^3 \\ 0 & \frac{3}{\sqrt{30}} z_3^2 z_4 & \frac{9}{5} |z_3^2 z_4|^2 + 25 |z_3^4|^2 & \frac{3}{5} z_3^2 z_4 \bar{z}_3^3 \\ \frac{2}{\sqrt{5}} z_3^3 & \frac{1}{\sqrt{30}} z_3^3 & \frac{3}{5} \bar{z}_3^2 \bar{z}_4 z_3^3 & |z_3^3|^2 + 25 |z_4^4|^2 \end{pmatrix} \quad \text{and}$$

perform the elementary operations $\mathrm{R}_4 - \frac{2}{\sqrt{5}} z_3^3 \mathrm{R}_1 \to \mathrm{R}_4$ and $\mathrm{C}_4 - \frac{2}{\sqrt{5}} \bar{z}_3^3 \mathrm{C}_1 \to \mathrm{C}_4$ on the matrix above to get

$$\xrightarrow[\mathrm{C}_4 - \frac{2}{\sqrt{5}} \bar{z}_3^3 \mathrm{C}_1 \to \mathrm{C}_4]{\mathrm{R}_4 - \frac{2}{\sqrt{5}} z_3^3 \mathrm{R}_1 \to \mathrm{R}_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{3}{\sqrt{30}} \bar{z}_3^2 \bar{z}_4 & \frac{1}{\sqrt{30}} \bar{z}_3^3 \\ 0 & \frac{3}{\sqrt{30}} z_3^2 z_4 & \frac{9}{5}|z_3^2 z_4|^2 + 25|z_3^4|^2 & \frac{3}{5} z_3^2 z_4 \bar{z}_3^3 \\ 0 & \frac{1}{\sqrt{30}} z_3^3 & \frac{3}{5} \bar{z}_3^2 \bar{z}_4 z_3^3 & \frac{1}{5}|z_3^3|^2 + 25|z_4^4|^2 \end{pmatrix} = \dot{\mathrm{A}}.$$

Both sets of elementary row operations correspond to the single polynomial transformation $\dot{z}_1 = \check{z}_1 + \frac{2}{\sqrt{5}} \check{z}_3^3 \check{z}_4$; $\dot{z}_k = \check{z}_k$ for $k = 2, 3, 4, 5$ by lemma 5.1.5. Substituting this change of variables into $\check{r}$ yields $\dot{r}$ given by

$$\dot{r} = 2\mathrm{Re}(\dot{z}_5) + |\dot{z}_1|^2 + \left|\dot{z}_2 + \frac{1}{\sqrt{30}} \dot{z}_3^3 \dot{z}_4\right|^2 + \frac{1}{6}|\dot{z}_3^3 \dot{z}_4|^2 + |\dot{z}_3^5|^2 + |\dot{z}_4^5|^2$$

whose Levi form is precisely the matrix $\dot{\mathrm{A}}$.

Decomposing the defining function $\dot{r}$ into $P_{\dot{r}}$ and $Q_{\dot{r}}$ shows that the hypothesis of lemma 5.1.5 is once again satisfied. Therefore, we continue by performing the elementary operations $\mathrm{R}_3 - \frac{3}{\sqrt{30}} z_3^2 z_4 \mathrm{R}_2 \to \mathrm{R}_3$ and $\mathrm{C}_3 - \frac{3}{\sqrt{30}} \bar{z}_3^2 \bar{z}_4 \mathrm{C}_2 \to \mathrm{C}_3$ on $\dot{\mathrm{A}}$ to get

$$\xrightarrow[\mathrm{C}_3 - \frac{3}{\sqrt{30}} \bar{z}_3^2 \bar{z}_4 \mathrm{C}_2 \to \mathrm{C}_3]{\mathrm{R}_3 - \frac{3}{\sqrt{30}} z_3^2 z_4 \mathrm{R}_2 \to \mathrm{R}_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{\sqrt{30}} \bar{z}_3^3 \\ 0 & 0 & \frac{3}{2}|z_3^2 z_4|^2 + 25|z_3^4|^2 & \frac{1}{2} z_3^2 z_4 \bar{z}_3^3 \\ 0 & \frac{1}{\sqrt{30}} z_3^3 & \frac{1}{2} \bar{z}_3^2 \bar{z}_4 z_3^3 & \frac{1}{5}|z_3^3|^2 + 25|z_4^4|^2 \end{pmatrix} \text{ and}$$

performing the elementary operations $\mathrm{R}_4 - \frac{1}{\sqrt{30}} z_3^3 \mathrm{R}_2 \to \mathrm{R}_4$ and $\mathrm{C}_4 - \frac{1}{\sqrt{30}} \bar{z}_3^3 \mathrm{C}_2 \to \mathrm{C}_4$ on the matrix above to get

$$\xrightarrow[\mathrm{C}_4 - \frac{1}{\sqrt{30}} \bar{z}_3^3 \mathrm{C}_2 \to \mathrm{C}_4]{\mathrm{R}_4 - \frac{1}{\sqrt{30}} z_3^3 \mathrm{R}_2 \to \mathrm{R}_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{2}|z_3^2 z_4|^2 + 25|z_3^4|^2 & \frac{1}{2} z_3^2 z_4 \bar{z}_3^3 \\ 0 & 0 & \frac{1}{2} \bar{z}_3^2 \bar{z}_4 z_3^3 & \frac{1}{6}|z_3^3|^2 + 25|z_4^4|^2 \end{pmatrix} = \ddot{\mathrm{A}}.$$

By lemma 5.1.5, these elementary operations correspond to the polynomial transformation $\ddot{z}_2 = \dot{z}_2 + \frac{1}{\sqrt{30}} \dot{z}_3^3 \dot{z}_4$; $\ddot{z}_k = \dot{z}_k$ for $k = 1, 3, 4, 5$. Applying this transformation to the defining function $\dot{r}$ gives a new defining function $\ddot{r}$ as

$$\ddot{r} = 2\mathrm{Re}(\ddot{z}_5) + |\ddot{z}_1|^2 + |\ddot{z}_2|^2 + \frac{1}{6}|\ddot{z}_3^3 \ddot{z}_4|^2 + |\ddot{z}_3^5|^2 + |\ddot{z}_4^5|^2$$

whose Levi form is precisely the matrix $\ddot{\mathrm{A}}$.

Remark 5.1.3. *From examples 4, 5, and 6, we observe that the elementary operations not only give the polynomial transformations needed at every step of the Kolář algorithm to reduce the number of variables in the leading polynomial but also provide a normalization of the defining function at the end of the procedure. It is instructive to compare this normalization to the one we obtained in the previous chapter in Proposition 4.1.4.*

5.2 Dependency and Allowable Polynomial Transformations

At every step of the Kolář algorithm, we are required to choose some polynomial transformation that makes the leading polynomial to be independent of the largest number of variables. We will construct this polynomial transformation as a composition of other polynomial transformations. We will refer to each factor of the composition as an allowable polynomial transformation. We now give the following definitions:

Let P_j be the leading polynomial and let Q_j be the leftover polynomial at step j of the Kolář algorithm for the computation of the multitype at the origin. Let A be the Levi matrix corresponding to the sum $P_j + Q_j$, and let A_{P_j} and A_{Q_j} be the Levi matrix corresponding to P_j and Q_j respectively.

Definition 5.2.1. A polynomial transformation is said to be *allowable* on P_j if it makes P_j to be independent of at least one of the variables contained in it.

Definition 5.2.2. Let i be given, where $1 \leq i \leq n$. An allowable polynomial transformation on P_j with respect to a variable z_i is a polynomial change of variables that makes P_j to be independent of the variable z_i.

We can apply Lemmas 5.1.3, 5.1.4, and 5.1.5 to obtain allowable polynomial transformations, if they exist. From the hypotheses of these lemmas, we see that they can be applied only if the Bloom-Graham type of $\mathcal{M}$ at the origin is 2. Note that the Bloom-Graham type of a sum of squares domain at origin is always even. If the Bloom-Graham type of $\mathcal{M}$ at the origin is greater than or equal to 4, however, the situation is a bit more complicated. Hence we seek a stronger notion that will address the general case. We seek an answer to the following question: Given the Levi matrix corresponding to a leading polynomial at some step of the Kolář algorithm, when can we obtain an allowable polynomial transformation via the elementary row and column operations regardless of what the Bloom-Graham type is?

At this point, we will give a more restrictive definition for dependency, which turns out to be the necessary and sufficient condition for the existence of an allowable polynomial transformation in the Kolář algorithm applied to a sum of squares domain.

Definition 5.2.3. For a given k, denote by R_k and C_k the k-th row and the k-th column of the matrix A_{P_j} respectively. Let $\mathscr{R}$ be the set of all rows of the matrix A_{P_j}.

1. The set $\mathscr{R}$ is said to be *dependent* if at least one of the rows can be written as a polynomial combination of the other rows. We shall also call an element R_k of $\mathscr{R}$ *dependent* if it satisfies the condition:

$$R_k = \sum_{\substack{l=1 \\ l \neq k}}^{n} \alpha_l R_l, \tag{5.9}$$

where $\alpha_l \in \mathbb{C}[z]$, $\alpha_l \neq 0$ for at least one l, and R_l is the l-th row of A_{P_j}.

