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Non-planar anomalous dimensions in super Yang–Mills theories

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Declaration

I declare that this thesis has not been submitted as an exercise for a degree at this or any other university and it is entirely my own work. I agree to deposit this thesis in the University's open access institutional repository or allow the Library to do so on my behalf, subject to Irish Copyright Legislation and Trinity College Library conditions of use and acknowledgement. I consent to the examiner retaining a copy of the thesis beyond the examining period, should they wish so.
Parts of this thesis are based on the publications [1, 2], the preprint [3], as well as on unpublished work [4], which contain work in collaboration with Riccardo Gonzo, Tristan McLoughlin, Diego Medrano and Raul Pereira.

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Summary

Conformal supersymmetric Yang–Mills theories play an important role in the gauge-gravity correspondence and, despite being highly non-physical, have been a driving force for many new approaches in more realistic theories like QCD and gravity. An important class of objects in conformal field theories is the spectrum of scaling dimensions of local operators, specifically their non-trivial coupling-dependent parts, the anomalous dimensions. The discovery of integrability in planar maximally supersymmetric Yang–Mills theory led to considerable advances in the computation of its anomalous-dimension spectrum. Less is known at the non-planar level where the theory is assumed to be non-integrable. In this thesis we consider non-planar anomalous dimensions in conformal supersymmetric Yang–Mills theories with gauge group $SU(N)$ and approach them by a number of means.

First, we use an on-shell form-factor approach based on the intimate connection between the dilatation operator and scattering amplitudes. The former gives rise to operator mixing and its diagonalisation gives the operators' anomalous dimensions. The latter are basic observables in any quantum field theory, describing its interactions and linking theoretical developments to experimental investigations. A lot of progress has been made in recent years in the study of scattering amplitudes due to the advent of on-shell methods which circumvent many difficulties of more traditional approaches, and we use some of these here to extract the dilatation operator in certain sectors of the theories considered. In particular, we study a set of dimension-4 operators in $\mathcal{N} = 4$ supersymmetric Yang–Mills theory that is relevant for the mixing of the theory's on-shell Lagrangian, and compute the spectrum of non-planar anomalous dimensions in this sector. Furthermore, we extract the general form of the one-loop dilatation operator in the sector of purely scalar operators in the β-deformed version of this theory.

In the planar limit of the theories considered in this thesis, the dilatation operator maps to a spin-chain Hamiltonian that can be diagonalised by integrability techniques, in particular a suitable Bethe ansatz. In this mapping the spectrum of anomalous dimensions becomes the energy spectrum of the corresponding spin chains. When going away from the planar limit, integrability is lost, but we can compute non-planar corrections to the planar spectrum using Rayleigh–Schrödinger perturbation theory.

Using the basis of Bethe states, we compute matrix elements of the deformed and undeformed dilatation operator relevant in this approach. We find compact expressions in terms of off-shell scalar products and hexagon-like functions. We then use non-degenerate perturbation theory to compute the leading $1/N^2$ corrections to operator dimensions and as an example compute the large R-charge limit for two-excitation states through subleading order.

Finally, we numerically study statistical properties of large sets of anomalous dimensions which we obtain from a direct diagonalisation of the dilatation operators discussed in this thesis. Specifically, we analyse the distribution of level spacings in these spectra and find universal features: in the planar limit it follows the Poisson distribution characteristic of integrable systems, and at finite values of N it transitions to the Wigner–Dyson distribution of the Gaussian orthogonal ensemble of random matrix theory. This provides numerical evidence that perturbative non-planar anomalous-dimension spectra are quantum-chaotic, which is further supported by similar findings in the spectral rigidity measuring long-range interactions in the spectra. We also demonstrate that the finite-N eigenvectors possess properties of chaotic states.

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Contents

Introduction

Quantum field theory (QFT) is the most powerful known theoretical framework to describe the elementary particles of nature and their interactions, and its predictions are being tested to ever higher precision in experiments. But despite having been studied for decades, only few of its predictions are exact results and most computations rely on perturbation theory. Toy models can pave a way towards new non-perturbative approaches, and can help us in overcoming problems in the more physically realistic setups. A prime example in this quest is $\mathcal{N} = 4$ supersymmetric Yang–Mills ($\mathcal{N} = 4$ sYM) theory which is the most symmetric renormalisable quantum field theory known in four dimensions. It is a quantum conformal field theory (CFT) with the maximal amount of supersymmetry, and is uniquely fixed up to the choice of the gauge group. Most commonly, it is studied with gauge group $SU(N)$. In the limit of infinite N, the so-called planar limit, the theory's symmetry structure is even further enhanced, and $\mathcal{N} = 4$ sYM theory becomes integrable. Here exact results can be achieved. But the planar limit is a highly unphysical limit and in this thesis we study $\mathcal{N} = 4$ sYM observables in the non-planar theory. By furthering our understanding of this theory we might not only hope to gain new techniques for the study of more generic QFTs, but due to its duality with a gravitational theory defined on an Anti-de-Sitter (AdS) background in the context of the AdS/CFT correspondence [5], we simultaneously learn more about the dual type IIB string theory, with strings propagating on $AdS_5 \times S^5$.

The basic objects of interest in conformal field theories like $\mathcal{N} = 4$ sYM theory are correlation functions. The conformal symmetry puts strong constraints on their structure and in particular the two-point correlation functions of local operators are completely determined up to the operators' scaling dimensions. An important class of operators in superconformal field theories are primary operators, which are annihilated by the lowering operators of the superconformal algebra, and they form the seeds from which descendant operators can be obtained by the action of the complementary raising operators. For scalar primary operators $\mathcal{O}_i(x)$, the correlation function can be brought into the form

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{\delta_{ij}}{|x_1 - x_2|^{2\Delta_i}} \ , \tag{1}$$

with Δ_i denoting the scaling dimension of operator $\mathcal{O}_i$. The conformal symmetry ensures that the scaling dimension of descendant states corresponds to that of the associated primary operator. Thus the spectrum of scaling dimensions of primaries, together with the structure constants of three-point functions, form the CFT data of the theory; any higher-point correlation function can be computed from this data via an operator product expansion (OPE).

Given the absence of any physical energy scales, the problem of finding the spectrum of scaling dimensions takes on the role of the Hamiltonian spectral problem in conformal sYM theories. The analogue of the Hamiltonian is the dilatation operator $\mathfrak{D}$ and we are interested in solving its eigenvalue problem

$$\mathfrak{D} \cdot \mathcal{O}_i(0) = \Delta_i \mathcal{O}_i(0) \ . \tag{2}$$

The Δ_i equal the bare dimensions of operators, Δ_i^0, at tree level, but are corrected in the quantum theory,

$$\Delta_i = \Delta_i^0 + \gamma_i(g_{\text{YM}}, N, ...) \ . \tag{3}$$

The coupling-dependent part γ_i is the so-called anomalous dimension and is a non-trivial function of the parameters of the theory, e.g. in $\mathcal{N} = 4$ sYM theory it is a function of the gauge coupling g_{YM} and the rank N of the gauge group. The problem of finding anomalous dimensions has been of continued interest due to its role as a proving ground for novel calculational techniques and because of its importance in the AdS/CFT correspondence, where the spectrum of scaling dimensions matches up with the string energy spectrum.

In this thesis we study the spectrum of anomalous dimensions perturbatively in the 't Hooft coupling $\lambda = g_{\text{YM}}^2 N$, and thus expand the dilatation operator as

$$\mathfrak{D} = \sum_k g^{2k} \mathfrak{D}_{2k} \qquad \text{with} \qquad g^2 = \frac{\lambda}{16\pi^2} \ . \tag{4}$$

There are a number of results for the $\mathcal{N} = 4$ sYM dilatation operator, and in particular it was found at one-loop order for the $\mathfrak{so}(6)$ sector in [6–9] and for the full theory in [10]. At each order in g^2 we can further consider the large-N expansion of the operator dimensions and a key development was the insight that in the planar limit $N \to \infty$ for the $\mathfrak{so}(6)$ sector of operators the one-loop, $\mathcal{O}(g^2)$, dilatation operator acts like an integrable spin-chain Hamiltonian [11]. Therefore the leading large-N anomalous dimensions can be computed by means of an integrable spin chain, making the problem in some sense exactly solvable. This was subsequently extended to the full one-loop theory [12] and to higher orders in perturbation theory [13] as well as being observed at strong coupling [14]. Thus the full planar theory is conjectured to be integrable,

and there are even non-perturbative results from the thermodynamic Bethe ansatz and the Quantum Spectral Curve (QSC) [15–18]. The latter interpolates between the weak-coupling gauge-theory regime and the strong-coupling string-theory regime and is conjectured to capture the finite-λ data of the planar theory.

The planar limit corresponds to the limit of an infinite number of gluon colour charges. In physical quantum field theories like QCD, this number is finite and thus it is important to understand whether and how the structures and techniques in the planar limit carry over to the non-planar regime of the theory. With the spectrum of $\mathcal{N} = 4$ sYM anomalous dimensions being the starting point of many advances in planar sYM theories, they are also a natural starting point for explorations towards a deeper understanding of non-planar theories, where considerably less is known. In particular there is no systematic direct approach to the computation of the non-planar anomalous dimension spectrum. There are a number of impressive perturbative results for specific operators, e.g. twist-two operators at four loops were studied and the four-loop non-planar correction to the cusp anomalous dimension was computed [19]. Furthermore, non-planar scaling dimensions can be obtained indirectly from the hexagon formalism for correlation functions [20–23] from OPE limits, or by evaluating four-point functions with two operators taken to be the identity [24]. In this work we obtain perturbative non-planar anomalous dimensions from diagonalising the dilatation operator itself by computing $1/N$ corrections about the planar limit using Rayleigh–Schrödinger perturbation theory. In order to do so, we split the dilatation operator into a leading planar part and subleading off-diagonal terms, which mix single- and multi-trace operators, and then compute matrix elements of the subleading terms in the basis of planar eigenstates. Such an approach was previously used to compute the large R-charge limit of non-planar dimensions of two-impurity BMN operators in the $\mathfrak{su}(2)$ sector at one- and two-loop level [6–9]. In this work we will generalise this approach for the one-loop dilatation operator in the $\mathfrak{su}(2)$ sector by making use of Bethe states describing arbitrary numbers of excitations or magnons.

Although the perturbative approach to non-planar data is promising at first sight, this procedure is made complicated by degeneracies in the planar spectrum [9, 13, 25]. Thus degenerate perturbation theory has to be employed, which requires solving a non-trivial mixing problem. We will instead consider deformations of $\mathcal{N} = 4$ sYM theory for which these degeneracies are lifted. In particular, we consider β-deformed $\mathcal{N} = 4$ sYM theory which is a marginal deformation of the maximally supersymmetric theory preserving $\mathcal{N} = 1$ supersymmetry. It is a special case of the Leigh–Strassler deformations [26] where products of fields get twisted by factors of $q = \exp(i\beta)$, and the Yang–Mills coupling g_{YM} in the superpotential gets deformed to κ. We concentrate on the case $\beta \in \mathbb{R}$ for which the theory is exactly conformal to all loop orders in the planar limit [27] with $|\kappa|^2 = g_{\text{YM}}^2$, and the planar spectral problem for the β-deformed

theory is described by an integrable twist of the undeformed spin chain.

The planar one-loop dilatation operator in the β-deformed theory has been previously studied in [28, 29] and in this work we derive it at finite N in the scalar sector and use it for quantum-mechanical perturbation theory also in the deformed theory. There are a number of techniques to obtain the dilatation operator in sYM theories, in particular the complete one-loop dilatation operator for $\mathcal{N} = 4$ sYM theory was found in [30] by a few Feynman diagram computations and symmetry arguments. Our approach of choice is a form-factor approach developed in [31], based on an earlier approach [32], relating the action of the dilatation operator to cuts of form factors. Importantly, this approach mainly relies on on-shell techniques that overcome difficulties related to gauge redundancies and the proliferation of diagrams in the usual perturbative approach by constructing observables directly from smaller on-shell building blocks. Here we will use it to find the one-loop deformed dilatation operator including non-planar double-trace contributions in the scalar sector of the theory to then study non-planar corrections to scaling dimensions also in the deformed theory. We find compact expressions for non-planar matrix elements relevant for the leading $1/N$-corrections and as an application of the method compute the large R-charge limit for two-excitation states.

In another attempt to further our understanding of non-planar sYM theories, in this thesis we also explore statistical behaviours of large sets of anomalous dimensions. Since the work of Wigner [33], as well as Porter and Rosenzweig [34] on the statistical properties of the energy levels of highly-excited nuclei, the study of spectral statistics has been a common technique to understand general properties of a system, while avoiding the exact analytic solution of the system's spectral problem. This is especially advantageous for very complex systems for which finding or solving the system's equations seems impossible. Here we will study spectra of planar, as well as finite-N anomalous dimensions of sYM theories. In particular, we will be interested in the phenomenon of level repulsion: It is characteristic of chaotic systems where energy levels are correlated and so avoid each other, while in integrable systems levels are uncorrelated and move independently, crossing on occasion. The phenomenon of (non-)repulsing energy levels can be studied by looking at the distribution of spacings between neighbouring energy levels. If one computes the probability $P(s)ds$ that the normalised spacing between adjacent levels lies in the interval between s and $s+ds$, one finds that for a generic, chaotic, quantum system $P(s) \to 0$ as $s \to 0$. For integrable systems it is generally the case that $P(s)$ goes to a constant as $s \to 0$ which reflects the presence of hidden symmetries in these models. In addition to the occurrence of level repulsion, energy spectra of chaotic systems can generally be described by ensembles of random matrix theory (RMT) [35], and the statistics of the nearest-neighbour level spacings are closely approximated by the corresponding Wigner–Dyson distribution.

This has often been taken as a defining property of chaotic quantum systems and has been seen in a wide variety of areas ranging from condensed matter physics to quantum gravity. By contrast, the statistics of spacings in integrable systems are generally described by the Poisson distribution [36]. In this work we numerically study the spectrum of both the deformed and undeformed $\mathcal{N} = 4$ sYM theory. This is particularly interesting in the context of $\mathcal{N} = 4$ sYM theory being the canonical example of a holographic theory, and many of the recent developments in quantum many-body chaos have come from the connection to black-hole physics [37–41]. We will find that in the planar limit the spectral distribution is Poisson, consistent with integrability, while at finite N the distribution is Wigner–Dyson and corresponds to that of the Gaussian orthogonal ensemble (GOE) random matrix theory.

Overview. This thesis is structured as follows. After this general introduction we review $\mathcal{N} = 4$ sYM theory and its marginal deformations in Chapter 1. This includes a discussion of the undeformed theory in Section 1.1 with its symmetries and field content, the dilatation operator and anomalous dimensions. Further important concepts necessary for subsequent chapters like scattering amplitudes and form factors are introduced. In Section 1.2 we discuss marginal deformations of $\mathcal{N} = 4$ sYM theory, including the Leigh–Strassler deformations, and dilatation operators and scattering amplitudes in the β-deformed theory. Finally in Section 1.3 we study infrared divergences of scattering amplitudes in non-abelian gauge theories, review concepts important for subsequent chapters and summarise the results of the paper [1].

In Chapter 2 we begin with a discussion of on-shell methods in the computation of scattering amplitudes and the dilatation operator in Section 2.1. Then we exploit this machinery to extract non-planar anomalous dimensions of dimension-4 operators in $\mathcal{N} = 4$ sYM theory in Section 2.2, and the one-loop dilatation operator of the β-deformed theory in the scalar sector in Section 2.3. These sections are mainly based on unpublished work [4].

In Chapter 3 we review results of the publication [2]. We first discuss in Section 3.1 how the spectrum of scaling dimensions in the planar limit of $\mathcal{N} = 4$ sYM theory can be obtained from integrability methods, focussing on the $\mathfrak{su}(2)$ sector at one-loop order. Then we compute non-planar anomalous dimensions perturbatively around the planar theory in the undeformed case in Section 3.2, and the β-deformed case in Section 3.3. Doing so, we find compact expressions for matrix elements of the leading non-planar length-changing terms in the one-loop dilatation operators in terms of ordered partitions, spin-chain scalar products and hexagon-like objects. These results are applied to specific examples for which we obtain the leading non-planar correction to one-loop anomalous dimensions, and in particular we evaluate our expressions in the BMN limit of the deformed theory.

In Chapter 4 we discuss further results of the publication [2], as well as of [3], about the numerical study of spectra in deformed and undeformed $\mathcal{N} = 4$ sYM theory. We begin with a review on random matrix theory in Section 4.1, and also discuss its relevance for quantum-chaotic models. Furthermore, we introduce level statistics observables in order to explore features of numerical spectra which we will then apply to spectra in sYM theories in Section 4.2.

We conclude this thesis with a summary of the results and an outlook on possible future directions of work. In addition, there are a few appendices included. In Appendix A we list form factors relevant for the mixing of marginal operators in $\mathcal{N} = 4$ sYM theory discussed in Section 2.2. In Appendix B we discuss the algebraic Bethe ansatz for the Heisenberg spin chain, including compact expressions for spin-chain scalar products following from this ansatz relevant for Chapter 3. Finally, in Appendix C we give more details on data preparation for the level statistics analysis performed in Chapter 4.

Chapter 1

$\mathcal{N} = 4$ super Yang–Mills theory and its marginal deformations

In this chapter we review $\mathcal{N} = 4$ sYM theory and its marginal deformations, introduce the dilatation operator in conformal field theories, and discuss scattering amplitudes and form factors. This will lay the foundations for the following chapters. In Section 1.1 we begin with a discussion of pure $\mathcal{N} = 4$ sYM theory, in particular its field content and symmetries, as well as scattering amplitudes and form factors in this theory. We furthermore introduce the main characters of this thesis: the dilatation operator and anomalous dimensions. In Section 1.2 we then move on to marginal deformations of $\mathcal{N} = 4$ sYM theory, mainly focussing on the β-deformed theory and its dilatation operator and scattering amplitudes. In the final Section 1.3 we give a review of infrared divergences in scattering amplitudes, and discuss soft theorems and coherent states.

1.1 Undeformed theory

1.1.1 Symmetries, field content and action

Field content and action. $\mathcal{N} = 4$ sYM theory [42, 43] has a rich symmetry structure. In four-dimensional spacetime it is the unique theory enjoying the maximal amount $\mathcal{N} = 4$ of supersymmetry allowed for renormalisable quantum field theories. Both its matter content and the form of the action are fixed by this symmetry.

The theory's matter fields are massless and contain the three complex scalar fields $\phi_{AB} = -\phi_{BA}$, the four fermionic fields $\psi_{A\alpha}$ and the four anti-fermionic fields $\bar{\psi}^A_{\dot{\alpha}}$. The indices $A, B = 1, 2, 3, 4$ correspond to the global R-symmetry group $SO(6) \simeq SU(4)$, and the spinor indices $\alpha, \dot{\alpha}$ range over $1, 2$ and $\dot{1}, \dot{2}$, respectively. Furthermore, there is a gauge field A_μ, with Lorentz index $\mu = 0, 1, 2, 3$ in mostly-minus signature, and we will work with $SU(N)$ as the gauge group with all fields transforming in the adjoint

representation. The $\mathcal{N}=4$ sYM action is given by

$$S_{\mathcal{N}=4} = \int \mathrm{d}^4x \,\mathrm{Tr}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}(\mathcal{D}^\mu\phi^{AB})(\mathcal{D}_\mu\phi_{AB}) + i\bar{\psi}^A_{\dot\alpha}\bar{\sigma}^{\dot\alpha\alpha}_\mu\mathcal{D}^\mu\psi_{A\alpha}\right.$$
$$\left. + g_{\mathrm{YM}}\psi^\alpha_A[\phi^{AB},\psi_{B\alpha}] + g_{\mathrm{YM}}\bar{\psi}^A_{\dot\alpha}[\phi_{AB},\bar{\psi}^{B\dot\alpha}] + \frac{g^2_{\mathrm{YM}}}{16}[\phi^{AB},\phi^{CD}][\phi_{AB},\phi_{CD}]\right) \quad (1.1)$$

with dimensionless Yang–Mills coupling constant g_{YM}. It contains the field strength $F_{\mu\nu}$ and covariant derivative $\mathcal{D}_\mu$ defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig_{\mathrm{YM}}[A_\mu, A_\nu]\ , \qquad \mathcal{D}_\mu X = (\partial_\mu X) - ig_{\mathrm{YM}}[A_\mu, X]\ , \quad (1.2)$$

and the matrix $(\bar{\sigma}_\mu)^{\dot\alpha\alpha} = (\mathbb{1}, \sigma_1, \sigma_2, \sigma_3)^{\dot\alpha\alpha}$ with Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\ , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ . \quad (1.3)$$

The equations of motion for the fundamental fields following from (1.1) are

$$\begin{aligned}
\mathcal{D}^\mu F_{\mu\nu} &= +2ig_{\mathrm{YM}}[\mathcal{D}_\nu\phi_{AB},\phi^{AB}] + g_{\mathrm{YM}}\bar{\sigma}^{\dot\alpha\alpha}_\nu\{\bar{\psi}^A_{\dot\alpha},\psi_{A\alpha}\} \\
i\bar{\sigma}^{\dot\alpha\alpha}_\mu\mathcal{D}^\mu\bar{\psi}^A_{\dot\alpha} &= +2g_{\mathrm{YM}}[\phi^{AB},\psi^\alpha_B]\ , \\
i\bar{\sigma}^{\dot\alpha\alpha}_\mu\mathcal{D}^\mu\psi_{\alpha A} &= -2g_{\mathrm{YM}}[\phi_{AB},\bar{\psi}^{B\dot\alpha}]\ , \\
\mathcal{D}^\mu\mathcal{D}_\mu\phi^{AB} &= \frac{g^2_{\mathrm{YM}}}{8}[[\phi^{AB},\phi^{CD}],\phi_{CD}] \\
&\quad + \frac{g_{\mathrm{YM}}}{4}\left(\varepsilon^{ABCD}\{\psi^\alpha_C,\psi_{D\alpha}\} + (\delta^A_C\delta^B_D - \delta^B_C\delta^A_D)\{\bar{\psi}^C_{\dot\alpha},\bar{\psi}^{D\dot\alpha}\}\right)\ .
\end{aligned} \quad (1.4)$$

Symmetries. $\mathcal{N}=4$ sYM theory has an exceptional amount of symmetry. This includes the maximal amount $\mathcal{N}=4$ of supersymmetry, with supercharges $\mathfrak{Q}^A_\alpha$ and $\dot{\mathfrak{Q}}_{A\dot\alpha}$, and the corresponding global $SO(6) \simeq SU(4)$ R-symmetry group, with R-charges $\mathfrak{R}^A{}_B$. Furthermore, it is a conformal field theory and so the Poincaré symmetry algebra, consisting of the Lorentz transformations $\mathfrak{L}^\alpha{}_\beta$ and $\dot{\mathfrak{L}}^{\dot\alpha}{}_{\dot\beta}$ and translations $\mathfrak{P}_{\alpha\dot\alpha}$, is enhanced to the conformal symmetry algebra $\mathfrak{so}(2,4) \simeq \mathfrak{su}(2,2)$, with additional dilatation generator $\mathfrak{D}$ and special conformal transformations $\mathfrak{K}^{\alpha\dot\alpha}$. At the classical level the conformal invariance of $\mathcal{N}=4$ sYM theory follows from the absence of masses and dimensionful couplings in its action (1.1), whereas its conservation at the quantum level is non-trivial [44–46]. The conformal symmetry algebra combines with the supersymmetry algebra to $\mathfrak{su}(2,2|4)$ whose irreducible part is the superconformal algebra $\mathfrak{psu}(2,2|4)$. This algebra additionally contains the superconformal charges $\mathfrak{S}^\alpha_A$ and $\dot{\mathfrak{S}}^{A\dot\alpha}$. The explicit commutation relations of the symmetry generators, as well as their action on composite operators can be found for example in [30].

Composite operators of the $\mathcal{N}=4$ sYM fundamental fields form primary states

	A_μ	ϕ_1	ϕ_2	ϕ_3	ψ_1	ψ_2	ψ_3	ψ_4
q^1	0	-1	0	0	$-1/2$	$+1/2$	$+1/2$	$-1/2$
q^2	0	0	-1	0	$+1/2$	$-1/2$	$+1/2$	$-1/2$
q^3	0	0	0	-1	$+1/2$	$+1/2$	$-1/2$	$-1/2$
Q^1	0	-1	$+1$	0	-1	$+1$	0	0
Q^2	0	0	-1	$+1$	0	-1	$+1$	0
r	0	$-2/3$	$-2/3$	$-2/3$	$+1/3$	$+1/3$	$+1/3$	-1

Table 1.1: $\mathfrak{u}(1)^3$ Cartan charges of the fundamental fields of $\mathcal{N}=4$ sYM theory, with the respective anti-fields carrying opposite charges.

when they are annihilated by all lowering operators

$$\{\mathfrak{K}^{\alpha\dot\alpha}, \mathfrak{S}^\alpha_A, \dot{\mathfrak{S}}^{A\dot\alpha}, \mathfrak{L}^\alpha{}_\beta(\alpha<\beta), \dot{\mathfrak{L}}^{\dot\alpha}{}_{\dot\beta}(\dot\alpha<\dot\beta), \mathfrak{R}^A{}_B(A<B)\} \ . \tag{1.5}$$

Descendant operators can be built by acting with the corresponding raising operators

$$\{\mathfrak{P}_{\alpha\dot\alpha}, \mathfrak{Q}^A_\alpha, \dot{\mathfrak{Q}}_{A\dot\alpha}, \mathfrak{L}^\alpha{}_\beta(\alpha>\beta), \dot{\mathfrak{L}}^{\dot\alpha}{}_{\dot\beta}(\dot\alpha>\dot\beta), \mathfrak{R}^A{}_B(A>B)\} \ . \tag{1.6}$$

The remaining symmetry generators form the Cartan subalgebra and their eigenvalues can be used to classify primary operators. Of particular importance for the organisation of operators will be the $\mathfrak{u}(1)^3$ Cartan charges of the $SU(4)$ R-symmetry group defined as

$$\mathfrak{q}_1 = \mathfrak{R}^2_2 - \mathfrak{R}^1_1 \ , \qquad \mathfrak{p} = \mathfrak{R}^3_3 - \mathfrak{R}^2_2 \ , \qquad \mathfrak{q}_2 = \mathfrak{R}^4_4 - \mathfrak{R}^3_3 \ . \tag{1.7}$$

The corresponding eigenvalues $[q_1, p, q_2]$ for a given state are the $\mathfrak{su}(4)$ Dynkin labels. In the context of marginal deformations of $\mathcal{N}=4$ sYM theory it will furthermore be convenient to introduce two alternative sets of labels (q^1, q^2, q^3) and (Q^1, Q^2, r) which are related to the Dynkin labels via [47]

$$\begin{aligned} q^1 &= \frac{1}{2}(q_1 - q_2) \ , & q^2 &= -\frac{1}{2}(q_1 + q_2) \ , & q^3 &= -\frac{1}{2}(q_1 + 2p + q_2) \ , \\ Q^1 &= q^1 - q^2 \ , & Q^2 &= q^2 - q^3 \ , & r &= \frac{2}{3}(q^1 + q^2 + q^3) \ . \end{aligned} \tag{1.8}$$

Their values for the $\mathcal{N}=4$ sYM fundamental fields are given in Table 1.1, where we define three complex scalars as

$$\phi_i := \phi_{i4} \ , i = 1, ..., 3 \ , \tag{1.9}$$

with conjugates $\bar\phi^i$. Another important label of a composite operator is their scaling

dimension Δ measured by the dilatation operator $\mathfrak{D}$. This quantity will take on a central role throughout this thesis and will be further elaborated on in Section 1.1.2.

Gauge group. The generators of the gauge group $SU(N)$ in the fundamental representation are $N \times N$ hermitian matrices obeying the commutation relation

$$[T^a, T^b] = \sqrt{2} i f^{abc} T^c \tag{1.10}$$

with structure constants f^{abc}. The generators are normalised to

$$\mathrm{Tr}(T^a T^b) = \delta^{ab} \ , \tag{1.11}$$

which implies

$$f^{abc} f^{abd} = N \delta^{cd} \ . \tag{1.12}$$

The structure constants moreover satisfy the Jacobi identity

$$f^{abe} f^{cde} + f^{ace} f^{dbe} + f^{ade} f^{bce} = 0 \ . \tag{1.13}$$

On-shell superfield and Lagrangian. In the following sections the Lagrangian formulation of $\mathcal{N} = 4$ sYM theory via (1.1) will play a secondary role. Instead, we will mainly make use of on-shell techniques to determine the quantities of interest. The main idea behind this is to build on-shell observables from other known on-shell observables. Of particular importance will be the $\mathcal{N} =$ sYM version of the Parke–Taylor amplitude [48] for tree-level MHV scattering of $\mathcal{N} = 4$ sYM fields. A compact version of this formula uses the fact that all on-shell states of the theory can be combined into a single on-shell superfield $\mathbf{\Phi}$ as a function of Grassmann variables $\tilde{\eta}$ [49]

$$\mathbf{\Phi}(p, \tilde{\eta}) = g_+(p) + \tilde{\eta}^A \psi_A(p) + \frac{1}{2!} \tilde{\eta}^A \tilde{\eta}^B \phi_{AB}(p) + \frac{1}{3!} \tilde{\eta}^A \tilde{\eta}^B \tilde{\eta}^C \bar{\psi}_{ABC}(p) + (\tilde{\eta})^4 g_-(p) \ . \tag{1.14}$$

It contains gluons $g_\pm$ (with helicity $h = \pm 1$), scalars ϕ_{AB} ($h = 0$), as well as fermions ψ_A ($h = +1/2$) and anti-fermions $\bar{\psi}^A$ ($h = -1/2$). They transform under conjugation as

$$\begin{aligned} g_\pm^\dagger &= g_\mp \ , & (\phi_{AB})^\dagger &= \bar{\phi}^{AB} = \phi^{AB} = \frac{1}{2} \varepsilon^{ABCD} \phi_{CD} \ , \\ (\psi_A^\alpha)^\dagger &= \bar{\psi}^{A\dot{\alpha}} = -\frac{1}{3!} \varepsilon^{ABCD} \bar{\psi}_{BCD}^{\dot{\alpha}} \ , & (\bar{\psi}_{ABC}^{\dot{\alpha}})^\dagger &= \psi^{ABC\alpha} = \varepsilon^{ABCD} \psi_D^\alpha \ , \end{aligned} \tag{1.15}$$

where the third relation is consistent with

$$\bar{\psi}^{\dot{\alpha}}_{ABC} = \varepsilon_{ABCD}\bar{\psi}^{D\dot{\alpha}} \ , \tag{1.16}$$

and ε_{ABCD} is the four-dimensional Levi-Civita symbol. The conjugation rules (1.15) are moreover consistent with $(\Phi^\dagger)^\dagger = 1$, where $\Phi \in \{g_\pm, \psi_A, \bar{\psi}^A, \phi_{AB}\}$ denotes an arbitrary component field of $\mathbf{\Phi}$.

Evaluating the Lagrangian (1.1) on-shell by using the equations of motion (1.4) yields the chiral on-shell Lagrangian of $\mathcal{N}=4$ sYM theory

$$\mathcal{L} = \mathrm{Tr}\left(F_{\alpha\beta}F^{\alpha\beta} - g_{\mathrm{YM}}\psi^\alpha_A[\phi^{AB},\psi_{\alpha B}] - \frac{g^2_{\mathrm{YM}}}{2^4}[\phi^{AB},\phi^{CD}][\phi_{AB},\phi_{CD}]\right) \ . \tag{1.17}$$

Here we dropped all total derivatives and a topological term $F^{\alpha\beta}F_{\alpha\beta} - \bar{F}^{\dot{\alpha}\dot{\beta}}\bar{F}_{\dot{\alpha}\dot{\beta}}$ in the pure-gauge part

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(F^{\alpha\beta}F_{\alpha\beta} + \bar{F}^{\dot{\alpha}\dot{\beta}}\bar{F}_{\dot{\alpha}\dot{\beta}}\right) \simeq F^{\alpha\beta}F_{\alpha\beta} \ . \tag{1.18}$$

In [50] this on-shell Lagrangian was introduced in the so-called Lagrangian insertion procedure to obtain loop corrections of correlation functions of half-BPS operators, and was later used to define a super-Wilson loop at loop level [51] which is dual to $\mathcal{N}=4$ sYM amplitudes with general helicity configurations.

1.1.2 Dilatation operator and anomalous dimensions

Classical conformal invariance. An important observable in four-dimensional CFTs is the spectrum of scaling dimensions of local operators. They are related to dilatations which scale spacetime coordinates by a scaling factor e^α as

$$x^\mu \to e^\alpha x^\mu. \tag{1.19}$$

In a free massless theory one can always find an appropriate transformation of the elementary fields such that the action of the theory is left invariant under dilatations. For example in a free massless scalar field theory, the action

$$S_0[\phi] = \int \mathrm{d}^4x \frac{1}{2}\left(\partial\phi(x)\right)^2 \tag{1.20}$$

is invariant under simultaneous transformation (1.19) and

$$\phi(x) \to \phi'(x) = e^{-\Delta^0_\phi \alpha}\phi(e^{-\alpha}x) \tag{1.21}$$

for $\Delta^0_\phi = 1$, which can be easily seen from

$$S_0[\phi] \to \int \mathrm{d}^4x' \frac{1}{2}(\partial'\phi'(x'))^2 = \int \mathrm{d}^4x e^{(4-2-2)\alpha} \frac{1}{2}(\partial\phi(e^{(1-1)\alpha}x))^2 = S_0[\phi] \ . \tag{1.22}$$

Δ^0_ϕ is the classical scaling dimension of the scalar field ϕ, which characterises the scaling properties of general operators under rescalings (1.19).

The scale-invariance of free massless theories is broken when adding mass terms to the Lagrangian. Similarly, deforming the action by adding interaction terms may break this symmetry, with the exception of marginal deformations. These are deformations

$$\delta\mathcal{L} \sim g\mathcal{O} \tag{1.23}$$

where the classical dimension of the operator $\mathcal{O}$ is $\Delta^0_\mathcal{O} = 4$ and the deformation parameter g is dimensionless. For unitary renormalisable classical field theories in four dimensions, scale invariance automatically implies conformal invariance [52, 53].

Conformal invariance at the quantum level. In general, it is not guaranteed that the conformal invariance of a classical theory is inherited by the quantum theory, but in the case of $\mathcal{N} = 4$ sYM theory it is exact also at the quantum level [44–46]. This is implied by the vanishing of the β-function

$$\beta(g^2_{\mathrm{YM}}) = \mu\partial_\mu g^2_{\mathrm{YM}} \ , \tag{1.24}$$

which encodes the dependence of the theory on the renormalisation scale μ, to all loop orders. Nevertheless, many composite operators are renormalised and their classical scaling dimensions Δ^0 receive quantum corrections. $\mathcal{N} = 4$ sYM theory with gauge group $SU(N)$ has two free parameters[1], the gauge coupling g_{YM} and the rank of the gauge group N, and a possible perturbative treatment of this theory corresponds to a double expansion of small 't Hooft coupling $\lambda = g^2_{\mathrm{YM}}N$ and large gauge parameter N [54]. In the large-N expansion it is useful to classify Feynman diagrams according to the topological surface they can be drawn on. Then diagrams are suppressed according to the genus of these surfaces, with graphs that can be drawn on the surface of a sphere, i.e. planar diagrams, being dominant. In this perturbative regime, the scaling dimension of a local operator $\mathcal{O}$ may be expanded as

$$\Delta_\mathcal{O} = \Delta^0_\mathcal{O} + \sum_{k=1}^{\infty} g^{2k}\gamma^{(k)}_\mathcal{O}(N) \ , \quad \text{with} \quad g^2 = \frac{\lambda}{16\pi^2} \ , \tag{1.25}$$

[1] In principle, as in any gauge theory, one could introduce a θ-term which would give rise to a complex coupling constant. Here we work in the $\theta = 0$ sector.

with the classical dimension $\Delta^0_{\mathcal{O}}$ receiving quantum corrections $\gamma^{(k)}_{\mathcal{O}}(N)$, the so-called anomalous dimensions. These are in general non-trivial functions of the rank N of the gauge group and can thus be further computed in an expansion of large N.

The operator generating dilatations in a CFT, and thus measuring scaling dimensions of local operators, is the dilatation operator $\mathfrak{D}$. A general operator $\mathcal{O}$ transforms under dilatations as, cf. (1.21),

$$\mathcal{O}(x) \to e^{-\Delta_{\mathcal{O}}\alpha}\mathcal{O}(e^{-\alpha}x) \tag{1.26}$$

generated by $\mathfrak{D}$ as

$$\mathcal{O}(x) \to e^{\alpha\mathfrak{D}}\mathcal{O}(x)e^{-\alpha\mathfrak{D}} \tag{1.27}$$

with

$$[\mathfrak{D}, \mathcal{O}(x)] = (\Delta_{\mathcal{O}} + x^\mu\partial_\mu)\,\mathcal{O}(x)\ . \tag{1.28}$$

In the perturbative regime we may expand the dilatation operator $\mathfrak{D}$ similar to (1.25) as[2]

$$\mathfrak{D} = \sum_k g^{2k}\mathfrak{D}_{2k}\ . \tag{1.29}$$

One-loop dilatation operator of $\mathcal{N} = 4$ sYM theory. Early computations of anomalous dimensions were based on the relation (1) between two-point functions and scaling dimensions and were performed by direct Feynman diagram calculations for each operator of interest. In order to efficiently compute anomalous dimensions for a number of operators, it is more useful to obtain them from diagonalisation of the dilatation operator. In [10], see also [30], the complete one-loop $\mathcal{N} = 4$ sYM dilatation operator was derived. This derivation relies on the computation of certain two-point functions in closed sectors via Feynman diagram calculations from which the dilatation operator in these sectors can be deduced. These results are then lifted to the full dilatation operator by symmetry arguments. After this first derivation of the dilatation operator in $\mathcal{N} = 4$ sYM theory, other methods were developed and we review an on-shell approach to the dilatation operator in Section 2.1.2.

In Chapters 3 and 4 we will consider anomalous dimensions of a particular class of local gauge-invariant operators which are given as products of traces of the covariant

[2]The full $\mathcal{N} = 4$ sYM dilatation operator has an expansion not only in even, but also odd powers of the coupling starting at $\mathcal{O}(g^3_{\text{YM}})$. These so-called "half-loop" contributions change the lengths of operators and this effect will be studied in more detail in Section 2.2. In the remaining parts of this thesis we restrict to closed sectors of $\mathcal{N} = 4$ sYM theory in which those length-changing effects do not occur and the dilatation operator can be expanded in even powers of the coupling only.

fields

$$\mathrm{Tr}(\Phi_1 \dots \Phi_L)(x) \ , \tag{1.30}$$

where Φ_i is either a scalar, fermion or field strength with possible insertions of covariant derivatives $\mathcal{D}_\mu$ and L is the operator length. The symmetry algebra of the theory, $\mathfrak{psu}(2,2|4)$, is non-compact and thus the operators organise themselves in infinite-dimensional representations. Of particular interest in this thesis are two rank-one sectors of the theory, the scalar $\mathfrak{su}(2)$ sector spanned by two complex scalars, e.g. $\{Z = \phi_{12}, X = \phi_{14}\}$, as well as the $\mathfrak{sl}(2)$ sector spanned by one complex scalar and one covariant light-cone derivative $\{Z, \mathcal{D}\}$.

$\mathfrak{su}(2)$ sector. First we discuss the $\mathfrak{su}(2)$ sector comprising operators made of products of traces of the two complex scalar fields Z and X, and so we consider operators such as

$$\mathrm{Tr}(Z^{\ell_1}) \ , \quad \mathrm{Tr}(XZ^{\ell_1}XZ^{\ell_2}) \ , \quad \mathrm{Tr}(XZ^{\ell_1}XZ^{\ell_2}Z^{\ell_3})\,\mathrm{Tr}(XZ^{\ell_4})\,\mathrm{Tr}(XX) \ . \tag{1.31}$$

These operators can be organised into $SO(6)$ representations with Dynkin labels $[M, L-2M, M]$, where L counts the total number of scalar fields in the operator, while M corresponds to the number of X-fields. This sector is known to be closed under the action of the dilatation operator and does not mix with operators containing other scalars, field strengths or fermions. To describe the action of the dilatation operator, we make use of the notation for functional derivatives of fields, for example

$$(\check{Z})^a{}_b := \frac{\delta}{\delta(Z)^b{}_a} \tag{1.32}$$

such that

$$(\check{Z})^a{}_b(Z)^c{}_d = \delta^c_b\delta^a_d - N^{-1}\delta^a_b\delta^c_d \ , \quad \text{and} \quad (\check{Z})^a{}_b(X)^c{}_d = 0 \ . \tag{1.33}$$

This can be used to derive the fusion and splitting formulas

$$\begin{aligned} \mathrm{Tr}(A\check{Z})\,\mathrm{Tr}(BZ) &= \mathrm{Tr}(AB) - N^{-1}\,\mathrm{Tr}(A)\,\mathrm{Tr}(B) \ , \\ \mathrm{Tr}(A\check{Z}BZ) &= \mathrm{Tr}(A)\,\mathrm{Tr}(B) - N^{-1}\,\mathrm{Tr}(AB) \ , \end{aligned} \tag{1.34}$$

where it is assumed that A and B do not contain any Z's. The N^{-1} terms are due to the fact that we are considering the $SU(N)$ gauge theory. This is not particularly important for $\mathcal{N}=4$ sYM theory and we could equally well consider a $U(N)$ gauge group, however it will become relevant when we subsequently consider the β-deformed theory.

Using this notation, the tree-level dilatation operator in the $\mathfrak{su}(2)$ sector can be

written as

$$\mathfrak{D}_0 = : \mathrm{Tr}(Z\check{Z} + X\check{X}) : \tag{1.35}$$

and simply counts the number of fields in a given operator. The normal-ordering markers : : indicate that the functional derivatives do not act on the fields enclosed by these colons. The one- and two-loop correction to the dilatation operator is then given by [6, 13]

$$\mathfrak{D}_2 = -\frac{2}{N} : \mathrm{Tr}([X, Z][\check{X}, \check{Z}]) : , \tag{1.36}$$

$$\mathfrak{D}_4 = -\frac{2}{N^2} : \mathrm{Tr}\left([[X, Z], \check{Z}]\,[[\check{X}, \check{Z}], Z] + [[X, Z], \check{X}]\,[[\check{X}, \check{Z}], X]\right.$$
$$\left. + [[X, Z], T^a]\,[[\check{X}, \check{Z}], T^a]\right) : \ . \tag{1.37}$$

Due to the residual $SU(2)$ R-symmetry we can arrange operators in terms of primary operators defined by $J_{-}\mathcal{O} = 0$, where the lowering operator acts as $J_{-}X = Z$, and descendant operators which can be obtained by acting with J_{+} on a primary. The above dilatation operator gives rise to a mixing problem which, for short operators, can be solved by hand to obtain the corresponding anomalous dimensions. For longer operators the dimension of the mixing problem gets very large very quickly and this calls for a more general approach to anomalous dimensions. In the planar limit the action of the one-loop dilatation operator reduces to the integrable XXX spin-chain Hamiltonian [11] and thus dimensions can be obtained using integrability techniques. In Chapter 3 we review the planar integrable theory and compute non-planar anomalous dimensions perturbatively around the solvable planar limit, and then we move on to an investigation of the statistical properties of anomalous dimensions at finite N in Chapter 4.

$\mathfrak{sl}(2)$ sector. A second sector we examine in this work is the $\mathfrak{sl}(2)$ sector. In this sector we are interested in operators consisting of traces of the scalar field Z with insertions of covariant light-cone derivatives. These operators are of particular interest as they are in some sense universal in non-abelian gauge theory [55], and can even be related to perturbative QCD [56] where there is an integrable $\mathfrak{sl}(2)$ sector at one-loop level in the planar limit [57, 58], despite the conformal symmetry being broken.

We denote a scalar excited with n derivatives by $Z^{(n)} \equiv \mathcal{D}^n Z/n!$. The one-loop dilatation operator is [10]

$$\mathfrak{D}_2 = -\frac{1}{N} \sum_{\substack{m,n \\ k+l=m+n}} C^{kl}_{mn} : \mathrm{Tr}\left([Z^{(k)}, \check{Z}^{(m)}][Z^{(l)}, \check{Z}^{(n)}]\right) : \tag{1.38}$$

with coefficients

$$C^{kl}_{mn} = \delta_{k=m}\big(h(m) + h(n)\big) - \frac{\delta_{k\neq m}}{|k-m|} \ , \tag{1.39}$$

and $h(n)$ being the harmonic sum. We organise the operators with respect to their number of fields L and derivatives S. Furthermore, since there is an $SL(2)$ symmetry, operators are arranged into primaries $\mathcal{O}$, obeying $S_-\mathcal{O} = 0$, and descendants obtained by the action of S_+ on $\mathcal{O}$. The action of the lowering operator is given by $S_- Z^{(n)} = nZ^{(n-1)}$. Note that in this case the representations are non-compact since we can act with the raising operator S_+ indefinitely, creating an infinite tower of descendants.

1.1.3 Scattering amplitudes and form factors

The scaling dimensions of $\mathcal{N} = 4$ sYM theory, together with the structure constants from three-point correlation functions, build the CFT data of the theory. In more general, non-conformal, quantum field theories we are often interested in an alternative set of observables to characterise the theory: the scattering amplitudes. As basic ingredients for cross sections, they form important data for experiment, but also provide insight more formally into structural properties of the theory. Their definition for theories with massless particles, and in particular conformal field theories, is subtle due to the invalidity of the assumption that interactions are negligible at large distances. This requires the introduction of an infrared regulator and we will discuss the associated infrared divergences in amplitudes in further detail in Section 1.3. Scattering amplitudes are an active field of physics and we will not attempt to give a complete overview, but only confine ourselves to aspects immediately relevant for the discussions in this work and refer to reviews and lectures such as [59–62] for a more complete treatment and list of references.

Scattering amplitudes are overlaps between on-shell states of elementary particles and thus probe their interactions. Choosing all particles to be outgoing by crossing symmetry, their general form is given by the overlap of an n-particle outgoing state and the vacuum, i.e.

$$\mathcal{M}_n(1^{a_1}, 2^{a_2}, ..., n^{a_n}) := \langle 1^{a_1}, 2^{a_2}, ..., n^{a_n}|0\rangle \ . \tag{1.40}$$

In this thesis we will also be interested in the more general overlaps between n-particle outgoing states and the off-shell state created by the action of a local operator $\mathcal{O}(x)$ on the vacuum, i.e.

$$\mathcal{F}_{\mathcal{O}}(1^{a_1}, 2^{a_2}, ..., n^{a_n}; x) := \langle 1^{a_1}, 2^{a_2}, ..., n^{a_n}|\mathcal{O}(x)|0\rangle \ . \tag{1.41}$$

These objects are the so-called form factors and they build a bridge between the purely on-shell scattering amplitudes and off-shell correlation functions.

