Quantum Coherence and Ergotropy

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Constraints on work extraction are fundamental to our operational understanding of the thermodynamics of both classical and quantum systems. In the quantum setting, finite-time control operations typically generate coherence in the instantaneous energy eigenbasis of the dynamical system. Thermodynamic cycles can, in principle, be designed to extract work from this non-equilibrium resource. Here, we isolate and study the quantum coherent component to the work yield in such protocols. Specifically, we identify a coherent contribution to the ergotropy (the maximum amount of unitarily extractable work via cyclical variation of Hamiltonian parameters). We show this by dividing the optimal transformation into an incoherent operation and a coherence extraction cycle. We obtain bounds for both the coherent and incoherent parts of the extractable work and discuss their saturation in specific settings. Our results are illustrated with several examples, including finite-dimensional systems and bosonic Gaussian states that describe recent experiments on quantum heat engines with a quantized load.

I. INTRODUCTION

The Thomson [1] formulation of the second law is a constraint on the ability of an external agent to extract work from a system. More precisely, it states that no work can be extracted from a closed equilibrium system during a cyclic variation of a parameter by an external source [2, 3]. This formulation was influential in mathematical physics, leading to a definition of the condition of thermal equilibrium for quantum states through the notion of passivity and complete passivity [4, 5]. A state $\hat{\rho}$ is said to be passive with respect to a Hamiltonian when no work can be extracted from it by means of a cyclical variation of a Hamiltonian parameter, while it can be shown that a Gibbs state is the unique completely passive state such that $\hat{\rho}^{\otimes N}$ remains passive for all N. In other words, passivity allows us derive Thomson's formulation of the second law as a constraint on unitary work extraction from quantum systems [6]. If a state is non-passive with respect to a Hamiltonian, work can be extracted and, upon maximization over the space of cyclical unitaries, the optimal yield is known as the *ergotropy* [7, 8]. The ergotropy has been established as an important quantity in the emerging field of quantum thermodynamics [10–14] and has recently been measured in two experiments which explore work deposition to external loads coupled to microscopic engines [15, 16]. In the limit of many copies, the ergotropy converges to the conventional non-equilibrium part of the free energy [17] and it has also been incorporated into an open system thermodynamic description of finite quantum systems, recovering first and second laws [18].

A central theme in the field of quantum thermodynamics over the last decade has been the identification of uniquely quantum signatures in thermodynamic settings. This includes the identification of quantum effects in thermal machines [19–34], in work extraction protocols [17, 35–48], in fluctuations of work [49–55], and in work deposition processes [56–62], to name but a few examples. Arguably the most fundamental of all non-classical features is quantum coherence, yet precise mathematical techniques for its quantification have only recently been formulated in quantum information theory [63, 64]. From the perspective of quantum thermodynamics, many studies have aimed at highlighting the non-trivial role that coherence may play [14, 65–74]. Coherence is a basis-dependent quantity that can be expressed in terms of the relative entropy between the state of the system at hand and its dephased counterpart in the relevant basis [63]. This provides a connection to the finite-time thermodynamics of quantum systems, where the relative entropy is ubiquitous in the assessment of irreversible entropy production of closed [75–77] and open systems [78–85]. This connection was recently exploited in order to isolate a coherent contribution to the entropy production in quantum dynamics [86–89]. Here, the relevant coherence is defined relative to the energy eigenbasis, which plays a distinguished role in thermodynamics.

In this work, we focus on the role of such coherence in ergotropic work extraction. We believe the simplicity of our approach together with its operational significance will be of particular interest to those interested in isolating non-classical signatures in quantum thermodynamics. We begin by introducing the basic notions of coherence and ergotropy in the following section. In Section III, we identify coherent and incoherent contributions to the ergotropy, while bounds for the coherent ergotropy are derived in Sec. IV. We then provide examples to illustrate our results in Sec. V and, finally, summarise in Sec. VI.

II. PRELIMINARIES

Given a quantum system in an initial state $\hat{\rho}$, and a Hamiltonian $\hat{H} = \sum_k \varepsilon_k |\varepsilon_k\rangle \langle\varepsilon_k|$, we are interested in the amount of coherence in the energy eigenbasis. In what follows, we will quantify the coherence with the *relative entropy of coher-*

ence $C(\hat{\rho})$ [63, 64]. This is motivated from the description of coherence as a quantum resource theory [64, 90].

A quantum resource arises when there is a naturally restricted set of operations $\mathcal{O}$ which are significantly easier to implement than operations outside $\mathcal{O}$ – e.g. local operations and classical communication (LOCC) in entanglement theory [91]. If these *free operations* $\mathcal{O}$ only allow some *free states* $\mathcal{F}$ to be created 'for free', all other states become a resource whose creation requires the (costly) implementation of operations outside $\mathcal{O}$. We may quantify the resourcefulness of a non-free state by means of a function μ that maps states to non-negative reals. We call μ a *resource monotone* if (i) its value cannot increase under application of any free operation $\Omega \in \mathcal{O}$ to any state $\hat{\rho}$: $\mu(\hat{\rho}) \geq \mu(\Omega(\hat{\rho}))$; and if (ii) $\mu(\hat{\varphi}) = 0$ for all $\hat{\varphi} \in \mathcal{F}$. One way of constructing a monotone μ is to minimize a (contractive) distance function d on the space of quantum states with respect to $\mathcal{F}$: $\mu_d(\hat{\rho}) := \min_{\hat{\varphi}\in\mathcal{F}} d(\hat{\rho}, \hat{\varphi})$. The usefulness of such a distance-based μ_d then depends not least on its ease of computation – i.e., if it can be expressed as a closed-form function.

Returning to coherence, various viable classes of free operations have been considered for which the free states $\mathcal{F}$ are the set of incoherent states I_H, i.e., density matrices $\hat{\delta}$ that are diagonal in the energy eigenbasis [64]. For all of these classes, valid coherence monontones may be obtained based on suitable distance measures such as Tsallis relative α-entropies D_α for which succinct expressions have been found [92]: $C_\alpha := \min_{\hat{\delta}\in I_H} D_\alpha(\hat{\rho}||\hat{\delta})$, where the normalized state $\hat{\delta}_{\rho,\alpha} \propto \sum \langle \varepsilon_j | \hat{\rho}^\alpha | \varepsilon_j \rangle^{1/\alpha} |\varepsilon_j\rangle\langle\varepsilon_j|$ obtains the minimum. We here focus on the limit $\alpha \to 1$ as, in this case, the minimal state $\hat{\delta}_\rho \equiv \hat{\delta}_{\rho,\alpha} = \Delta(\hat{\rho})$ is directly connected to the original state $\hat{\rho}$ by a physical operation – dephasing with Δ. In this limit, D_α becomes the standard quantum relative entropy $D(\hat{\rho}||\hat{\delta}) = \mathrm{Tr}\left\{\hat{\rho}(\log\hat{\rho} - \log\hat{\delta})\right\}$ and C_α becomes the entropy of coherence $C(\hat{\rho}) = S(\hat{\delta}_\rho) - S(\hat{\rho})$, with the Von Neumann entropy $S(\hat{\sigma}) = -\mathrm{Tr}\left\{\hat{\sigma}\log\hat{\sigma}\right\}$ [63].