Remark 5.2.1. *Since the matrix A_{P_j} is Hermitian, a similar definition holds for the k-th column C_k of A_{P_j} if*

$$C_k = \sum_{\substack{l=1 \\ l \neq k}}^{n} \bar{\alpha}_l C_l,$$

where C_l is the l-th column of A_{P_j}.

2. The set $\mathscr{R}$ is said to be *independent* if none of the rows can be written as a polynomial combination of the other rows in the more restrictive sense of (5.9).

The proposition that follows provides a general condition for the existence of an allowable polynomial transformation via the elementary row and column operations performed on the Levi matrix of a leading polynomial at some step of Kolář's algorithm.

Proposition 5.2.1. *Assume that the D'Angelo 1-type of $\mathcal{M}$ at 0 is finite. At step j of the Kolář algorithm applied to the defining function r_0 to compute the multitype at 0, let P_j be the leading polynomial and let Q_j be the leftover polynomial. Let k be given, $1 \leq k \leq n$. There exists an allowable polynomial transformation on P_j with respect to the variable z_k if and only if the k-th row of A_{P_j} is dependent.*

Remark 5.2.2. *The proof of this proposition is constructive in the sense that we will show that the allowable polynomial transformation on P_j arises as a composition of polynomial transformations corresponding to elementary row and column operations on A_{P_j}.*

Proof. Let k be given, and denote by R_k and C_k the k-th row and k-th column of the matrix A_{P_j} respectively. Suppose that the k-th row of A_{P_j} is dependent. This implies that C_k must also be dependent since A_{P_j} is Hermitian. Hence we can write both R_k and C_k respectively as:

$$R_k = \sum_{\substack{l=1 \\ l \neq k}}^{n} \beta^l R_l \text{ and } C_k = \sum_{\substack{l=1 \\ l \neq k}}^{n} \bar{\beta}^l C_l, \tag{5.10}$$

where $\beta^l \in \mathbb{C}[z]$ for every l, $1 \leq l \leq n$. As proven in Proposition 4.1.2, the leading polynomial P_j is a sum of squares, and so we write $P_j = \sum_{s=1}^{m} |\phi^s|^2$, where each ϕ^s is a nonzero polynomial with vanishing order greater than or equal to 1. Denote by $a_{k\bar{t}}$ the entry in the $(k, \bar{t})$ position of the matrix A_{P_j}, for $t = 1, \ldots, n$. Therefore, for $k \neq l$ the entries in R_k and R_l are given by

$$a_{k\bar{t}} = \sum_{s=1}^{m} \phi^s_k \bar{\phi}^s_t \quad \text{and} \quad a_{l\bar{t}} = \sum_{s=1}^{m} \phi^s_l \bar{\phi}^s_t, \tag{5.11}$$

and the entries in C_k and C_l are given by

$$a_{t\bar{k}} = \sum_{s=1}^{m} \phi^s_t \bar{\phi}^s_k \quad \text{and} \quad a_{t\bar{l}} = \sum_{s=1}^{m} \phi^s_t \bar{\phi}^s_l \tag{5.12}$$

respectively, where $\phi^s_j = \dfrac{\partial \phi^s}{\partial z_j}$.

We will show that there exist some elementary row and column operations that make R_k to be identically zero and also make every monomial in A_{P_j} to be independent of the variable z_k.

To that end, we see from (5.10) that every pair $\mathrm{R}_k - \beta^l \mathrm{R}_l;\ \mathrm{C}_k - \bar{\beta}^l \mathrm{C}_l$ for which the polynomial β^l is nonzero requires corresponding elementary row and column operations in the exact form expressed in the pair. If $\beta^l = 0$ or $R_l \equiv 0$, then no elementary row or column operation is required. Therefore, assume that β^l is nonzero and that R_l is not identically zero. Let $\beta^l(\zeta_k)$ be β^l with each factor of z_k replaced by a factor of ζ_k. Then $\displaystyle\int_0^{z_k} \beta^l(\zeta_k)\ d\zeta_k$ must contain the variable z_k together with all the other variables in β^l. Recall that $\beta^l \in \mathbb{C}[z]$, so β^l does not depend on any variable $\bar{z}_\nu$. Hence we shall investigate all monomials in A_{P_j} containing the variable z_k.

We recall at this point that by applying the operator $\partial_{z_k}\partial_{\bar{z}_t}$ to P_j, we obtain in row R_k the derivatives of all monomials containing the variable z_k. Let $a_{k\bar{t}}$ be an entry in row R_k. Now, because R_k is dependent, every monomial u in $a_{k\bar{t}}$ arises as the product of a monomial p in β^l for some l with a monomial q in entry $a_{l\bar{t}}$. So $u = pq$, but since u comes from differentiation by $\partial_{z_k}\partial_{\bar{z}_t}$, P_j must contain a monomial $m = u z_k \bar{z}_t$. If $u \in \mathbb{C}$, then no entries in A_{P_j}, except for those in R_k, contain derivatives from m. If u has positive vanishing order, then u depends on at least one variable z_ν or $\bar{z}_\nu$ for some ν. Since P_j is real-valued, it contains both m and $\bar{m}$. Therefore, without loss of generality, we can assume u depends on z_ν; otherwise, we work with $\bar{u}$. Since u depends on z_ν, the entry $a_{\nu\bar{t}}$ in the ν-th row R_ν contains the monomial $\partial_{z_\nu}\partial_{\bar{z}_t} m \neq 0$, which has at least one factor of z_k. We seek to eliminate all such monomials containing variable z_k from the matrix A_{P_j}.

Set $\gamma^l = \displaystyle\int_0^{z_k} \beta^l(\zeta_k)\ d\zeta_k$, and let $\gamma^l = \sum_{b=1}^{e} m^{l,b}$, where $m^{l,b}$ is a nonzero monomial containing the variable z_k for all $b \geq 1$. We recall from Lemma 5.1.1 that for any nonzero monomial m in the leading polynomial, if we perform the pair of elementary operations $\mathrm{R}_\nu - \partial_{z_\nu} m \mathrm{R}_\ell \to \mathrm{R}_\nu$ and $\mathrm{C}_\nu - \partial_{\bar{z}_\nu} \bar{m} \mathrm{C}_\ell \to \mathrm{C}_\nu$ for all variables z_ν in m, then this pair corresponds to the polynomial transformation $\tilde{z}_\ell = z_\ell + \displaystyle\int_0^{z_\nu} (\partial_{z_\nu} m)(\tau)\ d\tau = z_\ell + m$; $\tilde{z}_\omega = z_\omega$ for $\omega \neq \ell$, where $(\partial_{z_\nu} m)(\tau)$ is $\partial_{z_\nu} m$ with each factor of z_ν replaced by a factor of τ.

Now, for each monomial $m^{l,b}$ in γ^l, $b = 1, \ldots, e$, we perform the elementary row and column operations $\mathrm{R}_\nu - \partial_{z_\nu} m^{l,b} \mathrm{R}_l \to \mathrm{R}_\nu$ and $\mathrm{C}_\nu - \partial_{\bar{z}_\nu} \bar{m}^{l,b} \mathrm{C}_l \to \mathrm{C}_\nu$ for every variable z_ν in $m^{l,b}$. The composition of all of these polynomial transformations $\mathcal{S}$ is given by $\tilde{z}_l = z_l + \gamma^l$ for every l such that $\beta^l \neq 0$ and $\tilde{z}_\omega = z_\omega$ for all $\omega \neq l$, where $1 \leq \omega \leq n+1$. Note that (5.10) implies that γ^l has the same weight as z_l in Λ_j because P_j only contains terms of weight 1 with respect to Λ_j, so $\mathcal{S}$ is Λ_j-homogeneous as needed.

After all the elementary row and column operations corresponding to the polynomial transformation $\mathcal{S}$ have taken place, the entries in R_k are

$$a'_{k\bar{t}} = \sum_{s=1}^{m} \phi^s_k \bar{\phi}^s_t - \sum_{l=1}^{n} \beta^l \Big(\sum_{s=1}^{m} \phi^s_l \bar{\phi}^s_t \Big) = a_{k\bar{t}} - \sum_{l=1}^{n} \beta^l a_{l\bar{t}} \equiv 0 \tag{5.13}$$

as a consequence of (5.10). A similar argument holds for the entries in C_k, which we denote by $a'_{t\bar{k}}$, namely (5.10) implies that $a'_{t\bar{k}} \equiv 0$. Therefore, all entries in the k-th row and column of the matrix A_{P_j} are identically zero after the change of variables $\mathcal{S}$ has been performed. Now, assume that the leading polynomial P_j still contains the variable z_k after the given change of variables has been performed on it and some cancellation occurs. Since the leading polynomial P_j is a sum of squares, in the expansion of P_j besides the cross terms, which could possibly cancel each other, we would have at least two squares of monomials containing z_k. From the above discussion, it is clear that by performing these elementary row and column operations on A_{P_j}, all monomials containing the variable z_k in any of its entries will have been eliminated including any contribution from those squares. Thus, P_j could not possibly have contained the variable z_k, so $\mathcal{S}$ is an allowable polynomial transformation with respect to the variable z_k.