Colour ordering. In general, amplitudes of a given theory depend on the labels of the external particles and in non-abelian gauge theories we can separate the colour-dependence of amplitudes from the remaining degrees of freedom by a colour decomposition. In $\mathcal{N}=4$ sYM theory this decomposition of the full amplitude $\mathcal{M}_n$, depending on the colour indices $a_1, a_2, ..., a_n$, is achieved in terms of partial amplitudes M_n at tree-level via [63, 64]

$$\mathcal{M}_n^{(0)}(\{p_i, a_i\}) = g_{\mathrm{YM}}^{n-2} \sum_{\sigma \in \mathcal{S}_n/\mathbb{Z}_n} \mathrm{M}_n^{(0)}(p_{\sigma(1)}, p_{\sigma(2)}, ..., p_{\sigma(n)}) \, \mathrm{Tr}(a_{\sigma(1)} a_{\sigma(2)} ... a_{\sigma(n)}) \ , \quad (1.42)$$

where we leave the dependence on any other particle quantum numbers besides the momenta and colour indices implicit. The elements a_i in the colour trace represent the traceless generators T^{a_i} of $SU(N)$ in the fundamental representation, obeying the relation (1.10) with normalisation (1.11). The sum in (1.42) is over all permutations $\mathcal{S}_n$ of the colour labels up to cyclically invariant combinations, i.e. we sum over $\mathcal{S}_{n-1} = \mathcal{S}_n/\mathbb{Z}_n$. This reflects the (graded) cyclicity of partial amplitudes:

$$\mathrm{M}_n(1, 2, 3, ..., n) = \mathrm{M}_n(2, 3, ..., n, 1) \ . \quad (1.43)$$

Using the reflection and $U(1)$ decoupling identity,

$$\begin{aligned} &\mathrm{M}_n(1, 2, ..., n) = (-1)^n \mathrm{M}_n(n, ..., 2, 1) \ , \\ &\mathrm{M}_n(1, 2, 3, ..., n) + \mathrm{M}_n(2, 1, 3..., n) + ... + \mathrm{M}_n(2, 3..., n, 1) = 0 \ , \end{aligned} \quad (1.44)$$

as well as the more general Kleiss–Kuijf relations [65] of $\mathcal{N}=4$ sYM partial amplitudes, the relation (1.42) can be reformulated in terms of $(n-2)!$ partial amplitudes as [66]

$$\begin{aligned} \mathcal{M}_n^{(0)}(\{p_i, a_i\}) = (\sqrt{2} i g_{\mathrm{YM}})^{n-2} \sum_{\sigma \in \mathcal{S}_{n-2}} &\mathrm{M}_n^{(0)}(p_1, p_{\sigma(2)}, ..., p_{\sigma(n-1)}, p_n) \\ &\cdot f^{a_1 a_{\sigma(2)} e_1} f^{e_1 a_{\sigma(3)} e_2} ... f^{e_{n-3} a_{\sigma(n-1)} a_n} \ , \end{aligned} \quad (1.45)$$

which for the four-point amplitude yields the simple relation

$$\mathcal{M}_4^{(0)} = -2 g_{\mathrm{YM}}^2 \left(\mathrm{M}_4^{(0)}(1, 2, 3, 4) f^{a_1 a_2 e} f^{a_3 a_4 e} + \mathrm{M}_4^{(0)}(1, 3, 2, 4) f^{a_1 a_3 e} f^{a_2 a_4 e} \right) \ . \quad (1.46)$$

A further reduction of the basis of tree-level partial amplitudes to $(n-3)!$ independent amplitudes can be achieved via the BCJ relations [67], but as they have not been established for the β-deformed theory we discuss further below, we will not make use of them in this work.

Spinor-helicity variables. With $\mathcal{N} = 4$ sYM theory being a quantum conformal theory of only massless fields, it is particularly useful to describe amplitudes in momentum space in terms of spinor-helicity variables. They are introduced by decomposing on-shell massless momenta p^μ, $p^2 = 0$, into two two-component Weyl spinors λ and $\tilde{\lambda}$ as

$$p^{\dot{\alpha}\alpha} = p^\mu \bar{\sigma}_\mu^{\dot{\alpha}\alpha} = \tilde{\lambda}^{\dot{\alpha}} \lambda^\alpha = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix} . \tag{1.47}$$

For physical momenta with real and positive energy the two spinors are related via

$$(\lambda^\alpha(p))^* = \tilde{\lambda}^{\dot{\alpha}}(p) . \tag{1.48}$$

We adopt the convention where under crossing a momentum, $p \to k = -p$, the spinors satisfy

$$\lambda(-p) = -\lambda(p), \quad \tilde{\lambda}(-p) = \tilde{\lambda}(p) \tag{1.49}$$

and thus for crossed momenta k with real and negative energy, the relation (1.48) becomes

$$(\lambda^\alpha(k))^* = -\tilde{\lambda}^{\dot{\alpha}}(k) . \tag{1.50}$$

For a given momentum $p^{\dot{\alpha}\alpha}$ the spinors are only fixed up to a phase φ,

$$(\lambda^\alpha, \tilde{\lambda}^{\dot{\alpha}}) \to (e^{-i\varphi}\lambda^\alpha, e^{i\varphi}\tilde{\lambda}^{\dot{\alpha}}) \quad \Rightarrow \quad p^{\dot{\alpha}\alpha} \to p^{\dot{\alpha}\alpha} . \tag{1.51}$$

This invariance is related to the $SO(2) \simeq U(1)$ little-group symmetry which leaves the momentum invariant in the Lorentz frame in which it is given by $p^\mu = (p^0, 0, 0, p^0)$ with particle energy p^0. The associated little-group transformation rotates particles as a function of their helicity h as

$$\langle p, h| \to e^{2i\varphi h} \langle p, h| . \tag{1.52}$$

Thus the fundamental fields of $\mathcal{N} = 4$ sYM theory transform as

$$\begin{aligned} \langle g_+| &\to e^{2i\varphi} \langle g_+| , & \langle g_-| &\to e^{-2i\varphi} \langle g_-| , \\ \langle \psi| &\to e^{i\varphi} \langle \psi| , & \langle \bar{\psi}| &\to e^{-i\varphi} \langle \bar{\psi}| , \\ \langle \phi| &\to \langle \phi| , \end{aligned} \tag{1.53}$$

in particular the negative-helicity fermions $\bar{\psi}$ scale like λ, while positive-helicity fermions ψ scale like $\tilde{\lambda}$. Lorentz-invariant combinations of spinor-helicity variables

can be built as

$$\langle ij\rangle := \lambda_i^\alpha \lambda_{j\alpha}, \quad [ij] := \tilde{\lambda}_{i\dot{\alpha}} \tilde{\lambda}_j^{\dot{\alpha}} \ , \tag{1.54}$$

where spinor indices are raised and lowered with the help of the two-dimensional Levi-Civita symbol

$$\lambda_\alpha = \varepsilon_{\alpha\beta}\lambda^\beta, \quad \tilde{\lambda}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}^{\dot{\beta}} \ . \tag{1.55}$$

They are related to Mandelstam invariants as

$$s_{ij} = (p_i + p_j)^2 = \langle ij\rangle\,[ji] \ . \tag{1.56}$$

Parke–Taylor superamplitude. $\mathcal{N}=4$ sYM amplitudes of the elementary gluon, fermion and scalar fields can be cast into a particularly compact form by combining them into amplitudes of on-shell superfields $\mathbf{\Phi}$ given in (1.14). The MHV component amplitudes at tree-level combine to the $\mathcal{N}=4$ superfield version of the Parke–Taylor amplitude, the so-called Parke–Taylor superamplitude,

$$\mathrm{M}_n^{(0,\mathrm{MHV})}(\mathbf{\Phi}_1, \mathbf{\Phi}_2, ..., \mathbf{\Phi}_n) = \frac{\delta^{(8)}(\sum_{i=1}^n \lambda_i^\alpha \tilde{\eta}_i^A)}{\langle 12\rangle\,\langle 23\rangle\,...\,\langle n1\rangle} = \frac{\prod_{A=1}^4 \sum_{1\le i<j\le n} \langle ij\rangle\,\tilde{\eta}_i^A \tilde{\eta}_j^A}{\langle 12\rangle\,\langle 23\rangle\,...\,\langle n1\rangle} \ . \tag{1.57}$$

From this expression a component amplitude can be extracted by identifying the combination of $\tilde{\eta}_i$'s corresponding to the helicities h_i as in (1.14).

Originally the expression in (1.57) was conjectured for purely gluonic MHV tree-level amplitudes by Parke and Taylor [48] from expressions up to $n=5$ obtained essentially by direct Feynman diagram calculations. Only a year later it was proven for arbitrary numbers of external legs by Berends and Giele [68] in terms of an off-shell recursion. In $\mathcal{N}=4$ sYM theory, supersymmetry relates this purely gluonic amplitude to amplitudes with general external field content via Ward identities which result in the Parke–Taylor superamplitude (1.57). The particularly simple form of the Parke–Taylor amplitude is remarkable, especially in light of the huge number of Feynman diagrams contributing to its computation via traditional methods. In a quest to make this simplicity manifest directly in the derivation of scattering amplitudes, new approaches to such calculations were developed, with the main idea being to work with exclusively on-shell degrees of freedom at every stage of the computation. We elaborate on some of these so-called on-shell methods further in Section 2.1.

Form factors. A form factor measures the overlap of an on-shell external state and an off-shell state created by the action of a local operator $\mathcal{O}(x)$ on the vacuum $|0\rangle$ as in (1.41). Form factors in position space can be transformed into momentum space

via a Fourier transformation as

$$\begin{aligned}\mathcal{F}_{\mathcal{O}}(1,...,n;q) &= \int \mathrm{d}^4x e^{-iq\cdot x}\,\langle 1,...,n|\mathcal{O}(x)|0\rangle \\ &= \int \mathrm{d}^4x e^{-iq\cdot x}\,\langle 1,...,n|e^{ix\cdot\mathfrak{P}}\mathcal{O}(0)e^{-ix\cdot\mathfrak{P}}|0\rangle \\ &= (2\pi)^4\delta^4\big(q-\textstyle\sum_i p_i\big)\,\langle 1,...,n|\mathcal{O}(0)|0\rangle \ , \end{aligned} \tag{1.58}$$

where we use the momentum operator $\mathfrak{P}_\mu$ to translate the operator $\mathcal{O}(x)$ to the origin.

Similar to the colour decomposition (1.42) of scattering amplitudes, we can strip off the colour-dependence of form factors by defining colour-ordered partial form factors. For form factors of length-L single-trace operators $\mathcal{O}$ in $\mathcal{N}=4$ sYM theory this decomposition is achieved via

$$\mathcal{F}^{(0)}_{\mathcal{O}}(\{p_i,a_i\};q) = g_{\mathrm{YM}}^{n-L}\sum_{\sigma\in\mathcal{S}_n/\mathbb{Z}_n}\mathrm{F}^{(0)}_{\mathcal{O}}(p_{\sigma(1)},p_{\sigma(2)},...,p_{\sigma(n)};q)\,\mathrm{Tr}(a_{\sigma(1)}a_{\sigma(2)}...a_{\sigma(n)}) \tag{1.59}$$

at tree-level. For multi-trace operators there exists a similar decomposition where the sum over permutations splits into a sum over all possible partitions of colour indices over the different traces and sums over non-cyclic permutations for each individual trace.

An important class of form factors is formed by those for which the field content in the external state corresponds to the field content in the local operator. These so-called minimal form factors have a non-vanishing contribution even in the free-theory limit and will take on a particularly important role in this thesis. The minimal form factors of the fundamental fields are

$$\begin{aligned} \langle g_-(p)|F^{\alpha\beta}|0\rangle &= \lambda^\alpha\lambda^\beta \ , & \langle g_+(p)|\bar{F}^{\dot\alpha\dot\beta}|0\rangle &= \tilde\lambda^{\dot\alpha}\tilde\lambda^{\dot\beta} \ , \\ \langle\bar\psi^A(p)|\psi^\alpha_B|0\rangle &= \delta^A_B\lambda^\alpha \ , & \langle\psi_A(p)|\bar\psi^{\dot\alpha B}|0\rangle &= \delta^B_A\tilde\lambda^{\dot\alpha} \ , \\ \langle\phi_{AB}(p)|\phi_{CD}|0\rangle &= \varepsilon_{ABCD} \ . \end{aligned} \tag{1.60}$$

1.2 Deformations

$\mathcal{N}=4$ sYM theory has many exceptional features, including exact conformal invariance, a maximal amount of supersymmetry, planar integrability, and duality with a gravitational theory. This makes this theory stand out from all other four-dimensional quantum field theories and an ideal testing ground for new approaches to computations and conceptional explorations in more generic theories. Nevertheless, this amount of symmetry also makes $\mathcal{N}=4$ sYM theory highly unphysical, and this motivates the study of its deformations and, in particular, of how concepts developed in $\mathcal{N}=4$ sYM theory carry over to less-symmetric theories. One natural starting point in this

quest are exactly marginal deformations which we will discuss in Section 1.2.1. We will mainly focus our attention on the β-deformation, and discuss this theory, its dilatation operator and scattering amplitudes in Section 1.2.2.

1.2.1 Leigh–Strassler deformations

$\mathcal{N} = 4$ sYM theory is a representative of a larger family of supersymmetric CFTs in four dimensions that were classified by Leigh and Strassler in [26]. They can be obtained from $\mathcal{N} = 4$ sYM theory by deforming its superpotential in $\mathcal{N} = 1$ superspace formulation. In this formulation the $\mathcal{N} = 4$ elementary fields discussed in Section 1.1.1 are arranged into one vector superfield V, and three chiral superfields $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \boldsymbol{\phi}_3$. The corresponding superspace Lagrangian contains the superpotential

$$W_{\mathcal{N}=4} = g_{\text{YM}} \operatorname{Tr}([\boldsymbol{\phi}_1, \boldsymbol{\phi}_2]\boldsymbol{\phi}_3) \ . \tag{1.61}$$

Leigh and Strassler [26] showed that the two-parameter deformation

$$W_{\text{LS}} = \kappa \operatorname{Tr}\left(\boldsymbol{\phi}_1\boldsymbol{\phi}_2\boldsymbol{\phi}_3 - q\boldsymbol{\phi}_2\boldsymbol{\phi}_1\boldsymbol{\phi}_3 + \frac{h}{3}(\boldsymbol{\phi}_1^3 + \boldsymbol{\phi}_2^3 + \boldsymbol{\phi}_3^3)\right) \tag{1.62}$$

gives rise to a family of $\mathcal{N} = 1$ supersymmetric conformal field theories which remain conformal at the quantum level given a single constraint equation on the new parameters κ, q and h. This constraint can be obtained perturbatively and through two loops is given by the relation

$$2g_{\text{YM}}^2 = |\kappa|^2 \left(\frac{2}{N^2}\left|q + q^{-1}\right|^2 + \left(1 - \frac{4}{N^2}\right)\left(|q|^2 + |q^{-1}|^2 + |h|^2\right)\right) \ , \tag{1.63}$$

see e.g. [69]. The Leigh–Strassler superpotential (1.62) breaks the $SU(4)$ R-symmetry of $\mathcal{N} = 4$ sYM theory to the $U(1)$ R-symmetry of the $\mathcal{N} = 1$ supergroup. One well-studied subclass of these Leigh–Strassler theories is the q-deformation for which $h = 0$, and in this case the Cartan $\mathfrak{u}(1)^3$ of $SU(4)$ remains unbroken.

The Leigh–Strassler theories have been of continued interest in the context of possible deformations of $\mathcal{N} = 4$ sYM theory which inherit its planar integrability. For generic values of q and h in (1.62) the theory in the planar limit is non-integrable, and in fact integrability poses a strong constraint which is only satisfied at very special values [70–72]. One class of integrable models has parameters $|q| = 1$ and $h = 0$. If also $q^n = 1$, then these correspond to orbifolds of the undeformed theory. We can introduce a parameter $\beta \in \mathbb{R}$ with $q = \exp(i\beta)$ and the corresponding Leigh–Strassler theory is also known as (real-)β-deformed $\mathcal{N} = 4$ sYM theory. In this case

the conformality condition (1.63) becomes

$$g_{\rm YM}^2 = |\kappa|^2 \left(1 - \frac{4}{N^2}\sin^2\beta\right) \ , \tag{1.64}$$

and is further simplified to $g_{\rm YM}^2 = |\kappa|^2$ in the planar limit, where this constraint receives no higher-loop corrections [27].

1.2.2 β-deformed $\mathcal{N}=4$ sYM theory

In this work our main focus will be on the real β-deformed $\mathcal{N}=4$ sYM theory. This rather simple-looking deformation has rich physics and in this thesis we explore a few of its properties.

Lagrangian. In $\mathcal{N}=1$ superspace formalism the β-deformation of the $\mathcal{N}=4$ sYM Lagrangian corresponds to a single-trace deformation of the superpotential (1.62). However, when written in terms of the component fields, this results in both single-trace and double-trace deformations of the Lagrangian [47, 73]. Furthermore, all interaction terms originating from the superpotential appear with deformed coupling κ. The Lagrangian is given by

$$\begin{aligned}\mathcal{L}^\beta = {\rm Tr}\Bigg(&-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - (\mathcal{D}^\mu\phi^i)(\mathcal{D}_\mu\phi_i) + i\bar\psi^i_{\dot\alpha}\bar\sigma^{\dot\alpha\alpha}_\mu\mathcal{D}^\mu\psi_{i\alpha} + i\bar\psi^4_{\dot\alpha}\bar\sigma^{\dot\alpha\alpha}_\mu\mathcal{D}^\mu\psi_{4\alpha}\\
&- 2g_{\rm YM}\psi_4^\alpha[\bar\phi^i,\psi_{i\alpha}] - 2g_{\rm YM}\bar\psi^4_{\dot\alpha}[\phi_i,\bar\psi^{i\dot\alpha}] - \kappa\varepsilon^{ijk}\psi_i^\alpha[\phi_j,\psi_{k\alpha}]_\beta - \kappa^*\varepsilon_{ijk}\bar\psi^i_{\dot\alpha}[\bar\phi^j,\bar\psi^{k\dot\alpha}]_\beta\\
&-\frac{g_{\rm YM}^2}{2}[\phi_i,\bar\phi^i][\phi_j,\bar\phi^j] + |\kappa|^2[\phi_i,\phi_j]_\beta[\bar\phi^i,\bar\phi^j]_\beta\Bigg)\\
&- \frac{|\kappa|^2}{N}{\rm Tr}\left([\phi_i,\phi_j]_\beta\right){\rm Tr}\left([\bar\phi^i,\bar\phi^j]_\beta\right) \ ,\end{aligned} \tag{1.65}$$

where $i=1,...,3$ and the scalars are defined in (1.9). This expression contains deformed commutators

$$[\Phi_i,\Phi_j]_\beta = e^{i\beta(Q_i^1Q_j^2 - Q_j^1Q_i^2)}\Phi_i\Phi_j - e^{i\beta(Q_j^1Q_i^2 - Q_i^1Q_j^2)}\Phi_j\Phi_i \ , \tag{1.66}$$

where the charges Q_i^1 and Q_i^2 associated to the field Φ_i are two of the $U(1)^3$ Cartan charges and their values for all elementary fields are given in Table 1.1. Note that the Lagrangian (1.65) does not only conserve the $U(1)$ R-symmetry of general Leigh–Strassler theories corresponding to the quantum number r, but the full Cartan $U(1)^3$. The double-trace scalar term in (1.65) is important for the theory to be exactly conformal [47]. This is true even in the planar limit as, although this term occurs at subleading order in N, the corresponding interaction contributes to the leading planar anomalous dimensions of short operators.

In the undeformed action (1.1) the interactions are all of commutator-type. Due to the form of colour contractions in traces, this in particular results in anomalous-dimension spectra that are independent of whether we use $SU(N)$ or $U(N)$ as the gauge group. In the β-deformed theory the $U(N)$ gauge group is no longer conformal at the quantum level due to the couplings of $U(1)$ scalars. These degrees of freedom decouple at the infrared fixed point corresponding to the $SU(N)$ theory and we will thus consider only the $SU(N)$ gauge group.

Dilatation operator. The form of the single-trace part of the β-deformed dilatation operator is simply inherited from the undeformed theory and is found by replacing the commutators in the undeformed dilatation operator by the β-deformed commutator (1.66) defined via the R-charges of the fields. In the planar limit and for sufficiently large operators ($L > 2$) the anomalous dimensions are completely fixed by this single-trace part of the dilatation operator which maps to an integrable Hamiltonian discussed in Section 3.1.3. The planar dilatation operator for the deformed theory has been previously studied using both integrability methods [28] and direct field-theory computations [29].

The non-planar dilatation operator can in principle be directly computed from the deformed Lagrangian using standard Feynman diagrammatics. In this thesis we will derive it in the scalar subsector using on-shell methods based on [31, 32], cf. Section 2.3. The result in the $\mathfrak{su}(2)$ sector spanned by $X = \phi_{14}$ and $Z = \phi_{12}$ is given by

$$\mathfrak{D}_2^\beta = -\frac{2\tilde{\kappa}^2}{N}\left(:\mathrm{Tr}([X,Z]_\beta[\check{X},\check{Z}]_\beta): - \frac{(e^{i\beta}-e^{-i\beta})^2}{N}:\mathrm{Tr}(XZ)\mathrm{Tr}(\check{X}\check{Z}):\right) , \quad (1.67)$$

where the deformed commutator is

$$[X,Z]_\beta = e^{i\beta}XZ - e^{-i\beta}ZX , \quad (1.68)$$

and $\tilde{\kappa} = \kappa/g_{\mathrm{YM}}$. Note that the $U(1)^3$ symmetry of the β-deformed Lagrangian guarantees that the number of X- and Z-fields is preserved by interactions and thus this sector continues to be closed also in the deformed theory. The double-trace contribution in (1.67) is necessary to make the theory exactly conformal [47] and ensures that the operator $\mathrm{Tr}(XZ)$ is a protected operator

$$\mathfrak{D}_2^\beta \,\mathrm{Tr}(XZ) = 0 . \quad (1.69)$$

This has been shown perturbatively at one- and two-loop level by direct calculation in [74, 75]. While the double-trace contribution is relevant at leading order in a $1/N$ expansion when acting on short operators such as $\mathrm{Tr}(XZ)$, for long operators it is subleading due to its suppression by a factor of $1/N$.

Colour-ordered amplitudes. The deformation of the $\mathcal{N} = 4$ sYM Lagrangian results in a deformation of the corresponding amplitudes starting already at tree-level. The most striking difference is the occurrence of multi-trace terms in the amplitudes. In particular for the four-point amplitude the colour decomposition is achieved by

$$\begin{aligned}\mathcal{M}_4^{(0)}&(1^{a_1},2^{a_2},3^{a_3},4^{a_4})\\ &= g_{\text{YM}}^2 \sum_{\sigma\in\mathcal{S}_4/\mathbb{Z}_4} \text{M}_{4;1}^{(0)}(p_{\sigma(1)},p_{\sigma(2)},p_{\sigma(3)},p_{\sigma(4)})\,\text{Tr}(a_{\sigma(1)}a_{\sigma(2)}a_{\sigma(3)}a_{\sigma(4)})\\ &\quad + g_{\text{YM}}^2 \sum_{\sigma\in\mathcal{S}_4/\mathbb{Z}_2\times\mathbb{Z}_2} \text{M}_{4;2}^{(0)}(p_{\sigma(1)},p_{\sigma(2)}|p_{\sigma(3)},p_{\sigma(4)})\,\text{Tr}(a_{\sigma(1)}a_{\sigma(2)})\,\text{Tr}(a_{\sigma(3)}a_{\sigma(4)})\ , \end{aligned} \tag{1.70}$$

where the double-trace contribution is generated by the double-trace interaction term in the Lagrangian (1.65) and is thus suppressed by a factor of $1/N$ compared to the single-trace contribution. The decomposition of amplitudes with a larger number of external legs involves higher multi-trace terms, but we will mainly focus on the four-point amplitude in the following. Note that we pull out a factor of g_{YM}^2 in the four-point partial amplitudes in analogy to the undeformed case, however some interaction vertices in the deformed theory appear with a complexified coupling κ which is related to the undeformed gauge coupling g_{YM} via (1.64).

Planar amplitudes. In the planar limit, the deformed amplitudes only contain single-trace colour structures like their undeformed analogues and due to (1.64) the coupling κ is essentially given by the undeformed gauge coupling: $|\kappa|^2 = g_{\text{YM}}^2$. Moreover, the corresponding deformed and undeformed partial amplitudes are very closely related and, remarkably, they only differ by an overall phase factor that is entirely determined by the flavour structure of external legs and does not probe the explicit internal interaction structure [76]. This property holds for all planar amplitudes at arbitrary loop order and can be particularly simply phrased in an alternative approach to planar β-deformed $\mathcal{N} = 4$ sYM theory achieved by replacing all products in the undeformed action (1.1) by a Moyal-like $\star$-product [77]. It is defined as

$$\Phi_i \star \Phi_j = e^{i\beta(Q_i^1 Q_j^2 - Q_j^1 Q_i^2)}\Phi_i\Phi_j\ , \tag{1.71}$$

with $\Phi_i\Phi_j$ being the usual product of two elementary fields Φ_i and Φ_j. Using this product, the overall factor of deformed planar amplitudes can simply be determined by

$$\begin{aligned}\text{M}_n^{\beta}(\Phi_1,\Phi_2,...,\Phi_n) &= \text{M}_n^{\mathcal{N}=4}(\Phi_1\star\Phi_2\star...\star\Phi_n)\\ &:= e^{i\varphi(\Phi_1,...,\Phi_n)}\text{M}_n^{\mathcal{N}=4}(\Phi_1,\Phi_2,...,\Phi_n)\ , \end{aligned} \tag{1.72}$$

where we repeatedly use (1.71), and the phase

$$\varphi(\Phi_1, ..., \Phi_n) := \beta \sum_{1 \leq i < j \leq n} (Q_i^1 Q_j^2 - Q_j^1 Q_i^2) \tag{1.73}$$

measures the sum over the Cartan charge differences of all ordered pairs of fields in the amplitude. This similarity of the planar deformed and undeformed scattering amplitudes is a direct consequence of Filk's theorem [78] which states that in a non-commutative field theory the deformed version of a planar scattering amplitude corresponds to its undeformed version times a phase factor. Note that this theorem was originally formulated for spacetime non-commutative field theories with the Moyal $\star$-product, but due to its similarity with (1.71) it holds here too. As the phase factor only depends on the ordering of the fields in the partial amplitude, it can also be moved into the colour part of the full amplitude.

In general, the phase factor in (1.72) is non-cyclic, e.g.

$$\varphi(\Phi_n, \Phi_1, ..., \Phi_{n-1}) - \varphi(\Phi_1, ..., \Phi_n) = 2 \sum_{i=1}^{n-1} (Q_n^1 Q_i^2 - Q_i^1 Q_n^2) \ . \tag{1.74}$$

Nevertheless, since an amplitude forms a $U(1)^3$ singlet, the product of fields $\Phi_1, ..., \Phi_n$ must itself be uncharged under the corresponding Cartan Q^1, Q^2 and r and this implies

$$\sum_{i=1}^{n} Q_i^1 = \sum_{i=1}^{n} Q_i^2 = 0 \tag{1.75}$$

and thus the right-hand side in (1.74) vanishes, making the planar partial amplitudes inherit the cyclicity property of undeformed amplitudes. In contrast, for a general particle configuration the partial amplitude is not invariant under reflection. Instead, due to $\varphi(\Phi_1, \Phi_2, ..., \Phi_n) = -\varphi(\Phi_n, ..., \Phi_2, \Phi_1)$, planar partial amplitudes are related to their parity-conjugated analogues via

$$\mathrm{M}_n^\beta(\Phi_n, ..., \Phi_2, \Phi_1) = e^{-2i\varphi(\Phi_1, \Phi_2, ..., \Phi_n)} \mathrm{M}_n^\beta(\Phi_1, \Phi_2, ..., \Phi_n) \ . \tag{1.76}$$

Furthermore, the $U(1)$-decoupling identity is modified to

$$\begin{aligned} \mathrm{M}_n^\beta(\Phi_1, \Phi_2, \Phi_3, ..., \Phi_n) + e^{2i\varphi(\Phi_1, \Phi_2)} \mathrm{M}_n^\beta(\Phi_2, \Phi_1, \Phi_3, ..., \Phi_n) + ... \\ + e^{2i(\varphi(\Phi_1, \Phi_2) + ... + \varphi(\Phi_1, \Phi_n))} \mathrm{M}_{\beta,1}^{(0)}(\Phi_2, \Phi_3, ..., \Phi_n, \Phi_1) = 0 \ , \end{aligned} \tag{1.77}$$

where the amplitudes pick up a factor of $e^{2i\varphi(\Phi_1, \Phi_i)}$ for every Φ_i the field Φ_1 permutes through. Using these relations the full four-point planar tree amplitudes in the β-

deformed theory can be simplified to

$$\mathcal{M}_4^{\beta}(1^{a_1},2^{a_2},3^{a_3},4^{a_4}) = \mathrm{M}_4^{\mathcal{N}=4}(1,2,3,4)\,\mathrm{Tr}([a_1,a_2][a_3,a_4])_\beta + \mathrm{M}_4^{\mathcal{N}=4}(1,3,2,4)\,\mathrm{Tr}([a_1,a_3][a_2,a_4])_\beta \ . \tag{1.78}$$

The index β on both trace factors implies that each term in the trace is accompanied by the respective phase, e.g. $\mathrm{Tr}(a_1a_2a_3a_4)$ gets accompanied by $e^{i\varphi(\Phi_1,\Phi_2,\Phi_3,\Phi_4)}$. Thus all β-dependence is moved from the kinematic part to the colour part of the amplitude and the four-point amplitude can be written in a very similar fashion to the undeformed case in (1.46).

Non-planar amplitudes. At the non-planar level amplitudes not only receive double-trace terms, but also the single-trace partial amplitudes differ from their planar version (1.72) due to the appearance of the deformed coupling κ and, in particular, they are generally not proportional to the undeformed $\mathcal{N}=4$ sYM amplitudes. Non-planar amplitudes in the deformed theory were studied in [73] at tree-, one- and partially at two-loop order. Here we concentrate on scalar four-point amplitudes at tree-level as they will be important in the derivation of the scalar dilatation operator in Section 2.3. Their colour decomposition is achieved via (1.70). As the phase $\varphi(\Phi_i,\Phi_j)$ vanishes for $\Phi_i=\Phi_j$ as well as $\Phi_i=\bar\Phi_j$, and the interaction vertices that couple Φ_i and Φ_j remain unmodified in these cases, scalar amplitudes like $\mathcal{M}_4^{(0)}(\phi_1,\phi_1,\bar\phi^1,\bar\phi^1)$ simply correspond to the $\mathcal{N}=4$ sYM amplitudes and can be obtained from the Parke–Taylor superamplitude (1.57), i.e.

$$\mathcal{M}_4^{(0)}(\phi_1^{a_1},\phi_1^{a_2},\bar\phi^{1a_3},\bar\phi^{1a_4}) = -2g_{\mathrm{YM}}^2\left(\frac{\langle12\rangle\langle34\rangle}{\langle23\rangle\langle41\rangle}f^{a_1a_2e}f^{a_3a_4e} + \frac{\langle13\rangle^2\langle24\rangle^2}{\langle12\rangle\langle23\rangle\langle34\rangle\langle41\rangle}f^{a_1a_3e}f^{a_2a_4e}\right) \ . \tag{1.79}$$

If the amplitudes contain different types of scalars the partial amplitudes can get deformed, e.g.

$$\begin{aligned}
\mathrm{M}_{4;1}^{(0)}(\phi_1,\phi_2,\bar\phi^1,\bar\phi^2) &= +|\tilde\kappa|^2e^{-2i\beta} \ ,\\
\mathrm{M}_{4;1}^{(0)}(\phi_1,\phi_2,\bar\phi^2,\bar\phi^1) &= \frac{\langle13\rangle\langle24\rangle}{\langle23\rangle\langle41\rangle}+1-|\tilde\kappa|^2 \ ,\\
\mathrm{M}_{4;1}^{(0)}(\phi_1,\bar\phi^2,\phi_2,\bar\phi^1) &= \frac{\langle13\rangle\langle24\rangle}{\langle23\rangle\langle41\rangle} \ ,
\end{aligned} \tag{1.80}$$

where again $\tilde\kappa=\kappa/g_{\mathrm{YM}}$, and there occur double-trace partial amplitudes

$$\mathrm{M}_{4;2}^{(0)} = -\frac{|\tilde\kappa|^2}{N}(e^{i\beta}-e^{-i\beta})^2 \ . \tag{1.81}$$

They combine into a colour-dressed amplitude

$$\mathcal{M}_4(\phi_1^{a_1}, \phi_2^{a_2}, \bar{\phi}^{1a_3}, \bar{\phi}^{2a_4}) = -2g_{\text{YM}}^2 \left(f^{a_1 a_2 e}_{\varphi(a_1,a_2)} f^{a_3 a_4 e}_{\varphi(a_3,a_4)} + \frac{\langle 12\rangle \langle 34\rangle}{\langle 13\rangle \langle 24\rangle} f^{a_1 a_3 e} f^{a_2 a_4 e} \right) \quad (1.82)$$

with deformed structure constants[3]

$$f^{abc}_{\varphi(a,b)} = |\tilde{\kappa}| \operatorname{Tr}([T^a, T^b]_{\varphi(a,b)} T^c) \ . \quad (1.83)$$

This formulation of scalar tree-amplitudes in terms of deformed structure constants will be useful for the derivation of the scalar dilatation operator in Section 2.3.

1.3 Infrared divergences

In the traditional approach to scattering amplitudes, a scattering experiment is decomposed into three regions: an initial and a final asymptotic region, where the particles are well-separated and their interactions negligible, and an interaction region, where the particles are microscopically close and interact over a finite amount of time. Based on this notion, we can compute scattering amplitudes perturbatively for example by Feynman diagram calculations where the external particle lines denote asymptotically free states. However, interactions mediated by massless particles are known to be long-ranged so that parts of the interaction survive even at macroscopic distances. Moreover, in conformal field theories like $\mathcal{N} = 4$ sYM theory the notion of localised particles, as well as asymptotic regions breaks down completely due to the absence of a length-scale. It is then necessary to introduce an infrared (IR) regulator to define amplitudes and they are generally dependent on the regularisation scheme and diverge order by order when removing the regulator. Similarly, infrared divergences appear in form factors due to the asymptotic on-shell state, and will also occur in the on-shell approach to the dilatation operator used in the next chapter. In order to obtain the dilatation operator, which is finite, they have to be subtracted and we will do so by exploiting their universal structure. In this section we discuss infrared divergences more broadly, wandering off the topic of non-planar sYM theories briefly because of the inherent interest in infrared physics. In particular we discuss an IR-finite definition of the QCD S-matrix via dressed asymptotic states in Section 1.3.1, and the relation to asymptotic symmetries is schematically discussed in Section 1.3.2.

[3]For complex κ and general n-point amplitudes, one has to define two sets of deformed structure constants, one including the coupling κ and another including its conjugate κ^*.

1.3.1 Infrared finite S-matrix and coherent states

The study of the infrared behaviour of gauge-theory scattering amplitudes has a long history and in the case of QED has essentially been understood since the work of Bloch and Nordsieck [79], though it has been refined over the years [80–83]. The standard approach involves the computation of amplitudes which are formally singular, and they in fact vanish after exponentiation of the perturbative divergences. One then focuses on inclusive quantities involving arbitrary numbers of real soft photons which cancel the IR divergences from virtual photons in loops. An alternative approach to the infrared divergence problem is to directly formulate infrared-finite S-matrix elements by choosing appropriate asymptotic states. For QED this approach, where the asymptotic states are not eigenstates of the photon number operator but rather have the form of coherent states, was, starting from the work of Chung [84], developed by a number of authors, e.g. [85–88]. Faddeev and Kulish, building on the work of Dollard for the Coulomb problem in non-relativistic quantum mechanics [89], related the structure of these coherent states to the form of the large-time Hamiltonian [90]. The corresponding understanding of non-abelian infrared dynamics is significantly less complete. Compared to QED, the natural complication that arises is that gauge bosons self-interact in a non-trivial way and in particular collinear divergences appear. The persistence of these non-trivial self-interactions at early and late times is central to the infrared behaviour of QCD.

Asymptotic Hamiltonian and soft-evolution operator. The coherent-state approach was partially extended to the much more complicated non-abelian case in [91–101][4]. The starting point is the choice of an appropriate asymptotic Hamiltonian describing the parton dynamics in the far future and far past. One can carry out the Faddeev-Kulish approach in the non-abelian case [99] by splitting the standard QCD interaction Hamiltonian $H^I(t)$ in the interaction representation into soft and hard parts

$$H^I(t) = H_h^E(t) + H_s^E(t) \ , \tag{1.84}$$

which is done by introducing at each interaction vertex an energy transfer and defining the soft part of the Hamiltonian as containing only energies below the scale E. One can show that at leading order the four-gluon vertex as well as ghost contributions can be neglected, and the soft Hamiltonian is given solely by the cubic interaction

[4]See also [102] for a more recent application to perturbative gravity.

vertices between one soft gluon and two hard quarks or gluons, specifically

$$H_s^E(t) = -g_{\text{YM}} \int_{\omega_q} \widetilde{dp} \int^E \widetilde{dq}\ \rho^a(\mathbf{p}) \hat{p} \cdot [a^a(\mathbf{q}) e^{i\hat{p}\cdot qt} + h.c.]$$
$$\text{with} \quad \rho^a(\mathbf{p}) = \rho_f^a(\mathbf{p}) + \sum_h a_h^{\dagger b}(\mathbf{p}) t^a{}_{bc} a_h^c(\mathbf{p}) \quad \text{and} \quad \widetilde{dq} = \frac{1}{2} \frac{d^3q}{(2\pi)^3 (2\omega_q)} \ . \tag{1.85}$$

This expression contains the soft gluon creation/annihilation operators $a_h^{a\dagger}(\mathbf{q})/a_h^a(\mathbf{q})$, where a and h correspond to the colour and helicity of the inserted gluons with three-momentum $\mathbf{q}$ and energy $\omega_q < E$. t^a collectively denotes the $SU(N)$ gauge generators in the representation corresponding to the respective particle, and in this case for the gluon transforming in the adjoint representation, it is given by $t^a{}_{bc} = -\sqrt{2} i f^a{}_{bc}$. We furthermore use the notation $\hat{p}^\mu = p^\mu/\omega_p$, and the density ρ contains contributions from fermionic matter ρ_f and from hard gluons with energies $\omega_p > \omega_q$, .

Finite S-matrix and coherent states. The soft Møller operator, or soft-evolution operator, is defined as the time-ordered exponential

$$\Omega_\pm^E = \mathcal{T} \exp\left[-i \int_{\mp\infty}^0 H_s^E(\tau) d\tau \right] \ , \tag{1.86}$$

where Ω_+^E is an operator acting on an incoming Fock state, while Ω_-^E acts on outgoing states. It is useful to transform to frequency space and, using (1.85), the Møller operator becomes the energy-ordered exponential

$$\Omega^E = \mathcal{P}_\omega \exp\Big[\int^E \widetilde{dq}\ \mathcal{J}_q \cdot \Pi_q \Big] \ , \tag{1.87}$$

where $\Pi_\mu^a(q) = a_\mu^a(q) - a_\mu^{a\dagger}(q)$ is the displacement operator and

$$\mathcal{J}_q^a{}_\mu = g_{\text{YM}} \int_{\omega_q} \widetilde{dp}\ \rho^a(\mathbf{p}) \frac{p_\mu}{p \cdot q} \ . \tag{1.88}$$

The exponential is interpreted as being ordered in the soft gluon energies with smaller energies to the right.

The action of Ω_+^E/Ω_-^E on incoming/outgoing Fock states of hard particles dresses these states with clouds of soft gluons and thus takes into account their long-range interactions at leading order in the IR expansion. The soft evolution operators can then be used to define an IR-finite S-matrix S^E by removing the IR singularities due to these initial- and final-state interactions

$$S^E = \Omega_-^E\, S\, \Omega_+^{E\dagger} \tag{1.89}$$

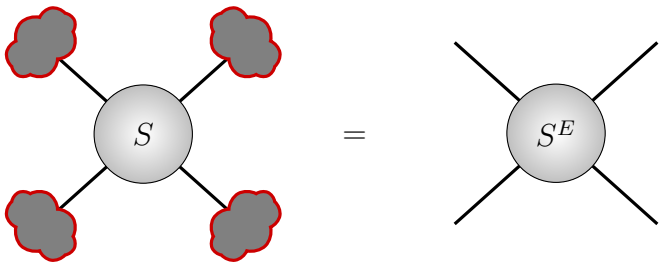


Figure 1.1: Dressing of external legs in an amplitude, represented here by the clouds, which removes the IR singularities and produces an IR-finite S-matrix.

as shown schematically in Figure 1.1. Alternatively, one can move the soft-evolution operator into the definition of asymptotic states by defining

$$\|\{p_i,\alpha_i\}\rangle\!\rangle = \Omega^{E\dagger}|\{p_i,\alpha_i\}\rangle \ , \tag{1.90}$$

where $\{p_i\}$ denotes the momenta and $\{\alpha_i\}$ collectively denotes the remaining quantum numbers of the hard particles. The state on the right-hand side is the usual Fock state $|\{p_i,\alpha_i\}\rangle = \prod_i b^\dagger_{\alpha_i}(p_i)|0\rangle$ with hard-particle generator $b^\dagger_\alpha$ generating a hard quark or gluon depending on the quantum numbers α. Then one computes IR-finite matrix elements of the traditional S-matrix between these dressed states (1.90). To leading order in the soft divergence, the dressing of external states factorises in colour space [96, 97, 99]

$$\Omega^{E\dagger}|\{p_i,\alpha_i\}\rangle = \prod_i \mathcal{U}^{p_iE}_{\alpha_i\beta_i}(\Pi) b^\dagger_{\beta_i}(p_i)|0\rangle \ , \tag{1.91}$$

where the coherent-state operator $\mathcal{U}^{p_iE}_{\alpha_i\beta_i}(\Pi)$ is a functional of the soft gluons only. The coherent-state operator for a parton in the gauge-group representation with generators $t^a_{\alpha\beta}$ is defined by the energy-ordered integral

$$\mathcal{U}^{pE}_{\alpha\beta} = \mathcal{P}_\omega \exp\Big[-g_{\rm YM} \int_\lambda^E \widetilde{dq}\ \frac{p\cdot\Pi^a_\omega(q)}{p\cdot q} t^a \Big]_{\alpha\beta} \ , \tag{1.92}$$

where the dressed gluon field is similarly defined by

$$\Pi^a_\omega(q) = \mathcal{U}^{qE}_{ab}\Pi^b(q) \tag{1.93}$$

and $\mathcal{U}^{pE}_{ab}$ is the adjoint coherent-state operator. These non-linear equations can be

solved iteratively so that to $\mathcal{O}(g_{\rm YM}^2)$ we have

$$\begin{aligned}\mathcal{U}^{pE}_{\alpha\beta} = \delta_{\alpha\beta} &- g_{\rm YM}\int_\lambda \widetilde{dq}\;\frac{p\cdot\Pi^e(q)}{p\cdot q}t^e_{\alpha\beta}\\ &+ g_{\rm YM}^2\int_\lambda \widetilde{dq}_1\int_{\omega_1}\widetilde{dq}_2\left(\frac{p\cdot\Pi^{e_2}(q_2)}{p\cdot q_2}t^{e_2}_{\alpha\gamma}\right)\left(\frac{p\cdot\Pi^{e_1}(q_1)}{p\cdot q_1}t^{e_1}_{\gamma\beta}\right)\\ &- g_{\rm YM}^2\int_\lambda \widetilde{dq}_1\int_{\omega_1}\widetilde{dq}_2\left(\frac{p\cdot\Pi^{e_2}(q_2)}{p\cdot q_2}\right)\left(\frac{q_2\cdot\Pi^{e_1}(q_1)}{q_2\cdot q_1}\right)\cdot[t^{e_2},t^{e_1}]_{\alpha\beta}\;. \end{aligned}\tag{1.94}$$

This dressing factor captures the leading-order effects of soft-gluon radiation of each of the hard partons. It includes all-order effects arising from arbitrary numbers of gluons being radiated, as well as loop effects which arise from normal ordering each of the terms.

Finite one-loop S-matrix. Let us proceed with a one-loop calculation illustrating how the dressing in coherent-state operators removes IR divergences. For now considering only incoming particles, we compute the one-loop matrix elements

$$\begin{aligned}\langle\!\langle 0\|\,S\,\|\{p_i,\alpha_i\}\rangle\!\rangle &= \langle 0|S\prod_i \mathcal{U}^{p_iE}_{\alpha_i\beta_i}|\{p_i,\beta_i\}\rangle\\ &= \langle 0|S|\{p_i,\alpha_i\}\rangle\\ &\quad - g_{\rm YM}\sum_j t_j^{e_1}\int_\lambda \widetilde{dq}\langle 0|S\;\tfrac{p_j\cdot\Pi^{e_1}(q)}{p_j\cdot q}|\{p_i,\alpha_i\}\rangle\\ &\quad + \frac{g_{\rm YM}^2}{2}\sum_{j\neq k}t_j^{e_1}t_k^{e_2}\int_\lambda\widetilde{dq}_1\int_\lambda\widetilde{dq}_2\langle 0|S\tfrac{p_j\cdot\Pi^{e_1}(q_1)}{p_j\cdot q_1}\tfrac{p_k\cdot\Pi^{e_2}(q_2)}{p_k\cdot q_2}|\{p_i,\alpha_i\}\rangle\\ &\quad + \text{one-parton terms}\;. \end{aligned}\tag{1.95}$$

t_i denotes the $SU(N)$ gauge generator associated to the hard particle i and for fields transforming in the fundamental representation it corresponds to $t^a=T^a$, whereas for fields in the adjoint it is again given by $(t^a)_{bc}=-\sqrt{2}if^a{}_{bc}$. It acts on the ith particle in a state $|\{p_j,\alpha_j\}\rangle$ as $t_i^a\,|\{p_j,\alpha_j\}\rangle := t^a_{\alpha_i\beta_i}\,|p_1,\alpha_1;...;p_i,\beta_i;...;p_n,\alpha_n\rangle$.

The first term in (1.95) is the usual IR-divergent S-matrix element which arises from diagrams such as Figure 1.2 (i). To one-loop order and leading IR accuracy in dimensional regularisation with parameter $\hat\epsilon=(d-4)/2$, it is known to be of the form [103]

$$\mathcal{M}_n = \Big[1+\frac{g_{\rm YM}^2}{16\pi^2\hat\epsilon^2}\sum_{j\neq k}t_j^e t_k^e\Big]\mathcal{M}_n^{(0)}\;. \tag{1.96}$$

The remaining terms are the compensating IR-divergent terms from the coherent state which are graphically represented in Figure 1.2 (ii) and (iii). The last term in

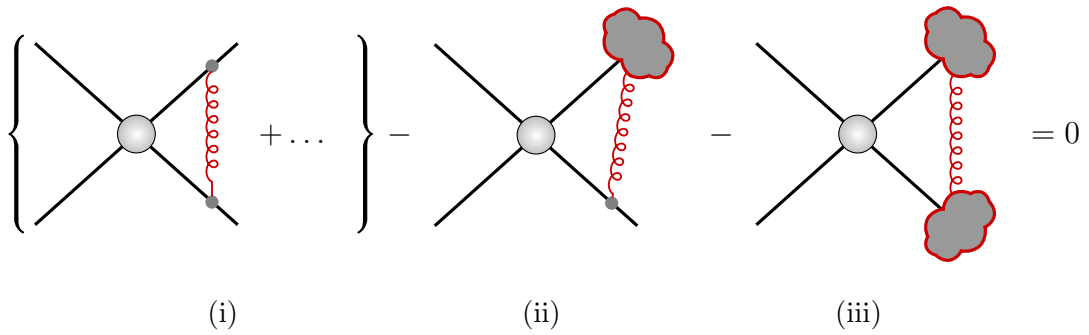


Figure 1.2: The IR-divergent contributions at one-loop involving two external partons.