Following the seminal paper [7], we are now interested in extracting work from the quantum system at hand by using a cyclic unitary transformation $\hat{U} \in \mathcal{U}_c$, where $\mathcal{U}_c$ denotes the set of unitary transformations generated in a given interval $(0, \tau)$ by a time dependent Hamiltonian $\hat{H}(t)$ such that $\hat{H}(0) = \hat{H}(\tau) = \hat{H}$. In this context, one typically assumes complete control over the system [70]: that is, the possibility of generating any unitary evolution through suitable control fields applied to the system, which are switched off at the end of the transformation. Under the action of the unitary $\hat{U}$, the state transforms as $\hat{\rho} \to \hat{U}\hat{\rho}\hat{U}^\dagger$, and the average work extracted from the system is $W(\hat{\rho}, \hat{U}) = \mathrm{Tr}\left\{\hat{H}(\hat{\rho} - \hat{U}\hat{\rho}\hat{U}^\dagger)\right\}$. The maximum of W over the set $\mathcal{U}_c$ is called ergotropy, $\mathcal{E}$. After ordering the labels of eigenstates of $\hat{H}$ and of $\hat{\rho}$ such that $\hat{H} = \sum_{k=1}^d \varepsilon_k |\varepsilon_k\rangle\langle\varepsilon_k|$, with $\varepsilon_k < \varepsilon_{k+1}$, and $\hat{\rho} = \sum_{k=1}^d r_k |r_k\rangle\langle r_k|$, with $r_k \geq r_{k+1}$, we define the optimal *ergotropic* transformation $\hat{E}_\rho$ as the one that maps $\hat{\rho}$ into the passive state $\hat{P}_\rho = \hat{E}_\rho\hat{\rho}\hat{E}_\rho^\dagger = \sum_k r_k |\varepsilon_k\rangle\langle\varepsilon_k|$. We notice that the optimal unitary $\hat{E}$ depends on the state $\hat{\rho}$, and that the ergotropy is then given by

$$\mathcal{E}(\hat{\rho}) = \max_{\hat{U}\in\mathcal{U}_c} W(\hat{\rho}, \hat{U}) \equiv W(\hat{\rho}, \hat{E}_\rho) = \mathrm{Tr}\left\{\hat{H}(\hat{\rho} - \hat{P}_\rho)\right\} \equiv \sum_k \varepsilon_k(\rho_{kk} - r_k)\,, \tag{1}$$

where ρ_{kk} (the population of $\hat{\rho}$ in the k-th energy eigenstate) can be expressed as $\rho_{kk} = \sum_{k'} r_{k'} |\langle r_{k'}|\varepsilon_k\rangle|^2$. Our main aim is to demonstrate a precise connection between $\mathcal{E}$ and the amount of coherence in the initial state $C(\hat{\rho})$ [93]. In the following section, we show how to split the ergotropy into two contributions, one of which directly connected to the presence of energetic coherence in the state $\hat{\rho}$.

III. COHERENT AND INCOHERENT CONTRIBUTIONS TO ERGOTROPY

We start by introducing the incoherent part of the ergotropy, $\mathcal{E}_i$, which can be defined in two equivalent ways. One can think of $\mathcal{E}_i$ as the maximum work extractable from $\hat{\rho}$ without altering its coherence. To formalize this idea, we can imagine breaking the transformation $\hat{E}_\rho$ into an incoherent operation followed by a second, coherence-consuming, cyclic unitary. To this end, we define the subset $\mathcal{U}_c^{(i)} \subset \mathcal{U}_c$ of incoherent, cyclic, unitary transformations, such that any $\hat{V} \in \mathcal{U}_c^{(i)}$ is coherence-preserving: $C(\hat{\rho}) = C(\hat{V}\hat{\rho}\hat{V}^\dagger)$. Such $\hat{V}$ is in fact a member of the class of *strictly incoherent operations* (SIOs) which admit a very operational structure [64, 94, 95]; $\hat{V}$ amounts to a reshuffling of the energy basis, up to irrelevant phase factors, of the form $\hat{V} = \sum_k e^{-i\varphi_k} |\varepsilon_k\rangle\langle\varepsilon_{\pi_k}| \equiv \hat{V}_\pi$, where π_k is the k-th element in the result of the permutation π of the indices [96]. The incoherent contribution to ergotropy is then defined as

$$\mathcal{E}_i = \max_{\hat{V}\in\mathcal{U}_c^{(i)}} W(\hat{\rho}, \hat{V}) \equiv \max_\pi W(\hat{\rho}, \hat{V}_\pi)\,. \tag{2}$$

The optimal permutation, $\tilde{\pi}$, realizing the maximum in the equation above, is the one that rearranges the populations $\{\rho_{kk}\}_{k=1,\ldots d}$ in descending order: $\rho_{\tilde{\pi}_j\tilde{\pi}_j} \geq \rho_{\tilde{\pi}_{j+1}\tilde{\pi}_{j+1}}, \forall j$. Letting $\hat{\sigma}_\rho = \hat{V}_{\tilde{\pi}}\hat{\rho}\hat{V}_{\tilde{\pi}}^\dagger = \sum_k\sum_{k'} \rho_{\tilde{\pi}_k,\tilde{\pi}_{k'}} |\varepsilon_k\rangle\langle\varepsilon_{k'}|$, the incoherent contribution to ergotropy is

$$\mathcal{E}_i(\hat{\rho}) = \mathrm{Tr}\left\{\hat{H}(\hat{\rho} - \hat{\sigma}_\rho)\right\} = \sum_k \varepsilon_k(\rho_{kk} - \rho_{\tilde{\pi}_k\tilde{\pi}_k}). \tag{3}$$

The state $\hat{\sigma}_\rho$ possesses the same coherence as $\hat{\rho}$, but less average energy. Therefore, $\mathcal{E}_i$ is the maximum amount of work that can be extracted from $\hat{\rho}$ without changing its coherence, and, among the states having the same amount of coherence as $\hat{\rho}$, $\hat{\sigma}_\rho$ is singled out as the one that possesses the least possible average energy [97]. In particular, we notice that, when trying to extract work from the state $\hat{\sigma}_\rho$ through the optimal cyclic unitary $\hat{E}_\sigma$, one arrives at the very same passive state that is obtained from $\hat{\rho}$. In our notation, $\hat{P}_\sigma = \hat{P}_\rho$. This is because $\hat{\sigma}_\rho$ has the same eigenvalues as $\hat{\rho}$.

An alternative (but equivalent) route to the identification of the incoherent contribution to ergotropy is provided by defining $\mathcal{E}_i$ as the maximum amount of work extractable from $\hat{\rho}$ after having erased all of its coherences via the dephasing map Δ. This amounts to defining $\mathcal{E}_i$ as the full ergotropy of the dephased state, $\mathcal{E}_i = \mathcal{E}(\hat{\delta}_\rho)$, where $\hat{\delta}_\rho = \Delta[\hat{\rho}]$ has the same energy populations as $\hat{\rho}$ (and, thus, the same average energy) but zero coherence. The ergotropy of $\hat{\delta}_\rho$ can be written by first defining the passive state $\hat{P}_\delta$ obtained from $\hat{\delta}_\rho$ after rearranging the populations in decreasing order, and then letting

$$\mathcal{E}_i(\hat{\rho}) \equiv \mathcal{E}(\hat{\delta}_\rho) = \mathrm{Tr}\left\{\hat{H}(\hat{\delta}_\rho - \hat{P}_\delta)\right\} . \tag{4}$$

This definition is fully equivalent to the one given in Eq. (3) [98]. Indeed, $\hat{\delta}_\rho$ has the same populations as $\hat{\rho}$ in the energy basis; consequently, the optimal reshuffling unitary that maps $\hat{\delta}_\rho$ into $\hat{P}_\delta$ is given by the very same $\hat{V}_{\tilde{\pi}}$ introduced above. This implies that $\hat{P}_\delta$ has the same populations as $\hat{\sigma}_\rho$ (in the same order!), but no coherence. As a result of these considerations, one immediately realizes that $\hat{P}_\delta$ can be obtained by applying the dephasing map to $\hat{\sigma}_\rho$, and that the two states share the same average energy:

$$\hat{P}_\delta \equiv \Delta[\hat{\sigma}_\rho] \quad \Rightarrow \quad \mathrm{Tr}\left\{\hat{H}\hat{\sigma}_\rho\right\} \equiv \mathrm{Tr}\left\{\hat{H}\,\hat{P}_\delta\right\} .$$

Having defined the incoherent part of $\mathcal{E}(\hat{\rho})$, the coherent contribution to ergotropy is simply given by the difference

$$\mathcal{E}_c = \mathcal{E} - \mathcal{E}_i = \mathrm{Tr}\left\{\hat{H}(\hat{\sigma}_\rho - \hat{P}_\rho)\right\} \equiv \sum_k \varepsilon_k(\rho_{\tilde{\pi}_k\tilde{\pi}_k} - r_k) . \tag{5}$$

This is a non-negative quantity as, in general, $\hat{\sigma}_\rho$ is an active state. Notice further, that it coincides with the full ergotropy of $\hat{\sigma}_\rho$.