Conversely, suppose that there exists an allowable polynomial transformation on P_j with respect to the variable z_k, and let $\mathcal{T}$ be this polynomial transformation, which we shall express as:

$$\tilde{z}_i = z_i + \gamma^i, \tag{5.14}$$

for $i = 1, \ldots, n+1$, where some of the γ^i may be zero. We note here that the transformation $\mathcal{T}$ is a Λ_j-homogeneous transformation, and so γ^i has the same weight with respect to Λ_j as z_i. Furthermore, we note that any Λ_j-homogeneous transformation can be written in this form.

We will prove that the k-th row R_k is dependent by showing that it satisfies the condition given in (5.9). Assume that the variable z_k is contained in γ^i for some $i \in \{1, \ldots, d\}$ with $d \leq n$. We know that each $\tilde{z}_i$ corresponds to the row and column operations

$$\mathrm{R}_k - \gamma^i_k \mathrm{R}_i \to \mathrm{R}_k \quad \text{and} \quad \mathrm{C}_k - \bar{\gamma}^i_k \mathrm{C}_i \to \mathrm{C}_k \tag{5.15}$$

respectively, for $i = 1, \ldots, d$, where $\gamma^i_k = \partial_{z_k}\gamma^i \neq 0$. Let $\tilde{P}_j$ be the leading polynomial P_j after the polynomial transformation $\mathcal{T}$ is applied to it. Since $\tilde{P}_j$ does not contain the variable z_k, the entries $\tilde{h}_{k\bar{t}}$ of R_k and $\tilde{h}_{t\bar{k}}$ of C_k of the matrix $A_{\tilde{P}_j}$ for all $t = 1, \ldots, n$ are zero entries.

Now, by simply reversing the signs involved in the elementary operations in (5.15), we can restore R_k and C_k to their previous forms before the transformation $\mathcal{T}$ was applied to P_j. Hence by performing the elementary row and column operations

$$\mathrm{R}_k + \gamma^i_k \mathrm{R}_i \to \mathrm{R}_k \quad \text{and} \quad \mathrm{C}_k + \bar{\gamma}^i_k \mathrm{C}_i \to \mathrm{C}_k$$

for all $i = 1, \ldots, d$ on the matrix $A_{\tilde{P}_j}$, the entries in R_k and C_k become

$$h_{k\bar{t}} = \sum_{i=1}^{d} \gamma^i_k h_{i\bar{t}} \quad \text{and} \quad h_{t\bar{k}} = \sum_{i=1}^{d} \bar{\gamma}^i_k h_{t\bar{i}}, \tag{5.16}$$

where $h_{i\bar{t}}$ and $h_{t\bar{i}}$ are the entries in the i-th row R_i and i-th column C_i respectively. Finally, we obtain that

$$h_{k\bar{t}} = \sum_{i=1}^{n} \gamma^i_k h_{i\bar{t}} \quad \text{and} \quad h_{t\bar{k}} = \sum_{i=1}^{n} \bar{\gamma}^i_k h_{t\bar{i}}, \tag{5.17}$$

where $\gamma^i_k = 0$ for all $i = d+1, \ldots, n$. Thus, both R_k and C_k are dependent. $\square$

It is important to note that for any k, if the diagonal $(k, \bar{k})$ entry of A_{P_j} is the only nonzero entry in its k-th row, then the k-th row cannot be dependent, where A_{P_j} is the Levi matrix of the leading polynomial P_j. This statement holds because of the Hermitian property of A_{P_j} and the fact that we are working with a sum of squares domain.

Lemma 5.2.2. *Let Γ be the set of $n \times n$ matrices with coefficients in the ring $\mathbb{C}[z, \bar{z}]$. Let $H \in \Gamma$ be Hermitian. For some given i and k, let B be the matrix obtained from H after the elementary row and column operations $R_k + \alpha R_i \to R_k$ and $C_k + \bar{\alpha} C_i \to C_k$, for some $\alpha \in \mathbb{C}[z]$, are performed on it. Then* $\det(B) = \det(H)$.

Proof. Let E be the matrix obtained from H by the elementary row operation $\mathrm{R}_k + \alpha\mathrm{R}_i \to \mathrm{R}_k$. Then B is the matrix obtained from E by the elementary column operation $\mathrm{C}_k + \bar{\alpha}\mathrm{C}_i \to \mathrm{C}_k$. It is obvious from the properties of the determinant that $\det(\mathrm{H}) = \det(\mathrm{E})$ and that $\det(\mathrm{E}) = \det(\mathrm{B})$.

□

Lemma 5.2.3. *Assume that the D'Angelo 1-type of $\mathcal{M}$ at 0 is finite. At step j of the Kolář algorithm applied to the defining function r_0 to compute the multitype at 0, let P_j be the leading polynomial and let Q_j be the leftover polynomial.*

If the determinant of A_{P_j} is nonzero, then P_j is independent of the largest number of variables, and no polynomial transformation needs to be performed on it before the next step in the Kolář algorithm.

Proof. We shall give a proof of the contrapositive of the statement of this lemma, which states that if there exists an allowable transformation and hence one of the rows of A_{P_j} is dependent by Proposition 5.2.1, then the determinant of A_{P_j} is zero. Suppose that $\mathscr{R} = \{R_1, \ldots, R_n\}$, the set of all rows of the matrix A_{P_j}, is dependent and that none of the rows is identically equal to zero. Thus for some k, we can write

$$R_k = \sum_{\substack{l=1 \\ l \neq k}}^{n} \alpha_l R_l,$$

where $\alpha_l \in \mathbb{C}[z]$ and R_l is the l-th row of A_{P_j}. From the proof of Proposition 5.2.1, we know that there must exist some elementary row and column operations that transform A_{P_j} into the matrix $\tilde{\mathrm{A}}_{P_j}$ whose k-th row and column have all zero entries. Since the matrix $\tilde{\mathrm{A}}_{P_j}$ has at least one row with all entries equal to zero, its determinant equals zero. From Lemma 5.2.2, we know that $\det(\tilde{\mathrm{A}}_{P_j}) = \det(\mathrm{A}_{P_j})$.

Thus, $\det(\mathrm{A}_{P_j}) = 0$, which is the result we need. □

Given the way the leading polynomial P_j and its Levi matrix A_{P_j} are constructed, it is possible that P_j could be independent of at least one of the variables. If that is the case, then the determinant of the Levi matrix corresponding to the leading polynomial P_j will always be zero since it will have at least one row that is identically zero. Therefore, we need a way to determine when a subset of all nonzero rows of A_{P_j} is independent. To address this situation, we shall consider the following:

Let $\mathrm{A}_{P_j} = (a_{i\bar{l}})_{1 \le i, l \le n}$ be the Levi matrix of the leading polynomial P_j at step j of the Kolář algorithm. Let m be the number of nonzero rows of the matrix A_{P_j}. Denote by $\mathrm{A}_{P_j|m}$ the principal submatrix obtained from A_{P_j} by removing all zero rows and columns to get precisely m rows and columns, for some $m \leq n$. Put differently, $\mathrm{A}_{P_j|m}$ is the submatrix consisting only of all nonzero rows and columns of A_{P_j}. If $m = n$, then none of the rows and columns are identically zero.

Also, via the elementary row and column operations, $\mathrm{A}_{P_j|m}$ can be transformed into a leading principal submatrix where the first m rows and columns are the ones that remain. In this case, $\mathrm{A}_{P_j|m} = (a_{i\bar{l}})_{1\leq i,l\leq m}$.

Let $\mathscr{R} = \{R_1, \ldots, R_n\}$ be the set of all rows of the matrix A_{P_j} and let $\mathscr{S} = \{R_{b_1}, \ldots, R_{b_m}\}$ be a subset of $\mathscr{R}$, for $b_e \in \{1, \ldots, n\}$, $e = 1, \ldots, m$ such that each element R_{b_e} is not identically zero. Then $\mathscr{S}$ is the set of all non zero rows of the submatrix $\mathrm{A}_{P_j|m}$.

We can now restate Proposition 5.2.1 and Lemma 5.2.3 as follows:

Proposition 5.2.4. *Assume that the D'Angelo 1-type of $\mathcal{M}$ at 0 is finite. At step j of the Kolář algorithm applied to the defining function r_0 to compute the multitype at 0, let P_j be the leading polynomial, and let Q_j be the leftover polynomial.*

There exists an allowable polynomial transformation on P_j with respect to the variable z_k via the elementary row and column operations if and only if the k-th row of $A_{P_j|m}$ is dependent.