(1.95) involves contracting two gluons from the dressing factors of two partons and the second term gives an $\mathcal{O}(g_{\rm YM}^2)$ contribution by using the tree-level soft limit of the S-matrix. Here we focus on those diagrams which involve two external partons and neglect one-parton contributions, where the soft gluon attaches on the leg/cloud of a single external parton. These one-parton contributions are subleading in the IR divergences and so are not needed. The tree-level soft-gluon theorem is given by[5]

$$\lim_{\omega_q\to 0}\mathcal{M}_{n+1}^{(0)}(q,a,h;\{p_i,\alpha_i\}) = g_{\rm YM}J_h^{(0)a}(q)\mathcal{M}_n^{(0)}(\{p_i,\alpha_i\}) \tag{1.97}$$

with the soft current, see e.g. [104],

$$J_h^{(0)a} = \sum_i \frac{p_i\cdot\varepsilon^h(q)}{p_i\cdot q}t_i^a\ , \tag{1.98}$$

where the soft gluon of momentum q, colour a and helicity h is taken to be outgoing. If the gluon was incoming, there would be an overall minus sign and the corresponding polarisation vector would be $\bar{\varepsilon}^h$. Incoming hard momenta in the amplitude give additional terms in the soft current similar to those of outgoing momenta but with an overall minus sign. The soft factor (1.98) is "universal" in the sense that it is independent of the type of the hard particle that the soft gluon couples to and, for example, is blind to the helicity or mass of the hard particle. This universality persists to all orders in $g_{\rm YM}$, i.e. at any order k the leading soft current $J_h^{(k)a}$ will only depend on the momenta p_i of external particles and the quantum numbers of the soft gluon. At subleading order in the IR expansion the universality does not survive and the soft factor probes more of the external data. Evaluating all coherent-state contributions to (1.95) explicitly in dimensional regularisation, one finds that the singular parts of (1.96) indeed get cancelled and the S-matrix evaluated in dressed states (1.90) is IR-finite, see [1] for more details.

[5]In this expression, and similar expressions below, as the limit does not strictly exist the notation $\lim_{\omega_q\to 0}$ should be understood as referring to the leading term in an expansion in small ω_q.

1.3.2 Asymptotic symmetries and soft theorems

The observation that asymptotic conservation laws pave the way to understanding the infrared dynamics of gauge theories [105,106][6] has led to a renewed interest in the study of coherent-state operators and soft dressing more generally [108–116]. These conservation laws follow from Noether's second theorem [117] for large gauge transformations and correspond to the soft theorems of amplitudes which were already explored by Weinberg [83]. It has been demonstrated that for QED [110] and perturbative gravity [112] the coherent states relevant to the construction of an infrared finite S-matrix follow from the symmetry of asymptotic charges. The existence of an infinite-dimensional symmetry group has led to the interpretation of the QED vacuum as being degenerate and that scattering processes are accompanied by a shift in the vacua. Infrared divergences due to massless particles which result in the vanishing of S-matrix elements are thus connected with the "wrong" choice of the in- and out-vacua, and a cure can be sought in a systematic way by considering the asymptotic charges. An analogous statement can be made for perturbative gravity, using BMS supertranslation charges to find suitable asymptotic states for an infrared-finite gravity S-matrix.

Asymptotic charge and soft gluon theorem. The behaviour of Yang–Mills amplitudes as individual gluons become soft is related to soft theorems which, as was shown in [105], are associated to the Ward identities of asymptotic symmetries. The corresponding asymptotic charge Q_ϵ can be defined, see [1] for a detailed discussion, and involves a soft contribution Q^s_ϵ containing soft gluon creation/annihilation operators $a_h^{a\dagger}(\omega_q)/a_h^a(\omega_q)$ with vanishing energy $\omega_q \simeq 0$. The subscript ϵ indicates the dependence on a transformation parameter. This charge has matrix elements with the scattering amplitude S of the form

$$\langle 1, 2, ..., n|Q^s_\epsilon S|0\rangle \sim \langle 1, 2, ..., n| \lim_{\omega_q \to 0} \omega_q a_h^a(\omega_q) S|0\rangle \tag{1.99}$$

and at tree level the limit in the definition of the charge is understood as

$$\langle 1, 2, ..., n| \lim_{\omega_q \to 0} \omega_q a_h^a(\omega_q) S|0\rangle = \lim_{\omega_q \to 0} \omega_q \langle 1, 2, ..., n|a_h^a(\omega_q) S|0\rangle \tag{1.100}$$

and can be computed using the tree-level soft gluon theorem (1.97). At loop level the issue of the soft limit is more subtle: after computing the perturbative terms defining the matrix elements, one may attempt to take the soft limit at the level of the integrands before performing loop integrations, or alternatively one may keep ω_q finite and take the limit only after performing the loop integrations. It is known

[6] See [107] for a review and more complete references.

from the case of subleading IR behaviour of graviton amplitudes [118, 119], that the order has important consequences for the interpretation of quantum corrections to the Ward identities for asymptotic symmetries. When taking the soft limit after the computation of matrix elements, i.e.

$$\langle 1,2,...,n|Q^s_\epsilon(0)S|0\rangle := \lim_{\omega_q\to 0}\langle 1,2,...,n|Q^s_\epsilon(\omega_q)S|0\rangle \ , \tag{1.101}$$

the insertion of the soft gluon operator can be computed by the usual soft gluon theorem, which to one-loop order is given by [103, 120]

$$\begin{aligned}\lim_{\omega_q\to 0}\mathcal{M}^{(1)}_{n+1}(q,a,h;\{p_i,\alpha_i\}) = g_{\rm YM}J^{(0)a}_h(q)\mathcal{M}^{(1)}_n(\{p_i,\alpha_i\}) \\ + g^3_{\rm YM}J^{(1)a}_h(q)\mathcal{M}^{(0)}_n(\{p_i,\alpha_i\}) \ .\end{aligned} \tag{1.102}$$

The first term on the right-hand side is the iterated tree result involving the tree-level soft current (1.98), while the second term is due to the one-loop soft current which is, to leading divergence in dimensional regularisation,

$$J^{(1)a}_h(q) = -\frac{C_A}{16\pi^2\hat{\epsilon}^2}J^{(0)a}_h(q) + \mathcal{O}\left(\hat{\epsilon}^{-1}\right) \tag{1.103}$$

with adjoint quadratic Casimir C_A. This leading double pole is due to both collinear and soft divergences at leading logarithmic accuracy and the coherent-state approach of [97, 100] deals with both of them at this order.

Ward identities for the dressed S-matrix. The soft-evolution operator (1.86) not only allows for an IR-finite definition of S-matrix elements, but also relates the free theory, where multi-particle asymptotic states can be described by the usual Fock states, to the interacting theory, where long-range interactions lead to IR divergences and can be tamed by coherent-state dressings (1.90). It similarly relates the soft asymptotic charge Q^s_ϵ, containing contributions linear in soft gluons, to the hard, or non-linear, part Q^h_ϵ of the asymptotic charge which imply a Ward identity for the S-matrix [1]

$$\langle\!\langle\{p_f,\alpha_f\}\|\,[Q^s_\epsilon,S]\,\|\{p_i,\alpha_i\}\rangle\!\rangle = -\langle\{p_f,\alpha_f\}|[Q^h_\epsilon,S^E]|\{p_i,\alpha_i\}\rangle \ . \tag{1.104}$$

Evaluating the expression on the left-hand side of (1.104) at tree-level and leading IR-divergence yields the tree-level hard charges for incoming and outgoing legs which are essentially given by the integrated tree-level soft current (1.98). The result of the analogous computation at one-loop order depends on the precise prescription for the order of soft limits in the definition of the charge and dressing factor which is analogous to the order of limits discussed in [118, 119]. In the first choice, corresponding to the

standard choice (motivated by its use in problems such as the computation of physical cross sections) in dimensionally-regularised soft limits, we find that the Ward identity receives no corrections at one-loop and leading IR-divergence. In the second choice we find a correction which is related to the one-loop soft current (1.102). See [1] for a more detailed discussion.

Chapter 2

Dilatation operator from on-shell methods

Recent years have seen much progress in the computation of scattering amplitudes due to the advent of on-shell methods. The basic idea behind these is to only work with physical degrees of freedom to obtain observables, and specifically amplitudes, in a given theory. Doing so, one avoids gauge redundancies and an in general large number of Feynman diagrams. The latter often times obfuscate the simplicity of the final expression and can be much less efficient and fast when compared to the new on-shell approaches. While most of these approaches were developed with the quest to compute amplitudes, they are useful for a wider range of physical observables, including form factors and the dilatation operator. We review some of these methods in Section 2.1, including an on-shell approach to the dilatation operator on which the remaining parts of this chapter will be based. This allows us to study the mixing of marginal operators in $\mathcal{N} = 4$ sYM theory on the basis of the Parke–Taylor superamplitude in Section 2.2, and obtain the non-planar anomalous dimensions in this sector. Then we move on to derive the one-loop β-deformed dilatation operator in the scalar sector in Section 2.3, which is the key result of this chapter, and this dilatation operator will be further analysed in subsequent parts of this thesis.

2.1 Review of on-shell techniques

2.1.1 On-shell recursion for amplitudes and form factors

An important step forward in the quest for more efficient methods than the traditional Feynman diagram techniques is the CSW approach [121]. Here amplitudes are constructed by taking lower-point MHV amplitudes as vertices and linking them by scalar propagators in a so-called MHV vertex expansion. This approach contains a considerably smaller number of more compact expressions than the analogous Feyn-

man diagram calculations, however with increasing number of external states the computation of amplitudes can still be rather involved. The most well-known on-shell approach to tree-level scattering amplitudes are the on-shell recursion relations due to Britto, Cachazo, Feng and Witten [122, 123].

The underlying idea behind on-shell recursion relations is to construct general n-point amplitudes $\mathcal{M}_n$ from a number of lower-point amplitudes which thus serve as the basic building blocks in this approach. A relation between amplitudes with different numbers of legs can be obtained by Cauchy's theorem in the following way: One considers a general amplitude $\mathcal{M}_n$ as a function of shifted external momenta, the shift being a complex deformation of the momenta which preserves their on-shellness and the overall momentum conservation. This deformation is parametrised by one complex parameter $z \in \mathbb{C}$ and is chosen in such a way that the original amplitude corresponds to the complexified amplitude $\hat{\mathcal{M}}_n(z)$ at $z = 0$, i.e.

$$\begin{aligned} \mathcal{M}_n &\equiv \hat{\mathcal{M}}_n(z=0) \\ &\equiv \oint \frac{dz}{2\pi i} \frac{\hat{\mathcal{M}}_n(z)}{z} \ , \end{aligned} \tag{2.1}$$

where in the second line we have rewritten the original amplitude as a contour integral of the complexified amplitude with the contour encircling $z = 0$ in the complex plane. By applying Cauchy's theorem one can deform this contour to infinity while picking up simple poles of the amplitude at complex $z = z_R$:

$$\mathcal{M}_n = -\sum_{z_R} \mathrm{Res}_{z=z_R}\Big(\frac{\hat{\mathcal{M}}_n(z)}{z}\Big) + \mathcal{B}_n \ . \tag{2.2}$$

The boundary term $\mathcal{B}_n$ is usually not easy to determine, but in a number of theories one can show that it vanishes for certain amplitudes and deformations, see e.g. [124–126]. Once the vanishing of this boundary contribution is established, the original amplitude $\mathcal{M}_n$ can be computed entirely on the basis of its poles. These poles occur whenever one of the deformed propagators $\hat{P}^\mu$ in $\mathcal{M}(z)$ goes on-shell, i.e. $\hat{P}_R^2 = 0$. The residue of the deformed amplitude then factorises into two lower-point on-shell amplitudes as

$$\mathrm{Res}_{z=z_R}\Big(\frac{\hat{\mathcal{M}}_n(z)}{z}\Big) = -\hat{\mathcal{M}}_{n_l}(z_R)\frac{1}{P_R^2}\hat{\mathcal{M}}_{n_r}(z_R) \ , \quad n_l + n_r = n \ , \tag{2.3}$$

where $P_R^2 = \hat{P}_R^2(z=0)$. Thus, from a given set of usually three- and four-point amplitudes, higher-point tree-level amplitudes can be recursively computed at complex factorisation channels.

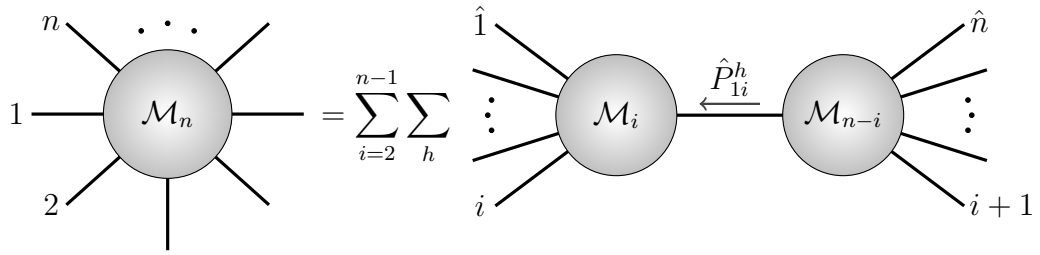


Figure 2.1: On-shell recursion relations for colour-ordered amplitudes.

BCFW recursion. The BCFW recursion is a specific version of the on-shell recursions where only two external momenta p_i and p_j get deformed. It is achieved by shifting the corresponding spinor-helicity variables as

$$\begin{aligned} \lambda_i &\to \hat{\lambda}_i(z) = \lambda_i - z\lambda_j \ , & \tilde{\lambda}_i &\to \hat{\tilde{\lambda}}_i(z) = \tilde{\lambda}_i \ , \\ \lambda_j &\to \hat{\lambda}_j(z) = \lambda_j \ , & \tilde{\lambda}_j &\to \hat{\tilde{\lambda}}_j(z) = \tilde{\lambda}_j + z\tilde{\lambda}_i \ . \end{aligned} \tag{2.4}$$

This shift is known as the BCFW-shift or, to be explicit about the shifted legs, is also called $[j,i\rangle$-shift. Importantly, the corresponding shifted momenta $\hat{p}_i^{\dot{\alpha}\alpha} = \hat{\tilde{\lambda}}_i^{\dot{\alpha}} \hat{\lambda}_i^{\alpha}$ and $\hat{p}_j^{\dot{\alpha}\alpha} = \hat{\tilde{\lambda}}_j^{\dot{\alpha}} \hat{\lambda}_j^{\alpha}$ remain on-shell, i.e. $\hat{p}_i^2 = \hat{p}_j^2 = 0$, and do not modify the overall momentum conservation due to $p_i^\mu + p_j^\mu = \hat{p}_i^\mu + \hat{p}_j^\mu$. Assuming a vanishing boundary contribution under a $[1,n\rangle$-shift, the colour-ordered component of an amplitude $\mathcal{M}_n$ with outgoing external states $\langle 1,2,...,n|$ can be obtained by BCFW recursion via the relation [122, 123]

$$\langle 1,2,...,n|\mathcal{M}|0\rangle = \sum_{i=2}^{n-1} \sum_h \left. \frac{\langle \hat{1},2,...,i|\mathcal{M}|\hat{P}_{1i}^h\rangle \, \langle \hat{P}_{1i}^h, i+1,...,n-1,\hat{n}|\mathcal{M}|0\rangle}{P_{1i}^2} \right|_{z=z_{1i}} . \tag{2.5}$$

Here the first sum is over all planar partitions of external momenta to the two lower-point amplitudes, and the second sum is over all possible helicities h of the intermediate state with momentum $\hat{P}_{1i} := \hat{p}_1 + p_2 + ... + p_i$. An illustration of this relation can be found in Figure 2.1.

Whether or not a specific amplitude $\mathcal{M}_n$ in a given theory can be constructed from a shift (2.4) depends on the scaling of $\hat{\mathcal{M}}_n(z)$ at the boundary, i.e. for $z \to \infty$. In [124] the BCFW shift for $z \to \infty$ was interpreted as a limit in which the two shifted lines become hard and scatter against a soft background of the remaining unshifted external states. This led to a proof that general tree-amplitudes in pure gauge, as well as in pure gravity theories are constructible by a BCFW shift, i.e. the boundary terms vanish. Furthermore, it was proven that for general tree-level amplitudes in $\mathcal{N} = 4$ sYM theory, one can always find a valid BCFW shift and thus they are

on-shell constructible [127].

Recursion relations beyond BCFW. The BCFW shift is the simplest shift to recursively construct amplitudes. Other shifts involve the deformation of more legs, and examples are the Risager shift [128] with three shifted legs, and the all-line shift [129]. Although generally shifting a higher number of legs makes the construction of amplitudes more complicated, these shifts are interesting because they can lead to an improved scaling of the deformed amplitude $\hat{\mathcal{M}}_n(z)$ at large z and thus to the on-shell constructibility of the corresponding amplitude. In particular, all tree amplitudes with $n > 4$ external particles in four-dimensional renormalisable theories have been shown to be on-shell constructible via all-line holomorphic or anti-holomorphic shifts [125]. In [126] the on-shell constructibility was further explored by a study of the simplest recursion relations (i.e. those with the smallest number of shifted legs) for constructing the amplitudes in a given theory. A constraint on the minimum number of shifted legs for a given amplitude was obtained and in general renormalisable field theories all amplitudes can be constructed from shifts involving at most five lines. For the Leigh–Strassler deformations that we are interested in in this thesis, the criteria of [126] even allow for the three-line shift constructibility of all amplitudes with $n > 4$ due to the presence of the $U(1)$ R-symmetry.

BCFW recursion for form factors. Form factors in $\mathcal{N} = 4$ sYM theory inherit many of the structures and techniques of their purely on-shell analogues, the scattering amplitudes, and specifically the BCFW recursion was shown to be applicable to the construction of tree-level form factors [130]. Assuming a vanishing boundary term for a $[1, n\rangle$-shift, i.e. the deformed form factor goes as $\hat{\mathcal{F}}_{\mathcal{O}}(z) \to 0$ for $z \to \infty$, the colour-ordered component of a form factor $\mathcal{F}_{\mathcal{O}}$ is given by the sum of its various factorisation channels as

$$\begin{aligned}\langle 1, 2, ..., n|\mathcal{O}|0\rangle = \sum_{i=2}^{n-1}\sum_{h}\Bigg[& \frac{\langle \hat{1}, 2, ..., i|\mathcal{M}|\hat{P}^h_{1i}\rangle \, \langle \hat{P}^h_{1i}, i+1, ..., n-1, \hat{n}|\mathcal{O}|0\rangle}{P^2_{1i}}\Bigg|_{z=z_{1i}} \\ & + \frac{\langle \hat{1}, 2, ..., i|\mathcal{O}| - \hat{P}^h_{i+1,n}\rangle \, \langle -\hat{P}^h_{i+1,n}, i+1, ..., n-1, \hat{n}|\mathcal{M}|0\rangle}{P^2_{i+1,n}}\Bigg|_{z=z_{i+1,n}}\Bigg] . \end{aligned} \tag{2.6}$$

Note that in this case the momenta running in the poles, $\hat{P}_{1i}$ and $-\hat{P}_{i+1,n}$, do not satisfy $\hat{P}^2_{1i} = \hat{P}^2_{i+1,n}$ due to the modified momentum conservation that relates the off-shell momentum q of the local operator $\mathcal{O}(q)$ to the sum of external on-shell momenta as $\sum_i p_i = q$. The relation (2.6) is represented in Figure 2.2.

In [131] the (super-)BCFW recursion for form factors was used to obtain the MHV

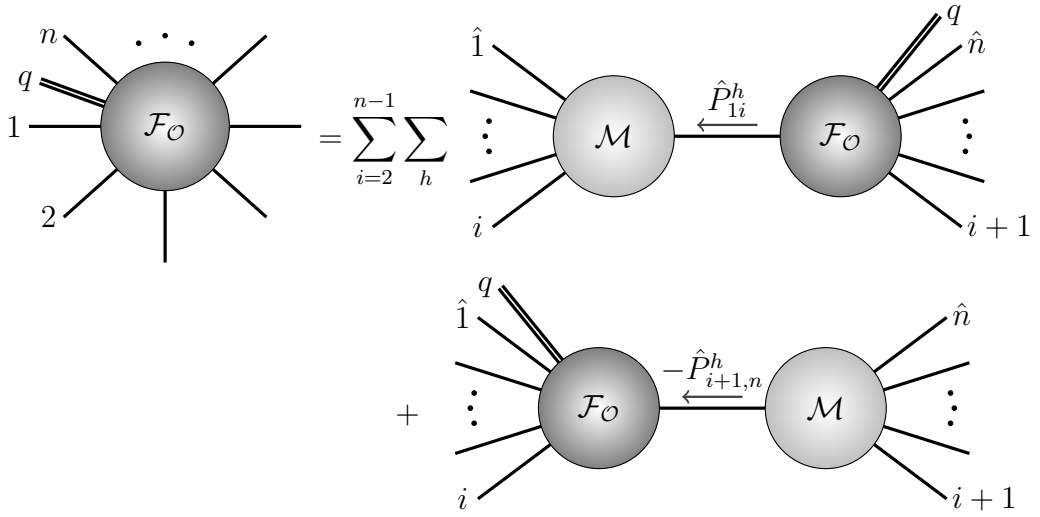


Figure 2.2: On-shell recursion relations for colour-ordered form factors.

form factor of the on-shell Lagrangian (1.17) and found to be

$$\langle 1, ..., n|\mathcal{L}(0)|0\rangle = \frac{\delta^{(8)}(\sum_{i=1}^n \lambda_i^\alpha \tilde{\eta}_i^A)}{\langle 12\rangle \langle 23\rangle ... \langle n1\rangle} , \tag{2.7}$$

which corresponds to the Parke–Taylor amplitude (1.57) up to the modified momentum-conserving δ-function.

2.1.2 Dilatation operator meets scattering amplitudes

We now move on to the discussion of an on-shell method that allows one to extract the dilatation operator $\mathfrak{D}$ of a conformal field theory. This method was developed in [32] by Zwiebel, and then reformulated in terms of form factors and proven in [31] by Caron-Huot and Wilhelm.

Review of proof. Since the relation between the dilatation operator $\mathfrak{D}$ and scattering amplitudes takes on a crucial role in this part of the thesis, we begin this section with a review of the corresponding proof of [31]. We start with the renormalisation group equation for a form factor $\mathcal{F}_\mathcal{O}(1, ..., n; q)$

$$\left[\mu\partial_\mu + \beta(g^2_{\text{YM}})\frac{\partial}{\partial g^2_{\text{YM}}} + \gamma_\mathcal{O} - \gamma_{\text{IR}}\right] \mathcal{F}_\mathcal{O}(1, ..., n; q) = 0 . \tag{2.8}$$

The first term measures the explicit dependence of $\mathcal{F}_\mathcal{O}$ on the renormalisation scale μ. It appears in dimensionless ratios p_i^2/μ^2 in $\mathcal{F}_\mathcal{O}$ such that we may replace it with the derivative with respect to the external momenta as

$$\mu\partial_\mu \simeq -\textstyle\sum_i p_i \partial_{p_i} . \tag{2.9}$$

The second term in the bracket measures the dependence of $\mathcal{F}_\mathcal{O}$ on the renormalisation scale from a running coupling constant g_{YM}. In conformal field theories with $\beta(g^2_{\text{YM}}) \equiv 0$ we may drop this term, while in general QFTs this term is subleading in the perturbative expansion in g^2_{YM}. The anomalous dimension $\gamma_\mathcal{O}$ measures the μ-dependence of the renormalisation factor of $\mathcal{O}$. Note that there does not occur a similar term for the external states since their wave-function renormalisation factor is stripped off from the form factor by LSZ-reduction. However, potential IR divergences may not be captured in this and thus there occurs a term γ_{IR} which measures the dependence on the scale that regularises these divergences.

Let us further investigate the first term in (2.8). Via (2.9) it generates shifts in momenta, i.e. it acts as the dilatation operator $\mathfrak{D}$ in momentum-space on momentum-dependent functions and, in particular, acts on the external momenta of a form factors as

$$e^{i\alpha\mathfrak{D}_p}\mathcal{F}_\mathcal{O}(p_1,...,p_n;q) = \mathcal{F}_\mathcal{O}(e^{i\alpha}p_1,...,e^{i\alpha}p_n;e^{i\alpha}q)\ , \quad \alpha\in\mathbb{R}\ . \tag{2.10}$$

Here we put the label p on $\mathfrak{D}$ to emphasize that this operator only generates dilatations of the momenta occurring explicitly in $\mathcal{F}_\mathcal{O}$, but does not generate dilatations of the renormalisation of the operator $\mathcal{O}$ or the running coupling g_{YM}. Furthermore, we make the momentum-labels of the external particles explicit, but any other labels are left implicit. In (2.10) we also assume that we may analytically continue the function $\mathcal{F}_\mathcal{O}$ to the complex p_i plane. We can assume that the rotation of momenta goes smoothly until the angle $\alpha = \pi$ is reached where the momenta reverse their signs and the kinematic invariants $(p_i + p_j + ... + p_k)^2$ return to their original values. Note that $\mathcal{F}_\mathcal{O}$ has a small imaginary contribution that regularises singularities for real p_i via Feynman's prescription. Thus by a rotation of $\alpha = \pi$, $\mathcal{F}_\mathcal{O}$ does not return to its original value but we end up on the other side of the cut, i.e. the complex conjugate of $\mathcal{F}_\mathcal{O}$:

$$\mathcal{F}_\mathcal{O} = e^{-i\pi\mathfrak{D}_p}\mathcal{F}^*_\mathcal{O}\ . \tag{2.11}$$

Another way of relating a form factor to its complex conjugate is by using its interpretation as the correction to the S-matrix S in a deformed theory with Lagrangian $\mathcal{L} \to \mathcal{L} + g\mathcal{O}$. Then the S-matrix becomes $S \to S + g\delta S$ with $\delta S = i\int \mathrm{d}^4x\mathcal{F}_\mathcal{O}$. Using unitarity of the original S-matrix $S\cdot S^\dagger = 1$, where in the product we sum over all possible intermediate states, and moreover demanding unitarity for the deformed S-matrix, yields $\mathcal{F}_\mathcal{O} = S\mathcal{F}^*_\mathcal{O}S$ where we again sum over all intermediate states. Multiplying this relation into an initial vacuum state gives

$$\mathcal{F}_\mathcal{O} = S\mathcal{F}^*_\mathcal{O}\ . \tag{2.12}$$

Together with (2.11) this implies

$$e^{-i\pi\mathfrak{D}_p}\mathcal{F}^*_{\mathcal{O}} = S\mathcal{F}^*_{\mathcal{O}} \ . \tag{2.13}$$

In the perturbative regime we may expand the relations (2.8) and (2.13). After stripping off a momentum conserving δ-function setting the form-factor off-shell momentum to $q = p_1 + ... + p_n$ via (1.58), for the former one obtains at $\mathcal{O}(g^2_{\rm YM})$

$$\left[-\mathfrak{D}^{(1)}_p + \gamma^{(1)}_{\mathcal{O}} - \gamma^{(1)}_{\rm IR}\right] \langle 1,...,n|\mathcal{O}(0)|0\rangle^{(0)} = 0 \ , \tag{2.14}$$

and for the latter

$$\mathfrak{D}^{(1)}_p \langle 1,...,n|\mathcal{O}(0)|0\rangle^{*(0)} = -\frac{1}{\pi} \langle 1,...,n|\mathcal{M}\otimes\mathcal{O}(0)|0\rangle^{*(0)} \ , \tag{2.15}$$

where we used $S = \mathbb{1} + i\mathcal{M}$ and $\otimes$ denotes a convolution corresponding to a one-loop unitarity cut which we will discuss in more detail further below. Note that the expansion may also include an expansion of $\mathcal{O}$ in $g_{\rm YM}$ and at leading order only the leading terms of $\mathcal{O}$ contribute. In the following we will often use these relations for minimal tree-level form factors which are real and thus we may drop the complex conjugation to obtain [31], see also [32],

$$\gamma^{(1)}_{\mathcal{O}} \langle 1,...,n|\mathcal{O}(0)|0\rangle^{(0)} = -\frac{1}{\pi} \langle 1,...,n|\mathcal{M}\otimes\mathcal{O}(0)|0\rangle^{(0)} + \gamma^{(1)}_{\rm IR} \langle 1,...,n|\mathcal{O}(0)|0\rangle^{(0)} \ . \tag{2.16}$$

Note that the anomalous dimension $\gamma_{\mathcal{O}}$ is only defined for operators $\mathcal{O}$ that are eigenstates of the dilatation operator $\mathfrak{D}$. For general operators the relation (2.16) generalises to

$$\begin{aligned}(\gamma^{(1)})^B_A \ \langle 1,...,n|\mathcal{O}_B(0)|0\rangle^{(0)} = &-\frac{1}{\pi} \langle 1,...,n|\mathcal{M}\otimes\mathcal{O}_A(0)|0\rangle^{(0)} \\ &+ \gamma^{(1)}_{\rm IR} \langle 1,...,n|\mathcal{O}_A(0)|0\rangle^{(0)} \ .\end{aligned} \tag{2.17}$$

Here $(\gamma^{(1)})^B_A$ is the component of the mixing matrix that determines the mixing from the operator $\mathcal{O}_A$ to operator $\mathcal{O}_B$ and the anomalous dimensions can be obtained by diagonalising this matrix. In the following we leave the dependence of operators $\mathcal{O}$ in form factors on the position implicit.

Length-preserving convolution. We begin with the study of the mixing between operators $\mathcal{O}_A$ and $\mathcal{O}_B$ that are of the same length L. In order to extract $(\gamma^{(1)})^B_A$ from (2.17), one can choose external states $\langle 1,2,...,n|$ with $n = L$ such that all form factors in this relation are minimal. In this case the convolution term $\langle 1,...,n|\mathcal{M}\otimes\mathcal{O}_A(0)|0\rangle$ only contains length-preserving cuts at leading order in the coupling, i.e. the number

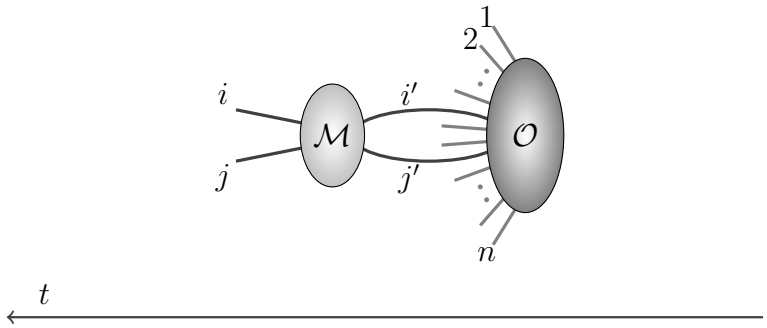


Figure 2.3: Length-preserving contribution to the one-loop unitarity cut of the form factor $\langle 1,2,...,n|\mathcal{O}|0\rangle^{(1)}$.

of particles in the intermediate state corresponds to the number of particles in the external state, cf. Figure 2.3.[1] It is defined as

$$\langle 1,...,n|\mathcal{M}\otimes\mathcal{O}|0\rangle^{(0)} = \frac{1}{16\pi}\sum_{1\leq i<j\leq n}\sum_{i',j'}\int\frac{\mathrm{d}\Omega}{4\pi}\,\langle i,j|\mathcal{M}|i',j'\rangle^{(0)}\cdot \\ \cdot\,\langle 1,...,i',...,j',...,n|\mathcal{O}|0\rangle^{(0)}\ . \quad (2.18)$$

Here we sum over all possible intermediate states i', j', e.g. in $\mathcal{N}=4$ sYM theory the sum runs over the set $\{g_\pm, \phi_{AB}, \psi_A, \bar{\psi}^A\}$. Furthermore, we integrate over all on-shell internal momenta p_i', p_j' with $p_i' + p_j' = p_i + p_j$ which can be parametrised by [32]

$$\begin{pmatrix}\lambda_i'\\ \lambda_j'\end{pmatrix} = \begin{pmatrix}\cos\vartheta & -\sin\vartheta e^{i\varphi}\\ \sin\vartheta e^{-i\varphi} & \cos\vartheta\end{pmatrix}\begin{pmatrix}\lambda_i\\ \lambda_j\end{pmatrix} \quad (2.19)$$

in terms of two angles ϑ and φ. This parametrisation implies the spinor brackets

$$\begin{aligned}&\langle i'j'\rangle = \langle ij\rangle\ , && \langle ij'\rangle = \langle i'j\rangle = \langle ij\rangle\cos\vartheta\ ,\\ &\langle i'i\rangle = \langle ij\rangle\sin\vartheta e^{i\varphi}\ , && \langle j'j\rangle = \langle ij\rangle\sin\vartheta e^{-i\varphi}\ ,\end{aligned} \quad (2.20)$$

which will be useful throughout this chapter. The summation over all on-shell momenta p_i', p_j' in this parametrisation corresponds to integrating over Ω with

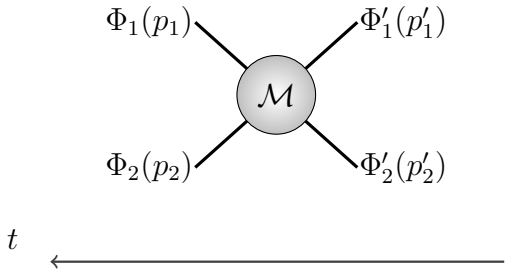
$$\int\frac{\mathrm{d}\Omega}{4\pi} = \int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}\int_0^{\pi/2}\mathrm{d}\vartheta\ 2\sin\vartheta\cos\vartheta\ . \quad (2.21)$$

Figure 2.3 contains an illustration of one of the terms in the sum on the right-hand side of (2.18).

[1] If the operators $\mathcal{O}_A$ and $\mathcal{O}_B$ have different lengths L_A and L_B, then choosing an external state $\langle 1,2,...,n|$ with $n = L_B$ results in a minimal form factor on the left-hand side of (2.17), but the convolution term on the right-hand side is a unitarity cut of a non-minimal form factor. In this case also length-changing cuts appear, where the number of intermediate particles does not correspond to the number of external particles. We discuss the case $L_B > L_A$ in Section 2.2.

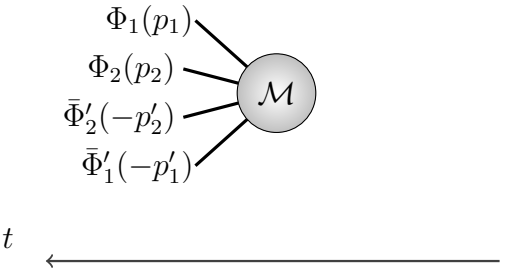
Crossing convention. In Chapter 1 we have given explicit expressions for amplitudes in pure and β-deformed $\mathcal{N}=4$ sYM theory and worked in the convention of all particles outgoing. To evaluate the amplitude in the convolution (2.18) using these expressions, we need to cross the incoming intermediate particles running in the cut over to the outgoing state and here we summarise our crossing conventions. By crossing symmetry of scattering amplitudes, we can relate amplitudes with crossed legs and in particular for $2 \to 2$ scattering the amplitudes

$$\mathcal{M}(\Phi_1(p_1), \Phi_2(p_2)|\Phi_1'(p_1'), \Phi_2'(p_2')) := \langle \Phi_1(p_1), \Phi_2(p_2)|\mathcal{M}|\Phi_1'(p_1'), \Phi_2'(p_2')\rangle$$



and

$$\mathcal{M}(\Phi_1(p_1), \Phi_2(p_2, \bar{\Phi}_2'(-p_2'), \bar{\Phi}_1'(-p_1')) := \langle \Phi_1(p_1), \Phi_2(p_1), \bar{\Phi}_2'(-p_2'), \bar{\Phi}_1'(-p_1')|\mathcal{M}|0\rangle$$



are related. Here Φ_i denotes an arbitrary elementary field, e.g. in $\mathcal{N}=4$ sYM theory $\Phi \in \{\gamma_\pm, \phi_{AB}, \psi_A, \bar{\psi}^A\}$. In crossing the two legs we keep their relative ordering and crossing a particle conjugates its helicity and inverses its momentum. In the case of fermions one has to be careful about the sign convention for the external spinors which introduces signs in the relation between crossed amplitudes. Take for example the process $X + \bar{\psi}(p) \to Y$, where X and Y denote the remaining incoming and outgoing external particles, with amplitude

$$\mathcal{M}(Y|X, \bar{\psi}(p)) = \mathcal{M}(Y|X, p) \cdot u_-(p) = \mathcal{M}(Y|X, p) \cdot \begin{pmatrix} 0 \\ \tilde{\lambda}(p) \end{pmatrix} \tag{2.22}$$

and the process $X \to Y + \psi(k)$ with amplitude

$$\mathcal{M}(Y, \psi(k)|X) = \mathcal{M}(Y, k|X) \cdot v_+(k) = \mathcal{M}(Y, k|X) \cdot \begin{pmatrix} 0 \\ \tilde{\lambda}(k) \end{pmatrix} . \tag{2.23}$$

For $k = -p$ the stripped amplitudes $\mathcal{M}(Y|X, p)$ and $\mathcal{M}(Y, k|X)$ are the same and due to $\tilde{\lambda}(k) = \tilde{\lambda}(-p) = \tilde{\lambda}(p)$, cf. (1.49), amplitudes with a crossed outgoing fermion ψ are related as

$$\mathcal{M}(Y|X, \bar{\psi}(p)) = \mathcal{M}(Y, \psi(-p)|X) . \tag{2.24}$$

In the case of an incoming fermion crossing to an outgoing anti-fermion one replaces $\bar{v}^+(p)$ by $\bar{u}^-(-p)$, or $\lambda(p)$ by $\lambda(-p)$ which via (1.49) introduces a sign in the relation of the amplitudes. In total crossing symmetry for component amplitudes is thus

$$\mathcal{M}(Y|X, \Phi(p)) = (-1)^{n_\psi} \mathcal{M}(Y, \bar{\Phi}(-p)|X) \tag{2.25}$$

with $n_\psi = 1$ for $\Phi = \psi$ and $n_\psi = 0$ otherwise. For superfields $\mathbf{\Phi}$, crossing is simply achieved by

$$(\lambda(p), \tilde{\lambda}(p), \tilde{\eta}(p)) \to (-\lambda(p), \tilde{\lambda}(p), \tilde{\eta}(p)) \tag{2.26}$$

and all the subtle minus-signs from fermion-crossings are simply dealt with by the Grassmann-odd parameters $\tilde{\eta}$.

Infrared contribution. The form of the infrared contribution in (2.17) can be obtained by studying this relation for a protected operator. The resulting expression is universal in the sense that it does not depend on the explicit operators $\mathcal{O}_A$ and $\mathcal{O}_B$. We begin by studying (2.17) for the energy-momentum tensor $T^{\alpha\beta,\dot{\alpha}\dot{\beta}}$ which has vanishing anomalous dimension in any theory, repeating the discussion in [31]. Choosing two gluons of opposite helicity as external particles yields

$$\gamma^{(1)}_{\text{IR}} = \frac{1}{\pi} \frac{\langle g^{a_1}_-(p_1), g^{a_2}_+(p_2)|\mathcal{M} \otimes T^{\alpha\beta,\dot{\alpha}\dot{\beta}}|0\rangle^{(0)}}{\langle g^{a_1}_-(p_1), g^{a_2}_+(p_2)|T^{\alpha\beta,\dot{\alpha}\dot{\beta}}|0\rangle^{(0)}} . \tag{2.27}$$

The minimal form factor of $T^{\alpha\beta,\dot{\alpha}\dot{\beta}}$ can be fixed from its mass dimension, little group scaling and colour structure as

$$\langle g^{a_1}_-(p_1), g^{a_2}_+(p_2)|T^{\alpha\beta,\dot{\alpha}\dot{\beta}}|0\rangle^{(0)} = 2\delta^{a_1 a_2} \lambda_1^\alpha \lambda_1^\beta \tilde{\lambda}_2^{\dot{\alpha}} \tilde{\lambda}_2^{\dot{\beta}} , \tag{2.28}$$

where we also use [132]

$$\langle p|T^{\alpha\beta,\dot\alpha\dot\beta}|p\rangle = 2p^{\alpha\dot\alpha}p^{\beta\dot\beta} \tag{2.29}$$

to determine the overall factor.

In pure Yang–Mills theory the convolution term in (2.27) can be computed via the Parke–Taylor amplitude, which in the parametrisation (2.19) for the two contributions to the convolution is given by

$$\begin{aligned}\langle g_-^{a_1}(p_1), g_+^{a_2}(p_2)|\mathcal{M}|g_+^{a_1'}(p_1'), g_-^{a_2'}(p_2')\rangle^{(0)}\,\delta^{a_1'a_2'} &= 2g_{\text{YM}}^2 N\delta^{a_1a_2}\frac{\sin^2\vartheta}{\cos^2\vartheta}e^{4i\varphi}\ ,\\ \langle g_-^{a_1}(p_1), g_+^{a_2}(p_2)|\mathcal{M}|g_-^{a_1'}(p_1'), g_+^{a_2'}(p_2')\rangle^{(0)}\,\delta^{a_1'a_2'} &= 2g_{\text{YM}}^2 N\delta^{a_1a_2}\frac{\cos^2\vartheta}{\sin^2\vartheta}\ .\end{aligned} \tag{2.30}$$

After using (2.28) and performing the φ-integration in the cut, one finds

$$\gamma_{\text{IR}}^{(1)} = \frac{g_{\text{YM}}^2 N}{4\pi^2}\int_0^{\pi/2}\frac{\mathrm{d}\vartheta}{\cos\vartheta\sin\vartheta}(\cos^8\vartheta+\sin^8\vartheta) \tag{2.31}$$

which is divergent at the boundaries of the integral. Instead of introducing a regularisation scheme, in the following we will first combine both the infrared and convolution contribution of relation (2.17) at the integrand level before performing the remaining ϑ-integration. This yields infrared-finite results for the mixing matrix $(\gamma^{(1)})_A^B$.

In a gauge theory with scalar and fermionic matter, the sum over intermediate states in the convolution term receives additional contributions from their coupling to the energy-momentum tensor. The relevant minimal form factors again follow from the mass dimension, little group scaling and colour structure, as well as the conservation of $T^{\mu\nu}$ via $(p_1+p_2)_{\alpha\dot\alpha}T^{\alpha\beta,\dot\alpha\dot\beta}=0$ and its tracelessness. One finds

$$\begin{aligned}\langle\phi_{AB}^{a_1}(p_1),\phi_{CD}^{a_2}(p_2)|T^{\alpha\beta,\dot\alpha\dot\beta}|0\rangle^{(0)} &= \tfrac{1}{3}\delta^{a_1a_2}\varepsilon_{ABCD}\Big(p_1^{\alpha\dot\alpha}p_1^{\beta\dot\beta}+p_2^{\alpha\dot\alpha}p_2^{\beta\dot\beta}-p_1^{\alpha\dot\alpha}p_2^{\beta\dot\beta}-p_2^{\alpha\dot\alpha}p_1^{\beta\dot\beta}+\\ &\qquad -p_1^{\alpha\dot\beta}p_2^{\beta\dot\alpha}-p_2^{\alpha\dot\beta}p_1^{\beta\dot\alpha}\Big)\,,\\ \langle\bar\psi_A^{a_1}(p_1),\psi^{A,a_2}(p_2)|T^{\alpha\beta,\dot\alpha\dot\beta}|0\rangle^{(0)} &= \tfrac{1}{2}\delta^{a_1a_2}\Big(\lambda_1^\alpha\lambda_1^\beta\tilde\lambda_1^{\dot\alpha}\tilde\lambda_2^{\dot\beta}+\lambda_1^\alpha\lambda_1^\beta\tilde\lambda_1^{\dot\beta}\tilde\lambda_2^{\dot\alpha}+\\ &\qquad -\lambda_1^\alpha\lambda_2^\beta\tilde\lambda_2^{\dot\alpha}\tilde\lambda_2^{\dot\beta}-\lambda_1^\beta\lambda_2^\alpha\tilde\lambda_2^{\dot\alpha}\tilde\lambda_2^{\dot\beta}\Big)\,.\end{aligned} \tag{2.32}$$

In $\mathcal{N}=4$ sYM theory the amplitudes occurring in the convolution can be extracted from the superamplitude (1.57) and are given by

$$\begin{aligned}\langle g_-^{a_1}(p_1), g_+^{a_2}(p_2)|\mathcal{M}|\phi_{AB}^{a_1'}(p_1'),\phi_{CD}^{a_2'}(p_2')\rangle^{(0)}\,\delta^{a_1'a_2'} &= 2g_{\text{YM}}^2 N\delta^{a_1a_2}\varepsilon_{ABCD}e^{2i\varphi}\ ,\\ \langle g_-^{a_1}(p_1), g_+^{a_2}(p_2)|\mathcal{M}|\bar\psi^{A,a_1'}(p_1'),\psi_A^{a_2'}(p_2')\rangle^{(0)}\,\delta^{a_1'a_2'} &= 2g_{\text{YM}}^2 N\delta^{a_1a_2}\frac{\cos\vartheta}{\sin\vartheta}e^{i\varphi}\ ,\\ \langle g_-^{a_1}(p_1), g_+^{a_2}(p_2)|\mathcal{M}|\psi_A^{a_1'}(p_1'),\bar\psi^{A,a_2'}(p_2')\rangle^{(0)}\,\delta^{a_1'a_2'} &= 2g_{\text{YM}}^2 N\delta^{a_1a_2}\frac{\sin\vartheta}{\cos\vartheta}e^{3i\varphi}\ .\end{aligned} \tag{2.33}$$

Here we used the crossing conventions (2.25). For three scalars (and three anti-scalars) and four fermions (and four anti-fermions) in $\mathcal{N}=4$ sYM theory these contributions add up to

$$\gamma_{\text{IR}}^{(1)} = \frac{g_{\text{YM}}^2 N}{4\pi^2} \int_0^{\pi/2} \frac{\mathrm{d}\vartheta}{\cos\vartheta \sin\vartheta} \,, \tag{2.34}$$

where it is important to include a minus sign for fermion propagators in the sum of different contributions to the convolution. This generalises to arbitrary external states as

$$\gamma_{\text{IR}}^{(1)} \langle\{p_i, a_i\}|\mathcal{O}|0\rangle^{(0)} = -\frac{g_{\text{YM}}^2}{8\pi^2} \mathcal{I}_{\text{IR}} \sum_{j<k} t_j^a t_k^a \langle\{p_i, a_i\}|\mathcal{O}|0\rangle^{(0)} \tag{2.35}$$

with divergent integral

$$\mathcal{I}_{\text{IR}} = \int_0^{\pi/2} \frac{d\vartheta}{\cos\vartheta \sin\vartheta} \,. \tag{2.36}$$

This infrared structure corresponds to the one-loop infrared divergence of amplitudes, which in dimensional regularisation is given in (1.96), and we use the notation of Section 1.3, in particular $(t^a)_{bc} = -\sqrt{2} i f^a{}_{bc}$ for fields in the adjoint. The general form of (2.35) for $n > 2$ in $\mathcal{N}=4$ sYM theory is less conveniently obtained from (2.17) and $\mathcal{O} = T^{\alpha\beta,\dot{\alpha}\dot{\beta}}$ since for a higher number n of external states the form factors are non-minimal. Instead one can use another protected operator in this theory which is the half-BPS highest-weight state

$$\text{Tr}(\phi_{34}^n) \,, \tag{2.37}$$

and an external state $\langle \phi_{12}^{a_1}, ..., \phi_{12}^{a_n}|$ of length n. For this operator there is only one intermediate state giving a non-vanishing contribution to the convolution of (2.17) and the relevant amplitude is $\mathcal{M}(\phi_{12}^{a_i}, \phi_{12}^{a_j}, \bar{\phi}_{12}^{a'_j}, \bar{\phi}_{12}^{a'_i})$. This amplitude is divergent in the limit of $\vartheta \to 0$ and $\vartheta \to \pi/2$, specifically

$$\begin{aligned}
\mathcal{M}(\phi_{12}^{a_i}, \phi_{12}^{a_j}, \bar{\phi}_{12}^{a'_j}, \bar{\phi}_{12}^{a'_i})^{(0)}\Big|_{\vartheta\to 0} &= g_{\text{YM}}^2 \frac{f^{a_i a'_i a} f^{a_j a'_j a}}{2\vartheta^2} + \mathcal{O}(\vartheta^0) \,, \\
\mathcal{M}(\phi_{12}^{a_i}, \phi_{12}^{a_j}, \bar{\phi}_{12}^{a'_j}, \bar{\phi}_{12}^{a'_i})^{(0)}\Big|_{\vartheta\to \pi/2} &= g_{\text{YM}}^2 \frac{f^{a_i a'_j a} f^{a_j a'_i a}}{2(\pi/2-\vartheta)^2} + \mathcal{O}((\vartheta-\pi/2)^0) \,.
\end{aligned} \tag{2.38}$$

The colour structure of these divergences corresponds to those induced by (2.35) and the proportionality factor is fixed by demanding that $\text{Tr}(\phi_{34}^n)$ is protected. We will use the structure of the infrared contributions (2.31) and (2.35) to cancel IR divergences in the convolution term in (2.17) already at the integrand level.