The coherent ergotropy $\mathcal{E}_c$ can be understood as that part of extractable work which cannot be obtained by means of incoherent operations applied to state $\hat{\rho}$, and it is due to the presence of coherence in the initial state. Despite this, $\mathcal{E}_c$ is not a coherence monotone, as the inequality $\mathcal{E}_c(\hat{V}\hat{\rho}\hat{V}^\dagger) \leq \mathcal{E}_c(\hat{\rho})$ is not satisfied for every incoherent operation $\hat{V}$ (see Appendix for an illustrative example). Nevertheless, both the state $\hat{\sigma}_\rho$ and the coherent part of the ergotropy, $\mathcal{E}_c$, are uniquely defined once the state $\hat{\rho}$ and the Hamiltonian $\hat{H}$ are given, and they result entirely from the initial coherence, implying that $\hat{\sigma}_\rho$ is not passive.

Fig. 1 summarizes these considerations and relationships. It shows the various states and operations defined up to now in the coherence-versus-average-energy plane.

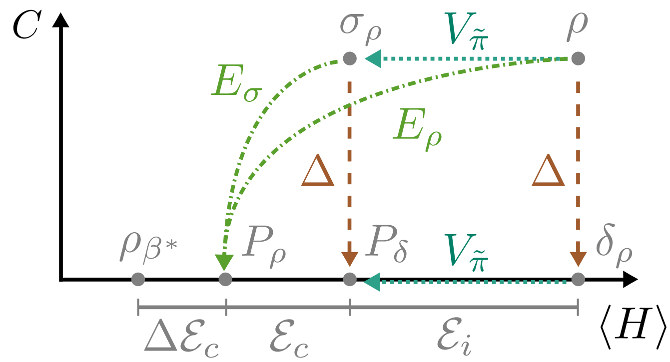

FIG. 1. Position of the various states (see main text) in a coherence-versus-average-energy diagram. Grey dots represent quantum states – e.g. arising from the initial state $\hat{\rho}$ after the transformations $\hat{E}_\rho, \Delta, \hat{V}_{\tilde{\pi}}$ are performed. The arrows representing transformations are intended merely to point from the initial to the final state, without implying a precise path in the plane. For example, the transformation $\hat{V}_{\tilde{\pi}}$ is represented by a horizontal line because it connects states with the same amount of coherence; however, coherence may change *during* the transformation. The horizontal distance $\Delta\mathcal{E}_c$ between the thermal state $\hat{\rho}_{\beta^*}$ and $\hat{P}_\rho$ is the *bound ergotropy* (see Sec. IV). It may be zero, depending on the system at hand (i.e., iff the eigenvalues of ρ and ρ_{β^*} are related by a permutation).

IV. BOUNDS FOR COHERENT ERGOTROPY

Given the form of the coherent ergotropy, we can provide upper and lower bounds to its value and show their tightness. Indeed, by introducing the Gibbs state $\hat{\rho}_\beta = e^{-\beta\hat{H}}/Z$ with inverse temperature β, we can exploit the identity $D(\hat{\sigma}||\hat{\rho}_\beta) = \beta\mathrm{Tr}\left\{\hat{H}(\hat{\sigma} - \hat{\rho}_\beta)\right\} - S(\hat{\sigma}) + S(\hat{\rho}_\beta)$, valid for any state $\hat{\sigma}$, in order to obtain the following chain of relations:

$$\begin{aligned}\beta\mathcal{E}_c = \beta(\mathcal{E} - \mathcal{E}_i) &= \beta\mathrm{Tr}\left\{\hat{H}\left(\hat{P}_\delta - \hat{P}_\rho\right)\right\} = \\ &= \beta\mathrm{Tr}\left\{\hat{H}\left(\hat{P}_\delta - \hat{\rho}_\beta\right)\right\} - \beta\mathrm{Tr}\left\{\hat{H}\left(\hat{P}_\rho - \hat{\rho}_\beta\right)\right\} = \\ &= \left[D(\hat{P}_\delta||\hat{\rho}_\beta) + S(\hat{P}_\delta) - S(\hat{\rho}_\beta)\right] \\ &\quad - \left[D(\hat{P}_\rho||\hat{\rho}_\beta) + S(\hat{P}_\rho) - S(\hat{\rho}_\beta)\right]\end{aligned}$$

After taking into account that $S(\hat{P}_\rho) = S(\hat{\rho})$, and that $S(\hat{P}_\delta) = S(\hat{\delta}_\rho)$ (due to the fact that they are connected by unitary transformations), and, finally, using $C(\hat{\rho}) = S(\hat{\delta}_\rho) - S(\hat{\rho})$, we obtain

$$\beta\mathcal{E}_c = C(\hat{\rho}) + D(\hat{P}_\delta||\hat{\rho}_\beta) - D(\hat{P}_\rho||\hat{\rho}_\beta) , \tag{6}$$

which is valid for every finite β.

From this relation, using the fact the $D \geq 0$, one easily obtains bounds for $\mathcal{E}_c(\hat{\rho})$:

$$C(\hat{\rho}) - D(\hat{P}_\rho||\hat{\rho}_\beta) \leq \beta\mathcal{E}_c(\hat{\rho}) \leq C(\hat{\rho}) + D(\hat{P}_\delta||\hat{\rho}_\beta) . \tag{7}$$

One can saturate the upper bound if $\hat{P}_\rho = \hat{\rho}_\beta$. This requires that the ergotropic transformation $\hat{E}_\rho$ takes $\hat{\rho}$ to the thermal state $\hat{\rho}_\beta$. Due to unitarity of this transformation, a necessary condition on β is that $S(\hat{\rho}) = S(\hat{\rho}_{\beta^*})$. We label the specific value of β for which this entropic equality holds β^*, and note that it exists for any $\hat{\rho}$. Moreover, for a single qubit, as well as for the important class of bosonic or fermionic states of Gaussian form, the condition $\beta = \beta^*$ is not just necessary, but also sufficient for the saturation of the upper bound in Eq. (7) (see examples in Sec. V).

More generally, however, the choice $\beta = \beta^*$ does not imply saturation of the bound. That is, the difference

$$\begin{aligned}\Delta\mathcal{E}_c &:= \frac{1}{\beta^*}\left[C(\hat{\rho}) + D(\hat{P}_\delta||\hat{\rho}_{\beta^*})\right] - \mathcal{E}_c(\hat{\rho}) \\ &= \frac{1}{\beta^*} D(\hat{P}_\rho||\hat{\rho}_{\beta^*}) \geq 0\end{aligned} \tag{8}$$

does not generally vanish. In fact, by expressing it as $\Delta\mathcal{E}_c = \mathrm{Tr}\left\{\hat{H}(\hat{P}_\rho - \hat{\rho}_{\beta^*})\right\}$ we note that it equates to what is called the *bound ergotropy* $\mathcal{E}_b$ [17] – i.e., the amount of additional ergotropy that a global unitary transformation could retrieve from $\hat{\rho}^{\otimes n}$, per system, in the limit $n \to \infty$ (in addition to the single-system ergotropy $\mathcal{E}$).

The saturation of the upper bound of Eq. (7) is, furthermore, equivalent to the results of Ref. [86] where the irreversible work W_{irr} *performed* on a quantum system was analyzed for a unitary transformation taking an initial thermal state $\hat{\rho}_{\beta^*}$ to a final state $\hat{\rho} = \hat{U}\hat{\rho}_{\beta^*}\hat{U}^\dagger$. It was shown that $\beta^* W_{irr} = C(\hat{\rho}) + D(\hat{\delta}_\rho||\hat{\rho}_{\beta^*})$. For a cyclic transformation, W_{irr} coincides with the average work performed on the system, whose absolute value, in turn, coincides with the work extracted from it by the cyclic unitary $\hat{U}^\dagger$, when it is prepared in the state $\hat{\rho}$. If we take $\hat{U}^\dagger = \hat{E}_\rho$, then the result of Ref. [86] is translated into our notation as

$$\beta^*\mathcal{E}(\hat{\rho}) = C(\hat{\rho}) + D(\hat{\delta}_\rho||\hat{\rho}_{\beta^*}),\ \text{if}\ \hat{E}_\rho\hat{\rho}\hat{E}_\rho^\dagger = \hat{\rho}_{\beta^*}\,. \tag{9}$$

But, with the same argument as given above, the incoherent ergotropy, Eq. (4), can be rewritten (for any β) as

$$\beta\mathcal{E}_i(\hat{\rho}) = D(\hat{\delta}_\rho||\hat{\rho}_\beta) - D(\hat{P}_\delta||\hat{\rho}_\beta)\,. \tag{10}$$