Lemma 5.2.5. *Assume that the D'Angelo 1-type of $\mathcal{M}$ at 0 is finite. At step j of the Kolář algorithm applied to the defining function r_0 to compute the multitype at 0, let P_j be the leading polynomial, and let Q_j be the leftover polynomial.*

If the determinant of $A_{P_j|m}$ is nonzero, then P_j is independent of the largest number of variables, and no polynomial transformation needs to be performed on it before the next step in the Kolář algorithm.

The proofs of Proposition 5.2.4 and Lemma 5.2.5 are identical to the proofs given for Proposition 5.2.1 and Lemma 5.2.3 respectively since the latter do not depend on rows being identically equal to zero. We also note here that the converses of Lemmas 5.2.3 and 5.2.5 do not hold. The reason is that the notion of dependency given in (5.10) is more restrictive than the standard notion of dependency in linear algebra, so there might not exist a row that is dependent according to our definition, but the set of rows may satisfy the standard notion of dependency, in which case the determinant of the Levi matrix would be identically equal to zero.

Now, the natural question to ask at this point is this: Given a Levi matrix of a leading polynomial with zero determinant, how can we tell whether or not it has dependent rows? Also, if there exist dependent rows, how can we identify such rows in order to determine the allowable polynomial transformations corresponding to these dependent rows? The answers to these questions lie in the formulation of an algorithm, which we will describe in the next section.

5.3 Gradient Ideals and Jacobian Modules

From section 5.1 of this chapter, we know that a row (column) operation on the Levi matrix is performed by a multiplication on the left (right) of the Levi matrix by an elementary row (column) matrix. The Levi matrix of a sum of squares domain can always be decomposed as the product of the complex Jacobian matrix of the holomorphic functions that generate the domain and its conjugate transpose. Therefore, every row operation on the Levi matrix could be performed on the complex Jacobian matrix, while every column operation on the Levi matrix is performed on the conjugate transpose of the complex Jacobian matrix. Let A be an $n \times n$ Levi matrix of a domain given by the sum of squares of N holomorphic functions. Then elementary matrices are $n \times n$ matrices, while the complex Jacobian matrix and its conjugate transpose must be $n \times N$ and $N \times n$ matrices respectively.

In our study of elementary row and column operations performed on the Levi determinant of a sum of squares domain, one particular property of the Levi matrix of a square of one of the generators drew our attention: All entries of any given column (row) have the same anti-holomorphic (holomorphic) parts, and so a study of the relationship between these entries narrows down to a study of the relationship between their holomorphic (anti-holomorphic) parts. In other words, we expect that the study of the Levi matrix will be much easier if we transition from the sum of squares to the underlying ideal of holomorphic functions that generate the domain as we already saw was the case for the computation of the multitype.

We start with a couple of definitions that we specialize to complex polynomials since those are the objects that appear in the Kolář algorithm when it is applied to a sum of squares domain instead of the full holomorphic generators:

Definition 5.3.1. Let $h \in \mathbb{C}[z_1, \ldots, z_n]$ be polynomial in the variables $z_1, \ldots, z_n$ with coefficients in $\mathbb{C}$. As in [25], we define the *gradient ideal of* h as the ideal generated by the partial derivatives of h :

$$\mathcal{I}_{grad}(h) = \langle \nabla h \rangle = \left\langle \frac{\partial h}{\partial z_1}, \cdots, \frac{\partial h}{\partial z_n} \right\rangle. \tag{5.18}$$

Definition 5.3.2. Given the ideal $\langle f \rangle = \langle f_1, \ldots, f_N \rangle \subset \mathbb{C}[z_1, \ldots, z_n]$, we define the *Jacobian module of* f as

$$\mathfrak{J}_{\langle f \rangle} = \left[\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n} \right], \tag{5.19}$$

where each $\frac{\partial f}{\partial z_j}$ is a vector. $\mathfrak{J}_{\langle f \rangle}$ is a module over the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$.

To every Jacobian module $\mathfrak{J}_{\langle f \rangle}$, we associate the complex Jacobian matrix $\mathrm{J}(f)$ given by

$$\mathrm{J}(f) = \begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_2}{\partial z_1} & \cdots & \frac{\partial f_N}{\partial z_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial z_n} & \frac{\partial f_2}{\partial z_n} & \cdots & \frac{\partial f_N}{\partial z_n} \end{pmatrix}. \tag{5.20}$$

Likewise, to each gradient ideal $\mathcal{I}_{grad}(f_i) = \left\langle \frac{\partial f_i}{\partial z_1}, \cdots, \frac{\partial f_i}{\partial z_n} \right\rangle$ of the generator $f_i \in \mathbb{C}[z_1, \ldots, z_n]$ of $\langle f \rangle$, we associate the i-th column of $\mathrm{J}(f)$ for $1 \leq i \leq n$. The reader should note that row operations on $\mathrm{J}(f)$ are precisely operations on the module $\mathfrak{J}_{\langle f \rangle}$.

Now, let $\langle f \rangle = \langle f_1, \ldots, f_N \rangle \subset \mathbb{C}[z_1, \ldots, z_n]$ be the leading polynomial ideal at some step of the Kolář algorithm. Then we are particularly interested in simplifying the Jacobian module $\mathfrak{J}_{\langle f \rangle}$ such that it is generated by the minimal number of generators. Every generator that is eliminated is a linear combination of some partial derivatives of f with coefficients in $\mathbb{C}[z_1, \ldots, z_n]$. Since every generator of the Jacobian module represents a row of the complex Jacobian matrix, every eliminated generator represents a dependent row in the complex Jacobian matrix. Owing to this connection, from every eliminated generator, we can construct a sequence of elementary row operations that corresponds to the linear combination of some partial derivatives of f as described in Proposition 5.2.1. Hence we obtain polynomial transformations corresponding to these row operations.

It is easy to observe this relationship if $N = 1$. Then reducing the number of generators of $\mathfrak{J}_{\langle f \rangle}$, if possible, reduces the number of nonzero rows of the associated complex Jacobian matrix. Thus, the minimal number of generators required to generate the Jacobian module is precisely the number of independent rows of the complex Jacobian matrix, which is the same as the number of variables on which the corresponding leading polynomial ideal $\langle f \rangle$ is dependent by Proposition 5.2.1 after the corresponding change of variables. Hence the number d_j at step j of the Kolář algorithm is given by $d_j = n - \#f$, where $\#f$ is the minimal number of generators generating the Jacobian module $\mathfrak{J}_{\langle f \rangle}$, n is the number of variables in the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$, and d_j is the largest number of variables of which the leading polynomial at step j is independent. This gives an algebraic characterization of the number d_j in the Kolář algorithm.

In the more general case where $N > 1$, reducing the number of generators of the Jacobian module implies reducing the generators of all gradient ideals by the same operations. Thus, $\frac{\partial f}{\partial z_\ell}$, for some ℓ, is a generator that is eliminated in the Jacobian module $\mathfrak{J}_{\langle f \rangle}$ if and only if the generator $\frac{\partial f_i}{\partial z_\ell}$ of the gradient ideal $\mathcal{I}_{grad}(f_i)$ for all $i = 1, \ldots, N$ is eliminated, namely reduced to 0. Clearly, if there exists at least one gradient ideal $\mathcal{I}_{grad}(f_i)$ with minimal number of generators equal to n, then the Jacobian module $\mathfrak{J}_{\langle f \rangle}$ cannot have fewer than n generators, i.e. no reduction via row operations is possible.

5.3.1 Row Reduction Algorithm

We shall now devise an algorithm that constructs explicitly the polynomial transformations required at each step of the Kolář algorithm when applied to the complex Jacobian matrix of the leading polynomial ideal.

The algorithm gives the conditions for characterizing the required elementary row operations that correspond to the polynomial transformations needed in the Kolář algorithm. The application of the algorithm to the complex Jacobian matrix corresponding to a given leading polynomial ideal will eliminate all dependent rows, if they exist, from the complex Jacobian matrix.

Let $\mathcal{M}_0 \subset \mathbb{C}^{n+1}$ be the model of a sum of squares domain defined by $\{r_0 < 0\}$, where

$$r_0 = 2\mathrm{Re}(z_{n+1}) + \sum_{i=1}^{N} |f_i(z_1, \ldots, z_n)|^2,$$

and $f_1, \ldots, f_N$ are holomorphic polynomial functions in the neighborhood of the origin. Let A be the $n \times n$ Levi matrix of the model $\mathcal{M}_0$, where we ignore the contribution of the $(n+1)^{st}$ coordinate as the holomorphic polynomials in the sum of squares do not depend on it by Lemma 4.1.3. For any $j \geq 1$, let P_j be the leading polynomial and Q_j the leftover polynomial at step j of the Kolář algorithm, and let J_{P_j} be the corresponding complex Jacobian matrix.