2.2 $\mathcal{N} = 4$ sYM theory from on-shell methods

We now move on to a discussion of the mixing of dimension-4 operators in $\mathcal{N} = 4$ sYM theory using the on-shell approach of (2.17). We start by reviewing the computation of the β-function in pure Yang–Mills theory to set up the notation. In the following chapters we will be interested in the computation of non-planar anomalous dimensions, and as a warm-up exercise we begin with a study of the mixing of $SU(4)$-invariant marginal operators and extract their one-loop anomalous dimensions directly from (2.17). In this sector we will find the known $\mathcal{N} = 4$ sYM on-shell Lagrangian (1.17), as well as descendants of the Konishi operator.

β-function in pure Yang–Mills. As a first exercise we use (2.17) to extract the β-function of pure YM theory following [31]. It starts by realising that the β-function is closely related to the anomalous dimension of the Lagrangian $\gamma_{\mathcal{L}}$ [31, 133, 134]

$$\gamma_{\mathcal{L}} = g_{\text{YM}}^2 \frac{\partial}{\partial g_{\text{YM}}^2} \left(\frac{\beta(g_{\text{YM}}^2)}{g_{\text{YM}}^2} \right) . \tag{2.39}$$

In pure Yang–Mills theory the Lagrangian is simply given by

$$\mathcal{L}_{\text{YM}} = \text{Tr} \left(-\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \right) , \tag{2.40}$$

where we used (1.18) to express it entirely in terms of the chiral part $F^{\alpha\beta}$ of the field strength. In a theory with only gluons the operator $\mathcal{L}_{\text{YM}}$ is the only on-shell Lorentz- and gauge-invariant operator with classical dimension 4 and thus the one-loop mixing in (2.17) is trivial and for minimal form factors reduces to

$$\gamma_{\mathcal{L}}^{(1)} \langle 1^{a_1}, 2^{a_2} | \mathcal{L}_{\text{YM}} | 0 \rangle^{(0)} = -\frac{1}{\pi} \langle 1^{a_1}, 2^{a_2} | \mathcal{M} \otimes \mathcal{L}_{\text{YM}} | 0 \rangle^{(0)} + \gamma_{\text{IR}}^{(1)} \langle 1^{a_1}, 2^{a_2} | \mathcal{L}_{\text{YM}} | 0 \rangle^{(0)} . \tag{2.41}$$

The only non-vanishing contribution to this relation at leading order in the coupling can be obtained from a purely gluonic external state, specifically $\langle g_-, g_- |$ and with (1.60) the tree-level minimal form factor of $\mathcal{L}_{\text{YM}}$ is

$$\langle g_-^{a_1}(p_1), g_-^{a_2}(p_2) | \mathcal{L}_{\text{YM}} | 0 \rangle^{(0)} = -\frac{1}{2} \delta^{a_1 a_2} \langle 12 \rangle^2 . \tag{2.42}$$

The convolution term in (2.41) is only non-zero if the intermediate state consists of two negative-helicity gluons, cf. (2.18). The relevant MHV amplitude in the parametrisation (2.19) including the colour-delta function from the form factor evaluates to

$$\mathcal{M}^{(0)}(g_-^{a_1}(p_1), g_-^{a_2}(p_2), g_+^{a_{2'}}(-p_2'), g_+^{a_{1'}}(-p_1'))^{(0)} \delta^{a_{1'} a_{2'}} = 2 g_{\text{YM}}^2 N \frac{\delta^{a_1 a_2}}{\cos^2\vartheta \sin^2\vartheta} . \tag{2.43}$$

Thus the anomalous dimension of the Yang–Mills Lagrangian is

$$\gamma_{\mathcal{L}}^{(1)} = -\frac{g_{\text{YM}}^2 N}{4\pi^2}\mathcal{I}_{\text{IR}} + \gamma_{\text{IR}}^{(1)} = -\frac{g_{\text{YM}}^2 N}{8\pi^2}\frac{11}{3}\,, \tag{2.44}$$

where in the second step we use (2.31) and combine the two terms at the level of the integrand before performing the ϑ-integration. With (2.39) this implies

$$\beta(g_{\text{YM}}^2) = -\frac{g_{\text{YM}}^4}{8\pi^2}\cdot\frac{11N}{3} + \mathcal{O}(g_{\text{YM}}^6) \tag{2.45}$$

in pure Yang–Mills theory as expected.

One-loop mixing of $SU(4)$-invariant marginal operators. Now we move on to the study of $\mathcal{N}=4$ sYM theory and the mixing problem of all chiral on-shell $SU(4)$-invariant dimension-4 operators at one-loop order. They are given by the basis

$$\begin{aligned}
&\mathrm{G} := \mathrm{Tr}(F_{\alpha\beta}F^{\alpha\beta})\,,\\
&\mathrm{F} := \mathrm{Tr}(\psi_A^\alpha[\phi^{AB},\psi_{B\alpha}])\,,\\
&\mathrm{S}_1 := \mathrm{Tr}([\phi^{AB},\phi^{CD}][\phi_{AB},\phi_{CD}])\,, \qquad \mathrm{S}_2 := \mathrm{Tr}(\phi^{AB}\phi^{CD}\phi_{AB}\phi_{CD})\,,\\
&\mathrm{S}_3 := \mathrm{Tr}(\phi^{AB}\phi^{CD})\,\mathrm{Tr}(\phi_{AB}\phi_{CD})\,, \qquad \mathrm{S}_4 := \mathrm{Tr}(\phi^{AB}\phi_{AB})\,\mathrm{Tr}(\phi^{CD}\phi_{CD})\,.
\end{aligned} \tag{2.46}$$

The corresponding tree-level minimal form factors are, using (1.60),

$$\begin{aligned}
&\langle g_-^{a_1}(p_1), g_-^{a_2}(p_2)|\mathrm{G}|0\rangle^{(0)} = 2\delta^{a_1a_2}\langle 12\rangle^2\,,\\
&\langle \bar\psi^{A,a_1}(p_1), \bar\psi^{B,a_2}(p_2), \phi^{CD,a_3}|\mathrm{F}|0\rangle^{(0)} = -2\sqrt{2}i f^{a_1a_2a_3}\varepsilon^{ABCD}\langle 12\rangle\,,\\
&\langle \phi_{AB}^{a_1}, \phi_{CD}^{a_2}, \phi_{EF}^{a_3}, \phi_{GH}^{a_4}|\mathrm{S}_1|0\rangle^{(0)} = 2^5\,[\varepsilon_{ABCD}\varepsilon_{EFGH}(-f^{a_1a_3e}f^{a_2a_4e} - f^{a_1a_4e}f^{a_2a_3e})\\
&\qquad\qquad + \varepsilon_{ABEF}\varepsilon_{CDGH}(-f^{a_1a_2e}f^{a_3a_4e} + f^{a_1a_4e}f^{a_2a_3e})\\
&\qquad\qquad + \varepsilon_{ABGH}\varepsilon_{CDEF}(+f^{a_1a_2e}f^{a_3a_4e} + f^{a_1a_3e}f^{a_2a_4e})]\,,\\
&\langle \phi_{AB}^{a_1}, \phi_{CD}^{a_2}, \phi_{EF}^{a_3}, \phi_{GH}^{a_4}|\mathrm{S}_2|0\rangle^{(0)} = 2^4\,[\varepsilon_{ABCD}\varepsilon_{EFGH}\,\mathrm{Tr}(a_1a_3a_2a_4 + a_1a_4a_2a_3)\\
&\qquad\qquad + \varepsilon_{ABEF}\varepsilon_{CDGH}\,\mathrm{Tr}(a_1a_2a_3a_4 + a_1a_4a_3a_2)\\
&\qquad\qquad + \varepsilon_{ABGH}\varepsilon_{CDEF}\,\mathrm{Tr}(a_1a_2a_4a_3 + a_1a_3a_4a_2)]\,,\\
&\langle \phi_{AB}^{a_1}, \phi_{CD}^{a_2}, \phi_{EF}^{a_3}, \phi_{GH}^{a_4}|\mathrm{S}_3|0\rangle^{(0)} = 2^4\,[\varepsilon_{ABCD}\varepsilon_{EFGH}(\delta^{a_1a_3}\delta^{a_2a_4} + \delta^{a_1a_4}\delta^{a_2a_3})\\
&\qquad\qquad + \varepsilon_{ABEF}\varepsilon_{CDGH}(\delta^{a_1a_2}\delta^{a_3a_4} + \delta^{a_1a_4}\delta^{a_2a_3})\\
&\qquad\qquad + \varepsilon_{ABGH}\varepsilon_{CDEF}(\delta^{a_1a_2}\delta^{a_3a_4} + \delta^{a_1a_3}\delta^{a_2a_4})]\,,\\
&\langle \phi_{AB}^{a_1}, \phi_{CD}^{a_2}, \phi_{EF}^{a_3}, \phi_{GH}^{a_4}|\mathrm{S}_4|0\rangle^{(0)} = 2^5\,[\varepsilon_{ABCD}\varepsilon_{EFGH}\;\delta^{a_1a_2}\delta^{a_3a_4} + \varepsilon_{ABEF}\varepsilon_{CDGH}\;\delta^{a_1a_3}\delta^{a_2a_4}\\
&\qquad\qquad + \varepsilon_{ABGH}\varepsilon_{CDEF}\;\delta^{a_1a_4}\delta^{a_2a_3}]\,.
\end{aligned} \tag{2.47}$$

At one-loop order only operators of the same length mix with each other and thus the operators G, F and $\vec{\mathrm{S}} := \{\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3, \mathrm{S}_4\}$ decouple. We compute the remaining mixing

via (2.17) and the superamplitude (1.57) in the superfield formalism for general MHV external states and find

$$
\begin{aligned}
(\Delta\gamma^{(1)})^{\mathrm{G}}_{\mathrm{G}} &= \frac{\lambda}{8\pi^2}\cdot(-2\,\mathcal{I}_{\mathrm{IR}})\ , \\
(\Delta\gamma^{(1)})^{\mathrm{F}}_{\mathrm{F}} &= \frac{\lambda}{8\pi^2}\cdot(-3\,\mathcal{I}_{\mathrm{IR}}+6)\ , \\
\big(\Delta\gamma^{(1)}\big)^{\vec{\mathrm{S}}}_{\vec{\mathrm{S}}} &= \frac{\lambda}{8\pi^2}\cdot\begin{pmatrix} -4\,\mathcal{I}_{\mathrm{IR}}+11 & -10 & -20/N & -10/N \\ 1 & -4\,\mathcal{I}_{\mathrm{IR}}+2 & -12/N & 2/N \\ -5/N & 0 & -4\,\mathcal{I}_{\mathrm{IR}} & 2 \\ -6/N & 0 & 0 & -4\,\mathcal{I}_{\mathrm{IR}}+12 \end{pmatrix}\ ,
\end{aligned}
\tag{2.48}
$$

where $\Delta\gamma=\gamma-\gamma_{\mathrm{IR}}$. Using (2.35) we can compute the infrared contribution γ_{IR} and it removes the infrared divergent integrals $\mathcal{I}_{\mathrm{IR}}$ in (2.48). Note that in the planar limit, the single- and double-trace scalar operators decouple. We can diagonalise this operator mixing and find the following one-loop anomalous dimensions and eigenoperators O in this sector:

$$
\begin{aligned}
\gamma_1^{(1)} &= 0 && \mathrm{O}_1 = \mathrm{G}\ , \\
\gamma_2^{(1)} &= \frac{\lambda}{8\pi^2}\cdot 6 && \mathrm{O}_2 = \mathrm{F}\ , \\
\gamma_3^{(1)} &= \frac{\lambda}{8\pi^2}\cdot\frac{1}{2}\Big(13+\sqrt{41}+\mathcal{O}(N^{-2})\Big) && \mathrm{O}_3 = \mathrm{S}_1-\frac{1}{2}\Big(9-\sqrt{41}\Big)\,\mathrm{S}_2+\mathcal{O}(N^{-1})\ , \\
\gamma_4^{(1)} &= \frac{\lambda}{8\pi^2}\cdot\frac{1}{2}\Big(13-\sqrt{41}+\mathcal{O}(N^{-2})\Big) && \mathrm{O}_4 = \mathrm{S}_1-\frac{1}{2}\Big(9+\sqrt{41}\Big)\,\mathrm{S}_2+\mathcal{O}(N^{-1})\ , \\
\gamma_5^{(1)} &= \frac{\lambda}{8\pi^2}\cdot\big(12+\mathcal{O}(N^{-2})\big) && \mathrm{O}_5 = \mathrm{S}_4+\mathcal{O}(N^{-1})\ , \\
\gamma_6^{(1)} &= \frac{\lambda}{8\pi^2}\cdot\big(0+\mathcal{O}(N^{-2})\big) && \mathrm{O}_6 = -6\mathrm{S}_3+\mathrm{S}_4+\mathcal{O}(N^{-1})\ .
\end{aligned}
\tag{2.49}
$$

For the first two eigenstates we give the exact one-loop anomalous dimensions and operator dependence, while for the remaining states we only give the leading terms in a $1/N$ expansion corresponding to the planar eigenstates and -values. Higher orders can be easily obtained from the diagonalisation of (2.48).

Note that the operator O_1 with vanishing one-loop anomalous dimension is the leading-coupling contribution to the $\mathcal{N}=4$ sYM Lagrangian and we will find the higher-order contributions in the next paragraph. Similarly, operator O_2 is the leading-order part of an operator from the Konishi multiplet, and in particular a descendant of the $\mathfrak{su}(4)$ Konishi operator $\mathcal{K}=\mathrm{Tr}(\phi_I\bar{\phi}_I)$. This multiplet contains the simplest operators whose scaling dimensions are not protected from quantum corrections and it has thus been of great interest in the study of the spectrum of anomalous dimensions

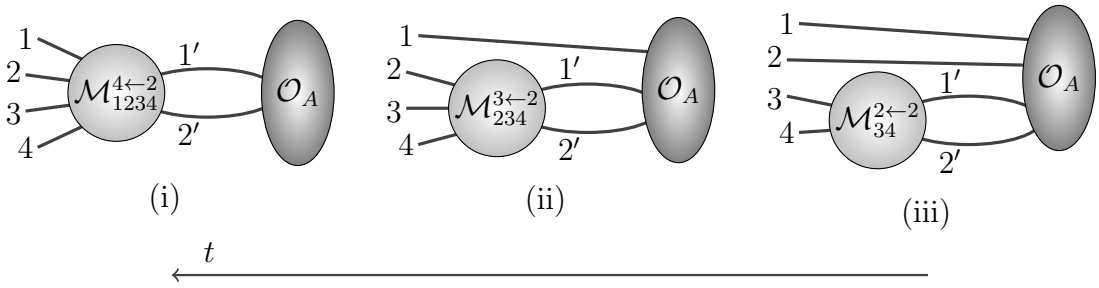


Figure 2.4: Length-increasing contributions to the mixing of a length-two operator $\mathcal{O}_A$ to a length-four operator.

and correlation functions in supersymmetric gauge theories. Moreover, the operator O_5 in the planar limit just corresponds to $\mathcal{K}^2$ and its planar anomalous dimension is twice that of the Konishi multiplet.

Upper off-diagonal mixing. The $\mathcal{N} = 4$ sYM theory dilatation operator has length-changing effects starting at $\mathcal{O}(g^3_{\rm YM})$. They were studied in the planar limit where they give rise to fluctuations in the lengths of the corresponding "dynamic" spin chains [135, 136]. We will study such effects here as they are relevant to the mixing in the sector spanned by the operators (2.46) and obtain them from (2.17) by focussing on this relation for operators $\mathcal{O}_A$ and $\mathcal{O}_B$ of different lengths. In this work, we only discuss the length-increasing cases, i.e. the mixing of G to F and $\vec{\rm S}$, as well as of F to $\vec{\rm S}$. This will allow us to extract the on-shell $\mathcal{N} = 4$ sYM Lagrangian as an eigenoperator of the mixing problem with vanishing anomalous dimension.

In the case of a length-two operator $\mathcal{O}_A$ mixing into a length-four operator $\mathcal{O}_B$, the convolution $\langle 1,2,3,4|\mathcal{M}\otimes\mathcal{O}_A|0\rangle^{(0)}$ obtains contributions from non-minimal form factors and length-changing amplitudes which are illustrated in Figure 2.4. While the intermediate momenta p'_1 and p'_2 in the cut of Figure 2.4 (iii) can be parametrised as in the length-preserving case, cf. (2.19), we need to find new parametrisations for the two other configurations. We begin with case (i) where the intermediate momenta satisfy $p'_1 + p'_2 = p_1 + p_2 + p_3 + p_4$ and parametrise them in terms of two on-shell momenta defined as

$$p_a = \frac{s_{1234}}{s_{12}+s_{13}+s_{14}}p_1 \ , \qquad p_b = p_2 + p_3 + p_4 - \frac{s_{234}}{s_{12}+s_{13}+s_{14}}p_1 \tag{2.50}$$

with spinor-helicity variables

$$\lambda_a = \lambda_1\sqrt{\frac{s_{1234}}{s_{12}+s_{13}+s_{14}}} \ , \quad \lambda_b = ([12]\lambda_2 + [13]\lambda_3 + [14]\lambda_4)\frac{1}{\sqrt{s_{12}+s_{13}+s_{14}}} \ . \tag{2.51}$$

The dependence of the intermediate momenta p'_1 and p'_2 on the new momenta p_a and p_b is then given by (2.19) in terms of the two angles ϑ and φ. For $p_4 \to 0$ these

formulas give a parametrisation for the case (ii) of Figure 2.4 and reproduce the $2 \to 3$ parametrisation discussed in [31].

We begin the study of length-increasing cuts in the sector spanned by (2.46) with a discussion of the cut from the operator G to F which we obtain from

$$(\Delta\gamma)^{\mathcal{O}}_{\mathrm{G}} \langle \Phi_1^{a_1}(p_1), \Phi_2^{a_2}(p_2), \Phi_3^{a_3}(p_3)|\mathcal{O}|0\rangle = -\frac{1}{\pi} \langle \Phi_1^{a_1}(p_1), \Phi_2^{a_2}(p_2), \Phi_3^{a_3}(p_3)|\mathcal{M} \otimes \mathrm{G}|0\rangle \ . \tag{2.52}$$

The external state again consists of arbitrary component fields $\Phi_i \in \{g_\pm, \phi_{AB}, \psi_A, \bar{\psi}^A\}$. At leading order in the coupling, the left-hand side of this relation has contributions from a minimal form factor of F and a non-minimal form factor of G

$$\begin{aligned}(\Delta\gamma)^{\mathcal{O}}_{\mathrm{G}} \langle \Phi_1^{a_1}(p_1), \Phi_2^{a_2}(p_2), \Phi_3^{a_3}(p_3)|\mathcal{O}|0\rangle\big|_{\mathcal{O}(g_{\mathrm{YM}}^3)} \\ = (\Delta\gamma^{(1)})^{\mathrm{G}}_{\mathrm{G}} \langle \Phi_1^{a_1}(p_1), \Phi_2^{a_2}(p_2), \Phi_3^{a_3}(p_3)|\mathrm{G}|0\rangle^{(0.5)} \\ + (\gamma^{(1.5)})^{\mathrm{F}}_{\mathrm{G}} \langle \Phi_1^{a_1}(p_1), \Phi_2^{a_2}(p_2), \Phi_3^{a_3}(p_3)|\mathrm{F}|0\rangle^{(0)} \ ,\end{aligned} \tag{2.53}$$

where $\langle \Phi_1^{a_1}(p_1), \Phi_2^{a_2}(p_2), \Phi_3^{a_3}(p_3)|\mathrm{G}|0\rangle^{(0.5)}$ denotes the tree-level non-minimal form factor of G which comes with one power of the coupling g_{YM}. At leading order on the right-hand side of (2.52), we have contributions from cuts with minimal and non-minimal form factors of G:

$$\begin{aligned}\langle \Phi_1^{a_1}(p_1), \Phi_2^{a_2}(p_2), \Phi_3^{a_3}(p_3)|\mathcal{M} \otimes \mathrm{G}|0\rangle\big|_{\mathcal{O}(g_{\mathrm{YM}}^3)} \\ = \langle \Phi_1^{a_1}(p_1), \Phi_2^{a_2}(p_2), \Phi_3^{a_3}(p_3)|(\mathcal{M}_{12}^{2\leftarrow 2} \otimes \mathrm{G} + \mathcal{M}_{13}^{2\leftarrow 2} \otimes \mathrm{G} + \mathcal{M}_{23}^{2\leftarrow 2} \otimes \mathrm{G} + \\ + \mathcal{M}_{123}^{3\leftarrow 2} \otimes \mathrm{G})|0\rangle^{(1.5)} \ .\end{aligned} \tag{2.54}$$

The first three contributions stem from the convolution of the non-minimal form factor $\langle \Phi_1, \Phi_2, \Phi_3|\mathrm{G}|0\rangle$ with a $2 \to 2$ scattering matrix. The subscripts on $\mathcal{M}^{2\leftarrow 2}$ indicate those external particles of the convolution that attach to the scattering amplitude. The last contribution corresponds to the convolution of the minimal form factor of G and the five-point MHV superamplitude (1.57). The three-point non-minimal form factors of G can be found in the Appendix A. By computing the mixing for arbitrary MHV external states we find

$$(\Delta\gamma^{(1)})^{\mathrm{G}}_{\mathrm{G}} = \frac{\lambda}{8\pi^2} \cdot (-3\mathcal{I}_{\mathrm{IR}}) \ , \qquad (\gamma^{(1.5)})^{\mathrm{F}}_{\mathrm{G}} = \frac{\lambda}{8\pi^2} \cdot 6g_{\mathrm{YM}} \ . \tag{2.55}$$

After subtracting the infrared divergence, the first term of (2.55) reproduces the one-loop anomalous dimension found in the length-preserving cuts, cf. (2.48). The anomalous dimension $(\gamma^{(1.5)})^{\mathrm{F}}_{\mathrm{G}}$ governs the mixing from the operator G to F at $\mathcal{O}(g_{\mathrm{YM}}^3)$.

For the mixing of F to operators $\vec{\mathrm{S}}$ we compute

$$(\Delta\gamma)^{\mathcal{O}}_{\mathrm{F}}\,\langle\Phi_1^{a_1}(p_1),\Phi_2^{a_2}(p_2),\Phi_3^{a_3}(p_3),\Phi_4^{a_4}(p_4)|\mathcal{O}|0\rangle = -\frac{1}{\pi}\,\langle\Phi_1^{a_1}(p_1),\Phi_2^{a_2}(p_2),\Phi_3^{a_3}(p_3),\Phi_4^{a_4}(p_4)|\mathcal{M}\otimes\mathrm{F}|0\rangle\ . \tag{2.56}$$

At leading order on the left-hand side, we have contributions from minimal form factors of $\vec{\mathrm{S}}$ and a non-minimal form factor of F

$$\begin{aligned}(\Delta\gamma)^{\mathcal{O}}_{\mathrm{F}}\,\langle\Phi_1^{a_1}(p_1),\Phi_2^{a_2}(p_2),\Phi_3^{a_3}(p_3),\Phi_4^{a_4}(p_4)|\mathcal{O}|0\rangle\big|_{\mathcal{O}(g_{\mathrm{YM}}^3)}&\\ = (\Delta\gamma^{(1)})^{\mathrm{F}}_{\mathrm{F}}\,\langle\Phi_1^{a_1}(p_1),\Phi_2^{a_2}(p_2),\Phi_3^{a_3}(p_3),\Phi_4^{a_4}(p_4)|\mathrm{F}|0\rangle^{(0.5)} +&\\ + (\gamma^{(1.5)})^{\vec{\mathrm{S}}}_{\mathrm{F}}\,\langle\Phi_1^{a_1}(p_1),\Phi_2^{a_2}(p_2),\Phi_3^{a_3}(p_3),\Phi_4^{a_4}(p_4)|\vec{\mathrm{S}}|0\rangle^{(0)}\ .&\end{aligned} \tag{2.57}$$

At leading order on the right-hand side, we have contributions from cuts including minimal and non-minimal form factors of F:

$$\begin{aligned}&\langle\Phi_1^{a_1}(p_1),\Phi_2^{a_2}(p_2),\Phi_3^{a_3}(p_3),\Phi_4^{a_4}(p_4)|\mathcal{M}\otimes\mathrm{F}|0\rangle|_{\mathcal{O}(g_{\mathrm{YM}}^3)}\\ &\quad=\langle\Phi_1^{a_1}(p_1),\Phi_2^{a_2}(p_2),\Phi_3^{a_3}(p_3),\Phi_4^{a_4}(p_4)|(\mathcal{M}_{12}^{2\leftarrow 2}\otimes\mathrm{F}+\mathcal{M}_{13}^{2\leftarrow 2}\otimes\mathrm{F}+\mathcal{M}_{23}^{2\leftarrow 2}\otimes\mathrm{F}\\ &\qquad+\mathcal{M}_{14}^{2\leftarrow 2}\otimes\mathrm{F}+\mathcal{M}_{24}^{2\leftarrow 2}\otimes\mathrm{F}+\mathcal{M}_{34}^{2\leftarrow 2}\otimes\mathrm{F}\\ &\qquad+\mathcal{M}_{123}^{3\leftarrow 2}\otimes\mathrm{F}+\mathcal{M}_{124}^{3\leftarrow 2}\otimes\mathrm{F}+\mathcal{M}_{134}^{3\leftarrow 2}\otimes\mathrm{F}+\mathcal{M}_{234}^{3\leftarrow 2}\otimes\mathrm{F})|0\rangle^{(1.5)}\ .\end{aligned} \tag{2.58}$$

Studying these expressions for general MHV external states and with the non-minimal form factors given in Appendix A, we find

$$\begin{aligned}&(\Delta\gamma^{(1)})^{\mathrm{F}}_{\mathrm{F}}=\frac{\lambda}{8\pi^2}(-4\mathcal{I}_{\mathrm{IR}}+6)\ ,\quad (\gamma^{(1.5)})^{\mathrm{S}_1}_{\mathrm{F}}=\frac{\lambda}{8\pi^2}\cdot\Big(-\frac{g_{\mathrm{YM}}}{2}\Big)\ ,\quad (\gamma^{(1.5)})^{\mathrm{S}_2}_{\mathrm{F}}=\frac{\lambda}{8\pi^2}\cdot g_{\mathrm{YM}}\ ,\\ &(\gamma^{(1.5)})^{\mathrm{S}_3}_{\mathrm{F}}=\frac{\lambda}{8\pi^2}\cdot\frac{2g_{\mathrm{YM}}}{N}\ ,\qquad (\gamma^{(1.5)})^{\mathrm{S}_4}_{\mathrm{F}}=\frac{\lambda}{8\pi^2}\cdot\frac{g_{\mathrm{YM}}}{N}\ .\end{aligned} \tag{2.59}$$

After subtracting the infrared divergence, the first expression in (2.59) reproduces result from the length-preserving cuts in (2.48), and the remaining terms govern the mixing from F to $\vec{\mathrm{S}}$.

Finally, for the mixing of G to length-four operators we look at

$$(\Delta\gamma)^{\mathcal{O}}_{\mathrm{G}}\,\langle\Phi_1^{a_1}(p_1),\Phi_2^{a_2}(p_2),\Phi_3^{a_3}(p_3),\Phi_4^{a_4}(p_4)|\mathcal{O}|0\rangle = -\frac{1}{\pi}\,\langle\Phi_1^{a_1}(p_1),\Phi_2^{a_2}(p_2),\Phi_3^{a_3}(p_3),\Phi_4^{a_4}(p_4)|\mathcal{M}\otimes\mathrm{G}|0\rangle\ . \tag{2.60}$$

The leading order contribution of this expression is $\mathcal{O}(g_{\mathrm{YM}}^4)$ and at this order the left-hand side contains minimal form factors of $\vec{\mathrm{S}}$ and non-minimal form factors of G and

F, specifically

$$
\begin{aligned}
(\Delta\gamma)^{\mathcal{O}}_{\mathrm{G}}\,\langle\Phi_1^{a_1}(p_1),\Phi_2^{a_2}(p_2),\Phi_3^{a_3}(p_3),\Phi_4^{a_4}(p_4)|\mathcal{O}|0\rangle\big|_{\mathcal{O}(g_{\mathrm{YM}}^4)}& \\
&= (\gamma^{(1)})^{\mathrm{G}}_{\mathrm{G}}\,\langle\Phi_1^{a_1}(p_1),\Phi_2^{a_2}(p_2),\Phi_3^{a_3}(p_3),\Phi_4^{a_4}(p_4)|\mathrm{G}|0\rangle^{(1)} \\
&\quad + (\Delta\gamma^{(1.5)})^{\mathrm{F}}_{\mathrm{G}}\,\langle\Phi_1^{a_1}(p_1),\Phi_2^{a_2}(p_2),\Phi_3^{a_3}(p_3),\Phi_4^{a_4}(p_4)|\mathrm{F}|0\rangle^{(0.5)} \\
&\quad + (\gamma^{(2)})^{\vec{\mathrm{S}}}_{\mathrm{G}}\,\langle\Phi_1^{a_1}(p_1),\Phi_2^{a_2}(p_2),\Phi_3^{a_3}(p_3),\Phi_4^{a_4}(p_4)|\vec{\mathrm{S}}|0\rangle^{(0)}\ . \qquad (2.61)
\end{aligned}
$$

At leading order on the right-hand side, we have contributions from cuts with minimal and non-minimal form factors of G:

$$
\begin{aligned}
&\langle\Phi_1^{a_1}(p_1),\Phi_2^{a_2}(p_2),\Phi_3^{a_3}(p_3),\Phi_4^{a_4}(p_4)|\mathcal{M}\otimes\mathrm{G}|0\rangle|_{\mathcal{O}(g_{\mathrm{YM}}^4)} = \\
&\quad = \langle\Phi_1^{a_1}(p_1),\Phi_2^{a_2}(p_2),\Phi_3^{a_3}(p_3),\Phi_4^{a_4}(p_4)|(\mathcal{M}_{12}^{2\leftarrow 2}\otimes\mathrm{G}+\mathcal{M}_{13}^{2\leftarrow 2}\otimes\mathrm{G}+\mathcal{M}_{23}^{2\leftarrow 2}\otimes\mathrm{G}+ \\
&\qquad + \mathcal{M}_{14}^{2\leftarrow 2}\otimes\mathrm{G}+\mathcal{M}_{24}^{2\leftarrow 2}\otimes\mathrm{G}+\mathcal{M}_{34}^{2\leftarrow 2}\otimes\mathrm{G}+ \\
&\qquad + \mathcal{M}_{123}^{3\leftarrow 2}\otimes\mathrm{G}+\mathcal{M}_{124}^{3\leftarrow 2}\otimes\mathrm{G}+\mathcal{M}_{134}^{3\leftarrow 2}\otimes\mathrm{G}+\mathcal{M}_{234}^{3\leftarrow 2}\otimes\mathrm{G}+ \\
&\qquad + \mathcal{M}_{1234}^{4\leftarrow 2}\otimes\mathrm{G})|0\rangle\ , \qquad (2.62)
\end{aligned}
$$

which contains an MHV six-point amplitude. The non-minimal form factors can again be found in Appendix A and the mixing can be computed and we find

$$
\begin{aligned}
&(\Delta\gamma^{(1)})^{\mathrm{G}}_{\mathrm{G}} = \frac{\lambda}{8\pi^2}\cdot(-4\mathcal{I}_{\mathrm{IR}})\ ,\quad (\gamma^{(1.5)})^{\mathrm{F}}_{\mathrm{G}} = \frac{\lambda}{8\pi^2}\cdot 6g_{\mathrm{YM}}\ ,\quad (\gamma^{(2)})^{\mathrm{S}_1}_{\mathrm{G}} = \frac{\lambda}{8\pi^2}\cdot\frac{3g_{\mathrm{YM}}^2}{16}\ , \\
&(\gamma^{(2)})^{\mathrm{S}_2}_{\mathrm{G}} = \frac{\lambda}{8\pi^2}\cdot\frac{3g_{\mathrm{YM}}^2}{8}\ ,\qquad (\gamma^{(2)})^{\mathrm{S}_3}_{\mathrm{G}} = \frac{\lambda}{8\pi^2}\cdot\frac{3g_{\mathrm{YM}}^2}{4N}\ ,\quad (\gamma^{(2)})^{\mathrm{S}_4}_{\mathrm{G}} = \frac{\lambda}{8\pi^2}\cdot\frac{3g_{\mathrm{YM}}^2}{8N}\ . \qquad (2.63)
\end{aligned}
$$

The first two expressions reproduce previous results in (2.48) and (2.55), and the remaining terms specify the mixing of G to $\vec{\mathrm{S}}$.

Gathering together all obtained results, we find the length-preserving and -increasing part of the mixing matrix

$$
\gamma^{(1)} = \frac{\lambda}{8\pi^2}\left(\begin{array}{c|c|cccc}
0 & 6g_{\mathrm{YM}} & 3g_{\mathrm{YM}}^2/16 & 3g_{\mathrm{YM}}^2/8 & 3g_{\mathrm{YM}}^2/4N & 3g_{\mathrm{YM}}^2/8N \\
\hline
 & 6 & -g_{\mathrm{YM}}/2 & g_{\mathrm{YM}} & 2g_{\mathrm{YM}}/N & g_{\mathrm{YM}}/N \\
\hline
 & & 11 & -10 & -20/N & -10/N \\
 & & 1 & 2 & -12/N & 2/N \\
 & & -5/N & 0 & 0 & 2 \\
 & & -6/N & 0 & 0 & 12
\end{array}\right) \qquad (2.64)
$$

which allows us to obtain the full one-loop eigenstate corresponding to a vanishing anomalous dimension

$$
\gamma_1^{(1)} = 0 \qquad\qquad \mathrm{O}_1 = \mathrm{G} - g_{\mathrm{YM}}\mathrm{F} - \tfrac{g_{\mathrm{YM}}^2}{2^4}\mathrm{S}_1\ , \qquad (2.65)
$$

which is precisely the on-shell Lagrangian (1.17). From this we obtain the $\mathcal{N}=4$ sYM β-function via (2.39) and find that it vanishes at one-loop order as expected. The operator O_1 is the only length-four $SU(4)$-invariant operator with vanishing anomalous dimension, which is consistent with $\mathcal{N}=4$ sYM theory being the unique four-dimensional CFT with $\mathcal{N}=4$ supersymmetry. The remaining entries in (2.64) can be extracted from length-decreasing terms in the dilatation operator.

2.3 Scalar β-deformed dilatation operator

We now move on to the calculation of the one-loop dilatation operator of the scalar subsector of β-deformed $\mathcal{N}=4$ sYM theory including its non-planar corrections. In order to set up the notation we start with the known undeformed case in Section 2.3.1 and then move on to the β-deformed case in Section 2.3.2.

2.3.1 Undeformed dilatation operator

We derive the undeformed dilatation operator from the on-shell approach of [31, 32], focusing on the scalar subsector of the theory. We extract it from (2.17) by studying local operators $\mathcal{O}_A$ composed of scalar fields only, i.e.

$$\mathcal{O}_A \sim \mathrm{Tr}(\phi_{I_1}\phi_{I_2}\dots)\dots\mathrm{Tr}(\dots\phi_{I_n}) \ . \tag{2.66}$$

The index I can take values $1,2,...,6$ corresponding to the set of six scalars $\{\phi_{14}, \phi_{24}, \phi_{34}, \phi_{23}, \phi_{31}, \phi_{12}\}$.

One-loop closedness. We begin by re-deriving that purely scalar operators (2.66) only mix among themselves at one-loop order, i.e. that the scalar sector is closed. This can be seen from the convolution term of (2.17) which determines the mixing between operators in this on-shell approach. At one-loop order only length-preserving mixing occurs and thus we study this relation for minimal form factors. For the (i,j)th component of the sum in the convolution term (2.18) to be non-vanishing, the external fields of the minimal form factor must be scalars, i.e. all the fields $1^{a_1}, 2^{a_2}, ..., n^{a_n}$ excluding i^{a_i} and j^{a_j}, as well as the intermediate states $i'^{a'_i}$ and $j'^{a'_j}$. This reduces the sum over intermediate states in the convolution to only include two scalar fields $\phi^{a'_i}_{I'_i}(p'_i)$ and $\phi^{a'_j}_{I'_j}(p'_j)$. The mixing of $\mathcal{O}_A$ is then governed via (2.18) by the one-loop amplitude

$$\langle i^{a_i}(p_i), j^{a_j}(p_j)|\mathcal{M}|\phi^{a'_i}_{I'_i}(p'_i), \phi^{a'_j}_{I'_j}(p'_j)\rangle^{(2)} \ . \tag{2.67}$$

The sum over intermediate scalars now includes three different cases:

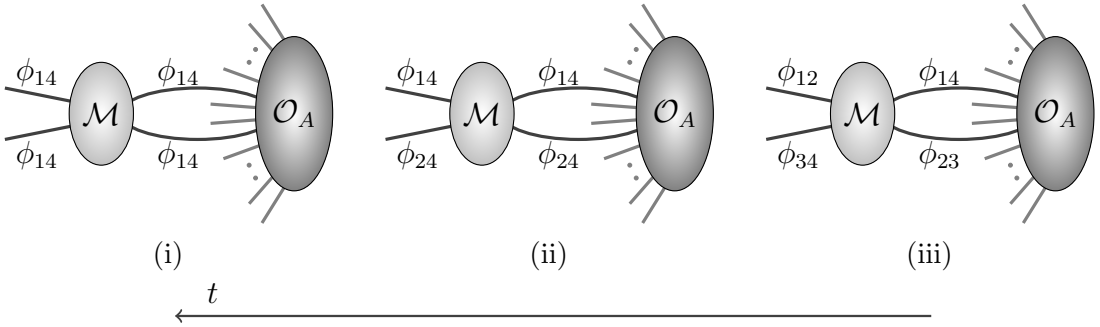


Figure 2.5: Non-vanishing contributions to one-loop mixing of the purely scalar operator $\mathcal{O}_A$ for (i) case 1, (ii) case 2, and (iii) case 3.

1. the two scalars are of the same type, e.g. $\phi_{I'_i} = \phi_{I'_j} = \phi_{14}$,
2. the two scalars span an $\mathfrak{su}(2)$ scalar sector, e.g. $\phi_{I'_i} = \phi_{14}$ and $\phi_{I'_j} = \phi_{24}$,
3. the two scalars form a singlet, e.g. $\phi_{I'_i} = \phi_{14}$ and $\phi_{I'_j} = \phi_{23}$.

In the first case, the amplitude (2.67) is only non-vanishing for external particles i, j being identical to the scalars in the intermediate state, e.g. $\langle \phi_{14}^{c_i}, \phi_{14}^{c_j} | \mathcal{M}^{(0)} | \phi_{14}^{a_i}, \phi_{14}^{a_j} \rangle$ and we illustrate such a convolution term in Figure 2.5 (i). For an $\mathfrak{su}(2)$ intermediate state there exist purely scalar amplitudes, cf. Figure 2.5 (ii), and mixed amplitudes with both scalars and fermions, cf. Figure 2.6 (i). Components of the one-loop MHV superamplitude (1.57) of this latter type are proportional to $e^{\pm i\varphi}$ by the parametrisation (2.19) and, since the minimal form factor in the convolution only depends on the φ-independent total sum of momenta, these terms vanish in the integral (2.21) of the convolution. Finally, if the sum over intermediate scalar states runs over a singlet state, we may have an amplitude (2.67) where the external state $|i, j\rangle$ corresponds to a scalar, fermionic or gluonic singlet state, cf. Figure 2.5 (iii) and 2.6 (ii,iii). Amplitudes of the form $\langle \psi, \bar{\psi} | \mathcal{M}^{(0)} | \phi, \bar{\phi} \rangle$ are again proportional to $e^{\pm i\varphi}$ and amplitudes of the form $\langle g_+, g_- | \mathcal{M} | \phi, \bar{\phi} \rangle$ are proportional to $e^{\pm 2i\varphi}$, and thus these cases all vanish in the integral of the convolution. This implies that (2.17) is non-trivial only for purely scalar external states. For a non-vanishing tree-level minimal form factor on the left-hand side of this equation, the fields in the external state must correspond to the fields in $\mathcal{O}_B$ and thus $\mathcal{O}_B$ is purely scalar. This proves that there only occurs mixing in the scalar subsector at one-loop level.

Convolution term. In the following we make the mixing between scalar operators explicit. The one-loop scalar four-point amplitude can be extracted from the MHV

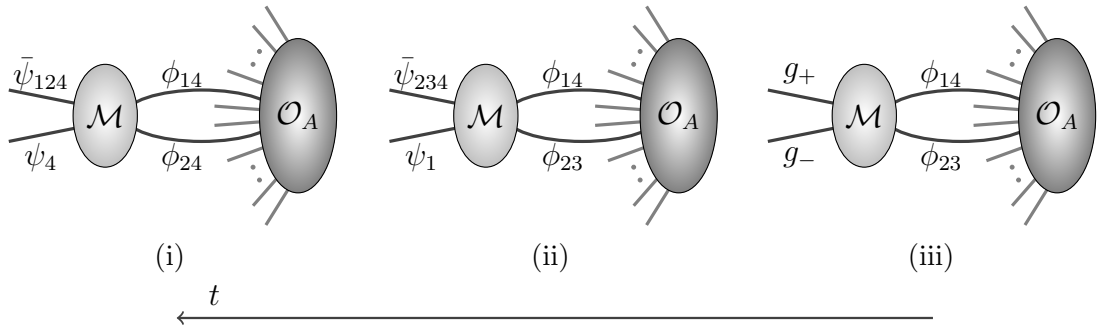


Figure 2.6: Vanishing contributions to one-loop mixing of the purely scalar operator $\mathcal{O}_A$ for (i) case 2, and (ii,iii) case 3.

superamplitude (1.57) and with the crossing convention (1.49) it is given by

$$\begin{aligned}
&\langle \phi_{I_i}^{a_i}(p_i), \phi_{I_j}^{a_j}(p_j)|\mathcal{M}|\phi_{I_i'}^{a_i'}(p_i'), \phi_{I_j'}^{a_j'}(p_j')\rangle^{(0)} \\
&\quad = -2g_{\text{YM}}^2 \left(\frac{f^{a_i a_j e} f^{a_j' a_i' e}}{\sin^2\vartheta} - \frac{f^{a_i a_j' e} f^{a_j a_i' e}}{\cos^2\vartheta \sin^2\vartheta} \right) \\
&\qquad \times \left(\cos^2\vartheta\ \delta_{I_i I_i'} \delta_{I_j I_j'} + \sin^2\vartheta\ \delta_{I_i I_j'} \delta_{I_j I_i'} - \sin^2\vartheta \cos^2\vartheta\ \delta_{I_i \overline{I_j}} \delta_{I_i' \overline{I_j'}} \right) ,
\end{aligned} \tag{2.68}$$

with $\phi_{\bar{I}} := \overline{\phi_I}$. Inserted into the convolution (2.18), we can perform the sum over intermediate scalars and the integration over Ω. Rearranging terms using the Jacobi identity and furthermore using

$$\begin{aligned}
&: \text{Tr}[\phi_I, \phi_J][\check{\phi}^I, \check{\phi}^J] : \phi_{I_i}^{a_i} \phi_{I_j}^{a_j} = -4 f^{a_i a_j e} f^{a_i' a_j' e} \phi_{I_i}^{a_i'} \phi_{I_j}^{a_j'} , \\
&: \text{Tr}[\phi_I, \check{\phi}^I][\phi_J, \check{\phi}^J] : \phi_{I_i}^{a_i} \phi_{I_j}^{a_j} = -4 f^{a_i a_i' e} f^{a_j a_j' e} \phi_{I_i}^{a_i'} \phi_{I_j}^{a_j'} , \\
&: \text{Tr}[\phi_I, \check{\phi}^J][\phi_{\bar{I}}, \check{\phi}^{\bar{J}}] : \phi_{I_i}^{a_i} \phi_{I_j}^{a_j} = -4 f^{a_i a_i' e} f^{a_j a_j' e} \delta_{I_i \bar{I}_j} \textstyle\sum_I \phi_I^{a_i'} \phi_{\bar{I}}^{a_j'} ,
\end{aligned} \tag{2.69}$$

the convolution terms can be cast into the form of an operator acting on the external states of a form factor. This yields

$$\begin{aligned}
&(\gamma^{(1)})_A^B \langle 1^{a_1}, ..., n^{a_n}|\mathcal{O}_B|0\rangle^{(0)} \\
&= \Bigg[-\frac{g_{\text{YM}}^2}{16\pi^2} \Bigg(: \text{Tr}[\phi_I, \phi_J][\check{\phi}^I, \check{\phi}^J] : + \frac{1}{2} \text{Tr}[\phi_I, \check{\phi}^J][\phi_{\bar{I}}, \check{\phi}^{\bar{J}}] : \\
&\qquad\qquad - \mathcal{I}_{\text{IR}} : \text{Tr}[\phi_I, \check{\phi}^I][\phi_J, \check{\phi}^J] : \Bigg) + \gamma_{\text{IR}}^{(1)} \Bigg] \langle 1^{a_1}, ..., n^{a_n}|\mathcal{O}_A|0\rangle^{(0)} .
\end{aligned} \tag{2.70}$$

IR divergence. What is left is to determine the form of γ_{IR}. Using (2.69) and translating its colour structure into an operator structure via (2.69) yields

$$\gamma_{\text{IR}}^{(1)} = -\frac{g_{\text{YM}}^2}{16\pi^2} \mathcal{I}_{\text{IR}} : \text{Tr}[\phi_I, \check{\phi}^I][\phi_J, \check{\phi}^J] : . \tag{2.71}$$

Dilatation operator in scalar sector. Inserting the result (2.71) into (2.70) we find that the IR divergence exactly cancels the divergent part of the convolution term and thus we obtain

$$\begin{aligned}&(\gamma^{(1)})_A^B \,\langle 1^{a_1},...,n^{a_n}|\mathcal{O}_B|0\rangle^{(0)} \\ &= -\frac{g_{\rm YM}^2}{16\pi^2}\left(:\mathrm{Tr}[\phi_I,\phi_J][\check\phi^I,\check\phi^J]: +\frac{1}{2}:\mathrm{Tr}[\phi_I,\check\phi^J][\phi_{\bar I},\check\phi^{\bar J}]:\right)\langle 1^{a_1},...,n^{a_n}|\mathcal{O}_A|0\rangle^{(0)}\ .\end{aligned} \tag{2.72}$$

In order to convert this mixing relation of form factors into a mixing relation of local operators, we note that at tree level the field content of the external state must match the field content of the local operator of a minimal form factor. Furthermore, we may map on the trace-structure of the local operator by multiplying the external state by the appropriate colour-trace structure. Thus we can extract the one-loop mixing matrix for purely scalar operators which is generated by the dilatation operator

$$\mathfrak{D}_2 = -\frac{g_{\rm YM}^2}{16\pi^2}\left(:\mathrm{Tr}[\phi_I,\phi_J][\check\phi^I,\check\phi^J]: +\frac{1}{2}:\mathrm{Tr}[\phi_I,\check\phi^J][\phi_{\bar I},\check\phi^{\bar J}]:\right)\ , \tag{2.73}$$

which is the known one-loop $\mathfrak{so}(6)$ dilatation operator of $\mathcal{N}=4$ sYM theory [6].

2.3.2 Deformed dilatation operator

We now turn to the β-deformed theory and derive the scalar dilatation operator. The deformation enters its computation via the form factor on-shell approach by deforming the four-point single-trace amplitudes and introducing double-trace terms. The mixed amplitudes contributing to the diagrams in Figure 2.6 only pick up a non-kinematic phase factor in the deformed theory and thus the scalar sector remains closed at one-loop order.