Taking $\beta = \beta^*$, and subtracting this relation from Eq. (9), we obtain the saturation of the upper bound of Eq. (7):

$$\beta^*\mathcal{E}_c(\hat{\rho}) = C(\hat{\rho}) + D(\hat{P}_\delta||\hat{\rho}_{\beta^*}),\ \text{if}\ \hat{E}_\rho\hat{\rho}\hat{E}_\rho^\dagger = \hat{\rho}_{\beta^*}\,. \tag{11}$$

The lower bound in Eq. (7), on the other hand, is saturated iff $\hat{P}_\delta = \hat{\rho}_\beta$ for some inverse temperature β. For $\mathcal{E}_c > 0$, this requires that the populations of $\hat{\rho}$ in the energy basis (coinciding with those of $\hat{\delta}_\rho$) are indeed thermal, but in the wrong order, and that the state $\hat{\rho}$ contains some coherence in the energy basis. An example is provided by the following qutrit density matrix, written in the energy basis:

$$\hat{\rho} = \begin{pmatrix} g_1 & c & 0 \\ c^* & g_3 & 0 \\ 0 & 0 & g_2 \end{pmatrix}, \quad g_i = \frac{e^{-\beta\varepsilon_i}}{\sum_j e^{-\beta\varepsilon_j}},\ |c| \leq \sqrt{g_1 g_3}\,.$$

For such a state, the three populations r_i are obtained by decreasingly ordering the set of numbers $\left\{\frac{g_1+g_3}{2} + \sqrt{\frac{(g_1-g_3)^2}{4} + |c|^2}; g_2; \frac{g_1+g_3}{2} - \sqrt{\frac{(g_1-g_3)^2}{4} + |c|^2}\right\}$, and the passive state $\hat{P}_\rho$ is obtained by taking the ordered set as energy level populations. On the other hand, $\hat{P}_\delta \equiv \hat{\rho}_\beta = \mathrm{diag}\{g_1, g_2, g_3\}$; but this thermal state does not have the same entropy as $\hat{\rho}$ (and β has nothing to do with β^*). Using the definitions above, we obtain $\mathcal{E} = \varepsilon_1(g_1 - r_1) + \varepsilon_2(g_3 - r_2) + \varepsilon_3(g_2 - r_3)$, while $\mathcal{E}_i = (\varepsilon_3 - \varepsilon_2)(g_2 - g_3)$. The difference between these two quantities gives $\mathcal{E}_c$, which saturates the lower bound in Eq. (7) (i.e., for these states, $D(\hat{P}_\delta||\hat{\rho}_\beta) = 0$).

Lastly, we can exploit Eq. (6) to investigate the convertibility of the states $\hat{P}_\delta$ and $\hat{P}_\rho$ under thermal operations, and endow this problem with an operational interpretation thanks to the definition of ergotropy. Since both these states commute with the Hamiltonian and are passive, their convertibility may be addressed within the resource theory of athermality [101–103]. In particular, if a thermal operation [102, 104] exists that takes $\hat{P}_\delta$ to $\hat{P}_\rho$ ($\hat{P}_\rho$ to $\hat{P}_\delta$), it follows that $D(\hat{P}_\delta||\hat{\rho}_\beta) - D(\hat{P}_\rho||\hat{\rho}_\beta) \equiv \beta\mathcal{E}_c - C(\hat{\rho}) \geq 0$ (≤ 0, respectively).

V. EXAMPLES

A. Qudits

In order to illustrate our results, we consider first the simple case of a qubit, having energy eigenvalues $\varepsilon_1 = 0$ and ε_2. In this case, any initial state $\hat{\rho}$ is transformed by the ergotropic transformation $\hat{E}$ into a passive state with a thermal structure $\hat{P}_\rho \equiv \hat{\rho}_{\beta^*}$, for a suitably chosen inverse temperature β^*. Then, $\Delta\mathcal{E}_c$ vanishes and the upper bound in Eq. (7) is saturated. Moreover, in this case, the coherent part of ergotropy can be directly expressed in terms of the purity of the state, $p(\hat{\rho}) = \mathrm{Tr}\left\{\hat{\rho}^2\right\}$ and of another coherence quantifier, the l_1 norm of coherence [64], defined as $C_{l_1}(\hat{\rho}) = 2\,|\langle\varepsilon_1|\,\hat{\rho}\,|\varepsilon_2\rangle|$. Indeed, some simple manipulations lead to

$$\mathcal{E}_c(\hat{\rho}) = \frac{\varepsilon_2}{2}\left(\sqrt{2p(\hat{\rho}) - 1} - \sqrt{2p(\hat{\rho}) - 1 - C_{l_1}^2(\hat{\rho})}\right). \tag{12}$$

This is proved by noticing that $\mathcal{E}_c(\hat{\rho}) = \varepsilon_2(\rho_{22} - r_2)$, where the smallest eigenvalue of $\hat{\rho}$ is $r_2 = \frac{1}{2}(1 - \sqrt{2p(\hat{\rho}) - 1})$, and where the smallest population of $\hat{\rho}$ is $\rho_{22} = \frac{1}{2} - \frac{1}{2}\sqrt{2p(\hat{\rho}) - 1 - C_{l_1}^2}$.

It follows from Eq. (12) that the ergotropy increases for any operation Ω with $p(\Omega(\hat{\rho})) < \frac{1}{2} + \frac{1}{2}\left(\frac{\mathcal{E}_c(\hat{\rho})}{\varepsilon_2} + \frac{1}{4}C_{l_1}^2(\Omega(\hat{\rho}))\frac{\varepsilon_2}{\mathcal{E}_c(\hat{\rho})}\right)^2$. In the Appendix we provide an example of an incoherent such operation – generalized amplitude damping – to prove that E_c is not a coherence monotone.

For a given value of the purity p, the coherence takes its maximum value for mixed states $\hat{\rho}$ with equal populations, $\rho_{11} = \rho_{22} = 1/2$, for which $p = (1 + C_{l_1}^2)/2$ and $\mathcal{E}_c = C_{l_1}/2$. It follows that $\mathcal{E}_c(\rho)$ is maximized if the initial state is a maximally coherent pure state with $C_{l_1} = 1$ and $p = 1$.

This latter observation is, in fact, more general: for a d-level system, we get the maximum value of $\mathcal{E}_c(\hat{\rho})$ (with, correspondingly, a null incoherent contribution $\mathcal{E}_i$) when $\hat{\rho}$ is a maximally coherent pure state, $\hat{\rho} = |\psi\rangle\langle\psi|$, with $|\psi\rangle =$

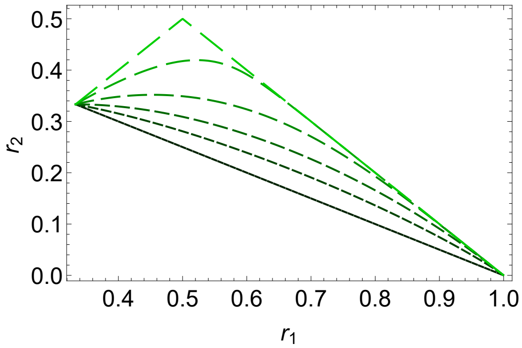

FIG. 2. In a three-level system, we identify the class of states which allow to saturate the right inequality in Eq. (7) by looking at a pair of eigenvalues, r_1 and r_2, for which $\Delta\mathcal{E}_c = 0$. The various lines refer to the cases in which the ratio of the second and third energy eigenvalues is given by $R = 0, 0.1, 0.3, 0.5, 0.7, 1$ (lighter dashed to darker solid lines).

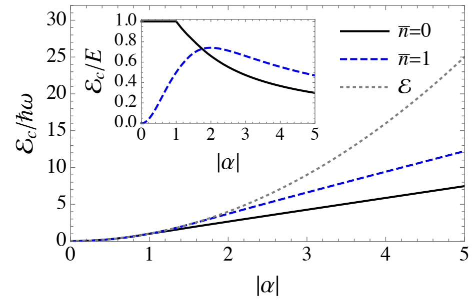

FIG. 3. Coherent ergotropy plotted in units of energy quanta (main) and as a fraction of the total energy (inset), for a displaced thermal state [Eq. (13)] of a single bosonic mode as a function of the displacement, α. The solid black line shows a pure coherent state, the dashed blue line shows a state with a thermal occupation of $\bar{n} = 1$, while the dotted grey line shows the total ergotropy (equal for both states).