GRADIENT IDEALS: Let $P_j = \sum_{i=1}^{N} |h_i|^2$. Then the leading polynomial ideal is given by

$\mathcal{I}_{P_j} := \langle h \rangle = \langle h_1, \ldots, h_N \rangle$, and the gradient ideal of h_i is $\mathcal{I}_{grad}(h_i) = \left\langle \frac{\partial h_i}{\partial z_1}, \cdots, \frac{\partial h_i}{\partial z_n} \right\rangle$, $i = 1, \ldots, N$. Note that the complex Jacobian matrix of P_j is given by $\mathrm{J}_{P_j} = \mathrm{J}(\mathcal{I}_{P_j}) = \mathrm{J}(h)$ and the Levi matrix A_{P_j} of P_j is the product of the complex Jacobian matrix of P_j with its conjugate transpose $\mathrm{J}^*(h)$: $\mathrm{A}_{P_j} = \mathrm{J}(h)\mathrm{J}^*(h)$. We shall reduce, if possible, the number of generators of each gradient ideal one at a time and control the changes that occur in other gradient ideals as a result of these reduction operations. By control, we mean setting appropriate conditions on the reduction operations used.

STRUCTURE OF THE ALGORITHM: If $\det \mathrm{A}_{P_j} = \det(\mathrm{J}(h)\mathrm{J}^*(h))$ is nonzero, then no change of variables is required by Lemma 5.2.3. Thus, assume that $\det(\mathrm{J}(h)\mathrm{J}^*(h)) = 0$. Then:

1. We begin the process by first considering the gradient ideal $\mathcal{I}_{grad}(h_i) = \left\langle \frac{\partial h_i}{\partial z_1}, \cdots, \frac{\partial h_i}{\partial z_n} \right\rangle$ for any i. Simplify the gradient ideal $\mathcal{I}_{grad}(h_i)$ such that it consists of the minimal number of generators, namely if a generator can be expressed as a linear combination of the other generators with coefficients in the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$ then it can be eliminated. Assume that at least one such generator can be eliminated, i.e.
$$\frac{\partial h_i}{\partial z_k} = \sum_{u=1}^{v} \gamma_{c_u} \frac{\partial h_i}{\partial z_{c_u}}, \tag{5.21}$$
for some k, $c_u \neq k$, $v < n$, and γ_{c_u} a nonzero polynomial in $\mathbb{C}[z_1, \ldots, z_n]$ for every u. Then perform the following elementary row operations on the complex Jacobian matrix $\mathrm{J}(h)$:
$$\mathrm{R}_\ell - \frac{\partial \zeta_{c_u}}{\partial z_\ell} \mathrm{R}_{c_u} \to \mathrm{R}_\ell, \tag{5.22}$$
for all $u = 1, \ldots, v$ and for all variables z_ℓ in $\zeta_{c_u} = \int_0^{z_k} \gamma_{c_u}(t)\, dt$. By Lemma 5.1.1, the row operations in (5.22) correspond to the polynomial transformation given by
$$\tilde{z}_{c_u} = z_{c_u} + \int_0^{z_k} \gamma_{c_u}(t)\, dt; \;\; \tilde{z}_\omega = z_\omega, \tag{5.23}$$
for all $\omega \neq c_u$ and for all $u = 1, \ldots, v$. The generator $\frac{\partial h_i}{\partial z_k}$ vanishes in $\mathcal{I}_{grad}(h_i)$ after the row operations in (5.22) are performed on $\mathrm{J}(h)$. In other words, after these changes of variables, h_i no longer depends on the variable z_k.

   We say row R_{c_u} is used as a *central row* in the sequence of row operations and the generator $\frac{\partial h_i}{\partial z_{c_u}}$ is used as a *central generator* in the simplification of the gradient ideal $\mathcal{I}_{grad}(h_i)$ for all i. We remark here that for all subsequent row operations performed on the complex Jacobian matrix, the row R_{c_u} cannot be used as a central row and $\frac{\partial h_e}{\partial z_{c_u}}$ cannot be used as a central generator in the simplification of any other gradient ideal $\mathcal{I}_{grad}(h_e)$ for $e \neq i$. This condition is imposed due to Proposition 5.2.1, which is an equivalence. Reusing a central row or a central generator might reintroduce a variable that has been eliminated from the leading polynomial.

2. Next, consider another gradient ideal $\mathcal{I}_{grad}(h_s) = \left\langle \frac{\partial h_s}{\partial z_1}, \cdots, \frac{\partial h_s}{\partial z_n} \right\rangle$ for $s \neq i$.

Clearly, the k-th generator of this gradient ideal is

$$\frac{\partial h_s}{\partial z_k} - \sum_{u=1}^{v} \gamma_{c_u} \frac{\partial h_s}{\partial z_{c_u}} \tag{5.24}$$

due to the row operations given in (5.22). We simplify the ideal $\mathcal{I}_{grad}(h_s)$ such that it has the minimal number of generators while ensuring that the generators $\frac{\partial h_s}{\partial z_{c_u}}$ for $u = 1, \ldots, v$ are not used as central generators in the simplification of $\mathcal{I}_{grad}(h_s)$. Perform the related row operations.

3. Proceed similarly by considering other gradient ideals different from the previous ones. Since there are only finitely many gradient ideals and finitely many generators that generate each of them, the process will terminate after a finite number of steps.

We will show in the lemma that follows that the polynomial transformation in (5.23) corresponding to the row operations given in (5.22) is Λ_j-homogeneous. Thus, we state the following:

Lemma 5.3.1. *Assume that the D'Angelo 1-type of $\mathcal{M}$ at 0 is finite. At step j of the Kolář algorithm for the computation of the multitype at 0, let Λ_j be the weight, P_j the leading polynomial, and $\mathcal{I}_{P_j}$ the corresponding leading polynomial ideal. Let $\mathcal{I}_{grad}(\psi)$ be the gradient ideal of some generator ψ of the ideal $\mathcal{I}_{P_j}$. Assume that*

$$\frac{\partial \psi}{\partial z_k} = \sum_{u=1}^{v} \gamma_{c_u} \frac{\partial \psi}{\partial z_{c_u}}, \tag{5.25}$$

for some k, where $k \neq c_u$, $v < n$, and γ_{c_u} is a nonzero polynomial in $\mathbb{C}[z_1, \ldots, z_n]$. Let $\zeta_{c_u} = \int_0^{z_k} \gamma_{c_u}(t)\, dt$.

Then the polynomial transformation given by $\tilde{z}_{c_u} = z_{c_u} + \zeta_{c_u}$; $\tilde{z}_\omega = z_\omega$ for all $\omega \neq c_u$ corresponding to the elementary row operations $R_\ell - \frac{\partial \zeta_{c_u}}{\partial z_\ell} R_{c_u} \to R_\ell$ for all variables z_ℓ in ζ_{c_u} and for all $u = 1, \ldots, v$ performed on the complex Jacobian matrix J_{P_j} is Λ_j-homogeneous for all $u = 1, \ldots, v$.

Proof. We start the proof by recalling from Proposition 4.1.2 that the leading polynomial is a sum of squares at every step of the Kolář algorithm. Hence P_j is a sum of squares. Let $\Lambda_j = (\lambda_1, \ldots, \lambda_n)$. Since variables are not ordered in increasing weight order, we let $\phi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be the bijection $\phi = (\phi_1, \ldots, \phi_n)$ such that the variable z_i, $1 \leq i \leq n$, has weight λ_{ϕ_i}.

We will show that the term γ_{c_u} in the polynomial transformation is of weighted degree $(\lambda_{\phi_{c_u}} - \lambda_{\phi_k})$. Let ν be the weighted degree of γ_{c_u} with respect to the weight Λ_j. By our hypothesis, the weighted degrees of $\frac{\partial \psi}{\partial z_{c_u}}$ and $\frac{\partial \psi}{\partial z_k}$ are $\frac{1}{2} - \lambda_{\phi_{c_u}}$ and $\frac{1}{2} - \lambda_{\phi_k}$ respectively since all generators of the leading polynomial ideal $\mathcal{I}_{P_j}$ are of weighted degree $\frac{1}{2}$ with respect to Λ_j. The weighted degree of the right hand side of the expression given in (5.25) is $\nu + \frac{1}{2} - \lambda_{\phi_{c_u}}$. Hence solving the equation in (5.25) for ν gives $\nu = \lambda_{\phi_{c_u}} - \lambda_{\phi_k}$. Thus, the weighted degree of $\zeta_{c_u} = \int_0^{z_k} \gamma_{c_u}(t)\, dt$ is $\lambda_{\phi_{c_u}}$ as required. □

Remark 5.3.1. *The polynomial γ_{c_u} cannot depend on the variable z_{c_u} because if it were to depend on z_{c_u}, then its weighted degree would satisfy $\nu \geq \lambda_{\phi_{c_u}}$, but $\nu = \lambda_{\phi_{c_u}} - \lambda_{\phi_k}$, which gives a contradiction because $\lambda_{\phi_k} > 0$.*

Lemma 5.3.2. *Assume that the D'Angelo 1-type of $\mathcal{M}$ at 0 is finite. At step j of the Kolář algorithm for the computation of the multitype at* 0, *let P_j be the leading polynomial, and let $\mathcal{I}_{P_j} = \langle h \rangle \subset \mathbb{C}[z_1, \dots, z_n]$ be the corresponding leading polynomial ideal.*

If the Row Reduction algorithm is applied to the complex Jacobian matrix $J(h)$, then every dependent row of $J(h)$ vanishes. In other words, the leading polynomial ideal $\mathcal{I}_{P_j}$ is independent of the largest number of variables after the Row Reduction algorithm is applied to $J(h)$.