$\mathfrak{so}(6)$ sector. In the $\mathfrak{so}(6)$ sector the deformed amplitude is given in terms of the deformed structure constants (1.83) by

$$\begin{aligned}&\langle\phi_{I_i}^{a_i}(p_i),\phi_{I_j}^{a_j}(p_j)|\mathcal{M}|\phi_{I_i'}^{a_i'}(p_i'),\phi_{I_j'}^{a_j'}(p_j')\rangle^{(0)} \\ &= -2g_{\rm YM}^2\Bigg[\left(\frac{f^{a_ia_je}_{\varphi(I_i,I_j)}f^{a_i'a_j'e}_{\varphi(I_i,I_j)}}{\cos^2\vartheta}-\frac{f^{a_ia_i'e}_{\varphi(I_i,I_i')}f^{a_ja_j'e}_{\varphi(I_j,I_j')}}{\sin^2\vartheta\cos^2\vartheta}\right) \\ &\qquad\qquad\times\left(\cos^2\vartheta\delta_{I_iI_i'}\delta_{I_jI_j'}-\sin^2\vartheta\cos^2\vartheta\delta_{I_i\bar I_j}\delta_{I_i'\bar I_j'}n^{I_iI_i'}\right) \\ &\quad+\left(\frac{f^{a_ia_je}_{\varphi(I_i,I_j)}f^{a_j'a_i'e}_{\varphi(I_i,I_j)}}{\sin^2\vartheta}-\frac{f^{a_ia_j'e}_{\varphi(I_i,I_j')}f^{a_ja_i'e}_{\varphi(I_j,I_i')}}{\sin^2\vartheta\cos^2\vartheta}\right) \\ &\qquad\qquad\times\left(\sin^2\vartheta\delta_{I_iI_j'}\delta_{I_jI_i'}-\sin^2\vartheta\cos^2\vartheta\delta_{I_i\bar I_j}\delta_{I_i'\bar I_j'}n^{I_iI_j'}\right)\Bigg]\ ,\end{aligned} \tag{2.74}$$

where $f^{abc}_{\varphi(I_a,I_b)}\delta_{I_aI_b} = f^{abc}$, so setting both the phase to 1 and sending $\tilde{\kappa} \to 1$. n^{IJ} is the particle-/anti-particle number and vanishes if I corresponds to a scalar index, i.e. $\phi_I \in \{\phi_{14}, \phi_{24}, \phi_{34}\}$, and J to an anti-scalar index, i.e. $\phi_J \in \{\phi_{23}, \phi_{31}, \phi_{12}\}$, and vice versa, while n^{IJ} is 1 if both indices correspond to either scalars or anti-scalars. The occurrence of such a parameter in the four-point scalar amplitude makes the breaking of the symmetry between chiral and anti-chiral scalars due to the breaking of the $SU(4)$ R-invariance in the β-deformed theory manifest.

The deformed structure constants satisfy

$$f^{abe}_{\varphi(a,b)}f^{cde}_{\varphi(c,d)} = -\frac{1}{2}|\tilde{\kappa}|^2\Big(\operatorname{Tr}\left([a,b]_{\varphi(a,b)}[c,d]_{\varphi(c,d)}\right) - \frac{1}{N}\operatorname{Tr}[a,b]_{\varphi(a,b)}\operatorname{Tr}[c,d]_{\varphi(c,d)}\Big)\,, \tag{2.75}$$

where $\varphi(a,b)$ is the phase of the particles corresponding to colour indices a and b. The deformed commutators on the right-hand side can be related to operators via the deformed version of (2.69),

$$\begin{aligned}
&:\operatorname{Tr}[\phi_I,\phi_J]_\varphi[\check{\phi}^I,\check{\phi}^J]_\varphi: \phi^{a_i}_{I_i}\phi^{a_j}_{I_j} = 2\operatorname{Tr}\left([a_i,a_j]_{\varphi(I_i,I_j)}[a'_i,a'_j]_{\varphi(I_i,I_j)}\right)\phi^{a'_i}_{I_i}\phi^{a'_j}_{I_j}\,,\\
&:\operatorname{Tr}[\phi_I,\check{\phi}^J]_\varphi[\phi_{\bar{I}},\check{\phi}^{\bar{J}}]_\varphi: \phi^{a_i}_{I_i}\phi^{a_j}_{I_j} = 2\delta_{I_i\bar{I}_j}\textstyle\sum_I \operatorname{Tr}\left([a_i,a'_i]_{\varphi(I_i,I)}[a_j,a'_j]_{\varphi(I_i,I)}\right)\phi^{a'_i}_{I}\phi^{a'_j}_{\bar{I}}\,,
\end{aligned} \tag{2.76}$$

and analogue relations for the double-trace colour structures. Inserting the amplitude (2.74) into the convolution (2.18) and using (2.76) and (2.69) yields

$$\begin{aligned}
&(\gamma^{(1)})^B_A\,\langle 1^{a_1},...,n^{a_n}|\mathcal{O}_B|0\rangle^{(0)}\\
&= \Bigg[-\frac{g^2_{\mathrm{YM}}}{16\pi^2}\Bigg(|\tilde{\kappa}|^2\Big(:\operatorname{Tr}[\phi_I,\phi_J]_\varphi[\check{\phi}^I,\check{\phi}^J]_\varphi: -\frac{1}{N}:\operatorname{Tr}[\phi_I,\phi_J]_\varphi\operatorname{Tr}[\check{\phi}^I,\check{\phi}^J]_\varphi:\Big)\\
&\quad+\frac{|\tilde{\kappa}|^2}{2}\Big(:\operatorname{Tr}[\phi_I,\check{\phi}^J]_\varphi[\phi_{\bar{I}},\check{\phi}^{\bar{J}}]_\varphi: -\frac{1}{N}:\operatorname{Tr}[\phi_I,\check{\phi}^J]_\varphi\operatorname{Tr}[\phi_{\bar{I}},\check{\phi}^{\bar{J}}]_\varphi:\Big)\\
&\quad-\mathcal{I}_{\mathrm{IR}}:\operatorname{Tr}[\phi_I,\check{\phi}^I][\phi_J,\check{\phi}^J]:\Bigg)+\gamma^{(1)}_{\mathrm{IR}}\Bigg]\langle 1^{a_1},...,n^{a_n}|\mathcal{O}_A|0\rangle^{(0)}\,.
\end{aligned} \tag{2.77}$$

Comparing with the undeformed result (2.70), we see that all finite terms get deformed, but the IR-divergent contribution is unchanged. This latter point is in accordance with the one-loop soft divergence being universal and taking the form (2.35) also in the deformed theory. Thus we again subtract (2.71) and obtain

$$\begin{aligned}
\mathfrak{D}^\varphi_2 = -\frac{|\kappa|^2}{16\pi^2}\Bigg(&:\operatorname{Tr}[\phi_I,\phi_J]_\varphi[\check{\phi}^I,\check{\phi}^J]_\varphi: +\frac{1}{2}:\operatorname{Tr}[\phi_I,\check{\phi}^J]_\varphi[\phi_{\bar{I}},\check{\phi}^{\bar{J}}]_\varphi:\\
&-\frac{1}{N}\Big(:\operatorname{Tr}[\phi_I,\phi_J]_\varphi\operatorname{Tr}[\check{\phi}^I,\check{\phi}^J]_\varphi: +\frac{1}{2}:\operatorname{Tr}[\phi_I,\check{\phi}^J]_\varphi\operatorname{Tr}[\phi_{\bar{I}},\check{\phi}^{\bar{J}}]_\varphi:\Big)\Bigg)\,.
\end{aligned} \tag{2.78}$$

In addition to deformed single-trace operators inherited from the undeformed theory,

this operator contains double-trace terms which are necessary to make the theory exactly conformal [47]. It is important to note that although they are formally suppressed by $1/N$, they can be relevant at leading order when acting on short operators which we discuss in the following. In Chapter 3 we compute non-planar anomalous dimensions from (2.78) perturbatively and in Chapter 4 we study statistical properties of its anomalous-dimension spectrum. Note that the calculation of $\mathfrak{D}_2^{\varphi}$ did not depend on the explicit form of the phase φ and thus is valid in more general deformed theories whose four-point scalar amplitudes can be brought into the form (2.74). However, as these deformed theories are not necessarily conformal, the dilatation operator $\mathfrak{D}_2^{\varphi}$ is not necessarily a symmetry generator.

$L = 2$ mixing. We now study the mixing of length-two operators induced by (2.78). The operators $\mathrm{Tr}(\phi_I\phi_J)$, where $\phi_I, \phi_J \in \mathfrak{su}(2)$, remain protected in the deformed theory. In particular when $I \neq J$ the first double-trace contribution of (2.78) is necessary for a vanishing anomalous dimension. The protectedness of these operators was shown previously at one- and two-loop level by direct calculations in [74, 75].

Also the Konishi operator $\mathcal{K} = \mathrm{Tr}(\phi_I\bar{\phi}_I)$ remains an eigenstate of the one-loop dilatation operator in the β-deformed theory ($\varphi(\phi_1,\phi_2) = \varphi(\phi_2,\phi_3) = \varphi(\phi_3,\phi_1) = \beta$) with anomalous dimension

$$\gamma_{\mathcal{K}}^{(1)} = \frac{g_{\mathrm{YM}}^2 N}{8\pi^2}\left(3 + 2\left[|\tilde{\kappa}|^2\left(1 - \frac{4}{N^2}\sin^2\beta\right) - 1\right]\right) , \tag{2.79}$$

reproducing the results in [74,75] from direct calculation. The expression in the square bracket corresponds to the condition on the deformed coupling κ of (1.64) which makes the deformed theory exactly conformal through two loops. Setting this expression to zero one finds that the Konishi operator has undeformed anomalous dimension

$$\gamma_{\mathcal{K}}^{(1)} = \frac{3\lambda}{8\pi^2} . \tag{2.80}$$

The remaining two eigenstates in this length-two sector of $U(1)^3$-uncharged operators are protected and can be chosen to be $\mathrm{Tr}(\phi_1\bar{\phi}_1 - \phi_2\bar{\phi}_2)$ and $\mathrm{Tr}(\phi_2\bar{\phi}_2 - \phi_3\bar{\phi}_3)$. They combine with the length-two operator in the $\mathfrak{su}(2)$ sector to the symmetric traceless operators $\mathcal{Q}_{IJ} := \mathrm{Tr}(\phi_I\phi_J) - \frac{1}{6}\eta_{IJ}\mathcal{K}$ with vanishing eigenvalues, corresponding to the chiral primaries of the $\mathbf{20'}$ representation of $SU(4)$.

In the more general γ_i-deformed theory, where $\gamma_1 := \varphi(\phi_1,\phi_2)$, $\gamma_2 := \varphi(\phi_2,\phi_3)$, $\gamma_3 := \varphi(\phi_3,\phi_1)$ and $\gamma_1 \neq \gamma_2 \neq \gamma_3$, the operator $\mathrm{Tr}(\phi_I\phi_J)$ for $\phi_I, \phi_J \in \mathfrak{su}(2)$ remains protected. Similarly the Konishi operator stays protected in the planar theory, however it mixes with the other uncharged operators at finite N. The corresponding

symmetric mixing matrix of operators $\{\mathrm{Tr}(\phi_1\bar{\phi}_1), \mathrm{Tr}(\phi_2\bar{\phi}_2), \mathrm{Tr}(\phi_3\bar{\phi}_3)\}$ is given by

$$\gamma^{(1)} = \frac{g^2_{\mathrm{YM}}N}{8\pi^2}\begin{pmatrix} 1 & |\tilde{\kappa}|^2(1-4\sin^2\gamma_1/N^2) & |\tilde{\kappa}|^2(1-4\sin^2\gamma_3/N^2) \\ & 1 & |\tilde{\kappa}|^2(1-4\sin^2\gamma_2/N^2) \\ & & 1 \end{pmatrix}. \tag{2.81}$$

Chapter 3

Perturbative non-planar anomalous dimensions

The action of the (β-deformed) $\mathcal{N} = 4$ sYM dilatation operator (1.36)-(1.38) and (2.78) on the space of multi-trace states gives rise to complicated mixing problems. Symmetries simplify the diagonalisation of this mixing by arranging states into closed sectors, with states belonging to different sectors not being allowed to mix. For example, only states that have the same classical scaling dimension can mix, and both the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ sector are closed. Despite these symmetries, with an increasing number of field insertions into states, the dimension of the mixing problem even in closed sectors increases very quickly. Thus, when going beyond short operators, its solution by direct diagonalisation quickly becomes inconceivable and a systematic analytic approach is desirable.

A step forward in this quest is the realisation that the action of the one-loop dilatation operator on a general multi-trace operator can be decomposed into a planar and non-planar piece. Take as an example the length-six single-trace operator $\mathrm{Tr}(X^2Z^4)$ on which the one-loop dilatation operator (1.36) acts as

$$\begin{aligned}\mathfrak{D}_2\,\mathrm{Tr}(X^2Z^4) = 4\left(\mathrm{Tr}(X^2Z^4) - \mathrm{Tr}(XZXZ^3)\right)\\ + \frac{4}{N}\left(\mathrm{Tr}(X^2Z^2)\,\mathrm{Tr}(Z^2) - \mathrm{Tr}(XZXZ)\,\mathrm{Tr}(Z^2)\right)\ ,\end{aligned} \tag{3.1}$$

where we repeatedly use the splitting and fusion identities (1.34). The leading term in a large-N expansion corresponds to the superposition of single-trace operators and the subleading term is a double-trace contribution. Similarly, we can decompose the action of the one-loop dilatation operator on general multi-trace operators into planar and non-planar pieces as [137]

$$\mathfrak{D}_2 = H^{(0)} + \frac{1}{N}H^- + \frac{1}{N}H^+\ . \tag{3.2}$$

The planar piece $H^{(0)}$ leaves the number of traces in an operator unchanged, while the non-planar corrections $H^{\pm}$, which are suppressed by a factor of $1/N$, increase or reduce the number of traces in a given operator. In order to find the eigenvalues of $\mathfrak{D}_2$, one can first solve the planar problem using integrability, and then use perturbation theory to find the $1/N^k$ corrections by treating $\frac{1}{N}H^{\pm}$ as a small perturbation. We review the solution of the planar problem in Section 3.1 and then derive expressions for perturbative non-planar corrections in the undeformed theory in Section 3.2. Although this approach to non-planar dimensions is promising at first sight, it is made complicated by degeneracies in the planar spectrum and thus degenerate perturbation theory has to be employed. In order to avoid this obstacle, one can consider the β-deformed $\mathcal{N} = 4$ sYM theory, in which most degeneracies in the $\mathfrak{su}(2)$ sector are lifted, and we will do so in Section 3.3.

3.1 Planar theory and integrability

3.1.1 Integrable structures in $\mathcal{N} = 4$ sYM theory

A seminal development in the understanding of $\mathcal{N} = 4$ sYM theory was the realisation that in its planar limit there occur integrable structures, first discovered in the scalar sector of $\mathcal{N} = 4$ sYM theory in [11]. Specifically, the one-loop mixing problem in the $\mathfrak{su}(2)$ sector can be mapped to the spectral problem of the Heisenberg spin chain. The latter was already studied long before its relevance for super Yang–Mills theories was realised, see e.g. [138, 139], and thus there were many tools at hand for its solution. We review these developments in the following, in particular as we will make use of the spin-chain notation and the results from integrability to organise the computation of non-planar corrections.

Heisenberg spin chain in $\mathfrak{su}(2)$ sector. In order to make the mapping between the Heisenberg spin chain and the planar $\mathfrak{su}(2)$ one-loop mixing problem explicit, we study the action of the dilatation operator $\mathfrak{D}_2$ given in (1.36) on the space of multi-trace operators composed of fields X and Z. In the large-N limit this action is enhanced by a factor of N when acting on adjacent fields X and Z in a trace. This action preserves the number of traces of an operator and corresponds to the leading term $H^{(0)}$ in (3.2). With the number of traces being conserved, multi-trace and single-trace operators decouple and the multi-trace mixing problem is essentially solved by the single-trace mixing problem: multi-trace eigenstates of $H^{(0)}$ are products of single-trace eigenstates and the corresponding eigenvalue is the sum of the eigenvalues of the constituent single-trace operators. On the space of single-trace operators the planar

one-loop dilatation operator (1.36) acts as

$$H_{\text{xxx}} = 2\sum_{i=1}^{L}\left(\mathbb{1}_{i,i+1} - \mathbb{P}_{i,i+1}\right) \ . \tag{3.3}$$

Here $\mathbb{P}_{i,i+1} = \frac{1}{2}(\mathbb{1}\otimes\mathbb{1} + \sigma^a \otimes \sigma^a)_{i,i+1}$ denotes the permutation operator acting on positions i and $(i+1)$ in a trace of L scalar fields and similarly $\mathbb{1}_{i,i+1} = (\mathbb{1}\otimes\mathbb{1})_{i,i+1}$ corresponds to the identity operator. The operator (3.3) is the famous Heisenberg Hamiltonian [140] of $\mathfrak{su}(2)$ spin chains. Thus we can map the problem of finding one-loop anomalous dimensions in the $\mathfrak{su}(2)$ sector to the problem of finding the energy spectrum of the cyclic Heisenberg spin chain.

Integrability of the Heisenberg spin chain. The solution of the Heisenberg spin chain can be achieved by various approaches going under the name of "Bethe ansatz". We will discuss the coordinate Bethe ansatz in Section 3.1.2, which is the original ansatz developed by Bethe [138], and will comment on the algebraic Bethe ansatz in Appendix B, which goes back to Faddeev [139]. The solution of the Heisenberg spin chain is characterised by a remarkable simplicity, specifically the energy of multi-particle eigenstates is just a sum of single-particle energies, and multi-particle scattering factorises into two-particle scattering. This simplicity is characteristic of integrable models and hints towards the existence of hidden symmetries. Indeed, one can show that the Heisenberg spin chain has a tower of conserved local charges which can be derived from its Lax operator, see e.g. [141]. An alternative formulation of this system's integrability for an infinitely long spin chain is based on a non-local symmetry algebra called Yangian, in particular $Y[\mathfrak{su}(2)]$ for the Heisenberg spin chain, introduced by Drinfel'd [142–145].

Integrability beyond $\mathfrak{su}(2)$ and one-loop order. The analogy between the one-loop dilatation operator and an integrable spin-chain Hamiltonian extends beyond the $\mathfrak{su}(2)$ sector to the full space of $\mathcal{N}=4$ sYM local operators, where the complete one-loop dilatation operator was found in [10]. The corresponding Hamiltonian is that of an integrable $\mathfrak{psu}(2,2|4)$ spin chain [12] and was shown to be invariant under the Yangian algebra $Y[\mathfrak{psu}(2,2|4)]$ [146], making the whole planar mixing problem solvable by integrability methods. At higher loop orders the planar dilatation operator is no longer of nearest-neighbour type, but remains integrable and can be solved via an asymptotic Bethe ansatz [147, 148]. It is asymptotic in the sense that it is only valid for local operators whose length is larger than the spin-chain interaction length at a given order in the perturbative g^2-expansion, and otherwise receives wrapping corrections [149]. An efficient way for explicit computations of the spectrum of planar $\mathcal{N}=4$ sYM theory at finite coupling is via the quantum spectral curve [150], see

also [151].

Integrable structures do not only appear in the spectral problem of planar $\mathcal{N}=4$ sYM theory, but were also unveiled in the context of other observables of this theory. For example, structure constants of three-point correlators can be computed non-perturbatively via an integrable bootstrap [20], and similarly planar scattering amplitudes can be described at finite coupling via a dual description in terms of polygonal null Wilson loops and an integrable flux-tube Hamiltonian [152, 153]. Moreover, integrable structures were also uncovered in the dual gravity theory and in particular superstrings on $AdS_5 \times S^5$ are classically integrable [14]. This integrability can be used to find explicit classical string solutions, like those of spinning strings [154, 155], and more generally to describe classical solutions as solutions of algebraic curves [156–161]. A comprehensive overview of integrability in the context of the AdS/CFT correspondence is given in the review articles of [16].

3.1.2 Coordinate Bethe ansatz for the Heisenberg spin chain

Spin-chain notation. In order to solve the mixing problem of $H^{(0)}$ on the space of $\mathfrak{su}(2)$ single-trace operators, it is first useful to introduce a notation which resembles the notation of spin chains: Single-trace operators with M insertions of X fields in a background of $(L-M)$ Z's will be denoted as

$$\mathrm{Tr}(\overbrace{Z\dots Z}^{n_1-1} X \overbrace{Z\dots Z}^{n_2-n_1-1} X\dots) \to |\overbrace{\uparrow\dots\uparrow}^{n_1-1}\downarrow\overbrace{\uparrow\dots\uparrow}^{n_2-n_1-1}\downarrow\dots\rangle_L \equiv |n_1,n_2,\dots,n_M\rangle_L \ . \quad (3.4)$$

Note that we do not transfer the cyclicity property of single-trace operators to the notation $|..\rangle$ on the right-hand side, where we only impose periodicity such as $|L+1\rangle_L := |1\rangle_L$, and so the two notations are not equivalent. Nevertheless, it will be more convenient to first work with states $|\{n\}\rangle$ to solve the planar mixing problem and then impose cyclicity of eigenstates at the very end. Multi-trace operators with K traces and M insertions of X fields, where $M=\sum_{k=1}^K M_k$, can then be denoted by products of such states

$$\prod_{k=1}^K |n_1^{(k)},\dots,n_{M_k}^{(k)}\rangle_{L_k} \ , \quad (3.5)$$

which is an element of the symmetrised tensor product. It will often be convenient to use the compressed notation $|\{n\}\rangle$ and $\prod_k |\{n^{(k)}\}\rangle$ for single- and multi-trace states, respectively. In the following we compute overlaps of states and thus we define the

natural dual basis $\langle\{m\}|$ with normalisation

$$ {}_{L_m}\langle m_1, m_2, \ldots, m_{M_m} | n_1, n_2, \ldots, n_{M_n} \rangle_{L_n} = \delta_{L_m,L_n} \delta_{M_m,M_n} \prod_{j=1}^{M_n} \delta_{m_j,n_j} \ . \tag{3.6} $$

In this basis of states, the action of the planar dilatation operator (3.3) is given by the well-known formula

$$ \begin{aligned} &H^{(0)} \, |n_1, n_2, \ldots\rangle_L \\ &\qquad = 2 \sum_{j=1}^{M} \Big(2 \, |\ldots, n_j, \ldots\rangle_L - |\ldots, n_j - 1, \ldots\rangle_L - |\ldots, n_j + 1, \ldots\rangle_L \Big) \ . \end{aligned} \tag{3.7} $$

This spin-chain Hamiltonian can now be diagonalised by means of the coordinate Bethe ansatz. It organises eigenstates by their number of excitations M based on the $SU(2)$ spin symmetry of (3.3). The lowest-energy state is degenerate with states

$$ |\emptyset\rangle = |\uparrow\uparrow \ldots \uparrow\rangle \qquad \text{and} \qquad |\bar{\emptyset}\rangle = |\downarrow\downarrow \ldots \downarrow\rangle \tag{3.8} $$

and vanishing energy. These states correspond to operators consisting of a single type of scalar field, $\mathrm{Tr}(Z^L)$ and $\mathrm{Tr}(X^L)$. They are half-BPS and due to supersymmetry their dimensions receive no quantum corrections. Choosing $|\emptyset\rangle$ as vacuum, we can build excited states as linear combinations of basis vectors (3.4) with fixed L and M as

$$ |\{p\}\rangle = \sum_{\{n\}} \psi_{\{n\}}^{\{p\}} \, |\{n\}\rangle \tag{3.9} $$

in terms of a wavefunction $\psi_{\{n\}}^{\{p\}}$ depending on the quantum numbers of the particular state. The states are characterised by the momenta $\{p\} = \{p_1, p_2, \ldots, p_M\}$ of the M excitations (or "magnons") and the sum in (3.9) is over the positions of excitations ranging over the nested values $1 \leq n_1 < n_2 < \cdots < n_M \leq L$. Using (3.6) the scalar product of two such states $|\{p\}\rangle$ and $|\{q\}\rangle$ is given by

$$ \langle\{q\}|\{p\}\rangle = \sum_{\{n\}} \big(\psi_{\{n\}}^{\{q\}}\big)^* \psi_{\{n\}}^{\{p\}} \ . \tag{3.10} $$

Bethe ansatz. The states (3.9) are eigenstates of the Heisenberg Hamiltonian (3.7) if the wavefunctions satisfy

$$ \psi_{\{n\}}^{\{p\}} \equiv \psi_{n_1,\ldots,n_M}^{p_1,\ldots,p_M} = \frac{1}{\prod_{j<k} \sqrt{S(p_j,p_k)}} \sum_{\sigma \in \mathcal{S}_M} e^{i \sum_{j=1}^{M} p_{\sigma(j)} n_j} \prod_{\substack{j>k \\ \sigma(j)<\sigma(k)}} S(p_{\sigma(j)}, p_{\sigma(k)}) \ , \tag{3.11} $$

which is a sum over all permutations σ of the M excitations, with permutation group $\mathcal{S}_M$. This expression furthermore contains the two-magnon S-matrix

$$S(p_j, p_k) = -\frac{e^{ip_j+ip_k} + 1 - 2e^{ip_k}}{e^{ip_j+ip_k} + 1 - 2e^{ip_j}} \ , \tag{3.12}$$

and we have made a particular choice for the overall, non-physical, phase of the wavefunction which is convenient for subsequent purposes. For the states (3.11) to satisfy periodic boundary conditions, the momenta must satisfy the Bethe equations, i.e. for each $j = 1, \ldots, M$

$$e^{i\phi_j} = 1 \ , \qquad \text{where} \qquad e^{i\phi_j} \equiv e^{ip_j L} \prod_{k\neq j}^{M} S(p_j, p_k) \ , \tag{3.13}$$

which quantises the momenta $\{p\}$ and implies that the wavefunctions satisfy the condition

$$\psi^{\{p\}}_{n_1,n_2,\ldots,n_M} = \psi^{\{p\}}_{n_2,\ldots,n_M,n_1+L} \ . \tag{3.14}$$

Each eigenstate corresponds to a solution of the algebraic equations (3.13) with pairwise distinct momenta and the energy eigenvalue is given as a sum over individual magnon energies

$$E^{(0)}(\{p\}) = \sum_{j=1}^{M} \varepsilon(p_j) \ , \qquad \varepsilon(p_j) = 4(1 - \cos p_j) \ . \tag{3.15}$$

The cyclicity of the trace for gauge-theory operators becomes the condition that the spin chain is invariant under the shift $n_j \to n_j + 1$ and so we consider only states which satisfy the condition

$$\prod_{j=1}^{M} e^{ip_j} = 1 \ . \tag{3.16}$$

Further useful notations. It is convenient to introduce rapidity variables u_j as

$$u_j = \frac{1}{2}\cot\frac{p_j}{2} \qquad \text{or} \qquad e^{ip_j} = \frac{u_j + i/2}{u_j - i/2} \tag{3.17}$$

for each excitation, and then the S-matrix (3.12) and the individual magnon energies (3.15) are given by

$$S(u_j, u_k) = \frac{u_j - u_k - i}{u_j - u_k + i} \qquad \text{and} \qquad \varepsilon(u_j) = \frac{2}{u_j^2 + \frac{1}{4}} \ , \tag{3.18}$$

and the periodicity (3.13) and cyclicity constraint (3.16) become

$$\left(\frac{u_j+\frac{i}{2}}{u_j-\frac{i}{2}}\right)^L \prod_{k\neq j}^{M}\frac{u_j-u_k-i}{u_j-u_k+i}=1 \qquad \text{and} \qquad \prod_{j=1}^{M}\frac{u_j+\frac{i}{2}}{u_j-\frac{i}{2}}=1\ , \tag{3.19}$$

respectively. The solutions $\{u\}$ to the first relation in (3.19) are called "Bethe roots". It will also be useful to define the quantity

$$h(u_j,u_k)=\frac{u_j-u_k}{u_j-u_k+i}\ , \tag{3.20}$$

so that the S-matrix is given as

$$S(u_j,u_k)=\frac{h(u_j,u_k)}{h(u_k,u_j)}\ , \tag{3.21}$$

and a normalisation factor for states with momenta $\{p\}$ corresponding to rapidities $\{u\}$ as

$$\mathcal{N}(p(u))=\frac{\prod_{i<j}h(u_i,u_j)}{\prod_{j<k}\sqrt{S(u_j,u_k)}} \tag{3.22}$$

and generalisations such as $\mathcal{N}(p,q)=\mathcal{N}(p)\mathcal{N}(q)$.

Finally, similar to [162], we will use the following short-hand notation for products

$$f^{\{a\}}=\prod_i f(a_i)\ , \qquad f^{\{a\}}_{<}=\prod_{i<j} f(a_i,a_j)\ , \qquad h^{\{a\}\{b\}}=\prod_{i,j}h(a_i,b_j) \tag{3.23}$$

and

$$\{z\}_{\hat{a}}=\{z_1,\ldots,\hat{z}_a,\ldots,z_n\}=\{z_1,\ldots,z_{a-1},z_{a+1},\ldots,z_n\} \tag{3.24}$$

for lists with a missing element. Using this notation a Bethe state can be written as

$$|\{p\}\rangle=\mathcal{N}(p)\sum_{\{n\}}\sum_{\sigma\in S_M}\frac{1}{h_{<}^{\{p_\sigma\}}}e^{ip_\sigma\cdot n}\,|\{n\}\rangle \tag{3.25}$$

and its conjugate as

$$\langle\{p\}|=\mathcal{N}(p^*)\sum_{\{n\}}\sum_{\sigma\in S_M}\frac{1}{h_{>}^{\{p^*_\sigma\}}}e^{-ip^*_\sigma\cdot n}\,\langle\{n\}| \tag{3.26}$$

where the functions $h_{<}^{\{p\}}$ etc. should be understood as being defined in terms of the set of rapidities $\{u\}$ corresponding to the momenta $\{p(u)\}$.

Singular Bethe states. The Bethe equations (3.19) allow for solutions

$$\left\{u_1 = \frac{i}{2}, u_2 = -\frac{i}{2}, u_3, ..., u_M\right\} , \tag{3.27}$$

which is apparent when casting (3.19) into the pole-free form

$$\left(u_j + \frac{i}{2}\right)^L \prod_{k\neq j}^{M}(u_j - u_k - i) = \left(u_j - \frac{i}{2}\right)^L \prod_{k\neq j}^{M}(u_j - u_k + i) . \tag{3.28}$$

The associated states are called "singular" as their wavefunctions (3.11) and energies (3.18) diverge. One can regularise both quantities by introducing a regulator ϵ in the rapidities $u_{1,2}$ as

$$u_1 = \frac{i}{2} + \epsilon + c \cdot \epsilon^L , \qquad u_2 = -\frac{i}{2} + \epsilon , \tag{3.29}$$

where c can be systematically found by demanding that the corresponding Bethe state is an eigenstate of the Heisenberg Hamiltonian [163][1]. Alternatively, one can regularise the Bethe equations by the introduction of a twist [165] which is equivalent to the deformation parameter β we introduce below.

Primary and descendant states. The Bethe states (3.9), with wavefunctions (3.11) and characterised by the Bethe roots, do not span the full space of eigenstates. Instead they are the highest-weight (or "primary") vectors as can be shown via the algebraic Bethe ansatz [139, 166]. The remaining eigenstates ("descendants") of a multiplet can be obtained from the highest-weight state by repeatedly acting with the spin-lowering operator J_-. This effectively adds magnons with infinite rapidity $u = \infty$, i.e. vanishing momentum $p = 0 \mod 2\pi$, to a given set of Bethe roots. The Bethe states and their descendants have been shown to span the complete space of 2^L eigenstates of the Heisenberg Hamiltonian (3.3) [164].

Degeneracies in the $\mathfrak{su}(2)$ spectrum. The plethora of degeneracies in the spin-chain energy spectrum translate into degeneracies in the planar spectrum of eigenoperators to the dilatation operator (1.36). For example, the $M = 1$ cyclic eigenstates are degenerate with the vacuum $|\emptyset\rangle$. Moreover, all descendants are degenerate with their corresponding highest-weight state. A further source of degeneracies is the invariance of the Heisenberg Hamiltonian (3.3) under parity transformations $\mathcal{P}$ which reverse the ordering of spins inside spin chains, e.g.

$$\mathcal{P}\,|\uparrow\downarrow\downarrow\uparrow\downarrow\rangle = |\uparrow\downarrow\uparrow\downarrow\downarrow\rangle \ , \tag{3.30}$$

[1]Note that not all combinations of L and M allow for singular solutions, for details see [163,164].

and this gives rise to degenerate parity pairs.

Interestingly, the energies following from the Bethe equations also demonstrate a particular degeneracy relevant to the mixing problem between multi-trace operators, see e.g. [9,13,25]. For $M = 2$ this can be easily illustrated as one can in fact solve the Bethe equations in the cyclic case (for which $e^{i(p_1+p_2)} = 1$) for any length L. In terms of the momenta such solutions are given by

$$p_1 = -p_2 = \frac{2\pi n}{L-1} \ , \tag{3.31}$$

with $n \in \mathbb{Z}$ and[2] $0 < n < \frac{L-1}{2}$. Thus states with different lengths L_a, L_b, and mode numbers n_a, n_b, but equal ratios

$$\frac{n_a}{L_a - 1} = \frac{n_b}{L_b - 1} \tag{3.32}$$

have equal energies. Therefore, in the planar limit, the single-trace state corresponding to the spin-chain state

$$\big|\, \big\{ \tfrac{2\pi m}{L-1}, -\tfrac{2\pi m}{L-1} \big\} \,\big\rangle_L \tag{3.33}$$

is degenerate with the double-trace state corresponding to the product of two spin chains

$$\big|\big\{ \tfrac{2\pi \tilde{m}}{L-L_1-1}, -\tfrac{2\pi \tilde{m}}{L-L_1-1} \big\}\big\rangle_{L-L_1} \big|\emptyset\big\rangle_{L_1} \tag{3.34}$$

for $\frac{\tilde{m}}{L-L_1-1} = \frac{m}{L-1}$. While it is less straightforward to show, analogous degeneracies generally also occur for higher excitation numbers and we discuss an example with $L = 8,\ M = 3$ below.

$\mathfrak{sl}(2)$ sector. Similar to the $\mathfrak{su}(2)$ case, in the $\mathfrak{sl}(2)$ sector the one-loop dilatation operator (1.38) reduces to an integrable nearest-neighbour Hamiltonian in the planar limit that corresponds to a non-compact version of the XXX Heisenberg Hamiltonian [139]. The corresponding Bethe equations are similar to the $\mathfrak{su}(2)$ case in (3.19) and given by

$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^L \prod_{k\neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i} = 1 \ . \tag{3.35}$$

[2]Note that $p_1 = -p_2 = 0$ does not correspond to an eigenstate of the Heisenberg Hamiltonian, as it is neither a highest-weight state (which have distinct Bethe roots), nor a descendant of a Bethe state.

3.1.3 Deformed planar theory

$\mathcal{N}=4$ sYM theory is contained in a larger class of supersymmetric four-dimensional CFTs which may be reached by marginal deformations, cf. Section 1.2. One example is the β-deformed $\mathcal{N}=4$ sYM theory where planar integrability is preserved at one-loop order [70, 71]. We discuss this integrable model in the closed $\mathfrak{su}(2)$ sector in the following.

Deformed spin chain. The action of the planar piece of the dilatation operator (2.78) on $\mathfrak{su}(2)$ single-trace operators of length $L>2$ is quite similar to the undeformed action (3.7), and is given by

$$\begin{aligned} &H_\beta^{(0)}\,|n_1,n_2,\ldots\rangle_L \\ &\quad=2\sum_{j=1}^{M}\Big(2\,|\ldots,n_j,\ldots\rangle-e^{2i\beta}\,|\ldots,n_j-1,\ldots\rangle-e^{-2i\beta}\,|\ldots,n_j+1,\ldots\rangle\Big)\,. \end{aligned} \tag{3.36}$$

It can be related to an integrable deformation of the Heisenberg Hamiltonian [70, 71]

$$H_{\text{XXX}}^{\beta}=\sum_{i=1}^{L}\left[\mathbb{1}_{i,i+1}-\sigma_i^z\sigma_{i+1}^z-2e^{2i\beta}\sigma_i^-\sigma_{i+1}^+-2e^{-2i\beta}\sigma_i^+\sigma_{i+1}^-\right]\,, \tag{3.37}$$

so that the planar spectrum can still be solved using integrability.

The $SU(2)$ symmetry of the undeformed spin chain (3.3) is broken to a $U(1)$ by the β-deformation, which corresponds to a conserved total spin. Thus the number of excitations $M=L-\sum_i\sigma_i^z$ is still a good quantum number and we can arrange eigenstates into sectors with fixed M. The lowest-energy states are the same as in the undeformed theory, i.e. (3.8), and we again choose $|\emptyset\rangle$ ($M=0$) as the vacuum state. The one-excitation eigenstate is given by the undeformed Bethe state, but its energy becomes

$$E_\beta^{(0)}(p)=\varepsilon_\beta(p)=4(1-\cos(p+2\beta))\,, \tag{3.38}$$

which is no longer degenerate with the vacuum energy for the cyclic case $p=0$. For a higher number of excitations, the eigenstates are given by the Bethe ansatz (3.25) but with deformed S-matrix

$$S_\beta(p_j,p_k)=-\frac{e^{i(p_j+p_k)}e^{2i\beta}+e^{-2i\beta}-2e^{ip_k}}{e^{i(p_j+p_k)}e^{2i\beta}+e^{-2i\beta}-2e^{ip_j}}\,. \tag{3.39}$$

The Bethe equations are the same as in the undeformed theory (3.13) but with the S-matrix replaced with S_β and the trace cyclicity condition (3.16) is unchanged.

The dependence of the S-matrix on the deformation parameter can be removed by

defining the shifted momenta

$$\tilde{p}_j = p_j + 2\beta \tag{3.40}$$

so that

$$S_\beta(p_j, p_k) = -\frac{e^{i(\tilde{p}_j+\tilde{p}_k)} + 1 - 2e^{i\tilde{p}_k}}{e^{i(\tilde{p}_j+\tilde{p}_k)} + 1 - 2e^{i\tilde{p}_k}} \ . \tag{3.41}$$

Thus, if we introduce the rapidity as

$$u = \frac{1}{2}\cot\frac{\tilde{p}}{2} \ , \tag{3.42}$$

both the β-deformed S-matrix S_β and the corresponding function h_β can be defined as in the undeformed case, i.e. via (3.18) and (3.20). Note however that this parametrisation makes the parameter β manifest in the Bethe equations and cyclicity condition as

$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^L \prod_{k\neq j}^{M} \frac{u_j - u_k - i}{u_j - u_k + i} = e^{2iL\beta} \quad \text{and} \quad \prod_{j=1}^{M} \frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}} = e^{2iM\beta} \ . \tag{3.43}$$

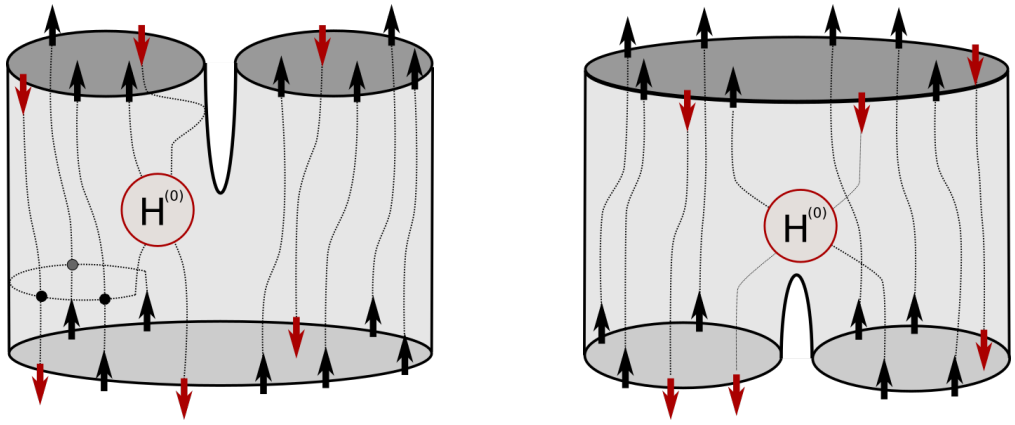
Fate of multi-trace degeneracies. One consequence of the deformation is that the degeneracy occurring in the undeformed theory between $M = 2$ single-trace (3.33) and double-trace operators (3.34) is lifted. This can be seen from the solution of the two-magnon Bethe equations, which is to first non-vanishing order in the deformation parameter given by

$$p(m, L) = \frac{2\pi n}{L-1} - \frac{2\beta^2}{L-1}\cot\left(\frac{n\pi}{L-1}\right) + \mathcal{O}(\beta^4) \ , \quad n \in \mathbb{Z} \ . \tag{3.44}$$

For generic real values of β there will be no integers $\tilde{m}$ and L_1 such that $p(\tilde{m}, L - L_1) = p(m, L)$ and hence the degeneracy between double- and single-trace operators is lifted. While we do not have a similar proof for states with more excitations, direct diagonalisation of the dilatation matrix for short operators shows that the degeneracy between single- and double-trace states is lifted in all cases of operators which were unprotected in the undeformed theory.

3.2 Perturbation theory in the undeformed theory

Mapping the problem of computing anomalous dimensions to that of computing integrable spin-chain energies proved to be an important step in solving the planar spectral problem. In this section we will make use of the planar results from inte-



(i) (ii)

Figure 3.1: The action of the non-planar dilatation operator on single-trace states (i) can be viewed as a simultaneous splitting of the spin-chain and an application of the planar Hamiltonian density, $H_j^{(0)}$, on pairs of non-adjacent spins. The action on double-trace operators (ii) is given by applying $H_j^{(0)}$ to pairs of spins, one drawn from each spin chain, while joining the two chains together.

grability to obtain one-loop non-planar corrections $E(N)$ to the scaling dimension perturbatively, i.e.

$$\Delta(g,N) = L + g^2 E(N) + \mathcal{O}(g^4) \ , \quad \text{with} \quad E(N) = \sum_{k=0}^{\infty} \frac{1}{N^k} E^{(k)} \ . \tag{3.45}$$

Doing so, we focus on operators in the scalar $\mathfrak{su}(2)$ sector. Since the subleading action $H^{\pm}$ of the dilatation operator is length-changing, there is no $\mathcal{O}(N^{-1})$ correction to planar anomalous dimensions of general single-trace operators, but instead non-planar corrections start at $\mathcal{O}(N^{-2})$. For a given Bethe state $|\{p\}\rangle$ with planar energy $E^{(0)}(\{p\})$ these leading contributions can be computed via Rayleigh-Schrödinger perturbation theory as

$$E^{(2)}(\{p\}) \sim \sum_{\{I\}} \frac{\langle\{p\}|H^-|\{I\}\rangle \langle\{I\}|H^+|\{p\}\rangle}{E^{(0)}(\{p\}) - E^{(0)}(\{I\})} \ , \tag{3.46}$$

up to a normalisation factor, which we make precise later. The sum in (3.46) is over intermediate double-trace states $\{I\}$. In Sections 3.2.1-3.2.3 we study the action of the non-planar dilatation operator on the planar eigenstates, schematically shown in Figure 3.1, and subsequent overlaps with other Bethe states, and give examples for non-planar corrections in Section 3.2.4. However, this procedure fails when there are degeneracies in the planar spectrum. In particular, when there is a degeneracy between a single- and a double-trace state, there can be $\mathcal{O}(N^{-1})$-corrections. We will give an explicit example of such a case in Section 3.2.4.

3.2.1 Matrix elements from ordered partitions

Matrix elements of H^-. We consider first the action of H^- on a double-trace operator corresponding to the product of a length L_q Bethe state $|\{q\}\rangle$ with Q excitations and a length L_r state $|\{r\}\rangle$ with R excitations. There are L_qL_r terms corresponding to the action of the dilatation operator on each pair of sites of the two spin chains. The terms where it acts on a Z field at the i-th site of one of them can be rewritten using the Bethe equations, so that they become equivalent to the action on a Z at the first site. This can be seen by gathering all the terms in the state (3.9) with a Z field at the i-th site and using the cyclicity and Bethe equation to write them with the Z field at the first position:

$$\sum_{l=0}^{M} \sum_{\substack{1\le n_1<\ldots<n_l\le i-1\\ i+1\le n_{l+1}<\ldots<n_M\le L}} \psi^{\{p\}}_{\{n\}} |n_1,\ldots,n_l\rangle_{i-1}\otimes|Z\rangle\otimes|n_{l+1},\ldots,n_M\rangle_{L-i} = \sum_{2\le n_1<\ldots<n_M\le L} \psi^{\{p\}}_{\{n\}}\ |Z\rangle\otimes|\{n\}\rangle_{L-1}\ . \tag{3.47}$$

Analogously, the action on an X field at the i-th site can be rewritten as the action on the same chain with the X field placed at the first site by realising that

$$\sum_{l=1}^{M} \sum_{\substack{1\le n_1<\ldots<n_l=i\\ i=n_l<\ldots<n_M\le L}} \psi^{\{p\}}_{n_1,\ldots,n_l=i,\ldots,n_M} |n_1,\ldots,n_l=i,\ldots,n_M\rangle = \sum_{1=n_1<\ldots<n_M\le L} \psi^{\{p\}}_{\{n\}}\ |\{n\}\rangle\ . \tag{3.48}$$

With these and similar simplifications the action of H^- on the double-trace state can be written as

$$\begin{aligned} H^-|\{q\}\rangle|\{r\}\rangle = 2L_qL_r\Bigg(&\sum_{\substack{1\le m_1<\ldots<m_Q=L_q\\ 2\le n_1<\ldots<n_R\le L_r}} \psi^{\{q\}}_{\{m\}}\psi^{\{r\}}_{\{n\}} |\{m\}_{\hat{Q}}\rangle\otimes|[X,Z]\rangle\otimes|\{n+L_q\}\rangle \\ &+ \sum_{\substack{1\le m_1<\ldots<m_Q\le L_q-1\\ 1=n_1<\ldots<n_R\le L_r}} \psi^{\{q\}}_{\{m\}}\psi^{\{r\}}_{\{n\}} |\{m\}\rangle\otimes|[Z,X]\rangle\otimes|\{n+L_q\}_{\hat{1}}\rangle \\ &+ \{\text{terms with } q\rightleftarrows r\}\Bigg)\ , \end{aligned} \tag{3.49}$$

where the terms on the right-hand side all correspond to single-trace operators.