$\sum_i |\varepsilon_i\rangle / \sqrt{d}$. In such a case, indeed, any incoherent unitary $\hat{V}_\pi$ preserves the average energy.

To discuss a less trivial case, where the upper bound in Eq. (7) is not always saturated, we now consider the behavior of the coherent part of ergotropy for a three-level system with energy eigenvalues $\varepsilon_1 = 0$, and $\varepsilon_2 = R\,\varepsilon_3$ (with $R \in (0,1)$). In particular, we ask under what conditions the bound is saturated (i.e., $\Delta\mathcal{E}_c = 0$). Selecting $\beta = \beta^*$ as required for saturation, Eq. 8 implies that once the energy values are fixed, what really matters are just the first two eigenvalues of the density matrix, r_1, r_2 (which fix the third one as $r_3 = 1 - r_1 - r_2$). For our three-level system, the bound ergotropy can be written as $\Delta\mathcal{E}_c = \varepsilon_3[r_2(R-1) + 1 - r_1 - Z^{-1}(Re^{-\beta^* R\varepsilon_3} + e^{-\beta^*\varepsilon_3})]$, where $Z = 1 + e^{-\beta^* R\varepsilon_3} + e^{-\beta^*\varepsilon_3}$. Looking for the values of r_1 and r_2 that give rise to a vanishing $\Delta\mathcal{E}_c$, we obtain the numerical results reported in Fig. 2, where we can appreciate that only under very stringent conditions on the eigenvalues of $\hat{\rho}$ one obtains a saturation of the inequality. For fixed R, all suitable eigenvalue pairs are confined to a single curve within the total (r_1, r_2)-plane.

B. Bosonic Gaussian states

Beyond finite-dimensional systems, our results can also be directly applied to bosonic Gaussian states. These states arise naturally in the description of weakly interacting fermions or bosons and are, by definition, related to a thermal state by a unitary transformation. As a consequence, they saturate the upper bound in Eq. (7).

Let us focus for simplicity on a single bosonic mode, with Hamiltonian $\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a}$, which is assumed to be in a Gaussian state of the form

$$\hat{\rho} = \hat{D}(\alpha)\hat{\rho}_\beta\hat{D}^\dagger(\alpha), \tag{13}$$

where $\hat{D}(\alpha) = e^{\alpha\hat{a} - \alpha^*\hat{a}^\dagger}$ is a unitary displacement operator. This could describe, for example, the mechanical motion of the trapped-ion heat engine reported in Ref. [15], whose vibrations act as a load or "flywheel" driven by a two-level working medium comprising the ion's electronic spin states. In that context, the displacement α arises from mechanical work performed by the engine on the load, while the thermal occupation $\bar{n} = (e^{\beta\hbar\omega} - 1)^{-1}$ is associated with random energy transfer due to thermal fluctuations of the working medium. The total energy of such a state is then given by $E = \mathrm{Tr}[\hat{H}\hat{\rho}] = \hbar\omega(|\alpha|^2 + \bar{n})$, while the total ergotropy is given simply by $\mathcal{E} = \hbar\omega|\alpha|^2$. We note that, since the dephasing operation Δ is non-Gaussian, it is difficult to obtain a simple closed-form expression for $\mathcal{E}_c$, but it can be readily computed numerically for small $|\alpha|$ and $\bar{n}$.

Fig. 3 displays the coherent part of the ergotropy evaluated for two different examples: a pure coherent state with $\bar{n} = 0$, and a displaced thermal state with $\bar{n} = 1$. For $\alpha \ll 1$, the population distribution (i.e., the dephased state $\hat{\delta}_\rho$) is passive and therefore all the ergotropy is coherent, i.e., $\mathcal{E}_c = \mathcal{E}$. Conversely, for large α, the coherent ergotropy is linear in the coherent displacement, $\mathcal{E}_c \propto |\alpha|$, while the total ergotropy is quadratic, $\mathcal{E} \propto |\alpha|^2$. Therefore, the energetics of Gaussian states with large displacement is dominated by the incoherent ergotropy, which is consistent with the quasi-classical nature of these states. The work content of such states derives primarily from the non-passivity of the population distribution.

Interestingly, increasing $\bar{n}$ for fixed α actually increases the coherent ergotropy. This is because, for a fixed value of α, thermal noise renders the population distribution more passive, thus decreasing $\mathcal{E}_i$ without changing the total ergotropy. This does not conflict with the obvious fact that, for fixed energy E, increasing $\bar{n}$ must imply that $|\alpha|$ is smaller and therefore both components of the ergotropy are reduced.

VI. SUMMARY AND CONCLUSIONS

In summary, in this paper we have highlighted the role of quantum coherence in work extraction processes, by identifying a contribution to the ergotropy that precisely corresponds to initial coherence in the energy basis. This is obtained by breaking the optimal, ergotropic, unitary cycle into an initial incoherent unitary operation, followed by a second unitary cycle through which one extracts work by exhausting the coherence. We have analyzed this coherent ergotropic contribution by exploring its range of possible values, which we have identified in terms of two bounds which can be saturated in specific cases. In particular, we discovered that the tightness of the upper bound is intimately related to the concept of bound ergotropy – a form of work potential that becomes available only when processing multiple identical copies of the system together. Finally, we have illustrated our results with the simplest non-trivial examples of a qubit and a qutrit, as well as a single-mode bosonic Gaussian state. The latter opens the possibility for future analysis of work extraction in continuous variable systems beyond unconstrained unitaries on single modes, considering, for instance, Gaussian operations, multiple modes, or both [105–109].

As quantum coherence is arguably the most primordial nonclassical effect in nature, we expect the framework described here to prove useful for the experimental characterisation of work production in quantum heat engines [15, 16], and, more generally, to help reveal and quantify the delicate fingerprints of genuinely quantum effects in non-equilibrium thermodynamic processes.

ACKNOWLEDGMENTS

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[1] Also known as Lord Kelvin outside of West Cork.

[2] W. Thomson, *XV.—On the Dynamical Theory of Heat, with numerical results deduced from Mr Joule's Equivalent of a Thermal Unit, and M. Regnault's Observations on Steam*, Earth and Enviromental Science Transactions of the Royal Society of Edinburgh, **20**, 261 (1853).

[3] J. Uffink, *Bluff your way in the second law of thermodynamics*, Stud. Hist. Phil. Mod. Phys., **32**, 305 (2001).

[4] W. Pusz and S. L. Woronowicz, *Passive states and KMS states for general quantum systems*, Commun. Math. Phys. **58**, 273 (1978).

[5] A. Lenard, *Thermodynamical proof of the Gibbs formula for elementary quantum systems*, J. Stat. Phys. **19**, 575 (1978).

[6] A. E. Allahverdyan and Th. M. Nieuwenhuizen, *A mathematical theorem as the basis for the second law: Thomson's formulation applied to equilibrium*, Physica A **305**, 542 (2002).

[7] A. E. Allahverdyan, R. Balian and Th. M. Nieuwenhuizen, *Maximal work extraction from finite quantum systems*, Europhys. Lett. **67**, 565 (2004).

[8] A similar quantity called *adiabatic availability* was introduced earlier in [9]. It corresponds to the ergotropy for separable processes.

[9] G. N. Hatsopoulos, and E. P. Gyftopoulos, *A unified quantum theory of mechanics and thermodynamics. Part IIa. Available energy*, Foundations of Physics **6**, 127 (1976).

[10] R. Kosloff, *Quantum Thermodynamics: A Dynamical Viewpoint* Entropy **15**, 2100 (2013).

[11] J. Goold, M. Huber, A. Riera, L del Rio, and P. Skrzypczyk, *The role of quantum information in thermodynamics*, J. Phys. A: Math. Theor. **49**, 143001 (2016).

[12] S. Vinjanampathy, and J. Anders, *Quantum thermodynamics*, Contemp. Phys. **57**, 545 (2016).

[13] M. T. Mitchison, *Quantum thermal absorption machines: refrigerators, engines and clocks*, Contemp. Phys. **60**, 164 (2019).

[14] *Thermodynamics in the Quantum Regime — Fundamental Aspects and New Directions*, (eds.) F. Binder, L. A. Correa, C. Gogolin, J. Anders, G. Adesso, Springer (2018).