Proof. From Proposition 4.1.2 the leading polynomial P_j is a sum of squares, and so let $P_j = \sum_{i=1}^{N} |h_i|^2$. Then the leading polynomial ideal $\mathcal{I}_{\mathrm{P}_j}$ is $\langle h \rangle = \langle h_1, \dots, h_N \rangle$, and let the gradient ideal of each generator h_i be $\mathcal{I}_{grad}(h_i) = \left\langle \frac{\partial h_i}{\partial z_1}, \cdots, \frac{\partial h_i}{\partial z_n} \right\rangle$.

Now, assume that R_k, the k-th row of $\mathrm{J}(h)$ is dependent. We will show that the generator $\frac{\partial h}{\partial z_k}$ of the Jacobian module given by $\mathfrak{J}_{\langle h \rangle} = \left[\frac{\partial h}{\partial z_1}, \cdots, \frac{\partial h}{\partial z_n} \right]$ vanishes after applying the Row Reduction algorithm on the complex Jacobian matrix $\mathrm{J}(h)$. Since R_k is dependent, we can write the generator $\frac{\partial h}{\partial z_k}$ as

$$\frac{\partial h}{\partial z_k} = \sum_{u=1}^{v} \gamma_{c_u} \frac{\partial h}{\partial z_{c_u}}, \tag{5.26}$$

for some $k,\ c_u \neq k,\ v < n$, and γ_{c_u} a nonzero polynomial in $\mathbb{C}[z_1, \dots, z_n]$ for every u. Hence every generator $\frac{\partial h_i}{\partial z_k}$ of the gradient ideal $\mathcal{I}_{grad}(h_i)$ can be written as

$$\frac{\partial h_i}{\partial z_k} = \sum_{u=1}^{v} \gamma_{c_u} \frac{\partial h_i}{\partial z_{c_u}}, \tag{5.27}$$

for $i = 1, \dots, N$ and the same polynomial coefficients γ_{c_u}. Thus, it suffices to show that $\frac{\partial h_i}{\partial z_k}$ vanishes at the termination of the algorithm for every $i = 1, \dots, N$. Consider the ideal $\mathcal{I}_{grad}(h_i)$ for some $i \in \{1, \dots, N\}$. If the generator $\frac{\partial h_i}{\partial z_k}$ is zero, then there is nothing to be done, and so we move to a different gradient ideal. Hence suppose that $\frac{\partial h_i}{\partial z_k}$ is nonzero. Then at least one of the generators $\frac{\partial h_i}{\partial z_{c_u}}$ is nonzero for some u. Suppose that $\frac{\partial h_i}{\partial z_{c_u}} \neq 0$ for all $u \in \{1, \dots, w\}$ for $w \leq v$. Since it satisfies the condition in (5.27), we perform the elementary row operations $\mathrm{R}_\ell - \frac{\partial \zeta_{c_u}}{\partial z_\ell} \mathrm{R}_{c_u} \to \mathrm{R}_\ell$, for all variables z_ℓ in $\zeta_{c_u} = \int_0^{z_k} \gamma_{c_u}(t)\ dt$ and for all $u = 1, \dots, w$. This eliminates the term $\sum_{u=1}^{w} \gamma_{c_u} \frac{\partial h_e}{\partial z_{c_u}}$, from the ideal $\mathcal{I}_{grad}(h_i)$. The generator $\frac{\partial h}{\partial z_k}$ of the Jacobian module becomes

$$\frac{\partial h}{\partial z_k} = \sum_{u=w+1}^{v} \gamma_{c_u} \frac{\partial h}{\partial z_{c_u}} \tag{5.28}$$

after the row operations have been performed on $\mathrm{J}(h)$. Note here that the generator $\frac{\partial h_e}{\partial z_{c_u}}$, for all $e \neq i$ and $u = 1, \dots, w$ cannot be central in any simplification process in the gradient ideal $\mathcal{I}_{grad}(h_e)$ after the row operations.

Next, consider another gradient ideal $\mathcal{I}_{grad}(h_e)$ for $e \neq i$. Then its k-th generator after the reduction operation is

$$\frac{\partial h_e}{\partial z_k} - \sum_{u=1}^{w} \gamma_{c_u} \frac{\partial h_e}{\partial z_{c_u}} = \sum_{u=w+1}^{v} \gamma_{c_u} \frac{\partial h_e}{\partial z_{c_u}}. \tag{5.29}$$

If the expression in (5.29) equals zero, then there is nothing left to be done. If the expression in (5.29) is nonzero, then $\frac{\partial h_e}{\partial z_{c_u}} \neq 0$ for some $u = w+1, \ldots, q$ with $q \leq v$. Perform the elementary row operations $\mathrm{R}_\ell - \frac{\partial \zeta_{c_u}}{\partial z_\ell} \mathrm{R}_{c_u} \rightarrow \mathrm{R}_\ell$, for all variables z_ℓ in $\zeta_{c_u} = \int_0^{z_k} \gamma_{c_u}(t)\, dt$ and for all $u = w+1, \ldots, q$ to eliminate the term $\sum_{u=w+1}^{q} \gamma_{c_u} \frac{\partial h_e}{\partial z_{c_u}}$, where $q \leq v$. We follow this process through in each of the distinct gradient ideals until all gradient ideals have been considered. The expression in (5.28) becomes zero at some point; otherwise, we get a contradiction to R_k being dependent.

□

Example 7. Let the hypersurface $\mathcal{M} \subseteq \mathbb{C}^5$ be given by the defining function

$$r = 2\mathrm{Re}(z_5) + |(z_1 + z_3^2 + z_3 z_4)^2 + (z_2 + z_3^2 + z_3 z_4)^2|^2 + |z_2^5|^2 + |z_3^6|^2 + |z_4^8|^2.$$

Let $B = 2(z_1 + z_3^2 + z_3 z_4)$, $C = 2(z_2 + z_3^2 + z_3 z_4)$, and $g = 2z_3 + z_4$. Let the ideal associated to the domain $\mathcal{M}$ be $\langle h \rangle = \langle (z_1 + z_3^2 + z_3 z_4)^2 + (z_2 + z_3^2 + z_3 z_4)^2, z_2^5, z_3^6, z_4^8 \rangle$. The complex Jacobian matrix is given by

$$\mathrm{J}(h) = \begin{pmatrix} B & 0 & 0 & 0 \\ C & 5z_2^4 & 0 & 0 \\ g(B+C) & 0 & 6z_3^5 & 0 \\ z_3(B+C) & 0 & 0 & 8z_4^7 \end{pmatrix}.$$

The Bloom-Graham type is 4 and $\Lambda_1 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ with $\mathcal{I}_{P_1} = \langle z_1^2 + z_2^2 \rangle$. Clearly, no changes of variables are required here. Hence $\max W_1 = \frac{1}{8}$ and $\Lambda_2 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})$ with the leading polynomial ideal $\mathcal{I}_{P_2} = \langle (z_1 + z_3^2 + z_3 z_4)^2 + (z_2 + z_3^2 + z_3 z_4)^2 \rangle$. The Jacobian ideal corresponding to $\mathcal{I}_{P_2} = \langle h_1 \rangle$ is the gradient ideal $\mathcal{I}_{grad}(h_1) = \langle B, C, g(B+C), z_3(B+C) \rangle$. Here

$$\frac{\partial h_1}{\partial z_3} = g\left(\frac{\partial h_1}{\partial z_1} + \frac{\partial h_1}{\partial z_2}\right) \quad \text{and} \quad \frac{\partial h_1}{\partial z_4} = z_3\left(\frac{\partial h_1}{\partial z_1} + \frac{\partial h_1}{\partial z_2}\right).$$

The ideal $\mathcal{I}_{grad}(h_1)$ can be simplified to $\mathcal{I}_{grad}(f_1) = \langle B, C \rangle$. So $\frac{\partial h_1}{\partial z_1}$, and $\frac{\partial h_1}{\partial z_2}$ are central generators in the simplification of $\mathcal{I}_{grad}(h_1)$.