The overlap with a dual state $\langle\{p\}|$, of length $L_p = L_q + L_r$ and with $P = Q + R$

excitations, can then be computed

$$\begin{aligned}
&\langle\{p\}|H^-|\{q\}\rangle|\{r\}\rangle \\
&= 2L_qL_r\,\mathcal{N}(p^*,q,r)\sum_{\rho,\sigma,\tau}\frac{1}{h_>^{\{p_\rho^*\}}h_<^{\{q_\sigma\}}h_<^{\{r_\tau\}}} \\
&\quad\times\Bigg(\delta_{Q\neq 0}(e^{ip^*_{\rho(Q)}}-1)e^{iL_qq_{\sigma(Q)}}e^{-i(L_q+1)(p^*_\rho)_Q^{Q+R}}P_{L_q}\Big(\{q_\sigma-p^*_\rho\}_1^{Q-1}\Big)P_{L_r}\Big(r_\tau-\{p^*_\rho\}_{Q+1}^{Q+R}\Big) \\
&\qquad+\delta_{Q\neq 0}(1-e^{ip^*_{\rho(R+1)}})e^{-i(L_r+1)(p^*_\rho)_{R+1}^{Q+R}}P_{L_q}\Big(\{q_\sigma\}_2^{Q}-\{p^*_\rho\}_{R+2}^{R+Q}\Big)P_{L_r}\Big(r_\tau-\{p^*_\rho\}_1^{R}\Big) \\
&\qquad+\{\text{terms with } q\rightleftarrows r\}\Bigg)\,,
\end{aligned}\tag{3.50}$$

where we define the sets $\{t_\lambda\}_a^b := \{t_{\lambda(a)},\dots,t_{\lambda(b)}\}$ and denote products of exponentials over such sets using the notation $e^{iL(t_\lambda)_a^b} := \prod_{i=a}^b e^{iLt_{\lambda(i)}}$. The factors of $P_L(z)$ in (3.50) correspond to the geometric sums of exponentials in the wavefunctions which can be rewritten as sums over ordered partitions

$$P_L(z) := \sum_{1\leq n_1<\ldots<n_{|z|}\leq L-1} e^{iz\cdot n} = \sum_{l=0}^{|z|}\prod_{k=1}^{l}\frac{1}{e^{-i\sum_{j=k}^l z_j}-1}\prod_{k=l+1}^{|z|}\frac{e^{iz_kL}}{e^{i\sum_{j=l+1}^k z_j}-1}\,.\tag{3.51}$$

Matrix elements of H^+. Using these notations we can write a similar expression for overlaps of H^+ as sums over ordered partitions:

$$\begin{aligned}
&\langle\{r\}|\langle\{q\}|H^+|\{p\}\rangle \\
&= 2L_p\,\mathcal{N}_+(p,q^*,r^*)\sum_{\rho,\sigma,\tau}\frac{1}{h_<^{\{p_\rho\}}h_>^{\{q^*_\sigma\}}h_>^{\{r^*_\tau\}}} \\
&\quad\times\Bigg(\delta_{Q\neq 0}(e^{iq^*_{\sigma(Q)}}-1)e^{-iL_qq^*_{\sigma(Q)}}e^{i(L_q-1)\{p_\rho\}_Q^{Q+R}}P_{L_q-1}\Big(\{p_\rho-q^*_\sigma\}_1^{Q-1}\Big)P_{L_r+1}\Big(\{p_\rho\}_{Q+1}^{Q+R}-r^*_\tau\Big) \\
&\qquad+\delta_{Q\neq 0}(1-e^{iq^*_{\sigma(1)}})e^{i(L_r+1)(p_\rho)_{R+1}^{R+Q}}P_{L_q-1}\Big(\{p_\rho\}_{R+2}^{R+Q}-\{q^*_\sigma\}_2^{Q}\Big)P_{L_r+1}\Big(\{p_\rho\}_1^{R}-r^*_\tau\Big) \\
&\qquad+\{\text{terms with } q\rightleftarrows r\}\Bigg)\,.
\end{aligned}\tag{3.52}$$

The normalisation in this case is defined slightly differently with $\mathcal{N}_+ = \mathcal{N}/S$, where S is a symmetry factor that equals 2 when the states in the double trace are equal and 1 otherwise.

Carrying out the geometric sums via (3.51) makes these formulas useful for analysing states of arbitrary lengths. However, while these expressions are reasonably compact, they involve sums over permutations for each of the sets of external momenta and so they quickly become impractical as the number of excitations grows. The same growth is known from the computation of spin-chain scalar products in the coordi-

nate Bethe ansatz and by making use of known results in this framework we can find further simplifications.

3.2.2 Matrix elements from spin-chain scalar products

The scalar product of two Bethe states

$$\langle\{l\}|\{k\}\rangle_L = \sum_{1\leq n_1<\ldots n_{|k|}\leq L} \psi^{*\{l\}}_{\{n\}}\psi^{\{k\}}_{\{n\}} = \mathcal{N}(k,l^*)\sum_{\rho,\sigma}\frac{P_{L+1}(k_\rho - l^*_\sigma)}{h_>^{\{l^*_\sigma\}}h_<^{\{k_\rho\}}} \tag{3.53}$$

involves double sums over permutations and so is generally complicated to evaluate. Fortunately, there are well-known formulas for such scalar products which were developed in the algebraic Bethe ansatz approach to integrable spin chains (see Appendix B for a brief review). In the case where both sets of momenta $\{k\}$ and $\{l\}$ do not satisfy the Bethe equations (i.e. they are off-shell), the scalar product can be written as a sum over partitions of the sets of momenta into subsets of equal cardinality [167], cf. (B.14). Similar simplifications can be used to rewrite the expressions (3.50) and (3.52). Each term in the formulas for the overlaps non-trivially involves one momentum of an excitation from the single-trace operator, which we label p_j, and one excitation momentum from the double-trace operator, i.e. from either $\{q\}$ or $\{r\}$, which we label as q_i or r_i. The remaining momenta are simply contracted using a rescaled spin-chain scalar product

$$(\{l\}|\{k\})_L := \frac{\langle\{l\}|\{k\}\rangle_L}{\mathcal{N}(k,l^*)} \ . \tag{3.54}$$

We can thus write the overlaps (3.50) and (3.52) in terms of the off-shell scalar products by splitting the single-trace excitation momenta into three subsets, $\{p\} = s\cup t\cup\{p_j\}$, with the cardinality of s equal to that of $\{q\}_{\hat{i}}$ (or $\{r\}_{\hat{i}}$) and the cardinality of t equal to that of $\{r\}$ (resp. $\{q\}$). In terms of off-shell scalar products, the overlap of H^- can then be written as a sum over all such splittings

$$\begin{aligned}
&\langle\{p\}|H^-|\{q\}\rangle|\{r\}\rangle \\
&\quad = 2L_qL_r\,\mathcal{N}(p^*,q,r)\sum_{\substack{i,j\\ s\cup t=\{p\}_{\hat{j}}}}\frac{e^{ip^*_j}-1}{h^{q_iq_{\hat{i}}}}\big[s_{\circlearrowleft}^{L_q+1\,*} - t_{\circlearrowleft}^{L_r+1\,*}\big](s|\{q\}_{\hat{i}})_{L_q-1}(t|\{r\})_{L_r-1} \\
&\qquad + \{\text{terms with } q\rightleftarrows r\}
\end{aligned} \tag{3.55}$$

and that of H^+ as

$$\begin{aligned}\langle\{r\}|\langle\{q\}|H^+|\{p\}\rangle \\ &= 2L_p\,\mathcal{N}_+(p,q^*,r^*)\sum_{\substack{i,j\\ s\cup t=\{p\}_{\hat{j}}}}\frac{e^{iq_i^*}-1}{h^{q_i^*q_i^*}}\big[s_{j\circlearrowleft}^{L_q-1}-t_{j\circlearrowleft}^{L_r+1}\big](\{q\}_{\hat{i}}|s)_{L_q-2}(\{r\}|t)_{L_r}\\ &\quad+\{\text{terms with } q\rightleftarrows r\}\ .\end{aligned}\tag{3.56}$$

In addition to the scalar products of Bethe states these expressions involve factors of the form $(e^{ip}-1)$, which are essentially the same as arise in the planar dilatation operator, and ordering factors for which we introduced the notation

$$s_{j\circlearrowleft}^{L}=\frac{e^{-iLs}}{h^{p_jt}h^{sp_j}h^{st}}\ ,\quad t_{j\circlearrowleft}^{L}=\frac{e^{-iLt}}{h^{p_js}h^{tp_j}h^{ts}}\ .\tag{3.57}$$

These terms account for the phase acquired by the p_j magnon as it is shifted around the chain before being contracted with a magnon from the double-trace operator. For each configuration there are two different ways to carry out this reordering and the overlap is a superposition of both.

Spin-chain scalar products have previously appeared in the context of $\mathcal{N}=4$ sYM theory in the computation of structure constants. In the all-order hexagon approach [20], structure constants are written as sums over partitions of the magnon excitations and it was noted that this formulation is related to the scalar-product formula of Korepin [167]. It is therefore convenient to use a tree-level version of the hexagon formulation of scalar products

$$(\{l(v)\}|\{k(u)\})=(-1)^M\prod_{j=1}^{M}(u_j+i/2)(v_j^*-i/2)\sum_{\substack{\alpha\cup\bar{\alpha}=\{k\}\\ \beta\cup\bar{\beta}=\{l^*\}}}\frac{e^{iL(\bar{\alpha}-\bar{\beta})}G(\alpha,\beta)G(\bar{\beta},\bar{\alpha})}{h^{\alpha\bar{\alpha}}h^{\bar{\beta}\beta}}\ ,\tag{3.58}$$

where

$$G(\alpha(u),\beta(v^*))=\frac{\det\left[\frac{i}{(u_j-v_k^*)(u_j-v_k^*+i)}\right]\prod_{j,k}(u_j-v_k^*+i)}{\prod_{j<k}(u_k-u_j)(v_j^*-v_k^*)}\ ,\tag{3.59}$$

in order to rewrite the overlaps (3.55) and (3.56).

3.2.3 A hexagon-like formulation

Hexagons in structure constants. In the previous section we obtained the non-planar dilatation operator matrix elements as sums over partitions of the rapidities, in a way that is reminiscent of the hexagon formulation of three-point correlation functions [20]. In that context, the partitions of the rapidities arise naturally in

the large-volume regime where the correlation function is broken down to its simplest building blocks, the hexagon form factors. Crucially, these form factors satisfy a set of axioms which, together with the diagonal symmetries and some educated guesswork, can be used to obtain an all-loop description of structure constants. A particular feature of the hexagon is its conical defect which is associated with the existence of three asymptotic regions and corresponds to a monodromy composed by three crossing operations. In the context of non-planar overlaps between a single-trace and a double-trace operator, a similar role seems to be played by the three distinct traces. In this section we investigate the properties of the objects arising from the action of H^+ and H^- and find that they satisfy some of the form factor axioms appearing in the context of correlation functions.

The sum over determinants occurring in our rewriting of off-shell scalar products (3.58) can be found in a straightforward way from the hexagonalisation of three-point functions [20]. To be precise, we consider the three-point function of two unprotected operators in the $SU(2)$ sector, one with X excitations and the other with $\bar{X}$, and one rotated half-BPS operator. The X and $\bar{X}$ fields must be Wick contracted at tree level in order to produce a non-vanishing contribution. If there are l Wick contractions between the excited operators, then the structure constant is

$$C^{\bar{X}|X}_{\{p\}|\{q\}} \propto \sum_{\substack{\alpha\cup\bar{\alpha}=\{p\}\\ \beta\cup\bar{\beta}=\{q\}}} \omega_l(\alpha,\bar{\alpha})\omega_{L_q-l}(\beta,\bar{\beta})\mathcal{H}(\alpha|\beta)\mathcal{H}(\bar{\beta}|\bar{\alpha}) \ , \tag{3.60}$$

with the splitting factor defined as

$$\omega_l(\alpha,\bar{\alpha}) = e^{i\bar{\alpha}l} \prod_{\substack{u_i\in\bar{\alpha},u_j\in\alpha\\ i<j}} S(u_i,u_j) \ . \tag{3.61}$$

The hexagon function $\mathcal{H}$ in this particular configuration is simply related to our determinant expression (3.59) by

$$\mathcal{H}(\alpha|\beta) = h^{\alpha}_{<}h^{\beta}_{<}G(\alpha,\beta) \ . \tag{3.62}$$

The hexagon description of three-point functions allows the evaluation of general configurations where all three operators are excited. If we now let two of the operators have $\bar{X}$ excitations, while the other is composed of X fields, then the structure constant becomes

$$C^{\bar{X}|X|\bar{X}}_{\{p\}|\{q\}|\{r\}} \propto \sum_{\substack{\alpha\cup\bar{\alpha}=\{p\}\\ \beta\cup\bar{\beta}=\{q\}\\ \gamma\cup\bar{\gamma}=\{r\}}} \omega_{l_{pq}}(\alpha,\bar{\alpha})\omega_{l_{qr}}(\beta,\bar{\beta})\omega_{l_{pr}}(\gamma,\bar{\gamma})\mathcal{H}(\alpha|\beta|\gamma)\mathcal{H}(\bar{\gamma}|\bar{\beta}|\bar{\alpha}) \ , \tag{3.63}$$

where l_{ij} denotes the number of Wick contractions between operators i and j at tree level and the sum over partitions is further restricted by the fact that $\mathcal{H}(\alpha|\beta|\gamma)$ is non-vanishing only when the cardinality of β matches that of $\alpha \cup \gamma$. It is interesting to note that while (3.63) is given as a sum over partitions of three sets of rapidities, a naive tree-level evaluation would give rise to geometric sums naturally yielding a sum over five partitions

$$C^{\bar{X}|X|\bar{X}}_{\{p\}|\{q\}|\{r\}} \propto \sum_{\substack{\alpha\cup\bar{\alpha}=\{p\}\\ \gamma\cup\bar{\gamma}=\{r\}}} \sum_{\substack{s\cup t=\{q\}\\ \beta\cup\bar{\beta}=s\\ \delta\cup\bar{\delta}=t}} \frac{e^{i(\bar{\alpha}-\bar{\beta})l_{12}} e^{i(\bar{\gamma}-\bar{\delta})l_{23}} e^{isl_{12}}}{h^{\alpha\bar{\alpha}} h^{\gamma\bar{\gamma}} h^{\bar{\delta}\delta} h^{\bar{\beta}\beta} h^{ts}} G(\alpha,\beta) G(\bar{\beta},\bar{\alpha}) G(\gamma,\delta) G(\bar{\delta},\bar{\gamma}) \ . \tag{3.64}$$

The equivalence of these descriptions follows from the tree-level relation

$$\mathcal{H}(\alpha|\beta|\gamma) = h^{\alpha}_{<} h^{\beta}_{<} h^{\gamma}_{<} \sum_{\mu\cup\nu=\beta} \frac{G(\alpha,\mu) G(\nu,\gamma)}{h^{\mu\nu}} \ . \tag{3.65}$$

Note that, computationally speaking, (3.63) is not necessarily a more efficient version of (3.64) as the objects $\mathcal{H}(\alpha|\beta|\gamma)$ do not have a known compact determinant description. Nevertheless, there is a conceptual advantage due to the fact that hexagon functions can be bootstrapped: their known analytical properties allow for an integrability-based framework to directly compute them at any coupling, thus circumventing the perturbative evaluation of three-point functions by methods such as Feynman diagrams. Specifically, the hexagon functions obey the Watson equation

$$\mathcal{H}(\ldots|\ldots,\beta_i,\beta_{i+1},\ldots|\ldots) = S(\beta_i,\beta_{i+1}) \mathcal{H}(\ldots|\ldots,\beta_{i+1},\beta_i\ldots|\ldots) \ , \tag{3.66}$$

which holds similarly for an exchange of excitations in the other edges, and they also satisfy the decoupling conditions

$$\begin{aligned} -i \operatorname*{Res}_{\alpha_{|\alpha|}=\beta_1} \left[\mathcal{H}(\ldots,\alpha_{|\alpha|}|\beta_1,\ldots|\ldots)\right] &= \mathcal{H}(\ldots,\alpha_{|\alpha|-1}|\beta_2,\ldots|\ldots) \ , \\ -i \operatorname*{Res}_{\beta_{|\beta|}=\gamma_1} \left[\mathcal{H}(\ldots|\ldots,\beta_{|\beta|}|\gamma_1,\ldots)\right] &= \mathcal{H}(\ldots,|\ldots,\beta_{|\beta|-1}|\gamma_2,\ldots) \ . \end{aligned} \tag{3.67}$$

Together with the diagonal symmetries of three-point functions, these form-factor axioms allow the determination of the hexagon functions at any value of the coupling [20].

Hexagons in non-planar overlaps. With this in mind, we can attempt a similar rewriting of the dilatation-operator overlaps. It is useful to work with normalised spin-chain states where we divide by the norms of on-shell Bethe states $\|\{p\}\| := \sqrt{\langle\{p\}|\{p\}\rangle}$. These can be conveniently calculated using the Gaudin formula (B.16)

which for the coordinate Bethe-ansatz normalisation is

$$\|\{p(u)\}\|^2 = (-1)^M \prod_j (u_j + i/2)(u_j - i/2) \det \partial_u \phi(u) \tag{3.68}$$

with ϕ defined in (3.13). This can be combined with the normalisation factors of the overlaps to define a new normalisation factor

$$\widetilde{\mathcal{N}}(p(u), q(v), r(w)) = \frac{\mathcal{N}(p,q,r)}{h_<^{\{p\}} h_<^{\{q\}} h_<^{\{r\}} \sqrt{\det \partial_u \phi(u) \det \partial_v \phi(v) \det \partial_w \phi(w)}} \ , \tag{3.69}$$

where we used the fact that solutions of the Bethe equations are invariant under complex conjugation, e.g. $\{u^*\} = \{u\}$ [168], and the cyclicity condition to simplify the expressions[3]. The overlap with normalised external states can then be written as

$$\begin{aligned} &V^-(q,r;p) \\ &= 2L_q L_r \widetilde{\mathcal{N}}(p,q,r) \sum_{\substack{\alpha \cup \bar{\alpha} = \{q\} \\ \beta \cup \bar{\beta} = \{p\} \\ \gamma \cup \bar{\gamma} = \{r\}}} \omega_{L_q}(\alpha, \bar{\alpha}) \omega_{L_r}(\beta, \bar{\beta}) \omega_0(\gamma, \bar{\gamma}) \\ &\times \Big[\mathcal{H}(\alpha|\beta|\gamma) \Big(\mathcal{H}_1^-(\bar{\gamma}|\bar{\beta}|\bar{\alpha}) + \mathcal{H}_2^-(\bar{\gamma}|\bar{\beta}|\bar{\alpha}) \Big) + \Big(\mathcal{H}_1^-(\alpha|\beta|\gamma) + \mathcal{H}_2^-(\alpha|\beta|\gamma) \Big) \mathcal{H}(\bar{\gamma}|\bar{\beta}|\bar{\alpha}) \Big] \ , \end{aligned} \tag{3.70}$$

where $\mathcal{H}$ is the same as in (3.65), and we define the new functions $\mathcal{H}_i^-$ as

$$\begin{aligned} \mathcal{H}_1^-(\alpha|\beta|\gamma) &= h_<^\alpha h_<^\beta h_<^\gamma \sum_{\substack{i,j \\ \mu \cup \nu = \beta_{\hat{j}}}} \frac{(e^{i\beta_j} - 1)(e^{-i\beta_j} - 1)(e^{i\alpha_i} - 1) G(\alpha_{\hat{i}}, \mu) G(\nu, \gamma)}{e^{i(\mu + \gamma - \alpha_{\hat{i}} - \nu)} h^{\alpha_i \alpha_i} h^{\mu \beta_j} h^{\beta_j \nu} h^{\mu \nu}} \ , \\ \mathcal{H}_2^-(\alpha|\beta|\gamma) &= h_<^\alpha h_<^\beta h_<^\gamma \sum_{\substack{i,j \\ \mu \cup \nu = \beta_{\hat{j}}}} \frac{(e^{i\beta_j} - 1)(e^{-i\beta_j} - 1)(e^{-i\gamma_i} - 1) G(\alpha, \mu) G(\nu, \gamma_{\hat{i}})}{e^{i(\mu + \gamma_i - \alpha - \nu)} h^{\gamma_i \gamma_i} h^{\mu \beta_j} h^{\beta_j \nu} h^{\mu \nu}} \ . \end{aligned} \tag{3.71}$$

We remind the reader that $\mu_{\hat{k}}$ denotes the set of rapidities μ without μ_k, following the notation introduced in (3.24), while the short-hand notations for products are defined in (3.23). This decomposition of the overlap in (3.70) seems to fit the splitting of Figure 3.1 (ii) particularly well. By cutting the pair of pants depicted in that figure, one would naively expect the side facing away to be represented by the original hexagon $\mathcal{H}$ of (3.65), while the side facing forward should lead to a new structure containing the action of the commutator from the dilatation operator. Similarly, the

[3]There is a potential ambiguity in our simplifications arising from square roots of S-matrices in $\mathcal{N}$. In principle, there exist combinations of rapidities such that products of S-matrices cross the square-root branch cut resulting in additional minus signs in the normalisation. Nevertheless, the same ambiguity seems to appear in the Gaudin norm and so these signs cancel. Moreover, the signs appear symmetrically in the overlaps of H^- and H^+ and thus certainly cancel in the calculation of energies.

overlap V^+ can be rewritten as

$$\begin{aligned}
&V^+(p;q,r)\\
&= 2L_p\widetilde{\mathcal{N}}_+(p,q,r)\sum_{\substack{\alpha\cup\bar{\alpha}=\{q\}\\ \beta\cup\bar{\beta}=\{p\}\\ \gamma\cup\bar{\gamma}=\{r\}}}\omega_{L_q}(\alpha,\bar{\alpha})\omega_{L_r}(\beta,\bar{\beta})\omega_0(\gamma,\bar{\gamma})\\
&\quad\times\Big[\mathcal{H}_0^+(\alpha|\beta|\gamma)\Big(\mathcal{H}_1^+(\bar{\gamma}|\bar{\beta}|\bar{\alpha})+\mathcal{H}_2^+(\bar{\gamma}|\bar{\beta}|\bar{\alpha})\Big)\\
&\qquad\qquad+\Big(\mathcal{H}_1^+(\alpha|\beta|\gamma)+\mathcal{H}_2^+(\alpha|\beta|\gamma)\Big)\mathcal{H}_0^+(\bar{\gamma}|\bar{\beta}|\bar{\alpha})\Big]\ ,
\end{aligned}\tag{3.72}$$

where the normalisation is now $\widetilde{\mathcal{N}}_+ = \widetilde{\mathcal{N}}/S$, with S a symmetry factor that equals 2 when the states in the double-trace are the same, and 1 otherwise, and we have further defined the functions $\mathcal{H}_i^+$

$$\begin{aligned}
\mathcal{H}_0^+(\alpha|\beta|\gamma) &= h_<^\alpha h_<^\beta h_<^\gamma\sum_{\mu\cup\nu=\beta}e^{i(\alpha-\mu)}\frac{G(\alpha,\mu)G(\nu,\gamma)}{h^{\mu\nu}}\ ,\\
\mathcal{H}_1^+(\alpha|\beta|\gamma) &= h_<^\alpha h_<^\beta h_<^\gamma\, e^{i(\alpha+\gamma-\beta)}\sum_{\substack{i,j\\ \mu\cup\nu=\beta_{\hat{j}}}}\frac{(e^{i\alpha_i}-1)(e^{-i\alpha_i}-1)(e^{-i\beta_j}-1)G(\alpha_{\hat{i}},\mu)G(\nu,\gamma)}{e^{i(\mu-\alpha_{\hat{i}})}h^{\alpha_i\alpha_{\hat{i}}}h^{\mu\beta_j}h^{\beta_j\nu}h^{\mu\nu}}\ ,\\
\mathcal{H}_2^+(\alpha|\beta|\gamma) &= h_<^\alpha h_<^\beta h_<^\gamma\sum_{\substack{i,j\\ \mu\cup\nu=\beta_{\hat{j}}}}\frac{(e^{i\gamma_i}-1)(e^{-i\gamma_i}-1)(e^{i\beta_j}-1)G(\alpha,\mu)G(\nu,\gamma_{\hat{i}})}{e^{i(\gamma_i-\nu)}h^{\gamma_i\gamma_i}h^{\mu\beta_j}h^{\beta_j\nu}h^{\mu\nu}}\ .
\end{aligned}\tag{3.73}$$

Unfortunately, in this case we are not able to write any of the new objects in terms of the original hexagon function $\mathcal{H}$, since the partitions of the rapidities β into μ and ν appear with a distinct structure. The decomposition is however very similar to that of V^-, and seems to match once again the intuition derived from Figure 3.1 (i), with the cutting producing a product between a simpler structure with a more complex ones. Although these formulas appear quite involved, once the rapidities are known they can be straightforwardly evaluated by using a suitable computer program such as Mathematica.

While the expressions (3.70) and (3.72) are a post hoc massaging of the expressions in (3.55) and (3.56), when written in this form they clearly resemble the formulas for structure constants. Importantly, the new objects $\mathcal{H}_i^+$ and $\mathcal{H}_i^-$ also obey the Watson equations (3.66) and decoupling conditions (3.67), which follows from the analogous properties of the object $\mathcal{H}(\alpha|\beta)$ defined in (3.62). This is non-trivial as it occurs only for certain functions of the rapidities in the summands of (3.71) and (3.73) and hints towards the possibility that non-planar dilatation-operator overlaps can be written in terms of hexagon-like objects and potentially bootstrapped, even at higher orders in perturbation theory. Alternatively, and given that the configuration of traces corresponds to an extremal setup, it might also be natural to consider a decomposition

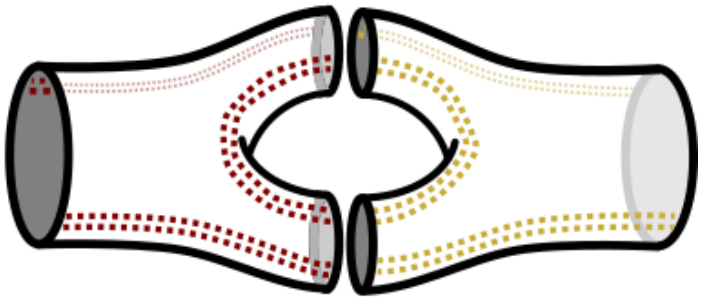


Figure 3.2: Both V^+ and V^- can be seen as a pair of pants where the asymptotic regions correspond to the three distinct traces involved in the overlap. We have found that each of them can be decomposed into hexagon-like objects satisfying the Watson and decoupling conditions. By glueing them together one can reconstruct the torus, thus finding the non-planar corrections to two-point functions.

into octagon form factors [169, 170]. One can massage the expressions in (3.70) and (3.72) such that they are of the form

$$V^{\pm} \propto \sum_{\alpha \cup \bar{\alpha} = \{p\}} \omega_{L_r}(\alpha, \bar{\alpha})\, \mathcal{O}(\{q\}|\alpha|\{r\}|\bar{\alpha}) \ , \tag{3.74}$$

and while there is no benefit in such a rewriting at tree-level, the octagon might be better suited for a higher-loop bootstrap.

The matrix elements $V^{\pm}$ are relevant for non-planar corrections of the one-loop anomalous dimensions according to (3.46). This expression contains a sum over intermediate double-trace operators, which has a natural representation of cutting a torus into two pairs of pants. The fact that the overlaps themselves have a decomposition into hexagon-like objects therefore seems to indicate a possible tessellation of the torus as depicted in Figure 3.2. The decomposition of each pair of pants into octagons would arise from cutting only along the seams that are adjacent to the single-trace operator, while keeping the zero-length bridges between the components of the double-trace operator intact.

There is an implicit notion of crossing that comes with the decoupling condition. It is natural to imagine that, once such an operation is defined, the excitations can be moved around, so that we relate the hexagon-like objects to a single function where all rapidities are on the same edge. It is upon crossing of the excitations in (3.67) to the same edge that a particle-antiparticle pair $\bar{X}(u^{2\gamma})X(u)$ can form in a manifest way and decouple from the corresponding form factor. Such a formulation of $\mathcal{H}_i^+$ and $\mathcal{H}_i^-$ with all excitations on the same edge would also be the ideal setup for implementing a bootstrap of those objects. Unfortunately, crossing operations do not commute with the perturbative expansion, and since our one-loop analysis gives access only to the more complicated form of these objects, we were not able to further explore the possibility of such a bootstrap programme.

3.2.4 Anomalous dimensions from overlaps

The main goal in calculating the above overlaps is to perturbatively compute the leading non-planar correction to operator anomalous dimensions using first-order Rayleigh–Schrödinger perturbation theory. We denote the planar energies as $E^{(0)}$ and their non-planar corrections at order N^{-k} as $E^{(k)}$. Given a single-trace operator characterised by momenta $\{p(u)\}$, satisfying the Bethe equations, and by its planar energy $E^{(0)}(\{p\})$, the non-planar correction is, cf. (3.46),

$$E^{(2)}(\{p\}) = \sum_{\{I\}} \frac{V^-(p;I)V^+(I;p)}{E^{(0)}(\{p\}) - E^{(0)}(\{I\})} \ . \tag{3.75}$$

The sum over I is taken over all intermediate double-trace states

$$|I\rangle = |\{q\}\rangle_{L_q} |\{r\}\rangle_{L_r} \tag{3.76}$$

where we must sum over all lengths $1 < L_q < L_p - 1$ and for each length sum also over all solutions $\{q\}$, $\{r\}$ of the Bethe equations corresponding to operators of lengths L_q and L_r and planar energy $E^{(0)}(\{I\}) = E^{(0)}(\{q\}) + E^{(0)}(\{r\})$.

A length-six example. As a simple example let us consider the unprotected operators of length six in the $[2,2,2]$ $SO(6)$ representation. There are two single-trace operators with planar energies and rapidities given by

$$\begin{aligned} E^{(0)}_{(6,2a)} &= 2(5+\sqrt{5}) \ , & u_{(6,2a),1} &= -u_{(6,2a),2} = \frac{1}{2}\sqrt{1-\frac{2}{\sqrt{5}}} \ , \\ E^{(0)}_{(6,2b)} &= 2(5-\sqrt{5}) \ , & u_{(6,2b),1} &= -u_{(6,2b),2} = \frac{1}{2}\sqrt{1+\frac{2}{\sqrt{5}}} \ , \end{aligned} \tag{3.77}$$

both of which mix with the double-trace operator with

$$E^{(0)}_{(4,2)} = 12 \ , \qquad u_{(4,2),1} = -u_{(4,2),2} = \frac{1}{2\sqrt{3}} \ . \tag{3.78}$$

The overlaps can be simply found from the general formulas (3.70) and (3.72)

$$\begin{aligned} &V^-(u_{(6,2a)};u_{(4,2)},\emptyset) = \frac{4}{3}(5+3\sqrt{5}) \ , \qquad V^-(u_{(6,2b)};u_{(4,2)},\emptyset) = \frac{4}{3}(5-3\sqrt{5}) \ , \\ &V^+(u_{(4,2)},\emptyset;u_{(6,2a)}) = V^+(u_{(4,2)},\emptyset;u_{(6,2b)}) = 6\sqrt{2} \ . \end{aligned} \tag{3.79}$$

The resulting non-planar corrections are

$$E^{(2)}_{(6,2a)} = 8(5+2\sqrt{5}) \quad \text{and} \quad E^{(2)}_{(6,2b)} = 8(5-2\sqrt{5}) \ . \tag{3.80}$$

These results are in agreement with those found by direct calculation [171], and also follow from directly diagonalising the dilatation operator [13].

Solution for $M = 2$ and BMN limit. If there are only two magnons in a cyclic spin chain, one can solve the Bethe equations via (3.31). Given such a complete set of solutions, it is possible to numerically carry out the sum over intermediate states so that one can quite efficiently compute the corrections to energies even for long states, e.g. $n = 1$ for $L = 100, 250, 400$, which to six digits gives

$$E^{(2)}_{L=\{100,250,400\}} = L^2\{0.758732, 0.770021, 0.772582\} \ . \tag{3.81}$$

From this and similar numerical examples it can be seen that the corrections to the energies of long operators scale as L^2/N^2. This is essentially the well-known BMN limit [172] where one considers operators with large R-charge J. The non-planar corrections to two-magnon states in the BMN limit were computed in [6, 25], see also [9, 173], and shown to be

$$\Delta_n = L + g'\Big[16\pi n^2 + g_2^2\left(\frac{1}{3} + \frac{35}{8\pi^2 n^2}\right)\Big] \ . \tag{3.82}$$

It is straightforward to check that our general expressions reproduce this result by substituting the large-L solution for the two-magnon rapidities,

$$u_{n,1} = -u_{n,2} = \frac{L-1}{2\pi n} + \mathcal{O}(L^{-1}) \ , \tag{3.83}$$

into (3.70) and (3.72) and taking the large-L limit. Doing so, we must consider the overlaps with all double-trace operators consisting of a vacuum state of length $(1-r)L$ and a two-magnon state with rapidities $u_{m,1} = -u_{m,2} = \frac{rL-1}{2\pi m}$. Following [9] we then expand in L, sum over $m = 0, \ldots, \infty$ and approximate the sum over intermediate lengths by an integral over r from 0 to 1. At leading order in $J = L-2$ this reproduces (3.82), while at subleading order we find the same result but with J replaced with $L - 1 = J + 1$ which is the natural parameter from the perspective of the Bethe equations.

Example for a degenerate parity pair. It is naturally interesting to consider higher numbers of excitations. For example at $L = 7$ with three excitations, i.e. for states in the $[3, 1, 3]$ representation, we have two single-trace operators with planar dimensions $E^{(0)}_{(7,3a/b)} = 10$. Due to the degeneracy of the states, a naive application of relation (3.75) will fail as it is not clear which linear combination of the Bethe states to use as planar eigenstates. We may use the fact that the two degenerate states are distinguished by their transformation under the parity operation (3.30) [13], which

commutes with the complete non-planar dilatation operator. Thus the non-planar eigenoperators must have definite parity, and consequently so do their planar limits. The rapidities for the two $L = 7$ and $M = 3$ Bethe solutions $u_{(7,3a)}$ and $u_{(7,3b)}$ can be easily found using the method (and Mathematica programme) of [174]. They can be seen to transform into each other under parity, which acts on finite rapidities by $u_i \to -u_i$, while rapidities at infinity are left invariant. The two parity eigenstates can then be formed from the corresponding Bethe eigenstates as

$$|\pm\rangle = \tfrac{1}{\sqrt{2}}(|u_{(7,3a)}\rangle \pm |u_{(7,3b)}\rangle) \ . \tag{3.84}$$

Having identified the proper planar linear combinations, we can proceed by computing the mixing with double-trace operators. We choose as our basis of double-trace operators

$$|u_{(5,3)}\rangle_5 \, |\emptyset\rangle_2 \ , \quad |u_{(5,2)}\rangle_5 \, |\infty\rangle_2 \ , \quad |u_{(4,2)}\rangle_4 \, |\infty\rangle_3 \ , \tag{3.85}$$

where we have labelled the Bethe states by the magnon rapidities rather than the momenta, and $u_{(5,3)} = \{\frac{1}{2}, -\frac{1}{2}, \infty\}$, and $u_{(5,2)} = \{\frac{1}{2}, -\frac{1}{2}\}$. The first two operators both have positive parity and the linear combination

$$\sqrt{\tfrac{2}{3}} \, |u_{(5,3)}\rangle_5 \, |\emptyset\rangle_2 - \sqrt{\tfrac{1}{3}} \, |u_{(5,2)}\rangle_5 \, |\infty\rangle_2 \tag{3.86}$$

is a non-protected operator in the $[3, 1, 3]$ representation. The other linear combination is a descendant of a two-excitation double-trace operator from $[2, 3, 2]$. The non-vanishing overlaps following from (3.70) and (3.72) are

$$\begin{aligned}
V^-(u_{(7,3a)}; u_{(5,3)}, \emptyset) \ &= V^-(u_{(7,3b)}; u_{(5,3)}, \emptyset) \ = +2\sqrt{\tfrac{14}{3}} \ , \\
V^-(u_{(7,3a)}; u_{(5,2)}, \infty) &= V^-(u_{(7,3b)}; u_{(5,2)}, \infty) = -2\sqrt{\tfrac{7}{3}} \ , \\
V^+(u_{(5,3)}, \emptyset; u_{(7,3a)}) \ &= V^+(u_{(5,3)}, \emptyset; u_{(7,3b)}) \ = +40\sqrt{\tfrac{2}{21}} \ , \\
V^+(u_{(5,2)}, \infty; u_{(7,3a)}) &= V^+(u_{(5,2)}, \infty; u_{(7,3b)}) = -40\sqrt{\tfrac{1}{21}} \ .
\end{aligned} \tag{3.87}$$

The overlaps involving $|u_{(4,2)}\rangle_4 \, |\infty\rangle_3$ are all zero as this operator is a descendant and thus cannot mix with the primary states $|\pm\rangle$. Now we apply (3.75) and find that the non-planar corrections arise from the mixing of the positive-parity eigenstate $|+\rangle$ with the double-trace state (3.86) (which has planar energy $E^{(0)} = 8$) and are given by

$$E^{(2)}_{(7,3+)} = 80 \ , \quad E^{(2)}_{(7,3-)} = 0 \ , \tag{3.88}$$

in agreement with [13, 175].

Example for a degeneracy between single- and double-trace states. The occurrence of degenerate parity pairs in the planar limit is quite general and so to use non-degenerate perturbation theory we must work within sectors of definite parity. Unfortunately, there are further degeneracies which are relevant to the mixing problem between multi-trace operators, of which we have given an example for $M = 2$ in (3.33) and (3.34). As another example we consider $L = 8$, $M = 3$ for which the Bethe equations have three solutions. Two of these, whose rapidities we denote $u_{(8,3a)}$ and $u_{(8,3b)}$, form a degenerate parity pair with energy $E^{(0)}_{(8,3a/b)} = 8$, while the third is a singular solution with energy $E^{(0)}_{(8,3s)} = 12$. There is a positive-parity double-trace state

$$\sqrt{\tfrac{3}{4}}\,|u_{(5,3)}\rangle_5\,|\emptyset\rangle_3 - \sqrt{\tfrac{1}{4}}\,|u_{(5,2)}\rangle_5\,|\infty\rangle_3 \tag{3.89}$$

which is degenerate with the parity pair and which mixes with the positive-parity single-trace state. The mixing matrix can be computed from the overlaps and is

$$\begin{pmatrix} 8 & -\frac{4\sqrt{15}}{N} \\ -\frac{32}{\sqrt{15}N} & 8 \end{pmatrix} \tag{3.90}$$

from which we can compute the leading corrections to the energies $E^{(1)} = \pm 8\sqrt{2}$. We can now proceed to use the corresponding eigenstates to find the subleading $1/N^2$ corrections. As we proceed to longer lengths and more impurities, the diagonalisation of the mixing matrix will rapidly become more difficult.

Deforming the theory. One way to remove above degeneracies and thus avoid this problem is to deform the theory. In principle, if we can completely solve the deformed problem, one can hope to remove the deformation parameter at the end to obtain the undeformed result. However, as this requires resumming the $1/N$-corrections before removing the deformation, we will only make preliminary steps in this direction.

Another reason for considering the deformed theory is related to the occurrence of singular solutions (3.27) of the Bethe equations. Already at $L = 6$ and $M = 3$ there is a solution $\{u_1 = i/2, u_2 = -i/2, u_3 = 0\}$ for which the Bethe wavefunction is singular and a naive application of the above formulas leads to unphysical infinities. It is possible to regularise the Bethe equations by the introduction of a twist, see [165] for a useful discussion and further references, and this twist is equivalent to the deformation parameter of the β-deformed theory. We can use the solutions of the twisted Bethe equations and the overlaps of the deformed theory to compute non-planar energies which reproduce the undeformed results in the limit of vanishing deformation.

3.3 Perturbation theory in the β-deformed theory

We now move on to an analogous discussion of perturbative non-planar anomalous dimensions in β-deformed $\mathcal{N}=4$ sYM theory, which we introduced in Section 1.2 and where due to the deformation of the coupling constant we do perturbation theory in the deformed 't Hooft coupling

$$\lambda_\kappa = |\kappa|^2 N \ . \tag{3.91}$$

The β-deformation modifies the action of both the planar piece and the length-changing pieces in the dilatation operator (2.78), but it also introduces new terms. To identify the precise decomposition of $\mathfrak{D}_2^\beta$, we first use the fusion and splitting formulas (1.34) to find the action on single-trace states

$$\begin{aligned}\mathfrak{D}_2^\beta \operatorname{Tr}(XAZB) = \frac{2}{N}\Big(e^{-i\beta}\ \operatorname{Tr}(A)\ \operatorname{Tr}([X,Z]_\beta B) - e^{i\beta}\ \operatorname{Tr}([X,Z]_\beta A)\ \operatorname{Tr}(B)\Big) \\ + \frac{2(e^{i\beta}-e^{-i\beta})}{N^2}\Big(\operatorname{Tr}([X,Z]_\beta\{A,B\}) + \operatorname{Tr}([X,Z]_\beta)\ \operatorname{Tr}(A)\ \operatorname{Tr}(B)\Big) \\ - \frac{4(e^{i\beta}-e^{-i\beta})}{N^3}\ \operatorname{Tr}([X,Z]_\beta)\ \operatorname{Tr}(AB)\ .\end{aligned} \tag{3.92}$$

Here the double-trace part of $\mathfrak{D}_2^\beta$ contributes the triple-trace term at order $1/N^2$. Similarly, for the action on double-trace states we find

$$\begin{aligned}\mathfrak{D}_2^\beta \operatorname{Tr}(XA)\operatorname{Tr}(ZB) = \frac{2}{N}\Big(e^{-i\beta}\operatorname{Tr}([X,Z]_\beta BA) - e^{i\beta}\operatorname{Tr}([X,Z]_\beta AB)\Big) \\ + \frac{2(e^{i\beta}-e^{-i\beta})}{N^2}\Big(\operatorname{Tr}(A)\ \operatorname{Tr}([X,Z]_\beta B) + \operatorname{Tr}([X,Z]_\beta A)\ \operatorname{Tr}(B) \\ + \operatorname{Tr}([X,Z]_\beta)\ \operatorname{Tr}(AB)\Big) \\ - \frac{4(e^{i\beta}-e^{-i\beta})}{N^3}\operatorname{Tr}([X,Z]_\beta)\ \operatorname{Tr}(A)\ \operatorname{Tr}(B)\ .\end{aligned} \tag{3.93}$$

These relations suggest that the deformed one-loop dilatation operator can be decomposed into planar and non-planar pieces similar to the undeformed case (3.2) with additional subleading contributions as

$$\mathfrak{D}_2^\beta = H_\beta^{(0)} + \frac{1}{N}H_\beta^- + \frac{1}{N}H_\beta^+ + \frac{1}{N^2}H_\beta^{(2)} + \frac{1}{N^3}H_\beta^{(3)}\ . \tag{3.94}$$

As for the undeformed case $H_\beta^{(0)}$ leaves the number of traces in an operator unchanged, while $H_\beta^\pm$ increases/reduces the number of traces. $H_\beta^{(2)}$ and $H_\beta^{(3)}$ are subleading terms which only occur in the deformed theory. $H_\beta^{(2)}$ has a contribution which leaves the number of traces unchanged such that we have an additional term in the leading

anomalous dimension correction

$$E_\beta^{(2)}(\{p\}) = \sum_{\{I\}} \frac{V_\beta^-(p;I)V_\beta^+(I;p)}{E_\beta^{(0)}(\{p\}) - E_\beta^{(0)}(\{I\})} + V_\beta^{(2)}(\{p\}) \ . \tag{3.95}$$

3.3.1 Matrix elements

The action of the non-planar dilatation operator on Bethe states and the corresponding overlaps can be computed by essentially the same methods as for the undeformed theory, the only difference being that there is an additional diagonal contribution $H_\beta^{(2)}$. We first compute the off-diagonal pieces $H_\beta^\pm$ and then move on to the new diagonal term.

Off-diagonal overlaps. We write the overlaps of $H_\beta^\pm$ using the same notation[4] as in Section 3.2. For H_β^- the overlaps are almost identical to (3.55) and given by

$$\begin{aligned}
&\langle\{p\}|H_\beta^-|\{q\}\rangle|\{r\}\rangle \\
&= 2L_qL_r\,\mathcal{N}(p^*,q,r) \sum_{\substack{i,j \\ s\cup t=\{p\}_{\hat{j}}}} \frac{e^{ip_j^*}e^{2i\beta}-1}{h^{q_iq_{\hat{i}}}}\big[e^{-2i\beta}s_{j\circlearrowleft}^{L_{q+1}\,*} - t_{j\circlearrowleft}^{L_r+1\,*}\big](s|\{q\}_{\hat{i}})_{L_q-1}(t|\{r\})_{L_r-1} \\
&\qquad + \{\text{terms with } q \rightleftarrows r\} \ .
\end{aligned} \tag{3.96}$$

The function h in this formula has exactly the same form (3.20) as in the undeformed theory when written in terms of the on-shell rapidities, and similarly for the S-matrices which are implicit in the scalar products. The on-shell rapidities themselves however depend on the deformation parameter through the Bethe equations.

The overlaps of H_β^+ now involve additional contributions whenever one of the traces has length two and are given by

$$\begin{aligned}
&\langle\{r\}|\langle\{q\}|H_\beta^+|\{p\}\rangle \\
&= 2L_p\,\mathcal{N}_+(p,q^*,r^*) \sum_{\substack{i,j \\ s\cup t=\{p\}_{\hat{j}}}} \frac{1}{h^{q_i^*q_i^*}}\Big[(e^{iq_i^*}e^{2i\beta}-1)(e^{-2i\beta}s_{j\circlearrowleft}^{L_q-1} - t_{j\circlearrowleft}^{L_r+1}) \\
&\qquad\qquad - 4\delta_{Q,1}\delta_{L_q,2}\sin^2\beta(e^{iq_i^*}s_{j\circlearrowleft}^{L_q-1} + t_{j\circlearrowleft}^{L_r+1})\Big](\{q\}_{\hat{i}}|s)_{L_q-2}(\{r\}|t)_{L_r} \\
&\qquad + \{\text{terms with } q \rightleftarrows r\} \ .
\end{aligned} \tag{3.97}$$

As in the undeformed theory, dividing by the norms of the external states we can define the normalised overlaps $V_\beta^\pm$.

[4]Note that also in the deformed case the solutions of the deformed Bethe equations are invariant under complex conjugation which can be used to simplify the expressions.

Diagonal overlaps. The contribution $H_\beta^{(2)}$ in (3.94), which does not occur in the undeformed theory, contains both length-preserving and -changing parts. Here we are only interested in the former, since the computation of non-planar corrections at $\mathcal{O}(N^{-2})$ to the anomalous dimensions requires solely the diagonal overlap $\langle\{p\}|H_\beta^{(2)}|\{p\}\rangle$. Using (3.47) one finds for the action of $H_\beta^{(2)}$ on a Bethe state (3.9)

$$\begin{aligned}
H_\beta^{(2)}\,|\{p\}\rangle \;&= 2L_p(e^{i\beta}-e^{-i\beta})\sum_{x=2}^{L_p}\sum_{l=1}^{P}\sum_{\substack{2\le n_1<\ldots<n_{l-1}<n_l=x\\ x<n_{l+1}<\ldots<n_P\le L_p}}\psi_{\{n\}}^{\{p\}}\\
&\times\Big(\,|[X,Z]_\beta\rangle\otimes|n_1+1,\ldots,n_{l-1}+1\rangle_{x-2}\otimes|n_{l+1},\ldots,n_P\rangle_{L_p-x}\\
&\quad+|n_1-1,\ldots,n_{l-1}-1\rangle_{x-2}\otimes|[X,Z]_\beta\rangle\otimes|n_{l+1},\ldots,n_P\rangle_{L_p-x}-2\delta_{L_p,2}\,|[X,Z]_\beta\rangle\,\Big)\\
&+\{\text{double-trace terms}\}\;,
\end{aligned}\tag{3.98}$$

where the $\delta_{L_p,2}$-term arises from the enhanced contribution of the last double-trace term in (3.92). The diagonal overlap can then be written in terms of ordered partitions (3.51) as

$$\begin{aligned}
&\langle\{p\}|H_\beta^{(2)}|\{p\}\rangle\\
&=2L_p\delta_{L_p\neq2}\,\mathcal{N}(p^*,p)(e^{i\beta}-e^{-i\beta})\sum_{\rho,\sigma}\frac{e^{i\beta}e^{ip^*_{\rho(1)}}-e^{-i\beta}}{h_>^{\{p^*_\rho\}}h_<^{\{p_\sigma\}}}\\
&\qquad\times\sum_{x=2}^{L_p}\sum_{l=1}^{P}\left(e^{i(x-1)p_{\sigma(1)}}\prod_{k=2}^{l}S_\beta\left(p_{\sigma(1)},p_{\sigma(k)}\right)+1\right)e^{i(x-1)(p_\sigma)_{l+1}^{P}}e^{-i(x-2)(p^*_\rho)_{l+1}^{P}}\\
&\qquad\times P_{x-1}\left(\{p_\sigma-p^*_\rho\}_2^l\right)P_{L_p-x+1}\left(\{p_\sigma-p^*_\rho\}_{l+1}^P\right)\;.
\end{aligned}\tag{3.99}$$

Note that $P_L(z)$ vanishes for $|z|\geq L$, cf. (3.51). In terms of scalar products of normalised Bethe states (3.53), the overlap can be written as

$$\begin{aligned}
&\langle\{p\}|H_\beta^{(2)}|\{p\}\rangle\\
&=2L_p\delta_{L_p\neq2}\,\mathcal{N}(p^*,p)(e^{i\beta}-e^{-i\beta})\\
&\times\sum_{\substack{k,l=1\\ \kappa\cup\bar\kappa=\{p\}_{\hat k}\\ \lambda\cup\bar\lambda=\{p^*\}_{\hat l}}}^{P}\sum_{x=2}^{L_p}\frac{e^{i\beta}e^{ip^*_l}-e^{-i\beta}}{h^{p_k\bar\kappa}h^{\kappa\bar\kappa}}\left(\frac{e^{-i(x-1)\kappa}}{h^{\kappa p_k}}+\frac{e^{i(x-1)\bar\kappa}}{h^{p_k\kappa}}\right)\frac{e^{-i(x-2)\bar\lambda}}{h^{p^*_lp^*_l}h^{\bar\lambda\lambda}}(\lambda|\kappa)_{x-2}(\bar\lambda|\bar\kappa)_{L_p-x}\;,
\end{aligned}\tag{3.100}$$

where κ and λ are of the same cardinality, which must be smaller than $x-1$. Finally, we can divide by the square of the norm of the external state, $\|p\|^2$, to define normalised overlaps $V_\beta^{(2)}(\{p\})$ which can be then used to compute the energy corrections of single-trace states.