[15] D. von Lindenfels, O. Gräb, C. T. Schmiegelow, V. Kaushal, J. Schulz, M. T. Mitchison, J. Goold, F. Schmidt-Kaler, and U. G. Poschinger, *Spin Heat Engine Coupled to a Harmonic-Oscillator Flywheel*, Phys. Rev. Lett. **123**, 080602 (2019).

[16] N. V. Horne, D. Yum, T. Dutta, P. Hänggi, J. Gong, D. Poletti, and M. Mukherjee, *Single-atom energy-conversion device with a quantum load*, npj Quantum Information **6**, 37 (2020).

[17] W. Niedenzu, M. Huber, and E. Boukobza, *Concepts of work in autonomous quantum heat engines*, Quantum **3**, 195 (2019)

[18] F. Binder, S. Vinjanampathy, K. Modi, Kavan, and J. Goold, John, *Quantum thermodynamics of general quantum processes*, Phys. Rev. E **91**, 032119 (2015)

[19] M. O. Scully, K. R. Chapin, K. E. Dorfman, M. B. Kim, and A. Svidzinsky, *Quantum heat engine power can be increased by noise-induced coherence*, Proc. Natl. Acad. Sci. U.S.A. **108**, 15097 (2011).

[20] S. Rahav, U. Harbola, and S. Mukamel, *Heat fluctuations and coherences in a quantum heat engine*, Phys. Rev. A **86**, 043843 (2012).

[21] N. Brunner, M. Huber, N. Linden, S. Popescu, R. Silva, and Paul Skrzypczyk, *Entanglement enhances cooling in microscopic quantum refrigerators*, Phys. Rev. E **89**, 032115 (2014)

[22] M. T. Mitchison, M. P. Woods, J. Prior, M. Huber, *Coherence-assisted single-shot cooling by quantum absorption refrigerators*, New J. Phys. **17**, 115013 (2015)

[23] J. Jaramillo, M. Beau, and A. del Campo, *Quantum supremacy of many-particle thermal machines*, New J. Phys. **18**, 075019 (2016).

[24] G. Watanabe, B. Prasanna Venkatesh, P. Talkner, A. del Campo, *Quantum Performance of Thermal Machines over Many Cycles*, Phys. Rev. Lett. **118**, 050601 (2017).

[25] K. Brandner, M. Bauer, and U. Seifert, *Universal Coherence-Induced Power Losses of Quantum Heat Engines in Linear Response*, Phys. Rev. Lett. **119**, 170602 (2017).

[26] J. Klaers, S. Faelt, A. Imamoglu, and E. Togan, *Squeezed Thermal Reservoirs as a Resource for a Nanomechanical Engine beyond the Carnot Limit*, Phys. Rev. X **7**, 031044 (2017)

[27] M. Kilgour and D. Segal, *Coherence and decoherence in quantum absorption refrigerators*, Phys. Rev. E **98**, 012117 (2018).

[28] V. Holubec and T. Novotný, *Effects of noise-induced coherence on the performance of quantum absorption refrigerators*, J. Low Temp. Phys. **192**, 147 (2018).

[29] J. Klatzow et al., *Experimental Demonstration of Quantum Effects in the Operation of Microscopic Heat Engines*, Phys. Rev. Lett. **122**, 110601 (2019).

[30] L. Buffoni, A. Solfanelli, P. Verrucchi, A. Cuccoli, and M. Campisi, *Quantum Measurement Cooling*, Phys. Rev. Lett. **122**, 070603 (2019).

[31] R. Dann, R. Kosloff, *Quantum signatures in the quantum Carnot cycle*, New J. Phys., **22**, 013055 (2020).

[32] B. Karimi, J. P. Pekola, *Otto refrigerator based on a superconducting qubit: Classical and quantum performance*, Phys. Rev. B **94**, 184503 (2016).

[33] J. P. Pekola, B. Karimi, G. Thomas, D. V. Averin *Supremacy of incoherent sudden cycles*, Phys. Rev. B **100**, 085405 (2019).

[34] J. P. S. Peterson et al., *Experimental Characterization of a Spin Quantum Heat Engine*, Phys. Rev. Lett. **123**, 240601 (2019).

[35] K. Funo, Y. Watanabe, and M. Ueda, *Thermodynamic work gain from entanglement*, Phys. Rev. A **88**, 052319 (2013).

[36] K. V. Hovhannisyan, M. Perarnau-Llobet, M. Huber, A. Acín, *Entanglement Generation is Not Necessary for Optimal Work Extraction*, Phys. Rev. Lett. **111**, 240401 (2013).

[37] P. Skrzypczyk, A. J. Short, and S. Popescu, *Work extraction and thermodynamics for individual quantum systems*, Nat. Commun. **5**, 4185 (2014).

[38] M. Perarnau-Llobet, K. V. Hovhannisyan, M. Huber, P. Skrzypczyk, N. Brunner, and A. Acín, *Extractable Work from Correlations*, Phys. Rev. X **5**, 041011 (2015).

[39] K. Korzekwa, M. Lostaglio, J. Oppenheim and D. Jennings, *The extraction of work from quantum coherence*, New J. Phys. 18 023045 (2016).

[40] C. Elouard, D. Herrera-Martí, B. Huard, A. Auffèves, *Extracting Work from Quantum Measurement in Maxwell Demon Engines*, Phys. Rev. Lett. **118**, 260603 (2017).

[41] N. Cottet, S. Jezouin, L. Bretheau, P. Campagne- Ibarcq, Q. Ficheux, J. Anders, A. Auffèves, R. Azouit, P. Rouchon, B. Huard, *Observing a quantum Maxwell demon at work*, Proc. Natl. Acad. Sci. U.S.A. **114**, 7561 (2017).

[42] G. Manzano, F. Plastina, R. Zambrini, *Optimal Work Extraction and Thermodynamics of Quantum Measurements and Correlations*, Phys. Rev. Lett. **121**, 120602 (2018).

[43] B. Morris, L. Lami, G. Adesso, *Assisted Work Distillation*, Phys. Rev. Lett. **122**, 130601 (2019).

[44] G. Vitagliano, C. Klöckl, M. Huber, and N. Friis *Trade-off Between Work and Correlations in Quantum Thermodynamics*, Thermodynamics in the Quantum Regime, Chap. 30, p. 731–750 (Springer, 2019)

[45] J. Monsel, M. Fellous-Asiani, B. Huard, A. Auffèves, *The Energetic Cost of Work Extraction* Phys. Rev. Lett. **124**, 130601 (2020).

[46] G.-L. Giorgi and S. Campbell, *Correlation approach to work extraction from finite quantum systems*, J. Phys. B **48**, 035501 (2015).

[47] G. Francica, J. Goold, F. Plastina, M. Paternostro, *Daemonic Ergotropy: Enhanced Work Extraction from Quantum Correlations*, npj Quantum Information **3**, 12 (2017).

[48] F. Bernards, M. Kleinmann, O. Gühne, M. Paternostro *Daemonic Ergotropy: Generalised Measurements and Multipartite Settings*, Entropy **21**, 771 (2019).

[49] A. E. Allahverdyan, *Nonequilibrium quantum fluctuations of work*, Phys. Rev. E **90**, 032137 (2014).

[50] P. Talkner and P. Hänggi, *Aspects of quantum work*, Phys. Rev. E **93**, 022131 (2016).

[51] M. Perarnau-Llobet, E. Bäumer, K. V. Hovhannisyan, M. Huber, A. Acin, *No-Go Theorem for the Characterization of Work Fluctuations in Coherent Quantum Systems*, Phys. Rev. Lett. **118**, 070601 (2017).

[52] P. Solinas, S. Gasparinetti, *Probing quantum interference effects in the work distribution*, Phys. Rev. A **94**, 052103 (2016).

[53] P. Solinas, H. J. D. Miller, and J. Anders, *Measurement-dependent corrections to work distributions arising from quantum coherences*, Phys. Rev. A **96**, 052115 (2017).

[54] M. Lostaglio, *Quantum Fluctuation Theorems, Contextuality, and Work Quasiprobabilities*, Phys. Rev. Lett. **120**, 040602 (2018).

[55] J. Åberg, *Fully Quantum Fluctuation Theorems*, Phys. Rev. X **8**, 011019 (2018).