Thus, we perform the elementary row operations $\mathrm{R}_3 - g\mathrm{R}_j \rightarrow \mathrm{R}_3$ and $\mathrm{R}_4 - z_3\mathrm{R}_j \rightarrow \mathrm{R}_4$ on the matrix $\mathrm{J}(h)$, where $j = 1, 2$ to obtain

$$\mathrm{J}(h) = \begin{pmatrix} B & 0 & 0 & 0 \\ C & 5z_2^4 & 0 & 0 \\ 0 & -g5z_2^4 & 6z_3^5 & 0 \\ 0 & -z_3 5z_2^4 & 0 & 8z_4^7 \end{pmatrix} \quad \text{and} \quad \mathcal{I}_{grad}(h_1) = \langle B, C \rangle.$$

These operations correspond to the polynomial transformation $\tilde{z}_1 = z_1 + z_3^2 + z_3 z_4$; $\tilde{z}_2 = z_1 + z_3^2 + z_3 z_4$; $\tilde{z}_\omega = z_\omega$ for $\omega \neq 1, 2$. Thus, $\mathcal{I}_{P_2} = \langle \tilde{z}_1^2 + \tilde{z}_2^2 \rangle$, and $\max W_2 = \frac{1}{12}$. The weight $\Lambda_3 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{12}, \frac{1}{12}\right)$. $\mathcal{I}_{P_3} = \langle \tilde{z}_1^2 + \tilde{z}_2^2, \tilde{z}_3^6 \rangle$, and $\max W_3 = \frac{1}{16}$. The multitype weight is $\Lambda_4 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{12}, \frac{1}{16}\right)$, and the final leading polynomial ideal is $\mathcal{I}_{P_4} = \langle \tilde{z}_1^2 + \tilde{z}_2^2, \tilde{z}_3^6, \tilde{z}_4^8 \rangle$.

Example 8. Let the hypersurface $\mathcal{M} \subseteq \mathbb{C}^5$ be given by the defining function

$$r = 2\mathrm{Re}(z_5) + |(z_1 + iz_2 + z_3)^2|^2 + |(z_1 + iz_2 + z_4)^2|^2 + |z_2^6|^2 + |z_3^4|^2.$$

Let $B = 2(z_1 + iz_2 + z_3)$, $C = 2(z_1 + iz_2 + z_4)$, and let the ideal associated to the domain $\mathcal{M}$ be $\langle h \rangle = \langle (z_1 + iz_2 + z_3)^2, (z_1 + iz_2 + z_4)^2, z_2^6, z_3^4 \rangle$. The complex Jacobian matrix is given by

$$\mathrm{J}(h) = \begin{pmatrix} B & C & 0 & 0 \\ iB & iC & 6z_2^5 & 0 \\ B & 0 & 0 & 4z_3^3 \\ 0 & C & 0 & 0 \end{pmatrix}.$$

The Bloom-Graham type is 4, $\mathcal{I}_{P_1} = \langle (z_1 + iz_2 + z_3)^2, (z_1 + iz_2 + z_4)^2 \rangle$, and $\Lambda_1 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. The complex Jacobian matrix corresponding to $\mathcal{I}_{P_1}$ is given by

$$\mathrm{J}(h_1, h_2) = \begin{pmatrix} B & C \\ iB & iC \\ B & 0 \\ 0 & C \end{pmatrix}.$$

Consider the gradient ideal $\mathcal{I}_{grad}(h_1) = \langle B, iB, B, 0 \rangle$. Note that

$$\frac{\partial h_1}{\partial z_2} = i \frac{\partial h_1}{\partial z_1} \quad \text{and} \quad \frac{\partial h_1}{\partial z_3} = \frac{\partial h_1}{\partial z_1},$$

so its simplification is $\mathcal{I}_{grad}(h_1) = \langle B \rangle$. $\frac{\partial h_1}{\partial z_1}$ is a central generator in the simplification of $\mathcal{I}_{grad}(h_1)$. We perform the elementary row operations $\mathrm{R}_2 - i\mathrm{R}_1 \to \mathrm{R}_2$ and $\mathrm{R}_3 - \mathrm{R}_1 \to \mathrm{R}_3$ on the matrix $\mathrm{J}(h)$. The matrices above become

$$\mathrm{J}(h) = \begin{pmatrix} B & C & 0 & 0 \\ 0 & 0 & 6z_2^5 & 0 \\ 0 & -C & 0 & 4z_3^3 \\ 0 & C & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{J}(h_1, h_2) = \begin{pmatrix} B & C \\ 0 & 0 \\ 0 & -C \\ 0 & C \end{pmatrix}.$$

These operations correspond to the polynomial transformation $\tilde{z}_1 = z_1 + iz_2 + z_3$: $\tilde{z}_\omega = z_\omega$ for $\omega \neq 1$, and now $B = 2\tilde{z}_1$ and $C = 2(\tilde{z}_1 - \tilde{z}_3 + \tilde{z}_4)$. Clearly, $\frac{\partial h_2}{\partial z_1}$ cannot be a central generator in the simplification of $\mathcal{I}_{grad}(h_2)$. Hence row 1 cannot be a central row in any subsequent elementary row operations. Consider the next gradient ideal after the row operations $\mathcal{I}_{grad}(h_2) = \langle C, 0, -C, C \rangle$. Clearly, the generators $-C$ and C

in the third and fourth components respectively can chosen as central generators in the simplification of the ideal $\mathcal{I}_{grad}(h_2)$. Thus, we can consider two simplifications of $\mathcal{I}_{grad}(h_2)$, which are $\langle 0,0,0,C\rangle$ or $\langle 0,0,-C,0\rangle$.

In the first case, we perform the elementary row operations $\mathrm{R}_1 - \mathrm{R}_4 \to \mathrm{R}_1$ and $\mathrm{R}_3 + \mathrm{R}_4 \to \mathrm{R}_3$ on the matrix $\mathrm{J}(h)$. The matrices become

$$\mathrm{J}(h) = \begin{pmatrix} B & 0 & 0 & 0 \\ 0 & 0 & 6z_2^5 & 0 \\ 0 & 0 & 0 & 4z_3^3 \\ 0 & C & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{J}(h_1,h_2) = \begin{pmatrix} B & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & C \end{pmatrix}.$$

These operations correspond to the polynomial transformation $\dot{z}_4 = \tilde{z}_4 + \tilde{z}_1 - \tilde{z}_3$; $\dot{z}_\omega = \tilde{z}_\omega$ for $\omega \neq 4$, and now $B = 2\dot{z}_1$ and $C = 2\dot{z}_4$. The leading polynomial ideal $\mathcal{I}_{P_1} = \langle \dot{z}_1^2, \dot{z}_4^2\rangle$. $\max W_1 = \frac{1}{8}$, and $\Lambda_2 = \left(\frac{1}{4},\frac{1}{4},\frac{1}{8},\frac{1}{8}\right)$. Again, $\max W_2 = \frac{1}{12}$, and $\Lambda_3 = \left(\frac{1}{4},\frac{1}{4},\frac{1}{8},\frac{1}{12}\right)$.

In the second case, we perform the elementary row operations $\mathrm{R}_1 + \mathrm{R}_3 \to \mathrm{R}_1$ and $\mathrm{R}_4 + \mathrm{R}_3 \to \mathrm{R}_4$ on the matrix $\mathrm{J}(h)$. The matrices become

$$\mathrm{J}(h) = \begin{pmatrix} B & 0 & 0 & 4z_3^3 \\ 0 & 0 & 6z_2^5 & 0 \\ 0 & -C & 0 & 4z_3^3 \\ 0 & 0 & 0 & 4z_3^3 \end{pmatrix} \quad \text{and} \quad \mathrm{J}(h_1,h_2) = \begin{pmatrix} B & 0 \\ 0 & 0 \\ 0 & -C \\ 0 & 0 \end{pmatrix}.$$

These operations correspond to the polynomial transformation $\dot{z}_3 = \tilde{z}_3 - \tilde{z}_1 - \tilde{z}_4$; $\dot{z}_\omega = \tilde{z}_\omega$ for $\omega \neq 3$, and now $B = 2\dot{z}_1$ and $C = -2\dot{z}_3$. The leading polynomial ideal $\mathcal{I}_{P_1} = \langle \dot{z}_1^2, \dot{z}_4^2\rangle$. $\max W_1 = \frac{1}{8}$, and $\Lambda_2 = \left(\frac{1}{4},\frac{1}{4},\frac{1}{8},\frac{1}{8}\right)$. Again, $\max W_2 = \frac{1}{12}$, and $\Lambda_3 = \left(\frac{1}{4},\frac{1}{4},\frac{1}{8},\frac{1}{12}\right)$.

Example 9. Let the hypersurface $\mathcal{M} \subseteq \mathbb{C}^5$ be given by the defining function

$$r = 2\mathrm{Re}(z_5) + |(z_1 + z_2 z_4)^2 + z_2^4|^2 + |(z_1 + z_2 z_3^2)^2|^2 + |z_2^9|^2 + |z_3^{10}|^2 + |z_4^{12}|^2.$$

Let $B = 2(z_1 + z_2 z_4)$, $C = 2(z_1 + z_2 z_3^2)$, and let the ideal associated to the domain $\mathcal{M}$ be $\langle h\rangle = \langle (z_1 + z_2 z_4)^2 + z_2^2, (z_1 + z_2 z_3^2)^2, z_2^9, z_3^{10}, z_4^{12}\rangle$. The complex Jacobian matrix is given by

$$\mathrm{J}(h) = \begin{pmatrix} B & C & 0 & 0 & 0 \\ z_4 B + 4z_2^3 & z_3^2 C & 9z_2^8 & 0 & 0 \\ 0 & 2z_2 z_3 C & 0 & 10z_3^9 & 0 \\ z_2 B & 0 & 0 & 0 & 12z_4^{11} \end{pmatrix}.$$

The Bloom-Graham type is 4, $\mathcal{I}_{P_1} = \langle z_1^2, z_1^2\rangle$, and $\Lambda_1 = (\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$. The maximum $\mathrm{W}_1 = \frac{1}{8}$, the leading polynomial ideal $\mathcal{I}_{P_2} = \langle (z_1 + z_2 z_4)^2 + z_2^4, z_1^2\rangle$, and $\Lambda_2 = (\frac{1}{4},\frac{1}{8},\frac{1}{8},\frac{1}{8})$.