3.3.2 Anomalous dimensions

As the deformation lifts many of the degeneracies present in $\mathcal{N}=4$ sYM theory, we can use the overlaps in the deformed theory to compute the corrections to energies for a wide range of states via (3.95). The additional input to such a calculation are the solutions to the deformed Bethe equations. Solving the deformed Bethe equations is generally a non-trivial task, however for short lengths it can be done either for specific numerical values of β, or by starting with the undeformed result and perturbatively solving for $\beta \ll 1$. The latter is particularly useful when we wish to use the deformation as a regulator of singular solutions of the undeformed Bethe equations. One must be careful with the order of limits as the one-loop anomalous dimensions are functions of both β and N. We may choose to first expand in large N and then small β, $E(\beta \gg N^{-1})$, by computing the energy as in (3.95) at finite β and then sending β to zero. Alternatively, we can expand first in small β and then large N, $E(\beta \ll N^{-1})$, by computing the relevant overlaps at small β and then computing the energies from the resulting expressions and (3.95). In general these two expansions will not commute.

Singular $L=6$ example. For example, let us consider the $L=6$, $M=3$ single-trace operator described in the planar undeformed theory by the roots $\{u_1 = i/2, u_2 = -i/2, u_3 = 0\}$ and planar energy $E^{(0)} = 12$. This solution is singular as it has rapidities separated by i. In the undeformed theory it has a vanishing $E^{(2)}$ contribution which is due to the $\mathfrak{su}(2)$ symmetry ensuring that there is no other operator with which it can mix. In the deformed theory, where the mixing problem of this operator is non-trivial, we find from direct diagonalisation that through $\mathcal{O}(N^{-4})$ (and keeping only the leading terms in the β-expansion) we have

$$E(\beta \gg N^{-1}) = (12 - 72\beta^2 + \mathcal{O}(\beta^4)) + \frac{1}{N^2}\Big(-\frac{2304}{23} + \mathcal{O}(\beta^2)\Big) + \frac{1}{N^4}\Big(\frac{400896}{12167\beta^2} + \mathcal{O}(\beta^0)\Big)\,. \quad (3.101)$$

In this expression we can see that the leading non-planar term does not reduce to the vanishing undeformed answer at $\mathcal{O}(N^{-2})$ in the $\beta \to 0$ limit, and in fact the $1/N^4$ term is singular. There will be additional singular terms at subsequent powers in the $1/N$ expansion. Resumming these is necessary to recover a smooth $\beta \to 0$ limit.

As the singular wavefunction in the deformed theory is perfectly regular, we can use (3.96), (3.97) and (3.100) to compute the overlaps between states relevant for the corresponding mixing problem, then take the $\beta \to 0$ limit and use the resulting expressions to perturbatively compute the undeformed non-planar correction for $\beta \ll N^{-1}$. In order to do so, we first regularise the wavefunction by solving the Bethe equations

for the deformed rapidities to $\mathcal{O}(\beta^6)$, where the first non-vanishing correction occurs[5], specifically

$$u_1^\beta - u_2^\beta = i + 24576i\beta^6 + \mathcal{O}(\beta^8) \ . \tag{3.102}$$

The corresponding state mixes with the double-trace operator $|u_{(4,2)}\rangle_4 |u_{(2,1)}\rangle_2$ of planar energy

$$E^{(0)}_{(4,2)} = 12 - \frac{32}{3}\beta^2 + \mathcal{O}(\beta^4) \tag{3.103}$$

with normalised overlaps

$$V^+_\beta = -48\sqrt{2}\beta \ , \qquad\qquad V^-_\beta = -64\sqrt{2}\beta \tag{3.104}$$

to leading order in the β-expansion. The deformation is particularly important when calculating the norm of the singular state, which diverges in the $\beta \to 0$ limit. However, the overlaps themselves are smoothly vanishing in this limit and so, consistent with the symmetries, there is no mixing in the undeformed theory. If we instead use the overlaps (3.104) in the perturbative formula (3.75) we find a cancellation between the powers of β in the overlaps and the energy differences. As the diagonal contribution is of order $\mathcal{O}(\beta^2)$ it gives no leading contribution, and we find

$$E^{(2)}(\beta \gg N^{-1}) = -\frac{2304}{23} + \mathcal{O}(\beta^2) \tag{3.105}$$

in agreement with the result from direct diagonalisation.

Singular $L = 8$ example. Let us now turn to the $L = 8, M = 3$ singular Bethe state $|u_{(8,3s)}\rangle$ with planar energy $E^{(0)}_{(8,3s)} = 12$. This state is not protected by symmetry in the undeformed theory. Instead it mixes with the double-trace operator $|u_{(6,3s)}\rangle_6 |\emptyset\rangle_2$ made up of the length-6 singular state and length-2 vacuum. From direct diagonalisation one can find the corrected spectrum

$$E_\pm = 12 \pm \frac{12}{N} \ , \tag{3.106}$$

where we see that due to the degeneracy the correction is $\mathcal{O}(N^{-1})$. Due to the singular nature of the Bethe solution we cannot directly use the overlap formulas of $\mathcal{N} = 4$ sYM theory to reproduce this result. Nevertheless, we can again compute the overlaps, and thus mixing matrix, using the regularised singular states in the deformed theory

[5]For general length-L singular states, one has to find deformed rapidities to $\mathcal{O}(\beta^L)$ to resolve the singularity.

and find that they are non-vanishing in the $\beta \to 0$ limit

$$V_\beta^+ = 4\sqrt{6} + \mathcal{O}(\beta^2) \ , \qquad V_\beta^- = 6\sqrt{6} + \mathcal{O}(\beta^2) \tag{3.107}$$

and give the correct $1/N$ corrections (3.106) for $\beta \ll N^{-1}$.

Alternatively, in the deformed theory as the degeneracy between the two states is lifted, with the planar energy of the single-trace state becoming $E^{(0)}_{(8,3s)} = 12 - 36\beta^2 + \mathcal{O}(\beta^4)$, we can use the same overlaps in non-degenerate perturbation theory in the small β-limit with $\beta \gg N^{-1}$ to find the $1/N^2$ corrections in the deformed theory. The contribution of the overlaps between the two regularised singular states to $E^{(2)}_{(8,3s)}$ is $+4/\beta^2$, i.e. it is singular in the limit $\beta \to 0$. There are additional overlaps with other double-trace states, however they are subleading as is the diagonal contribution which is $\mathcal{O}(\beta^4)$. Thus for $\beta \gg N^{-1}$ the non-planar corrections start at order N^{-2} but are singular in the $\beta \to 0$ limit. This demonstrates that, in general, the two limits $\beta \to 0$ and $N^{-1} \to 0$ do not commute.

3.3.3 BMN limit

In the following we focus on two-excitation single-trace solutions and their non-planar corrections in the BMN limit [172] of large R-charge, $J = L-2 \to \infty$, in the deformed theory, where the deformation parameter scales as

$$\beta = \pi b/L \tag{3.108}$$

with b fixed. Perturbation theory can be rewritten in terms of the effective loop- and genus-counting parameters

$$g' = \frac{|\kappa|^2 N}{16\pi^2 J^2} \quad \text{and} \quad g_2 = \frac{J^2}{N} \ , \tag{3.109}$$

and thus the anomalous dimensions of two-impurity BMN operators can be written in terms of rescaled energies $\tilde{E}^{(k)} = J^{2-2k} E^{(k)}$ as

$$\Delta_n(g', g_2, b, J) = L + g' \big[\tilde{E}^{(0)}_n(b, J) + g_2^2 \tilde{E}^{(2)}_n(b, J) + \mathcal{O}(g_2^4) \big] + \mathcal{O}((g')^2) \ , \tag{3.110}$$

labelled by the mode number n of the planar Bethe state. We compute $\tilde{E}^{(2)}_n(b, J)$ through $\mathcal{O}(J^{-1})$ in the following. We start by discussing the planar Bethe states, and then compute the leading non-planar corrections from perturbation theory via the overlap formulas (3.96), (3.97) and (3.100).

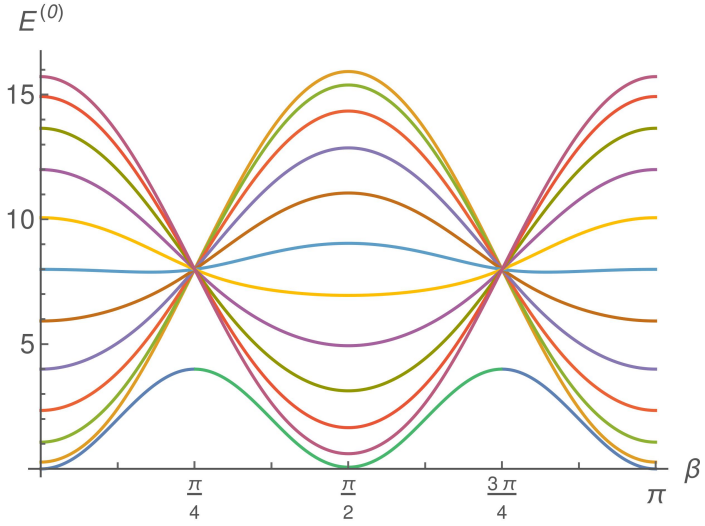


Figure 3.3: Planar energy levels for $L = 25$ and $M = 2$ in the deformed theory. Although in the BMN regime we take the limit of small β, this plot already hints at the different nature of the zero-mode solution corresponding to the lowest energy.

Planar states. The Bethe equations (3.13) in the deformed theory for two magnons with momenta $p_{1,2}$ are given by

$$2\pi n = Lp_1 - i\log(S_\beta(p_1, p_2)) \ , \tag{3.111}$$

and its solutions are parametrised by the integer-valued mode number n. With the deformed S-matrix (3.39) being periodic in β with period π, the solutions are also periodic and symmetric around $\beta = \pi/2$. As an example we plot the cyclic solutions ($p_1 = -p_2 \bmod 2\pi i$) of (3.111) for $L = 25$ in Figure 3.3. All but one of the energy levels become degenerate for $\beta = \pi/4$ and this point splits the spectrum into two regions $\beta \in [0, \pi/4)$ and $\beta \in (\pi/4, \pi/2]$, where the energy levels correspond to different sets of mode numbers $\{n\}$. In the first region, n takes integer values from the range $[0, \lfloor L/2 \rfloor - 1]$ and solutions with positive n correspond to deformations of primary operators in $\mathcal{N} = 4$ sYM theory, while the zero mode (with lowest energy in this interval) becomes a descendant in the undeformed theory. In the second region, n takes values in the range $[1, \lfloor L/2 \rfloor]$, with the lowest-energy state now corresponding to $n = \lfloor L/2 \rfloor$.

In the BMN limit the deformation parameter scales as (3.108) and thus we effectively concentrate on a regime of small deformations with mode number $n \in [0, \lfloor L/2 \rfloor - 1]$. We can solve the Bethe equations in this limit perturbatively, and

find that the rapidity for a strictly positive mode number n_+ is given by

$$u_{n_+} = \frac{L}{2(n_+ + b)\pi}\left(1 - \frac{n_+ - b}{n_+ L} - \frac{(n_+ + b)\left(3b(n_+ - b) + (n_+ + b)n_+^3\pi^2\right)}{3n_+^3 L^2} + \dots\right) . \tag{3.112}$$

Momentum conservation requires the second excitation to have rapidity u_{-n_+}. Meanwhile, the zero-mode solutions have a distinct expression where the expansion parameter now becomes the square root of the length

$$u_0^{\pm} = \frac{L}{2b\pi}\left(1 \pm \frac{i}{L^{1/2}} - \frac{1}{L} \pm \frac{i(b^2\pi^2 - 3)}{6L^{3/2}} - \frac{2b^2\pi^2}{3L^2} + \dots\right) . \tag{3.113}$$

The planar energies in the BMN limit can then be computed through (3.38), yielding

$$\begin{aligned} E_{n_+}^{(0)} &= \frac{16(n_+^2 + b^2)\pi^2}{L^2}\left(1 + \frac{2(n_+^2 - b^2)}{(n_+^2 + b^2)L}\right. \\ &\qquad \left. + \frac{3(3n_+^4 - 2b^2n_+^2 - b^4) - (n^4 + 6b^2n_+^2 + b^4)n_+^2\pi^2}{3(n_+^2 + b^2)n_+^2 L^2} + \dots\right) , \\ E_0^{(0)} &= \frac{16b^2\pi^2}{L^2}\left(1 - \frac{1}{L} - \frac{(3 + 2b^2\pi^2)}{3L^2} + \dots\right) . \end{aligned} \tag{3.114}$$

Note that despite the unusual expansion of the zero-mode rapidities $u_0^{\pm}$, the expansion of the corresponding energy is free of any square roots of L. Finally, while at the leading order the rapidities $u_0^{\pm}$ seem to be only a particular case $n_+ = 0$ of $u_{\pm n_+}$, the expression for the norm differs already at this order by a factor of two. Labelling the Bethe states associated to the rapidities (3.112) and (3.113) by the mode number n as $|n\rangle$, their Gaudin norms $N_n = \| |n\rangle \|^2$ can be computed and are given by

$$\begin{aligned} N_{n_+} &= L^2\left(1 - \frac{n_+^2 + b^2}{n_+^2 L} + \frac{2b^2(n_+^2 - b^2)}{n_+^4 L^2} + \dots\right) , \\ N_0 &= 2L^2\left(1 + \frac{b^2\pi^2 - 3}{3L} + \frac{2b^2\pi^2(4b^2\pi^2 - 15)}{45L^2} + \dots\right) . \end{aligned} \tag{3.115}$$

Scaling of non-planar energy corrections. Equipped with the Bethe solutions, we can now study the non-planar corrections to the energies in the BMN limit. The strategy is to expand the dilatation operator overlaps obtained in Section 3.3.1 and plug them into (3.95) written explicitly as

$$E_n^{(2)} = \sum_I \frac{1}{N_n N_I}\frac{\langle n|H_\beta^-|I\rangle\langle I|H_\beta^+|n\rangle}{E_n^{(0)} - E_I^{(0)}} + \frac{1}{N_n}\langle n|H_\beta^{(2)}|n\rangle . \tag{3.116}$$

We need to consider three contributions to this expression:

1. Off-diagonal matrix elements of $H_\beta^\pm$ with double-trace operators where
   (a) both excitations end up on the same trace, or
   (b) the excitations split over the two traces.
2. Diagonal matrix elements of $H_\beta^{(2)}$.

In general the two-excitation overlaps corresponding to H_β^- and H_β^+ scale at most as L and L^2, respectively. The sum over intermediate double-trace states I includes a sum over the splitting of the lengths $(L', L-L')$, which in the large-L limit can be approximated by the Euler–MacLaurin formula

$$\sum_{L'=l_1}^{l_2} f(L') \approx L \int_{l_1/L}^{l_2/L} \mathrm{d}r f(r) + \frac{f(l_1)+f(l_2)}{2} + \dots \,, \tag{3.117}$$

thus leading to a further factor of L. Combined with the scaling of the planar energies (3.114) and norms (3.115), the off-diagonal contribution to the one-loop non-planar energies $E_n^{(2)}$ scales at most as L^2. In contrast, the diagonal overlap grows linearly with L, and so its normalised contribution only starts at $\mathcal{O}(1/L)$ and is subleading. Combined with the associated colour factor $1/N^2$ of $E_n^{(2)}$, the leading contribution of (3.116) gives a contribution at $\mathcal{O}(g'g_2^2)$, cf. (3.109), in agreement with the expansion (3.110) of the scaling dimension. While in principle we can compute the off-diagonal and diagonal overlaps at any order in the BMN $1/J$-expansion, the expansion of $\tilde{E}_n^{(2)}(b,J)$ at $\mathcal{O}(g'g_2^2)$ eventually breaks down due to the approximation of the summation over intermediate states by an integration (3.117). More precisely, we find that for mode numbers $n>1$ the sub^k-leading BMN correction to the integrand with $k>1$ has simple poles at lengths $L'/L = n'/n$ with $n' = 1,\dots,n-1$. As shown in Figure 3.4 this failure of the BMN expansion is in fact expected and agrees with the numerical tests performed.

Off-diagonal elements of type (a) for states with positive mode number. We now move on to the explicit evaluation of overlaps in the BMN limit, starting with the configuration of uneven splitting in the off-diagonal case, and first consider the situation where the external state has positive mode number n_+. The intermediate double-trace state $|I\rangle$ has one trace with two excitations, whose length we denote by $L' = rL$, and another trace corresponding to a vacuum state. The former is furthermore characterised by the mode number n' of the Bethe roots. While $L'<L$, the deformation parameter for the double-trace solution can be expanded in terms of the length L of the single-trace operator as in (3.108), so the rapidity, energy and

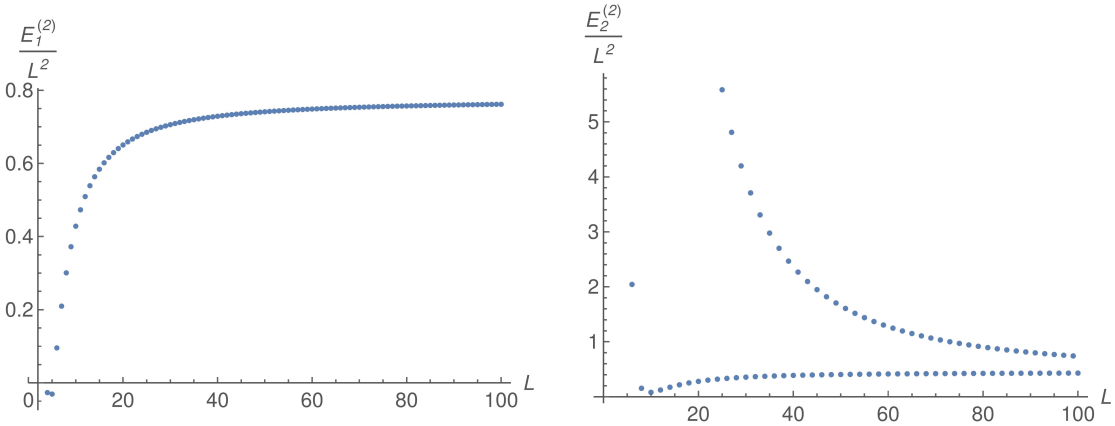


Figure 3.4: Non-planar correction to the energies $E_n^{(2)}$ for single-trace operators with varying length L, $L \leq 100$, computed from the dilatation-operator matrix elements at $b = 1/3$ for mode number (i) $n = 1$ and (ii) $n = 2$. We observe that for $n = 2$ the large-L limit is approached differently for even- and odd-length operators. However, fitting the two curves with polynomials in $1/L$ we find that the mismatch in the coefficients starts only at subsubleading order.

norm of the double-trace states are written as

$$u'_{n'}(L,b) = u_{n'}(rL,rb)\ , \qquad E'^{(0)}_{n'}(L,b) = E^{(0)}_{n'}(rL,rb)\ , \qquad N'_{n'}(L,b) = N_{n'}(rL,rb)\ . \tag{3.118}$$

In the case where the mode number of the intermediate double-trace state is a positive integer n'_+, the overlaps in the BMN limit become

$$\begin{aligned} H^-_{n_+n'_+} &= -\frac{32(1-r)r^3n_+^2\sin^2(\pi r n_+)L^2}{n'^2_+ - r^2n_+^2}\left(1 - 2\frac{(1-r)n_+^2n'^2_+ + r(n'^2_+ - rn_+^2)b^2}{rn_+^2(n'^2_+ - r^2n_+^2)L}\right. \\ &\qquad \left. - \pi\cot(\pi r n_+)\frac{(1-2r)n_+^2 - (3-2r)b^2}{n_+L} + \dots\right)\ , \\ H^+_{n'_+n_+} &= \frac{32n'^2_+\sin^2(\pi r n_+)L}{n'^2_+ - r^2n_+^2}\left(1 - 2r\frac{(1-r)n_+^2n'^2_+ + r(n'^2_+ - rn_+^2)b^2}{n'^2_+(n'^2_+ - r^2n_+^2)L}\right. \\ &\qquad \left. - \pi\cot(\pi r n_+)\frac{(3-2r)n_+^2 - (1-2r)b^2}{n_+L} + \dots\right)\ . \end{aligned} \tag{3.119}$$

On the other hand, if the intermediate double-trace operator consists of a zero-mode solution, then the H^+_β overlap is suppressed by a factor of L and we have

$$\begin{aligned} H^-_{n_+0} &= 32(1-r)r\sin^2(\pi r n_+)L^2 + \dots\ , \\ H^+_{0n_+} &= \frac{32b^2\sin^2(\pi r n_+)}{rn_+^2} + \dots\ . \end{aligned} \tag{3.120}$$

The contribution to the non-planar energies is the combination of those two cases, which leads to

$$E^{(2)}_{n_+(a)} = L\int_{2/L}^{(L-2)/L} \mathrm{d}r \left(\frac{H^-_{n_+0}H^+_{0n_+}}{N_{n_+}N'_0\left(E^{(0)}_{n_+} - E'^{(0)}_0\right)} + \sum_{n'_+=1}^{\infty} \frac{H^-_{n_+n'_+}H^+_{n'_+n_+}}{N_{n_+}N'_{n'_+}\left(E^{(0)}_{n_+} - E'^{(0)}_{n'_+}\right)} \right)$$
$$= \left(\frac{1}{3} + \frac{35}{8\pi^2 n^2}\right) L^2 \left(1 - \frac{2n^2 - 4b^2}{n^2 L} + \dots\right) . \tag{3.121}$$

Note that in this particular case the subleading correction of the Euler-McLaurin formula is vanishing and only the leading integral expression of (3.117) contributes. Furthermore, the contribution of the intermediate zero mode is crucial for the simplicity of this formula, which would otherwise be plagued by more complex functions such as $\int_0^z \mathrm{d}t \sin(t)/t$.

As explained above, the integral approximation at the subsubleading order is not well defined for $n_+ > 1$, which is manifested here by the presence of poles in the integrand. However, the expression for $n = 1$ appears to be well defined at any order, thus allowing us to obtain

$$E^{(2)}_{1(a)} = \left(\frac{1}{3} + \frac{35}{8\pi^2}\right) L^2 + \frac{(105 + 8\pi^2)(2b^2 - 1)}{12\pi^2} L$$
$$+ \frac{1}{24\pi^2}\big(105(1 - 12b^2 + 15b^4) - (1 + 144b^2 - 9b^4)\pi^2$$
$$-8(3 + 4b^2 + b^4)\pi^4 - 288(1 + b^2)^2\pi^2\zeta_3\big) . \tag{3.122}$$

Off-diagonal elements of type (a) for states with vanishing mode number. We now move on to the case where the external state is a zero mode for the uneven splitting case (a). If the intermediate operator is a zero mode as well, then each of the overlaps is suppressed by a power of L and given by

$$H^-_{00} = 32\pi^2(1 - r)(2 - r)r^2b^2L$$
$$\times \left(1 - \frac{6(2 - r + r^2) - \pi^2 r(2 - r)(3 - r(3 - 2r))}{6r(2 - r)L} + \dots\right) ,$$
$$H^+_{00} = 32\pi^2 r b^2 \left(1 - \frac{12 - \pi^2 r(3 - r(3 - 2r))}{6rL} + \dots\right) . \tag{3.123}$$

Notice that in this case the difference of planar energies $E^{(0)}_0 - E'^{(0)}_0$ vanishes at leading order, therefore enhancing the contribution of the intermediate zero-mode to $E^{(2)}_0$ by a factor of L. In principle, we also need the off-diagonal overlaps with positive-mode states, $H^-_{0n'_+}$ and $H^+_{n'_+0}$, but each of them starts contributing at $\mathcal{O}(L^0)$ and the difference $E^{(0)}_0 - E'^{(0)}_{n'_+}$ is once again $\mathcal{O}(L^2)$ and so we can safely ignore the contribution of these modes at the order we wish to consider. The non-planar correction to the

energy of the zero-mode solution is then given by

$$E^{(2)}_{0(a)} = L\int_{2/L}^{(L-2)/L} \mathrm{d}r \left(\frac{H^-_{00}H^+_{00}}{N_0 N_0' \left(E^{(0)}_0 - E_0'^{(0)}\right)} \right) + 8\pi^2 b^2$$
$$= \frac{20\pi^2 b^2 L}{3}\left(1 + \frac{2\pi^2 b^2 - 225}{25L} + \dots\right) , \tag{3.124}$$

where the second term of the first line corresponds to the subleading correction in the Euler–MacLaurin formula (3.117).

Off-diagonal elements of type (b) for states with positive mode number. In order to study the second splitting configuration (b), where H^+_β leads to double-trace operators with an excitation in each of the traces, we need to consider the single-excitation solution in more detail. The perturbative solution of the one-magnon Bethe equations for the rapidity is

$$u_{(1)} = \frac{L}{2b\pi}\left(1 - \frac{b^2\pi^2}{3L^2} + \dots\right) . \tag{3.125}$$

Note that this expression only depends on the length L of the external single-trace operator via its relation with the scaled deformation parameter (3.108), but is independent of the length of the state it characterises. The corresponding energy of this state is

$$E^{(0)}_{(1)} = \frac{8b^2\pi^2}{L^2}\left(1 - \frac{b^2\pi^2}{3L^2} + \dots\right) , \tag{3.126}$$

and the norm is given by the length of the operator. Importantly, the single-excitation solution of length 2, $\mathrm{Tr}(ZX)$, is an exception to this formula and is protected due to the contribution of the double-trace term to the planar dilatation operator.

The resulting overlaps for an external non-zero mode single-trace operator are

$$H^-_{n_+} = -32r(1-r)\sin^2(\pi r n_+)L^2 + \dots ,$$
$$H^+_{n_+} = \frac{16\pi^2 b^2}{L}\left(\cos(2\pi r n_+) - \frac{b^2}{n_+^2}\right) + \dots , \tag{3.127}$$

which will only contribute at subsubleading order as H^+_n is suppressed by two powers of L. Thus it suffices to consider the leading integral approximation which gives

$$E^{(2)}_{n_+(b)} = \frac{L}{2}\int_0^1 \mathrm{d}r \frac{H^-_n H^+_n}{N_{n_+}(rL)((1-r)L)(E^{(0)}_{n_+} - 2E^{(0)}_{(1)})}$$
$$= \frac{4b^2}{n_+^4}\left(n_+^2 + 2b^2\right) + \dots . \tag{3.128}$$

Off-diagonal elements of type (b) for states with vanishing mode number. Meanwhile, for an external zero mode, the overlaps are

$$\begin{aligned} H_0^- &= 16\pi^2 b^2 (1-r) r L \\ &\quad \times \left(1 - 2r(1-r) - \frac{12r(1-r) - (1 - 6r(1-r) + 4r^2(1-r)^2) b^2\pi^2}{6L} + \dots\right) , \\ H_0^+ &= 16\pi^2 b^2 \left(1 + \frac{\pi^2 b^2}{6L} + \dots\right) , \end{aligned} \tag{3.129}$$

which are both suppressed by one power in L. We now wish to perform the sum over intermediate states as in equation (3.116). The difference of planar energies $E_0^{(0)} - 2E_{(1)}^{(0)}$ vanishes at leading order, cf. (3.114) and (3.126), and so the resulting non-planar energy correction from these overlaps is $\mathcal{O}(L^1)$. However this reasoning does not apply when one of the traces has length 2, in which case the difference of planar energies has a non-vanishing leading-order contribution. This term separates out of the Euler–MacLaurin approximation (3.117) and is thus subsubleading. Therefore the non-planar correction to the energy in this configuration becomes

$$\begin{aligned} E_{0(b)}^{(2)} &= \frac{L}{2} \int_{3/L}^{(L-3)/L} \mathrm{d}r \frac{H_0^- H_0^+}{N_0(rL)((1-r)L)\left(E_0^{(0)} - 2E_{(1)}^{(0)}\right)} - 4\pi^2 b^2 \\ &= -\frac{8\pi^2 b^2 L}{3}\left(1 - \frac{120 + 7\pi^2 b^2}{15L} + \dots\right) , \end{aligned} \tag{3.130}$$

where, as before, the second term of the first line corresponds to a non-vanishing subleading contribution in the Euler-McLaurin approximation.

Final results in deformed and undeformed theory. As discussed above, the diagonal contributions in (3.116) are $\mathcal{O}(L^{-1})$ and thus do not contribute to the perturbative orders we consider. Therefore, the non-planar corrections to the energy of two-excitation single-trace operators is, at $\mathcal{O}(L^0)$ in the BMN limit, given by the sum of the off-diagonal uneven splittings, (3.121) and (3.124), and symmetric splittings, (3.128) and (3.130)

$$\begin{aligned} E_{n_+}^{(2)} &= E_{n_+(a)}^{(2)} + E_{n_+(b)}^{(2)} , \\ E_0^{(2)} &= E_{0(a)}^{(2)} + E_{0(b)}^{(2)} . \end{aligned} \tag{3.131}$$

Then the scaling dimensions of two-excitation states in the BMN limit of large R-charge $J = L - 2$ can be written as

$$\Delta_{n_+} = L + g' \Bigg[16\pi^2 (n_+^2 + b^2) \left(1 - \frac{2(n_+^2 + 3b^2)}{(n_+^2 + b^2) J} + \mathcal{O}(J^{-2}) \right) + g_2^2 \left(\frac{1}{3} + \frac{35}{8\pi^2 n_+^2} \right) \left(1 + \frac{2(n_+^2 + 2b^2)}{n_+^2 J} + \mathcal{O}(J^{-2}) \right) + \mathcal{O}(g_2^4) \Bigg] + \mathcal{O}((g')^2) \ , \quad (3.132)$$

for the non-zero modes, while for the zero mode we find

$$\Delta_0 = L + g' \Bigg[16\pi^2 b^2 \left(1 - \frac{5}{J} + \frac{51 - 2\pi^2 b^2}{3J^2} + \mathcal{O}(J^{-3}) \right) + 4\pi^2 b^2 g_2^2 \left(\frac{1}{J} + \frac{4\pi^2 b^2 - 69}{9J^2} + \mathcal{O}(J^{-3}) \right) + \mathcal{O}(g_2^4) \Bigg] + \mathcal{O}((g')^2) \ . \quad (3.133)$$

Taking the $b \to 0$ limit of the expressions above yields the non-planar corrections to the two-excitation energies of $\mathcal{N} = 4$ sYM theory. In this limit both $E^{(2)}_{n_+(b)}$ and $E^{(2)}_{0(b)}$ vanish, which is expected as single-excitation solutions correspond to descendants in the undeformed theory and thus they are not expected to contribute to the non-planar energies. In fact, all loop- and non-planar corrections in Δ_0 to the classical scaling dimension L vanish for $b \to 0$, where the zero-mode operator becomes a descendant of a chiral primary. For positive mode numbers n_+ the corrections in (3.132) reproduce the results of [6] at leading order.

Chapter 4

Statistical properties of sYM theory spectra

While we have analytic control over the spectrum of scaling dimensions in the planar limit of $\mathcal{N} = 4$ sYM theory due to integrability, at finite N there are only partial results, e.g. from perturbation theory as discussed in the previous chapter. Exact results can, in most cases, only be found for short operators by directly diagonalising the corresponding mixing problem. For a given integer rank N of the gauge group, such a direct diagonalisation gives access to numerical spectra of the theory which can be analysed by statistical means. Studying generic features of numerical spectra gives insight into the underlying theory where an analytical description is not yet achieved. Such a statistical analysis of spectra was first performed by Wigner to describe the spectra of large atomic nuclei, and, as they resemble spectra of random matrices, this in turn initiated the development of random matrix theory. In Section 4.1 we briefly review random matrix theory, its connection to quantum chaos, and the statistical analysis of numerical spectra. In Section 4.2 we then study spectra of both deformed and undeformed $\mathcal{N} = 4$ sYM theory and show that in the planar limit the spectral distribution is Poisson, consistent with integrability, while at finite N the distribution corresponds to that of random matrix theory.

4.1 Random matrix theory and quantum chaos

4.1.1 Review of random matrix theory

Random matrix theory for heavy nuclei. After being introduced by Wishart [176] in 1928, the study of random matrices gained impetus in the early 50's with Wigner's realisation that random matrix theory (RMT) can be used to describe statistical properties of spectra of large atomic nuclei [33]. With large nuclei being complex strongly-interacting many-body systems, analytical control over their spec-

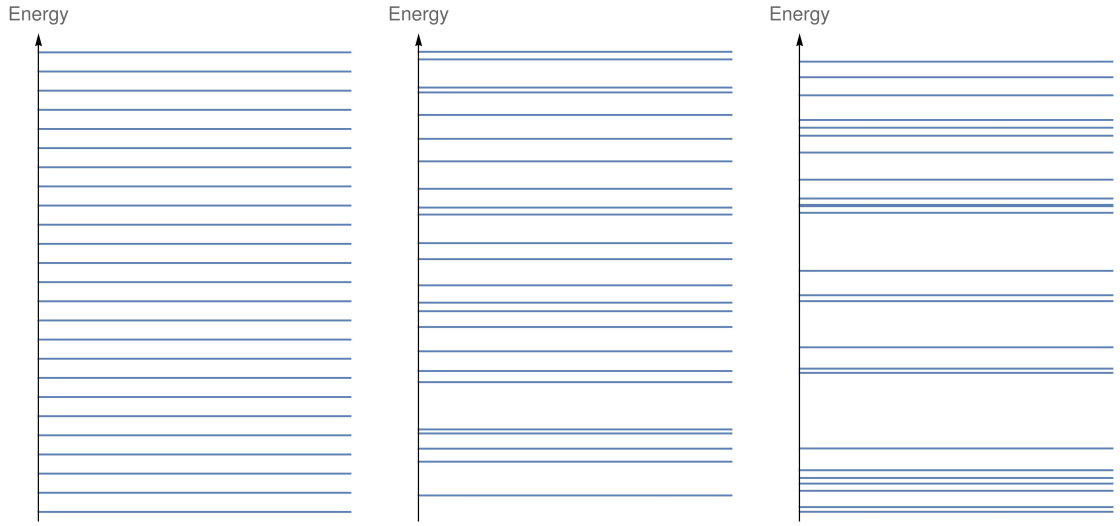


Figure 4.1: Examples of energy spectra corresponding to the (i) uniform, (ii) GOE and (iii) uncorrelated case, normalised to the same mean level spacing.

tra was and remains a seemingly impossible endeavour. Instead of aiming at a detailed understanding of the exact location of nuclear energy levels, Wigner concentrated his analysis on their overall statistical features and realised that certain properties of experimental nuclear spectra can be modelled by RMT. This initiated the development of RMT, in particular by Wigner [33,177–179], Dyson and Mehta [180–186] and others[1], and remains an important field of research with a broad range of applications.

One advantage of the method of spectral statistics is that it allows for the study of the general properties of a system, while avoiding the exact analytic solution of the spectral problem. As outlined by Dyson [180] the idea instead is *"to renounce exact knowledge [..] of the nature of the system itself"* and *"picture a complex nucleus as a black box in which a large number of particles are interacting according to unknown laws"*. Then the route to obtaining an understanding of the physics of the corresponding model is to find the ensemble of random matrices that contains the unknown Hamiltonian, and study the general properties of spectra in this ensemble. According to the ergodic principle, the average spectral properties of this ensemble (ensemble average) are the same as those of a single spectrum of a member in this ensemble (spectral average). If the system's Hamiltonian is a generic member of the ensemble, it will share these statistical features. For the case of heavy nuclei, and many more physical systems, the ensemble which gives a remarkably good description of the spectra is known as the Gaussian orthogonal ensemble (GOE). Figure (4.1) contains an example of a GOE spectrum in the middle, which is contrasted by a uniform spectrum on the left and a spectrum with uncorrelated energy levels on the right. In the following we will review the Gaussian orthogonal ensemble before discussing the

[1]See [187] for a collection of important publications in this field before 1965.

relevance of these spectra in the context of quantum chaos and integrability.

Gaussian ensembles. A general random matrix H is a square matrix whose entries H_{ij} are chosen randomly from a given probability distribution $\mathcal{P}[\{H_{ij}\}]$. Its eigenvalues $\{E_i\}$ are thus also random and the central problem in RMT is to relate the choice of distribution $\mathcal{P}[\{H_{ij}\}]$ to the resulting distribution of eigenvalues $\mathcal{P}[\{E_i\}]$ and its statistical properties. An important role in the development of RMT and its application to physical systems is played by the Gaussian ensembles and in particular the Gaussian orthogonal ensemble.

A general matrix H in the GOE is a real symmetric $n\times n$ matrix whose independent entries $H_{ij} = H_{ji}$, $i \leq j$, are randomly chosen from a Gaussian distribution, with the joint probability distribution of the entries being

$$\mathcal{P}_{\text{GOE}}[\{H_{ij}\}] = \mathcal{N} \exp\left(-\frac{\sum_i H_{ii}^2 + 2\sum_{i<j} H_{ij}^2}{(2\sigma)^2}\right) \tag{4.1}$$

with constants $\mathcal{N}$ and σ. Note that $\mathcal{P}_{\text{GOE}}$ factorises into Gaussian distributions for each entry of H which can thus be sampled independently. This property makes the GOE easy to study numerically as members of the ensemble can be generated rather simply via appropriate computer software. A further property of the distribution (4.1) is that it is invariant under orthogonal transformations R, with $R^{-1} = R^T$, acting as $H \to R^T H R$. This can easily be seen from rearranging the exponent of $\mathcal{P}_{\text{GOE}}$ as $\sum_i H_{ii}^2 + 2\sum_{i<j} H_{ij}^2 = \sum_{i,j} H_{ij}^2 = \text{Tr}(H^2)$. This symmetry property simplifies the analytical treatment of the GOE and indeed it is among those ensembles for which $P[\{E_i\}]$ can be found exactly. What makes the GOE exceptional in RMT is that it is the only ensemble which combines these two properties of independent entries and invariance under orthogonal transformations as was shown by Porter and Rosenzweig [188]. This makes it an ideal starting point for studying more generic ensembles. Of particular interest are orthogonal matrix models $\mathcal{P} \sim \exp[-\text{Tr}(V(H))]$ which, for general potentials $V(H)$, are harder to study numerically due to highly correlated matrix entries H_{ij}.

The Gaussian orthogonal ensemble is also a representative of the larger class of classical Gaussian ensembles [180] whose probability distributions for matrix entries H_{ij} all factorise and which are classified according to the symmetry group of R leaving $\mathcal{P}[\{H_{ij}\}]$ invariant as $H \to R^{-1} H R$. They are

- the Gaussian orthogonal ensemble (GOE):
  ensemble of real symmetric random matrices with $R \in O(n)$,
- the Gaussian unitary ensemble (GUE):
  ensemble of complex hermitian random matrices with $R \in U(n)$, and

ensemble	β_D	Hamiltonian	symmetry	physical system
GOE	1	real symmetric	$O(n)$	1) time-reversal invariant & even spin 2) time-reversal invariant & odd spin & rotational invariant
GUE	2	complex hermitian	$U(n)$	not time-reversal invariant
GSE	4	quaternionic	$Sp(n)$	time-reversal invariant & odd spin & not rotational invariant

Table 4.1: Classical Gaussian ensembles.

- the Gaussian symplectic ensemble (GSE):
  ensemble of complex quaternionic random matrices with $R \in Sp(n)$.

The symmetry properties of the ensembles correspond directly to physical properties of the systems they model: e.g. a Hamiltonian of a system with both time-reversal and rotational invariance can be chosen real symmetric in a suitable basis and is thus a member of the GOE. In contrast, if there is no rotational invariance and the system has half-integer spin, then the appropriate ensemble is GSE, and for a system with no time-reversal invariance it is GUE, cf. Table 4.1. The GOE, GUE and GSE are also known as the "classical" Gaussian ensembles and were supplemented by further "non-canonical" ensembles in [189].

The probability distribution function of the spectrum of eigenvalues of matrices in the three Gaussian ensembles can be computed from $\mathcal{P}[\{H_{ij}\}]$ and jointly written as [180, 188]

$$\mathcal{P}_{\beta_D}(E_1, ..., E_n) = \mathcal{N} \exp\left(-\frac{\beta_D}{2}\sum_{j=1}^{n} E_j^2\right) \prod_{j<k} |E_j - E_k|^{\beta_D} \tag{4.2}$$

with normalisation constant $\mathcal{N}$ and the Dyson index β_D taking different values for the different ensembles according to Table 4.1. This function (4.2) is the starting point for computing ensemble expectations for spectral observables which can then be compared with numerical spectra. We discuss spectral observables in Section 4.1.2, but firstly we review the occurrence of RMT in the context of quantum chaos.

BGS and Berry–Tabor conjecture. Since Wigner's realisation that RMT can be used to describe spectra of complex many-body systems like those of large nuclei, more applications both in physics and mathematics were found. For instance, in mathematics an interest in RMT arose from the realisation [190, 191] that the distribution of zeroes of the Riemann zeta function resemble the spectrum of a certain ensemble of random matrices. In physics RMT has become more popular due to its relation with quantum chaos [35].

In classical mechanics, the definition of chaos usually relies on a trajectory $X(t)$ describing the time evolution of a system and its dependence on the initial conditions. If for a small change in the initial conditions the difference between the original $X_1(t)$ and the new path $X_2(t)$ differs exponentially in time, i.e.

$$|X_1(t) - X_2(t)| \sim e^{\lambda_L t} \tag{4.3}$$

with so-called Lyapunov exponent $\lambda_L > 0$, the system is called classically chaotic. In contrast, for regular systems trajectories only separate linearly in time.

While chaos in most classical-mechanics systems is well-defined in terms of the Lyapunov exponent, the notion of chaos in quantum systems is less understood. Here the concept of a trajectory usually has no quantum analogue and thus a quantum version of the Lyapunov exponent – the out-of-time-order correlator (OTOC) – was only recently introduced [192]. A more traditional description of quantum chaos is based on the statistical features of a system's spectrum. This is connected with the observation that the transition to chaos in classical systems is usually accompanied by the development of level repulsion in the spectra of the corresponding quantum models, i.e. levels are correlated and avoid each other. In contrast, regular systems often have spectra with no level repulsion and so the levels are independent, see e.g. [193–195]. Figure 4.1 contains examples of spectra with different strengths of level repulsion: The uniform spectrum (i) and uncorrelated spectrum (iii) have maximal and zero level repulsion, respectively, while level repulsion in a GOE spectrum is intermediate. In addition to these observations, it was realised that the spectra of a number of quantum models with classically-chaotic limits are well-described by RMT, in a similar fashion to complex many-body systems like heavy nuclei. These developments culminated in the Bohigas–Giannoni–Schmidt (BGS) conjecture [35] that all spectra of quantum-chaotic systems with known chaos in the classical case are described by RMT, specifically the GOE for time-reversal invariant systems according to Table 4.1. Over the years, in lieu of a good definition of quantum chaos, this statement was reversed and the appearance of RMT features in spectra was, and is continued to be, used as a sign, or even definition, of quantum chaos. The complementary Berry–Tabor conjecture [36] relates integrable systems to the occurrence of uncorrelated energy levels in a system's spectrum. There are a number of successful attempts for proving these conjectures for specific models, e.g. in [36], but their scope is limited. Thus the connection between the statistical properties of spectra and quantum integrability/chaos of the underlying model remains conjectural. Nevertheless, they have been verified for a large number of cases like many integrable models such as the Heisenberg spin chain, the t-J model and the Hubbard model [196], as well as chaotic models such as quantum Sinai's billiard [35] or the Heisenberg chain with a random magnetic field [197].

Despite these successes, there are also a few exceptions to these conjectures, e.g. the Haldane–Shastry integrable spin chain whose spectrum is neither of the uncorrelated type, nor can it be described by one of the Gaussian ensembles [198]. Another example is the harmonic oscillator where the level repulsion is maximal and the levels are uniformly spaced as depicted in Figure 4.1 (i). Furthermore, in families of integrable models connected by continuous parameters, one can find non-integrable statistics at finely-tuned values of these parameters, e.g. in Richardson-Gaudin models [199] or the XXZ spin chain [200]. However in that case even small changes in the parameters result in a restoration of the integrable distribution.

To sum up, an understanding of the statistical features of a spectrum can give insights into the symmetry properties of the underlying system. In particular, uncorrelated levels in a spectrum are a sign of integrability, while the occurrence of level repulsion points towards quantum chaos. Moreover, since RMT relates the distribution of levels to invariances of the system as summarised in Table 4.1, identifying the RMT ensemble best describing a spectrum unveils a system's spacetime symmetries. In the following section we will move on to discussing observables that can be computed to obtain a better understanding of the statistical properties of a spectrum.

4.1.2 Level statistics observables

Degeneracies. When exploring a given numerical spectrum, one of the first properties one might identify is whether or not there are degeneracies. It was already noted in the early days of quantum mechanics by von Neumann and Wigner [201] that, given a generic theory depending on a number of parameters, it is necessary to tune at least two parameters to cause energy levels to cross and produce a degeneracy. Subsequently it was shown by Teller [202] that surfaces $E = E(\beta_1, \beta_2)$ representing energy levels depending on two such parameters β_1, β_2 are connected at points like the two sheets of a degenerate cone. The situation is quite different for systems with additional symmetries where degeneracies can occur between eigenstates with different quantum numbers with respect to these symmetries. Since these symmetries are often well-known, one desymmetrises a spectrum by separating it into sectors where eigenstates have the same quantum numbers before doing an analysis of its statistical features. Doing so, the symmetry-induced degeneracies do not overlay the correlations that can reveal the quantum chaos or integrability of the underlying model.

Global and local properties of spectra. To understand correlations in spectra we first introduce a quantity $\rho(E)$ that measures the level density at position E in

the spectrum and is defined as

$$\rho(E) = \frac{1}{n}\sum_{i=1}^{n}\delta(E - E_i) \tag{4.4}$$

for an ordered spectrum $E_1 \leq E_2 \leq ... \leq E_n$. It is normalised to $\int dE\ \rho(E) = 1$. Figure 4.2 (i) depicts the spectral density for a 2000×2000 real symmetric random matrix. Up to statistical fluctuations it follows Wigner's semi-circle law [179] which says that the limiting eigenvalue distribution $\bar{\rho}(E)$ of Gaussian ensembles for $n \to \infty$ is a semicircle

$$\bar{\rho}(E) = \frac{2}{\pi r^2}\sqrt{r^2 - E^2}\ . \tag{4.5}$$

However, this distribution of energy levels is uncommon for a physical system like for example a heavy nucleus. In fact, RMT does not describe the global properties of physical spectra which are theory-dependent and, for example, in a nuclear context differ for different nuclei. Similarly, the anomalous-dimension spectra in sYM theories, which we will discuss in detail in Section 4.2, do not follow the semi-circle law as illustrated in Figure 4.2 (ii) and (iii) which show the level density of a non-planar and planar spectrum, respectively, in the β-deformed theory. Instead, it is the fluctuations around the smoothed density which can be compared between different systems and which show universal features according to the BGS and Berry–Tabor conjecture.