[56] R. Alicki, M. Fannes, *Entanglement boost for extractable work from ensembles of quantum batteries*. Phys. Rev. E **87**, 042123 (2013).

[57] F. C. Binder, S. Vinjanampathy, K. Modi, J. Goold, *Quantacell: powerful charging of quantum batteries* , New J. Phys. **17**, 075015 (2015)

[58] Campaioli *et al.*, *Enhancing the Charging Power of Quantum Batteries*, Phys. Rev. Lett **118**, 150601 (2017).

[59] D. Ferraro, M. Campisi, G. M. Andolina, V. Pellegrini, M. Polini, *High-Power Collective Charging of a Solid-State Quantum Battery*, Phys. Rev. Lett. **120**, 117702 (2018).

[60] G. M. Andolina, M. Keck, A. Mari, M. Campisi, V. Giovannetti, and M. Polini, *Extractable Work, the Role of Correlations, and Asymptotic Freedom in Quantum Batteries*, Phys. Rev. Lett. 122, 047702 (2019).

[61] S. Julià-Farré, T. Salamon, A. Riera, M. N. Bera, M. Lewenstein, *Bounds on the capacity and power of quantum batteries*, Phys. Rev. Res. **2**, 023113 (2020).

[62] L. P. García-Pintos, A. Hamma, A. del Campo, *Fluctuations in stored work bound the charging power of quantum batteries*, preprint arXiv:1909.03558.

[63] T. Baumgratz, M. Cramer, and M.B. Plenio, *Quantifying Coherence*, Phys. Rev. Lett. **113**, 140401 (2014).

[64] A. Streltsov, G. Adesso, and M. B. Plenio, *Quantum coherence as a resource*, Rev. Mod. Phys. 89, 041003 (2017).

[65] J. Åberg, *Catalytic Coherence*, Phys. Rev. Lett. **113**, 150402 (2014).

[66] M. Lostaglio, D. Jennings, and T. Rudolph, *Description of quantum coherence in thermodynamic processes requires constraints beyond free energy*, Nat. Commun. **6**, 6383 (2015).

[67] M. Lostaglio, K. Korzekwa, D. Jennings, and T. Rudolph, *Quantum Coherence, Time-Translation Symmetry, and Thermodynamics*, Phys. Rev. X, **5**, 021001 (2015).

[68] R. Uzdin, A. Levy, and R. Kosloff, *Equivalence of Quantum Heat Machines, and Quantum-Thermodynamic Signatures*, Phys. Rev. X **5**, 031044 (2015)

[69] P. Kammerlander, J. Anders, *Coherence and measurement in quantum thermodynamics*, Sci. Rep. **6**, 22174 (2016).

[70] S. Kallush, A. Aroch, R. Kosloff, *Quantifying the unitary generation of coherence from thermal operations*, Entropy **21**, 810 (2019).

[71] A. Purkayastha, G. Guarnieri, M. T. Mitchison, R. Filip, J. Goold, *Tunable phonon-induced steady-state coherence in a double-quantum-dot charge qubit*, npj Quantum Information **6** (1), 1-7 (2020)

[72] G. Guarnieri, D. Morrone, B. Çakmak, F. Plastina, S. Campbell, *Non-equilibrium steady-states of memoryless quantum collision models*, Phys. Lett. A **384**, 126576 (2020).

[73] C. L. Latune, I. Sinayskiy, F. Petruccione, *Heat flow reversals without reversing the arrow of time: the role of internal quantum coherences and correlations*, Phys. Rev. Research **1**, 033097 (2019).

[74] B.Çakmak, *Ergotropy from coherences in an open quantum system*, arXiv:2005.08489.

[75] M. J. Donald, *Free energy and the relative entropy*, J. Stat. Phys. **49**, 81 (1987).

[76] S. Deffner and E. Lutz, *Generalised Claussius Inequality for Nonequillibrium Quantum Processes*, Phys. Rev. Lett. **105**, 170402 (2010).

[77] F. Plastina, A. Alecce, T.J.G. Apollaro, G. Falcone, G. Francica, F. Galve, N. Lo Gullo, and R. Zambrini, *Irreversible Work and Inner Friction in Quantum Thermodynamic Processes*, Phys. Rev. Lett. **113**, 260601 (2014).

[78] H. Spohn, *Entropy production for quantum dynamical semigroups*, J. Math. Phys. **19**, 1227 (1978).

[79] H. Spohn and J. Lebowitz, *Irreversible thermodynamics for quantum systems weakly coupled to thermal reservoirs*, Adv. Chem. Phys. **38** 109 (1978).

[80] M. Esposito, K. Lindenberg and C. Van den Broeck, *Entropy production as a correlation between systems and reservoir*, New Journal of Physics **12** 013013 (2010).

[81] S. Deffner and E. Lutz, *Nonequilibrium Entropy Production for Open Quantum Systems*, Phys. Rev. Lett. **107** 140404 (2011).

[82] G. Guarnieri, G.T. Landi, S. R. Clark and J. Goold, *Thermodynamics of precision in quantum nonequilibrium steady states*, Phys. Rev. Research **1** 033021 (2019).

[83] P. A. Camati et al., *Experimental Rectification of Entropy Production by Maxwell's Demon in a Quantum System*, Phys. Rev. Lett. **117**, 240502 (2016).

[84] M. H. Ansari, A. van Steensel, Yu. V. Nazarov, *Entropy Production in Quantum is Different*, Entropy **21**, 854 (2019).

[85] K. Ptaszyński, M. Esposito, *Entropy Production in Open Systems: The Predominant Role of Intraenvironment Correlations*, Phys. Rev. Lett. **123**, 200603 (2019).

[86] G. Francica, J. Goold, F. Plastina, *The role of coherence in the non-equilibrium thermodynamics of quantum systems*, Phys. Rev. E **99**, 042105 (2019).

[87] J.P. Santos, L.C. Céleri, G.T. Landi and M. Paternostro, *The role of quantum coherence in non-equilibrium entropy production*, npj Quantum Info. **5**, 23 (2019).

[88] P. M. Riechers, M. Gu, *Initial-State Dependence of Thermodynamic Dissipation for any Quantum Process*, arXiv:2002.11425 (2020).

[89] A. D. Varizi, A. P. Vieira, C. Cormick, R. C. Drumond, G. T. Landi, *Quantum coherence and criticality in irreversible work*, preprint arXiv:2004.00616 (2020).

[90] E. Chitambar, G. Gour, *Quantum resource theories*, Rev. Mod. Phys. **91**, 025001 (2019)

[91] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Quantum entanglement*, Rev. Mod. Phys. **81**, 865 (2009)

[92] A. Rastegin, *Quantum-coherence quantifiers based on the Tsallis relative α-entropies*, Phys. Rev. A **93**, 032136 (2016)

[93] If the Hamiltonian has degenerate levels, the passive state is defined up to unitaries acting in each degenerate subspace. Through such a unitary, it is possible to change the coherence without altering the ergotropy, which, thus, is independent of the coherence between degenerate states. To avoid counting this kind of coherence in $C(\hat{\rho})$, we can always choose the energy eigenbasis such that $\hat{\rho}$ is diagonal in every degenerate subspace.

[94] B. Yadin, J. Ma, D. Girolami, M. Gu, V. Vedral, *Quantum processes which do not use coherence*, Phys. Rev. X **6**, 041028 (2016)

[95] Y. Peng, Y. Jiang, and H. Fan, *Maximally coherent states and coherence-preserving operations*, Phys. Rev. A **93**, 032326 (2016).

[96] Just to fix the notation for the permutation, let $d = 4$, and take π such that $\pi(\{1,2,3,4\}) = \{3,2,4,1\}$. In this case, we write $\pi_1 = 3, \pi_2 = 2, \pi_3 = 4, \pi_4 = 1$. The inverse permutation, π^{-1}, is the one such that $\pi^{-1}(\pi(\{1,2,3,4\})) = \{1,2,3,4\}$. Thus, in this case, $\pi^{-1}(\{1,2,3,4\}) = \{4,2,1,3\}$, and $\pi_1^{-1} = 4, \pi_2^{-1} = 2, \pi_3^{-1} = 1, \pi_4^{-1} = 3$. It follows that, if $\pi_k = k'$, then, $k = \pi_{k'}^{-1}$.