The maximum $\mathrm{W}_2 = \frac{1}{16}$, $\mathcal{I}_{P_3} = \langle (z_1 + z_2z_4)^2 + z_2^4, (z_1 + z_2z_3^2)^2 \rangle$, and $\Lambda_3 = (\frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16})$. The complex Jacobian matrix corresponding to $\mathcal{I}_{P_3}$ is given by

$$\mathrm{J}(h_1, h_2) = \begin{pmatrix} B & C \\ z_4B + 4z_2^3 & z_3^2C \\ 0 & 2z_2z_3C \\ z_2B & 0 \end{pmatrix}.$$

Consider the gradient ideal $\mathcal{I}_{grad}(h_1) = \langle B, z_4B + 4z_2^3, 0, z_2B \rangle$. Since $\frac{\partial h_1}{\partial z_4} = z_2\frac{\partial h_1}{\partial z_1}$, its simplification is $\mathcal{I}_{grad}(h_1) = \langle B, 4z_2^3 \rangle$. We perform the elementary row operations $\mathrm{R}_2 - z_4\mathrm{R}_1 \to \mathrm{R}_2$ and $\mathrm{R}_4 - z_2\mathrm{R}_1 \to \mathrm{R}_4$ on the matrix $\mathrm{J}(h)$. The matrices above become

$$\mathrm{J}(h) = \begin{pmatrix} B & C & 0 & 0 & 0 \\ 4z_2^3 & (z_3^2 - z_4)C & 9z_2^8 & 0 & 0 \\ 0 & 2z_2z_3C & 0 & 10z_3^9 & 0 \\ 0 & -z_2C & 0 & 0 & 12z_4^{11} \end{pmatrix} \quad \text{and} \quad \mathrm{J}(h_1, h_2) = \begin{pmatrix} B & C \\ 4z_2^3 & (z_3^2 - z_4)C \\ 0 & 2z_2z_3C \\ 0 & -z_2C \end{pmatrix}.$$

These operations correspond to the polynomial transformation $\tilde{z}_1 = z_1 + z_2z_4 :\ \tilde{z}_\omega = z_\omega$ for $\omega \neq 1$, and now $B = 2\tilde{z}_1$ and $C = 2(\tilde{z}_1 + \tilde{z}_2(\tilde{z}_3^2 - \tilde{z}_4))$. The generator $\frac{\partial h_2}{\partial z_1}$ cannot be a central generator in the simplification of $\mathcal{I}_{grad}(h_2)$, and row 1 cannot be used as a central row in any subsequent elementary row operations. Consider the next gradient ideal $\mathcal{I}_{grad}(h_2) = \langle C, (\tilde{z}_3^2 - \tilde{z}_4)C, 2\tilde{z}_2\tilde{z}_3C, -\tilde{z}_2C \rangle$. Here the generator $-\tilde{z}_2C$ in the fourth component is the only central generator in $\mathcal{I}_{grad}(h_2)$. Thus, $\frac{\partial h_2}{\partial z_3} = -2\tilde{z}_3\frac{\partial h_2}{\partial z_4}$, and its simplification is $\langle C, (\tilde{z}_3^2 - \tilde{z}_4)C, -\tilde{z}_2C \rangle$.

We perform the elementary row operations $\mathrm{R}_3 + 2z_3\mathrm{R}_4 \to \mathrm{R}_3$ on the matrix $\mathrm{J}(h)$. The matrices become

$$\mathrm{J}(h) = \begin{pmatrix} B & C & 0 & 0 & 0 \\ 4z_2^3 & (z_3^2 - z_4)C & 9z_2^8 & 0 & 0 \\ 0 & 0 & 0 & 10z_3^9 & 24z_3z_4^{11} \\ 0 & -z_2C & 0 & 0 & 12z_4^{11} \end{pmatrix} \quad \text{and} \quad \mathrm{J}(h_1, h_2) = \begin{pmatrix} B & C \\ 4z_2^3 & (z_3^2 - z_4)C \\ 0 & 0 \\ 0 & -z_2C \end{pmatrix}.$$

This operation corresponds to the polynomial transformation $\dot{z}_4 = \tilde{z}_4 - \tilde{z}_3^2;\ \ \dot{z}_\omega = \tilde{z}_\omega$ for $\omega \neq 4$, and now $B = 2\dot{z}_1$ and $C = 2(\dot{z}_1 - \dot{z}_2\dot{z}_4)$. The leading polynomial ideal $\mathcal{I}_{P_3} = \langle \dot{z}_1^2 + \dot{z}_2^4, (\dot{z}_1 - \dot{z}_2\dot{z}_4)^2 \rangle$. The maximum $W_3 = \frac{1}{20}$, and $\Lambda_4 = \left(\frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{20}\right)$.

Chapter 6

Conclusion and Further Work

In his study of the boundary regularity properties for solutions to the $\bar{\partial}$-Neumann problem on finite-type domains, D. Catlin introduced an important CR invariant called the multitype. In its own right, the multitype has been extensively studied over the years in the field of several complex variables. As our contribution, we sought to answer a question posed by J.P. D'Angelo to Andreea Nicoara, namely how would the multitype level set stratification of the boundary look like in the simplest possible case of a sum of squares domain.

Our aim in this thesis was to introduce some preparatory tools and techniques necessary for tackling D'Angelo's question. In our quest for these preparatory tools, we obtained two crucial results that answer interesting questions pertaining to sums of squares domains in their own right. The first result shows that the model of a sum of squares domain is likewise a sum of squares domain. In the second result, the multitype of a domain given by a sum of squares of holomorphic functions is shown to be an invariant of the ideal of holomorphic functions defining the domain. Both results were obtained by relying on an algorithm devised by Martin Kolář for computing the multitype when it has finite entries. Another interesting development following these results is our modification of the Kolář algorithm for computing the multitype of a sum of squares domain by reformulating it in terms of ideals of holomorphic functions. We also show how to explicitly construct the polynomial transformations required at every step in Kolář's algorithm applied to a sum of squares domain in order to minimize the number of variables appearing in the leading polynomial.

To better understand the implications of our results, future studies could be focused on restating and proving the propositions and lemmas in this thesis in a more general setting by relaxing some of the existing assumptions. More specifically, it should be possible to relax the assumption of finite D'Angelo 1-type to just having all finite multitype entries without fundamentally affecting the statements and proofs given here.

Further research is also needed to answer the question posed by D'Angelo. The restatement of the Kolář algorithm in terms of ideals of holomorphic functions might make it easier to solve D'Angelo's problem since working with ideals aligns better with complex algebraic geometry. As such, we hope to obtain some commutative-algebraic invariants of the underlying ideals of holomorphic functions that would enable us to compute the multitype directly rather than following the Kolář algorithm. Hopefully, the stratification of the boundary of a sums of squares domain by multitype level sets could be understood if we succeed to relate the values of the multitype to invariants in algebraic geometry or commutative algebra. The characterization of the rank of the Levi determinant described in chapter three is the first step in this process. Also, owing to the ideal reformulation of the Kolář algorithm, the geometric significance of the Kolář algorithm could be fully understood in light of the extensive literature on the properties of ideals of holomorphic functions. A geometric interpretation of the Kolář algorithm is most likely to give a much clearer geometric picture in the sum of squares case, which possesses the nicest algebraic-geometric properties of any smooth pseudoconvex domain.

Following Catlin's result that the multitype and the commutator multitype are equal on pseudoconvex domains, it should also be possible to restate Catlin's algorithm for the computation of the commutator multitype of a sum of squares domain in terms of ideals of holomorphic functions. A natural connection could hopefully be found between Catlin's algorithm and Kolář's when both are restated in terms of ideals. More specifically, in the sum of squares case, it should be possible to identify and establish a connection between the polynomial transformations constructed in the Kolář algorithm and the choice of tangential vector fields needed to obtain the entries in the commutator multitype.

Another interesting question worth investigating is figuring out whether or not the Kolář algorithm could be extended to the case where there is at least one infinite entry in the multitype. As we saw in example 2 in chapter two, the Kolář algorithm in its current form can fail to terminate if there is at least one entry of the multitype that is infinite. We know from other examples we have constructed that the Kolář algorithm can terminate even if the multitype has one or more infinite entries. It would be very interesting to characterize the most general setting in which the Kolář algorithm can be used in its current form and how it can be generalized to a procedure that would work even when some of the multitype entries are infinite.

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