[97] We note that, when the state $\hat{\rho}$ does not commute with H it evolves in the time. Thus, the application of the ergotropic unitary cyle $\hat{E}$ requires a very precise timing. On the other hand, this is not the case for the incoherent operation $\hat{V}_\pi$.

[98] This equivalence is a direct consequence of SIOs being a subset of Dephasing-covariant Incoherent Operations (DIOs), which are those quantum channels that commute with full dephasing [99, 100]. Here, this implies that $\hat{V}_{\tilde{\pi}}$ followed by Δ and Δ followed by $\hat{V}_{\tilde{\pi}}$ both result in the same state $\hat{P}_\rho$, and hence the same definition of $\mathcal{E}_i$.

[99] E. Chitambar, and G. Gour, *Critical Examination of Incoherent Operations and a Physically Consistent Resource Theory of Quantum Coherence*, Physical Review Letters **117**, 030401 (2016)

[100] B. Yadin, *Resource theories of quantum coherence: foundations and applications*, DPhil thesis, Wolfson College, University of Oxford (2017)

[101] D. Janzing, P. Wocjan, R. Zeier, R. Geiss, and T. Beth, *Thermodynamic cost of reliability and low temperatures: Tightening Landauer's principle and the second law*, International Journal of Theoretical Physics **39**, 2717-2753 (2000)

[102] F.G.S.L. Brandão, M. Horodecki, J. Oppenheim, J.M Renes, and R.W. Spekkens, *Resource Theory of Quantum States Out of Thermal Equilibrium*, Phys. Rev. Lett. **111**, 250505 (2013)

[103] F. Brandão, M. Horodecki, N. Ng, J. Oppenheim, and S. Wehner, *The second laws of quantum thermodynamics*, PNAS **112**, 3275–3279 (2015)

[104] N.H.Y. Ng, and M.P. Woods, *Resource Theory of Quantum Thermodynamics: Thermal Operations and Second Laws*, Thermodynamics in the Quantum Regime, Chap. 26, p. 625 (Springer, 2019)

[105] E. G. Brown, N. Friis, and M. Huber. *Passivity and practical work extraction using gaussian operations*. New Journal of Physics, **18**(11), 113028 (2016)

[106] N. Friis, M. Huber, *Precision and Work Fluctuations in Gaussian Battery Charging*, Quantum **2**, 61 (2018)

[107] U. Singh, M. G. Jabbour, Z. Van Herstraeten, and N. J. Cerf, *Quantum thermodynamics in a multipartite setting: A resource theory of local Gaussian work extraction for multimode bosonic systems*, Phys. Rev. A **100**, 042104 (2019)

[108] V. Narasimhachar, S. Assad, F. C. Binder, J. Thompson, B. Yadin, and M. Gu, *Thermodynamic resources in continuous-variable quantum systems*, arXiv:1909.07364 (2019)

[109] A. Serafini, M. Lostaglio, S. Longden, U. Shackerley-Bennett, C.-Y. Hsieh, and G. Adesso, *Gaussian Thermal Operations and The Limits of Algorithmic Cooling*, Phys. Rev. Lett. **124**, 010602 (2020)

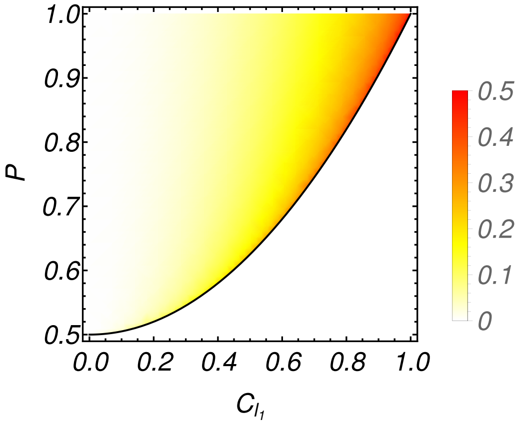

FIG. 4. The density plot of the work $\mathcal{E}_c$ as a function of the purity $p(\hat{\rho})$ and the coherence monotone C_{l_1}. The black line is $p(\hat{\rho}) = (1 + C_{l_1}^2)/2$.

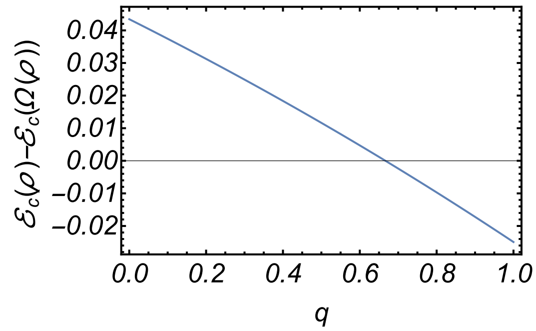

FIG. 5. The plot shows the difference $\mathcal{E}_c(\hat{\rho}) - \mathcal{E}_c(\Omega(\hat{\rho}))$ as a function of the parameter q.

Appendix A: $\mathcal{E}_c$ is not a coherence monotone

In this Appendix we show that $\mathcal{E}_c$ is not a coherence monotone, i.e., there is some incoherent operation Λ such that $\mathcal{E}_c(\Lambda(\hat{\rho})) \not\leq \mathcal{E}_c(\hat{\rho})$. We focus on the qubit example from the main text with for which, choosing $\varepsilon_2 = 1$,

$$\mathcal{E}_c(\hat{\rho}) = \frac{1}{2}\left(\sqrt{2p(\hat{\rho}) - 1} - \sqrt{2p(\hat{\rho}) - 1 - C_{l_1}^2(\hat{\rho})}\right) \quad \text{(A1)}$$

An operation Λ is an incoherent operation (IO) if it can be represented in terms of Kraus operators K_i such that $K_i \hat{\rho}_i K_i^\dagger$ is proportional to an incoherent state for all i and incoherent inputs $\hat{\rho}_i$ [64]. For any such Λ we have that $C_{l_1}(\Lambda(\hat{\rho})) \leq C_{l_1}(\hat{\rho})$. In contrast, as observed in the main text, if the purity $p(\Lambda(\hat{\rho}))$ is smaller than $p_c \equiv \frac{1}{2} + \frac{1}{2}\left(\mathcal{E}_c(\hat{\rho}) + \frac{C_{l_1}^2(\Lambda(\hat{\rho}))}{4\mathcal{E}_c(\hat{\rho})}\right)^2$ it turns out that $\mathcal{E}_c(\Lambda(\hat{\rho})) > \mathcal{E}_c(\hat{\rho})$ (see Fig. 4).

As an example of incoherent operations, we now consider the generalized amplitude damping map

$$\Omega(\hat{\rho}) = \sum E_j \hat{\rho} E_j^\dagger \quad \text{(A2)}$$

with Kraus operators

$$E_0 = \sqrt{q}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix},$$
$$E_1 = \sqrt{q}\begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix},$$
$$E_2 = \sqrt{1-q}\begin{pmatrix} \sqrt{1-\gamma} & 0 \\ 0 & 1 \end{pmatrix},$$
$$E_3 = \sqrt{1-q}\begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix}$$

Ω maps an initial state $\hat{\rho} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$ into

$$\Omega(\hat{\rho}) = (1-\gamma)\begin{pmatrix} \rho_{11} & 0 \\ 0 & \rho_{22} \end{pmatrix} + \sqrt{1-\gamma}\begin{pmatrix} 0 & \rho_{12} \\ \rho_{21} & 0 \end{pmatrix} + \gamma\begin{pmatrix} q & 0 \\ 0 & 1-q \end{pmatrix}. \quad \text{(A3)}$$

Here, we have separated the coherent and incoherent contribution from the state $\hat{\rho}$, as well as the state-independent contribution.

In Fig. 5 we study $\mathcal{E}_c(\hat{\rho}) - \mathcal{E}_c(\Omega(\hat{\rho}))$ as a function of the parameter q, for $\gamma = 1/10$, $\rho_{11} = 1/3$ and $\rho_{12} = \sqrt{\rho_{11}\rho_{22}}$. Notice that when the purity $p(\Omega(\hat{\rho}))$ is smaller than p_c, we have that $\mathcal{E}_c(\Omega(\hat{\rho})) > \mathcal{E}_c(\hat{\rho})$. Therefore, $\mathcal{E}_c$ is not a coherence monotone under incoherent operations.