In order to separate global and local properties of a spectrum we first introduce the integrated spectral density $I(E)$ as

$$I(E) = n\int_{-\infty}^{E}\rho(e)de \equiv \sum_{i=1}^{n}\Theta(E - E_i)\ . \tag{4.6}$$

It increases by one at the position of each level E_i and is thus also called the staircase function. Assuming that fluctuation properties of the spectrum do not depend on the region in the spectrum, we can separate global from local properties by decomposing $I(E)$ into an average and fluctuation part

$$I(E) = I_{\text{av}}(E) + I_{\text{fl}}(E)\ . \tag{4.7}$$

Next we map the original spectrum $\{E_i\}$ to a so-called unfolded spectrum $\{\varepsilon_i\}$ via

$$\varepsilon_i = I_{\text{av}}(E_i)\ . \tag{4.8}$$

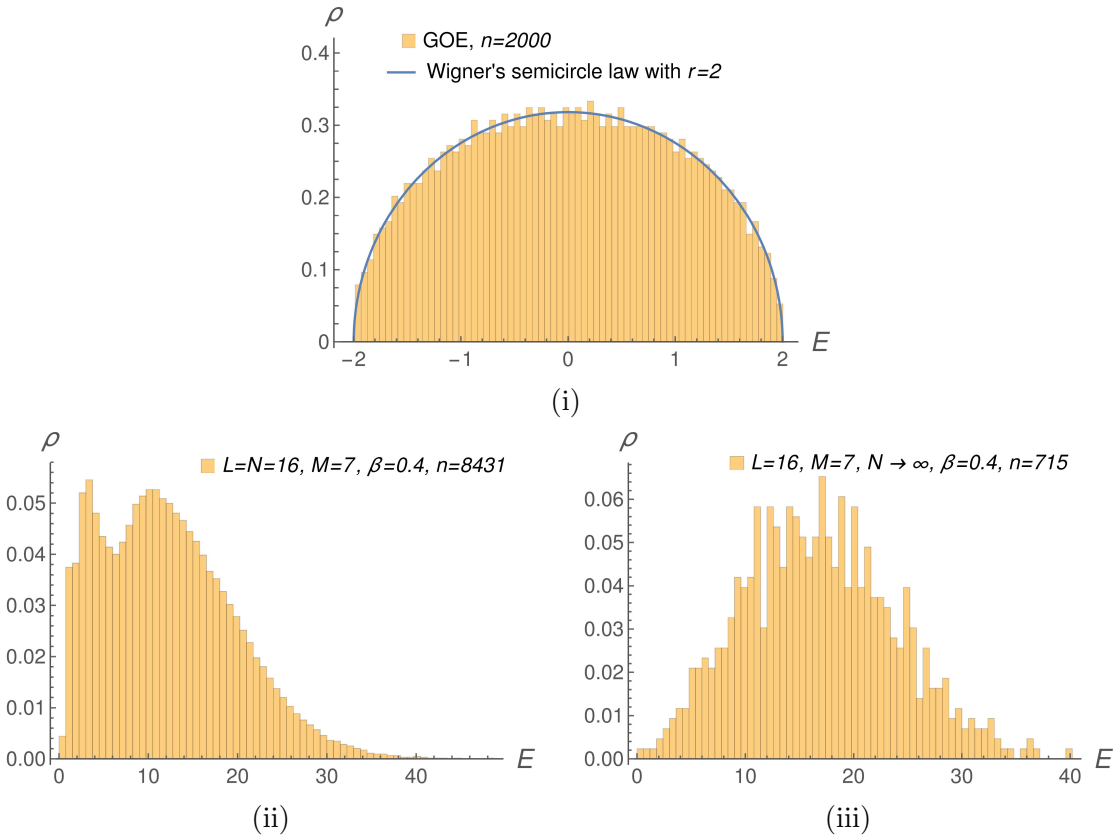


Figure 4.2: Spectral densities for (i) a real symmetric random matrix and Wigner's semicircle law, (ii) a non-planar and (iii) planar spectrum in the β-deformed theory.

These new variables capture the fluctuations of the original spectrum

$$\varepsilon_{i+1} - \varepsilon_i \simeq \frac{E_{i+1} - E_i}{D_i} \ , \tag{4.9}$$

with local spacing $D_i = (dI_{\rm av}(E_i)/dE)^{-1} = (n\rho_{\rm av}(E_i))^{-1}$, but have a constant mean level spacing over the whole spectrum

$$\langle \varepsilon_{i+1} - \varepsilon_i \rangle = 1 \ . \tag{4.10}$$

In the following we will concentrate on the statistical properties of this unfolded spectrum with unfolded level density $\hat{\rho}(\varepsilon)$ and staircase function $\hat{I}(\varepsilon)$ defined as in (4.4) and (4.6) but for the unfolded spectrum. Details of the unfolding procedure used for the anomalous dimension spectra are discussed in Appendix C.

Edges of spectra. RMT also makes predictions for the distribution of the largest and smallest eigenvalues of spectra in a given ensemble, the so-called Tracy–Widom distribution [203]. However, in a physical system the eigenstates with lowest and highest energies are usually more constrained than the states in the bulk of the spectrum, even when the system is chaotic. Therefore, in general they do not follow the

RMT prediction, but are more regular [204, 205]. We will observe this behaviour in the information entropy of anomalous dimension eigenstates in Section 4.2.4. In the meantime we focus on statistical properties in the bulk of spectra and discuss the clipping of low- and high-energy states for anomalous dimension spectra in Appendix C.

Nearest-neighbour spacing statistics. The main sign of the presence of chaos in a given spectrum is level repulsion. This phenomenon can be studied by looking at the distribution $P(s)$ of spacings

$$s_i = \varepsilon_{i+1} - \varepsilon_i \tag{4.11}$$

between consecutive levels in the desymmetrised and unfolded spectrum. $P(s)ds$ gives the probability for a nearest-neighbour spacing laying in the interval $[s, s + ds]$. For integrable systems it is generally the case that $P(s) \to 1$ as $s \to 0$ which reflects the presence of hidden symmetries in these models. They make eigenstates decouple and thus their energy levels become uncorrelated in line with the Berry–Tabor conjecture. Based on this independence of energy levels one can compute the corresponding nearest-neighbour spacing (NNS) distribution $P(s)$ as follows [206]: Given an energy level at position ε, the probability for finding the adjacent level at a spacing $[s, s + ds]$ is the product of the probability of finding no level in the interval $(\varepsilon, \varepsilon + s)$ and one level in $[\varepsilon + s, \varepsilon + s + ds]$. The latter is just given by the interval length ds for an unfolded spectrum with mean level spacing 1. The former can be computed by first partitioning the interval $(\varepsilon, \varepsilon + s)$ into m equal segments

$$\left[\varepsilon + k\frac{s}{m}, \varepsilon + (k+1)\frac{s}{m}\right] \ , \ k = 0...m-1 \ . \tag{4.12}$$

The probability for having no level in one of these segments is $(1 - s/m)$ and thus the probability for having no level in $(\varepsilon, \varepsilon + s)$ is given by

$$\lim_{m\to\infty} \left(1 - \frac{s}{m}\right)^m = \exp(-s) \ . \tag{4.13}$$

Putting everything together, the probability for finding two adjacent levels at a distance s is

$$P(s)ds = e^{-s}ds \tag{4.14}$$

and thus we find that the NNS distribution in the integrable case is given by the Poisson distribution

$$P(s) = e^{-s} \qquad \text{(integrable case)} \ . \tag{4.15}$$

That this distribution is a good description of integrable systems has been numerically shown in a range of models including many-body systems such as the Heisenberg spin chain, the t-J model at its integrable supersymmetric point, and the Hubbard model [196, 207]. There are significantly fewer analytical results, however one important result by Berry and Tabor [36] showed, on the basis of the existence of action-angle variables, that for a generic integrable semi-classical system the distribution of energy levels is indeed Poisson.

For a chaotic quantum system energy levels are correlated and so avoid each other, i.e. $P(s) \to 0$ as $s \to 0$. The exact NNS distributions follow from (4.2), however for most practical purposes they can be approximated by the $n = 2$ distributions ("Wigner's surmise"). Then $P(s)$ follows from

$$P(s) = \int dE_1 \; dE_2 \; \mathcal{P}(E_1, E_2)\delta(s - |E_1 - E_2|) \tag{4.16}$$

via simple integrations and, upon normalisation, one finds the distributions [187, 206]

$$\begin{aligned} P(s) &= \frac{\pi}{2} \; s \; \exp\left(-\frac{\pi}{4}s^2\right) && \text{(GOE)} \; , \\ P(s) &= \frac{32}{\pi^2} \; s^2 \; \exp\left(-\frac{4}{\pi}s^2\right) && \text{(GUE)} \; , \\ P(s) &= \frac{2^{18}}{3^6\pi^3} \; s^4 \; \exp\left(-\frac{64}{9\pi}s^2\right) && \text{(GSE)} \; . \end{aligned} \tag{4.17}$$

For small separations this implies $P(s) \sim s^{\beta_D}$ and thus the Dyson index β_D characterises the strength of the level repulsion in Gaussian ensembles, with GOE having the smallest and GSE having the strongest repulsion. Figure 4.3 shows the different NNS distributions for the Gaussian ensembles, as well as for the uncorrelated case and the uniform case, the latter having $P(s) = \delta(s - 1)$. In addition to a wide range of numerical evidence, there are also a small number of analytical results supporting the RMT predictions for physical models. In [208] the authors show that in chaotic semi-classical systems the NNS distribution can be approximated by (4.17). They also find an interpolating function $P(s)$ for semi-classical systems with both integrable and chaotic regions in phase space which interpolates between the Poisson distribution and Wigner's surmise. An alternative interpolation is given by the Brody-distribution [209]

$$P(s) = \Gamma\left(\frac{\omega+2}{\omega+1}\right)^{1+\omega}(1+\omega)s^{\omega}\exp\left(-\Gamma\left(\frac{\omega+2}{\omega+1}\right)^{1+\omega}s^{1+\omega}\right) \; , \tag{4.18}$$

which for $\omega \to 0$ gives the Poisson distribution (4.15), while for $\omega \to 1$ one finds the GOE prediction in (4.17).

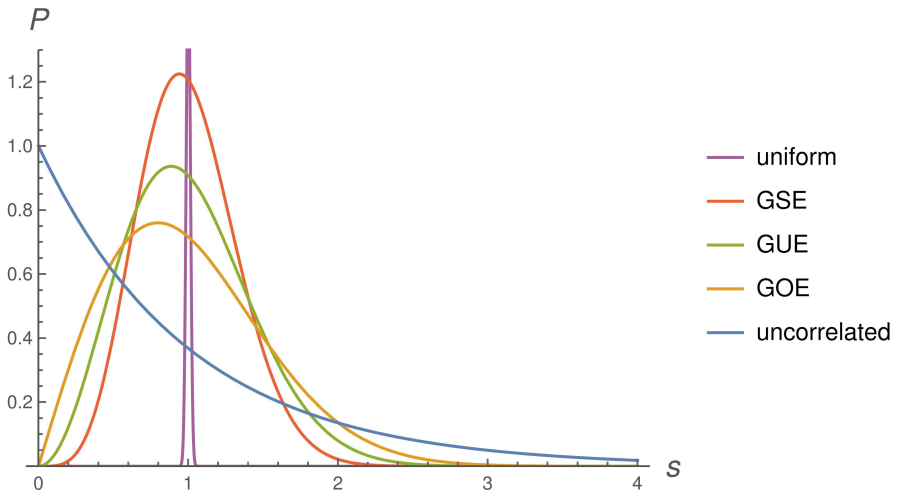


Figure 4.3: Nearest-neighbour spacing distribution $P(s)$ for the uncorrelated and uniform case, as well as the three classical Gaussian ensembles.

Spectral rigidity. While the NNS distribution is useful for probing short-range correlations in a spectrum, there are other observables measuring its long-distance properties, one of which is the spectral rigidity. In general a spectrum is said to be rigid when levels are regularly spaced, i.e. the uniform spectrum has maximum rigidity, while the Poisson spectrum is not rigid at all. This feature of spectra is, similarly to the NNS distribution, intertwined with the occurrence of level repulsion which makes spectra more regular and thus more rigid.

A measure for the rigidity of a spectrum is given by the Dyson–Mehta statistic Δ_3 introduced in [183][2]. It is defined as

$$\Delta_3(l) = \frac{1}{l} \left\langle \min_{A,B} \int_{\varepsilon_0}^{\varepsilon_0+l} d\varepsilon (\hat{I}(\varepsilon) - A\varepsilon - B)^2 \right\rangle \ , \tag{4.19}$$

where the expression inside the angle brackets computes the least-square deviation of the unfolded staircase function $\hat{I}(\varepsilon)$ from the best straight line $A\varepsilon + B$ fitting it in the interval $[\varepsilon_0, \varepsilon_0 + l]$. The bracket $\langle .. \rangle$ denotes an average over values of ε_0 taken from a discretisation of the interval $[\varepsilon_1, \varepsilon_m - l]$. Increasing the interval length l increases the number of probed energy levels and thus this measure probes long-distance correlations. In an unfolded spectrum $\{\varepsilon_i\}$ the mean level spacing is 1 and this makes the statistic $\Delta_3(l)$ only meaningful for sufficiently large interval lengths $l \gtrsim 1$.

Figure 4.4 shows the Dyson–Mehta statistic as a function of the interval length l

[2]The index on this quantity was introduced to distinguish three different measures for the rigidity proposed in this paper. Δ_3 was established as the most useful statistic and has since been used as the most common measure for rigidity. In this work we carry over this established convention. In this chapter all spectra of scaling dimensions $\{\Delta_i\}$ are denoted by $\{E_i\}$, or $\{\varepsilon_i\}$ for the unfolded case, to minimise confusion.

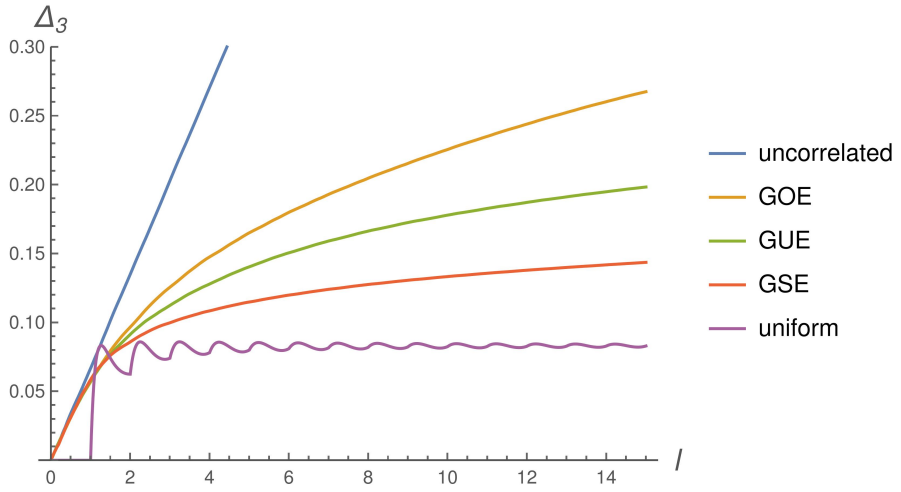


Figure 4.4: Dyson–Mehta statistic $\Delta_3(l)$ for the uncorrelated and uniform case, as well as the three classical Gaussian ensembles.

for an uncorrelated and uniform spectrum, as well as for the three Gaussian ensembles. For uncorrelated fluctuations Δ_3 grows linearly in l, specifically

$$\Delta_3(l) = \frac{l}{15} \qquad \text{(uncorrelated spectrum)} \ , \tag{4.20}$$

while for the uniform case it approaches a constant value for sufficiently large interval lengths l,

$$\Delta_3(l) \simeq \frac{1}{12} \qquad \text{(uniform spectrum)} \ . \tag{4.21}$$

For more general spectra the Dyson–Mehta statistic, and thus the spectral rigidity, lies between these two extremal cases. For the Gaussian ensembles the expectation for $\Delta_3(l)$ is shown in Figure 4.4. For small interval lengths $l < 1$ they follow the uncorrelated behaviour (4.20), but for large l one finds the slower growth [206]

$$\begin{aligned}
\Delta_3(l) &= \frac{1}{\pi^2}\left(\ln(2\pi l) + \gamma_E - \frac{5}{4} - \frac{\pi^2}{8}\right) + \mathcal{O}(l^{-1}) \qquad \text{(GOE)} \ , \\
\Delta_3(l) &= \frac{1}{2\pi^2}\left(\ln(2\pi l) + \gamma_E - \frac{5}{4}\right) + \mathcal{O}(l^{-1}) \qquad \text{(GUE)} \ , \\
\Delta_3(l) &= \frac{1}{4\pi^2}\left(\ln(2\pi l) + \gamma_E - \frac{5}{4} + \frac{\pi^2}{8}\right) + \mathcal{O}(l^{-1}) \qquad \text{(GSE)} \ .
\end{aligned} \tag{4.22}$$

The Dyson–Mehta statistic thus quantifies the increasing level repulsion which results in an increasing rigidity when going from GOE to GUE and GSE.

The Dyson–Mehta statistic is widely used as a measure for long-range correlations in spectra, see e.g. [183] for its application to atomic nuclei spectra, or [210] where it was used in the study of the XXZ spin chain. In [211] the expected behaviour (4.20)

and (4.22) was shown for integrable and chaotic semi-classical theories, respectively, up to some non-universal length $l_{\max}$ which grows with the number of degrees of freedom in the system.

For an efficient evaluation of the expression (4.19) over a large spectrum, it is useful to parametrise the levels

$$\varepsilon_0 + l z_i \tag{4.23}$$

by the real numbers $\{z_i\}$ with $0 < z_1 < ... < z_k < 1$, so that (4.19) simplifies to

$$\Delta_3(l) = \langle 6 s_1 s_2 - 4 s_1^2 - 3 s_2 + s_3 \rangle \tag{4.24}$$

with [212]

$$s_1 = \sum_{i=1}^{k} z_i \ , \qquad s_2 = \sum_{i=1}^{k} z_i^2 \ , \qquad s_3 = \sum_{i=1}^{k} (2k - 2i + 1) z_i \ . \tag{4.25}$$

Other spectral observables. The Dyson–Mehta statistic is related to other statistical measures like the number variance and spectral form factor. These relations can be made manifest by realising that all of these measures are essentially probing two-level correlations in the spectrum. The general k-level correlation function $R_k(E_1, E_2, ..., E_k)$ is the probability density for finding k energy levels at positions $E_1, E_2, ..., E_k$, irrespective of the positions of the remaining levels. Thus we obtain it by integrating out $n - k$ levels from the energy distribution function $\mathcal{P}(\{E_i\})$

$$R_k(E_1, E_2, ..., E_k) = \frac{n!}{(n-k)!} \int dE_{k+1} ... dE_n \ \mathcal{P}(E_1, E_2, ..., E_k, E_{k+1}, ..., E_n) \tag{4.26}$$

with a combinatorial prefactor accounting for the fact that R_k is insensitive to which of the n levels lie at positions $E_1, E_2, ..., E_k$.

The two-point correlation $R_2(E_1, E_2)$, which is the probability function for finding any two levels at positions E_1 and E_2, effectively only depends on a single variable $E = |E_2 - E_1|$, the distance between the two levels, in a translation-invariant unfolded spectrum. The Dyson–Mehta statistic is related to the two-point correlation function via

$$\Delta_3(l) = \frac{l}{15} - \frac{1}{15 l^4} \int_0^l dE \ (l - E)^3 (2l^2 - 9lE - 3E^2)(1 - R_2(E)) \ , \tag{4.27}$$

see e.g. [213].

Another measure that is closely related to both the Dyson–Mehta statistic and the two-point correlation function is the number variance $\Sigma^2(l)$, which measures the variance of the number of eigenvalues in an interval of length l and is thus another

measure for the rigidity of a spectrum. It is based on the level density (4.4) via $\eta(\varepsilon_0, l)$

$$\eta(\varepsilon_0, l) = \int_{\varepsilon_0}^{\varepsilon_0+l} d\varepsilon \ \hat{\rho}(\varepsilon) \ , \tag{4.28}$$

which measures the number of states in an interval $[\varepsilon_0, \varepsilon_0+l]$. In an unfolded spectrum the average of $\eta(\varepsilon_0, l)$ is independent of ε_0, i.e.

$$\langle \eta(\varepsilon_0, l) \rangle = 1 \ , \tag{4.29}$$

where we again average over values of ε_0 taken from a discretised interval $[\varepsilon_1, \varepsilon_n - l]$. The number variance is then

$$\Sigma^2(l) = \langle \eta(\varepsilon_0, l)^2 \rangle - l^2 \ . \tag{4.30}$$

In the uniform case it vanishes, while for uncorrelated spectra it is $\Sigma^2(l) = l$ and thus there are $l \pm \sqrt{l}$ levels in an interval of length l in the Poisson case. For the three Gaussian ensembles it lies between these two extremes and the exact RMT predictions can be found in [206]. The relation to the Dyson–Mehta statistic is given e.g. in [213].

Yet another measure that is often employed to measure correlations in a spectrum is the spectral form factor $K(t)$. It is based on the spectral auto-correlation function

$$C(l) = \langle \hat{\rho}(\varepsilon_0) \hat{\rho}(\varepsilon_0 + l) \rangle - 1 \tag{4.31}$$

and then the spectral form factor is the Fourier transform

$$K(t) = \int_{-\infty}^{+\infty} dl \ C(l) \exp(-ilt) \ . \tag{4.32}$$

Its relation to $\Delta_3(l)$ can be found in [213].

Statistics of eigenvectors. So far, we have only introduced observables on the set of eigenvalues of a spectral problem. If the set of eigenvectors is accessible in a given physical system, one might wonder whether they can also be used to explore its properties. RMT makes a number of predictions for the distribution of eigenstates, in particular that they spread out over any non-finely tuned reference basis, i.e. they are delocalised. As measure for this spreading one can use the information entropy (also known as Shannon entropy)

$$S_i = -\sum_{a=1}^{n} |c_{ia}|^2 \ln |c_{ia}|^2 \tag{4.33}$$

for a given eigenvector $|E_i\rangle$ decomposed with respect to a reference basis $\{|a\rangle\}$, i.e. $|E_i\rangle = \sum_{a=1}^{n} c_{ia} |a\rangle$ with coefficients c_{ia}. For a normalised eigenvector $|E_i\rangle = \sum_{a=1}^{n} \frac{1}{\sqrt{n}} |a\rangle$ which is equally distributed over an orthonormal basis $|a\rangle$ the information entropy is maximal

$$S_i = \ln(n) \qquad \text{(evenly spread over whole basis)} \ . \tag{4.34}$$

In contrast, if an eigenstate is pure, i.e. localised on one basis state such that $c_{ia} = \delta_{ia}$, the information entropy vanishes

$$S_i = 0 \qquad \text{(pure)} \ . \tag{4.35}$$

In the intermediate case in which $|E_i\rangle$ is exponentially localised over $1 \ll n_i \ll n$ basis states, the information entropy is

$$S_i = \ln(e\ n_i) + \mathcal{O}(n_i^{-1}) \qquad \text{(exponentially localised)} \ . \tag{4.36}$$

Thus, roughly speaking, the information entropy of a state is the logarithm of the number of basis vectors it is spread over. The RMT prediction is that

$$S_{\text{GOE}} = \ln(2e^{\gamma_E - 2} n) + \mathcal{O}(n^{-1}) \simeq \ln(0.48n) \qquad \text{(GOE)} \ , \tag{4.37}$$

with the Euler–Mascheroni constant γ_E, for all GOE eigenvectors in the large-n limit, see e.g. [214]. Thus, the information entropy of the eigenvectors of a spectral problem can be used to expose the possible chaotic nature of a system. A statistical observable which is based on the information entropy of eigenstates is the average localisation length $\langle n_S \rangle$ defined via

$$\langle n_S \rangle = \exp(\langle S_i \rangle - S_{\text{GOE}}) \ , \tag{4.38}$$

where $\langle S_i \rangle$ denotes the mean of the information entropy on the set of eigenvectors.

4.2 Level statistics for sYM theories

We now move on to the level statistics of one-loop anomalous dimension spectra in $\mathcal{N} = 4$ sYM theory and its β-deformed version. We begin by discussing the specific spectra to be analysed and how they are desymmetrised in Section 4.2.1. Appendix C contains further details on how they are set up for a level statistics analysis. Then we move on to the study of the short- and long-range correlations in these spectra in Sections 4.2.2 and 4.2.3, as well as of the eigenstates of the spectral problem in Section 4.2.4. We close with a discussion of the results in Section 4.2.5.

4.2.1 Desymmetrisation

In order to analyse the spectrum of anomalous dimensions, we must first focus on specific sectors of operators whose mixing is not forbidden due to the symmetries of the dilatation operator. We will henceforth restrict to the rank-one sectors $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$, which are both closed at one-loop order. The $\mathcal{N}=4$ sYM dilatation operator in these sectors is given in (1.36) and (1.38). It does not mix operators with different numbers of fields L and excitations M and S, respectively, and thus we organise operators with respect to these quantum numbers. Furthermore, the dilatation operator in the $\mathfrak{su}(2)$ sector has a residual $SU(2)$ R-symmetry which arranges operators in terms of primary and descendant fields and the former do not mix with the latter. Thus, in order to desymmetrise the spectrum, we only work with primary states $\mathcal{O}$, satisfying $J_{-}\mathcal{O}=0$ with $J_{-}X=Z$, which form the biggest sector and allow for better statistics. Similarly, the $\mathfrak{sl}(2)$ sector has an $SL(2)$ symmetry and so also here we only work with operators $\mathcal{O}$ with $S_{-}\mathcal{O}=0$, where $S_{-}Z^{(n)}=nZ^{(n-1)}$. An additional symmetry in both these sectors is parity $\mathcal{P}$ which reverses the order of fields within a trace

$$\mathcal{P}\ \mathrm{Tr}(\chi_1\dots\chi_L)(x)=\mathrm{Tr}(\chi_L\dots\chi_1)(x)\ , \tag{4.39}$$

generalising the spin-chain version given in (3.30), and so eigenstates with different parity are uncorrelated. Therefore, we consider operators with definite parity and all spectra analysed in this work correspond to the bigger positive-parity sector, with the similar analysis of negative-parity sectors yielding the same results but with poorer statistics. To complete the desymmetrisation of the mixing matrix, we remove all zero energies corresponding to protected states whose dimensions are fixed by supersymmetry.

In the planar limit, there are additional symmetries we must account for. In particular the number of traces in a given operator is conserved under the action of the dilatation operator and so we must work at fixed number of traces. In the single-trace sector this reduces the problem to essentially that of an integrable spin chain. In the $\mathfrak{su}(2)$ sector this is the usual Heisenberg spin chain (3.3) and in the $\mathfrak{sl}(2)$ sector its non-compact $s=0$ version. Therefore, in order to obtain the planar single-trace spectra, one can solve the corresponding Bethe equations (3.19) and (3.35). In the case of the $\mathfrak{sl}(2)$ sector this method allows one to go to relatively large sectors quite easily and specifically we will work with the $L=18,\ S=7$ primary spectrum comprising 6 804 states. In contrast, the $\mathfrak{su}(2)$ Bethe equations cannot be solved for high L and M that easily and direct diagonalisation is more efficient.

Next to an analysis of pure $\mathcal{N}=4$ sYM theory spectra, we will also consider spectra in the β-deformed theory in the $\mathfrak{su}(2)$ sector with dilatation operator (1.67). Also here the excitation number M is a conserved quantity, however the corresponding

$U(1)$ symmetry is not part of a larger $SU(2)$ symmetry. Thus, sectors of fixed L and M do not further decompose into primary and descendant states and this allows for better statistics than in the undeformed case.

Level repulsion in the β-deformed theory. The lifting of degeneracies in the spectrum of one-loop dimensions by the β-deformation is related to the phenomenon of level repulsion. In general, energy surfaces depending on a number of parameters are connected only at special points where multiple parameters are tuned [201]. For the spectrum of $\mathcal{N} = 4$ sYM theory this implies that operator dimensions depending on parameters λ and N avoid crossing for generic fixed values of N as λ is varied, as was borne out in [215]. In our case, being at one-loop order, the λ-dependence is trivial, however we can study the spectrum as a function of both the deformation parameter β and the rank N of the gauge group. By numerically solving for the eigenvalues of specific families of operators, we can see the behaviour of the scaling dimensions as we vary β and as an example we consider the $L = 6,\ M = 3$ states in Figure 4.5. This sector has twelve eigenstates, three of which are protected in the β-deformed theory and we only plot the eigenvalues of the remaining nine states. The top (purple) line and fourth from the top (brown) correspond in the undeformed planar limit to descendants of two single-trace two-impurity states. The second and third lines (yellow and light blue) correspond to the single-trace three-impurity singular solution and a degenerate double-trace operator. The remaining operators are protected in the undeformed theory but acquire non-vanishing anomalous dimensions for non-vanishing β. For finite values of N the energy levels mostly repel and even at points where they appear to come close they do not in fact cross, maintaining a separation of $\sim 1/N^2$. There is one obvious exception which clearly does cross other levels at finite N. This is a double-trace state that does not mix with other operators, receives no $1/N$ corrections and so is effectively uncorrelated with the other states. This is due to the fact that at half-filling the charge conjugation transformation $Z \leftrightarrow X$ combined with the parity transformation (4.39) is a symmetry which commutes with both the impurity number and the one-loop non-planar dilatation operator. This double-trace operator is the only $L = 6, M = 3$ state with negative charge with respect to this transformation. This again points to the fact that, in order to avoid trivial crossings, we must consider operators which have the same quantum numbers, i.e. desymmetrise the spectrum. At large values of N we can see the appearance of crossings at special values where $\beta/\pi \in \mathbb{Q}$; for example in Figure 4.5 there are crossings in the planar limit at $\beta = \pi/4$ and $\beta = \pi/6$. These points correspond to values where the β-deformed theory becomes equivalent to an orbifold of $\mathcal{N} = 4$ sYM theory, see e.g. [216].

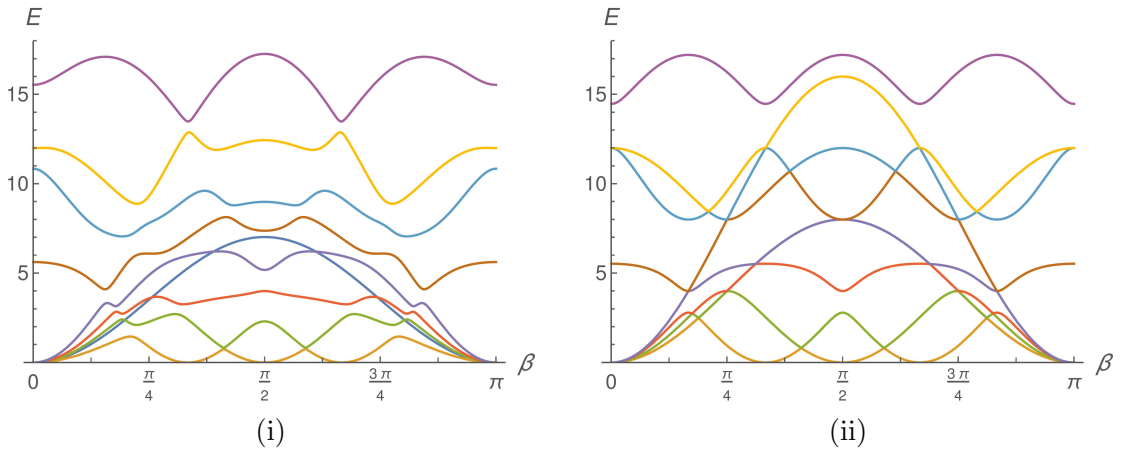


Figure 4.5: Nine eigenvalues corresponding to states with $L = 6, M = 3$ as functions of $\beta \in [0, \pi]$ with (i) $N = 7$ and (ii) $N = 10^6$.

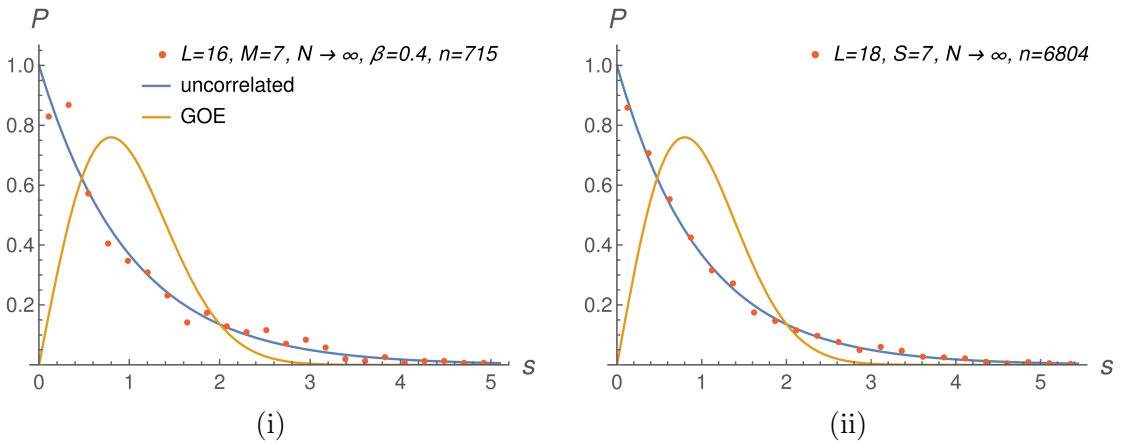


Figure 4.6: Nearest-neighbour spacing distribution for planar spectra in the (i) β-deformed $\mathfrak{su}(2)$ and (ii) undeformed $\mathfrak{sl}(2)$ sector.

4.2.2 Short-range correlations

After numerically computing desymmetrised spectra $\{E_1, E_2, ..., E_n\}$ by direct diagonalisation or integrability techniques, we perform an unfolding procedure to separate the fluctuations from the overall energy behaviour of the spectrum. Details about this data preparation are given in Appendix C. Then we label the unfolded spectrum of anomalous dimensions from smallest to largest, i.e. $\varepsilon_1 \leq \varepsilon_2 \leq ... \leq \varepsilon_n$, and estimate the distribution of level spacings between consecutive levels by computing nearest-neighbour spacings $s_i = \varepsilon_{i+1} - \varepsilon_i$, then sorting the data into bins and calculating the fraction that occur in each bin. The results naturally depend on the unfolding procedure and bin size and a choice is made such that small changes do not significantly affect the overall results, see Appendix C for more details. The estimate of the probability distribution naturally improves with larger numbers of states and so one must compute the dimensions for relatively long operators.

M	n	ω	α
2	400	0.70	0.44
3	1035	0.91	0.90
4	2316	0.96	0.89
5	4198	0.98	0.91
6	6539	0.97	0.98
7	8431	0.95	0.93

Table 4.2: NNS distribution of the β-deformed spectrum ($\beta = 0.4$) of $L = 16$ states with M excitations and size n of the Hilbert space fitted to the Brody distribution with parameter ω and the Wigner–Dyson distribution with parameter α.

Planar results. We first look at the strict planar limit and look at spectra of single-trace operators in the β-deformed $\mathfrak{su}(2)$ and undeformed $\mathfrak{sl}(2)$ sector in Figure 4.6. The numerical data seem to be well described by the Poisson distribution, reflecting the integrability of the planar theory which has additional hidden symmetries that make eigenstates decouple and thus their eigenvalues are independent. To be more quantitative, we fit the data to the Brody distribution (4.18), which interpolates between the GOE distribution ($\omega = 1$) and the Poisson distribution ($\omega = 0$). We find that the best fit data are

$$\omega_{\mathfrak{su}(2)} = 0.05 \ , \qquad\qquad \omega_{\mathfrak{sl}(2)} = 0.06 \ , \tag{4.40}$$

compatible with the Poisson distribution. One can perform a similar level statistics analysis for multi-trace operators and also in these sectors we find that the NNS distribution is Poissonian.

Non-planar results for different excitations M. In Figure 4.7 we present the results for $L = N = 16$ states in the β-deformed theory and increasing excitation number $M = 2...7$. By visual inspection it is apparent that the GOE Wigner–Dyson distribution closely matches the data for most values of M. To be more quantitative, one can fit the data to the Brody distribution (4.18) and the Wigner–Dyson distribution

$$P_\alpha(s) = 2\frac{\Gamma(1+\alpha/2)^{1+\alpha}}{\Gamma((1+\alpha)/2)^{2+\alpha}} s^\alpha \exp\left(-\frac{\Gamma(1+\alpha/2)^2}{\Gamma((1+\alpha)/2)^2} s^2\right) \ , \tag{4.41}$$

which interpolates between the different Gaussian ensembles (4.17) for $\alpha = \beta_D$, in particular $\alpha = 1$ for GOE. In Table 4.2 we show the best fit values of ω for different values of M which are generally close to 1 suggesting that this is the appropriate value for the distribution at relatively small values of N. This fit captures the Gaussian behaviour of the exponential decay of the tail and the fact that the distribution goes

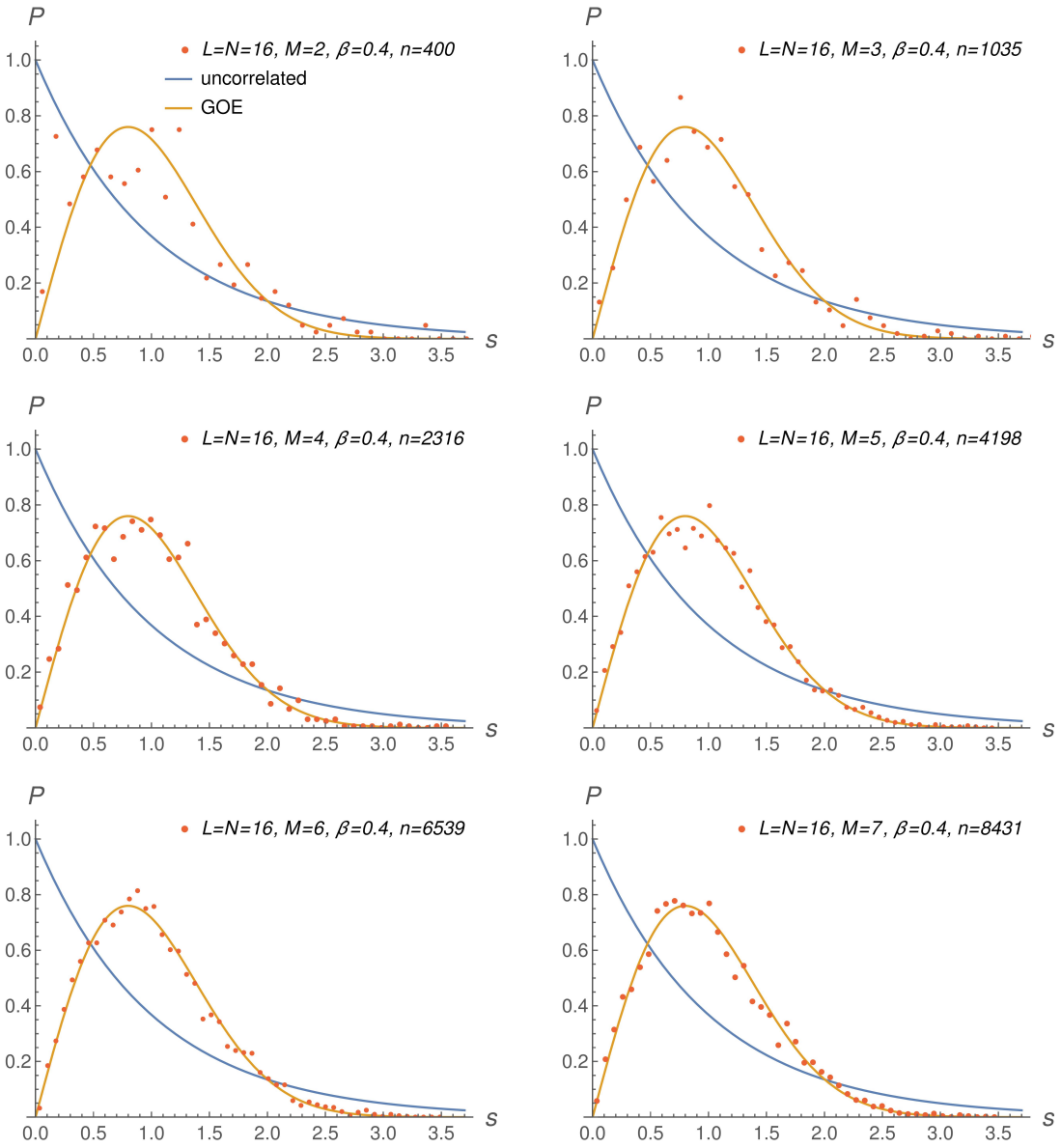


Figure 4.7: Nearest-neighbour spacing distribution for non-planar spectra in the β-deformed $\mathfrak{su}(2)$ sector and increasing excitation number M.

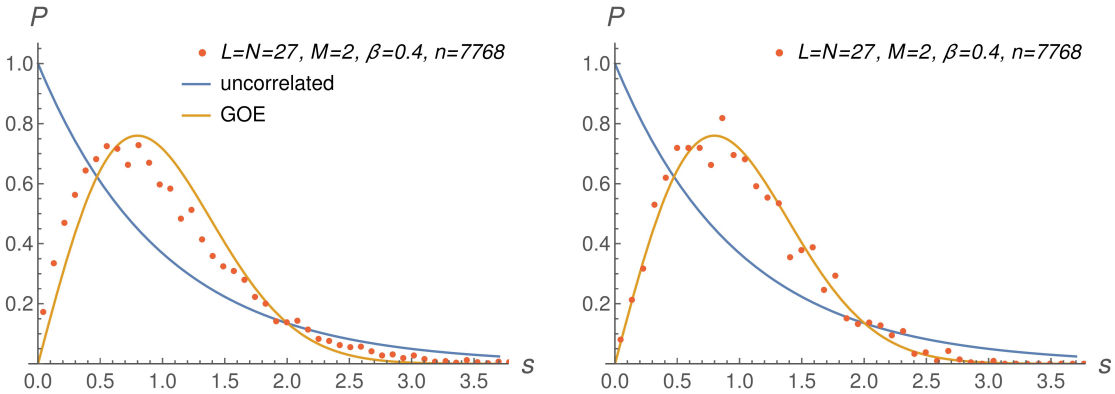


Figure 4.8: Nearest-neighbour spacing distribution for a non-planar β-deformed spectrum with different clippings of states at the boundary: In (i) 8% (4%) of low-(high-) energy states and in (ii) 35% on both sides of the spectrum are clipped.

to zero as $s \to 0$, i.e. the levels are correlated. If we assume the distribution is of the Wigner–Dyson form we can perform a fit to the general form (4.41) and find the best fit value of α which from Table 4.2 can again be seen to be approximately ~ 1. It is clear that the fit is better for higher excitation number M as the values for $M = 2$ are furthest from those of the GOE. The values for $M = 0$, which are protected operators, and $M = 1$, which are protected in the undeformed theory, clearly do not fit the Wigner–Dyson distribution. In the case of $L = 16$, $M = 2$ the poor fit of the Wigner–Dyson distribution is not just an effect of the poor statistics, as one can go to longer operators with $M = 2$ where one finds similar behaviour, see Figure 4.8 (i). In fact, the distribution seems to be a superposition of the Poisson distribution and the Wigner–Dyson distribution with $\omega = 0.71$. Interestingly, when clipping more states from the edges of the spectrum, one finds the GOE distribution becomes a good fit, see Figure 4.8 (ii) where we only do statistics on a third of the states in the middle of the spectrum and find $\omega = 0.92$. The fact that states at the edges of spectra of chaotic systems typically are more regular than the ones in the bulk is well-known, but we do not have a clear explanation for why the fraction of regular states is so high in the case of small excitation number M. In general however we find that the GOE describes the non-planar distribution of energy levels in the $\mathfrak{su}(2)$ sector of deformed $\mathcal{N} = 4$ sYM theory.

Non-planar results for different undeformed sectors. We can repeat the computation for the undeformed theory where we look at both the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ sector in Figure 4.9. Similarly to the deformed spectra, we encounter level repulsion and, more precisely, the fluctuations behave like those of the GOE and can be approximated by the Wigner–Dyson distribution with parameters

$$\alpha_{\mathfrak{su}(2)} = 1.01 \ , \qquad \alpha_{\mathfrak{sl}(2)} = 0.99 \ . \tag{4.42}$$

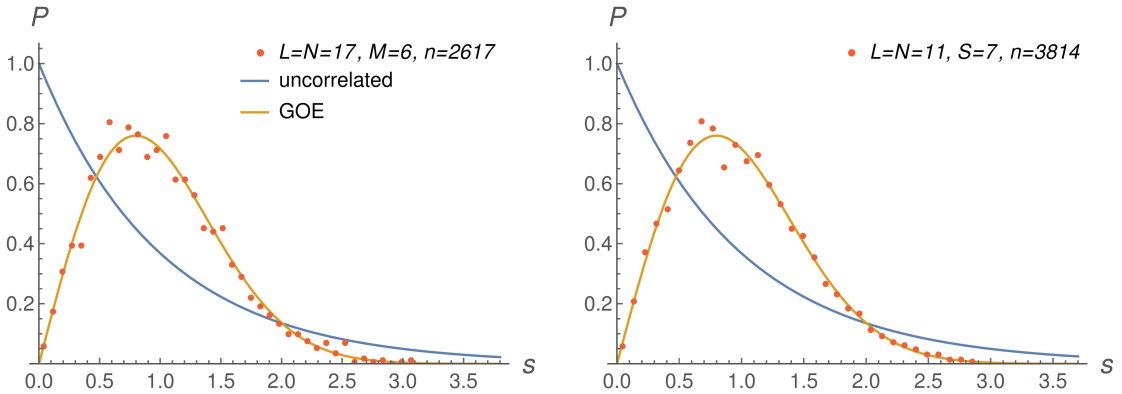


Figure 4.9: Nearest-neighbour spacing distribution for non-planar spectra in the (i) $\mathfrak{su}(2)$ and (ii) $\mathfrak{sl}(2)$ sector of $\mathcal{N}=4$ sYM theory.

Non-planar results at small N. So far we have only looked at non-planar spectra where the rank N of the gauge group is chosen to be $N=L$. In this case (and for $N>L$) the dependence of the spectrum on N is due solely to its appearance in the matrix elements of the dilatation operator. When $N<L$ there are also relations between single- and multi-trace operators which effectively reduce the size of the Hilbert space. Looking for example at the extreme case of the $SU(2)$ gauge theory, the only surviving states are those built from length-2 traces

$$\mathrm{Tr}(ZZ)^{\frac{L-M-n}{2}}\,\mathrm{Tr}(ZX)^n\,\mathrm{Tr}(XX)^{\frac{M-n}{2}}\ . \tag{4.43}$$

We compute all such identities when analysing the energy fluctuations for theories with $2<N<L$. For given L and M the spectral statistics become poorer for small N due to the shrinking of the basis of states. It is interesting to note that the dilatation operator in the $N=2$ case is particularly simple, and the mixing problem becomes solvable in the $\mathfrak{su}(2)$ sector at one loop, yielding the spectrum of energies

$$E_n = 4\cos^2\beta(L+1-2n)n\ , \tag{4.44}$$

with $n=0,...,\lfloor M/2\rfloor$. Meanwhile, for $N>2$ and looking at the β-deformed theory in the length-16 sector, we find

$$\alpha_{N=16}=0.93\ , \qquad\qquad \alpha_{N=4}=0.74\ , \tag{4.45}$$

showing that the GOE distribution is still a good approximation as we decrease the value of N, see Figure 4.10. The fit for $N=4$ is clearly worse, but this is likely due to the poorer statistics inherent to the smaller size of the Hilbert space.

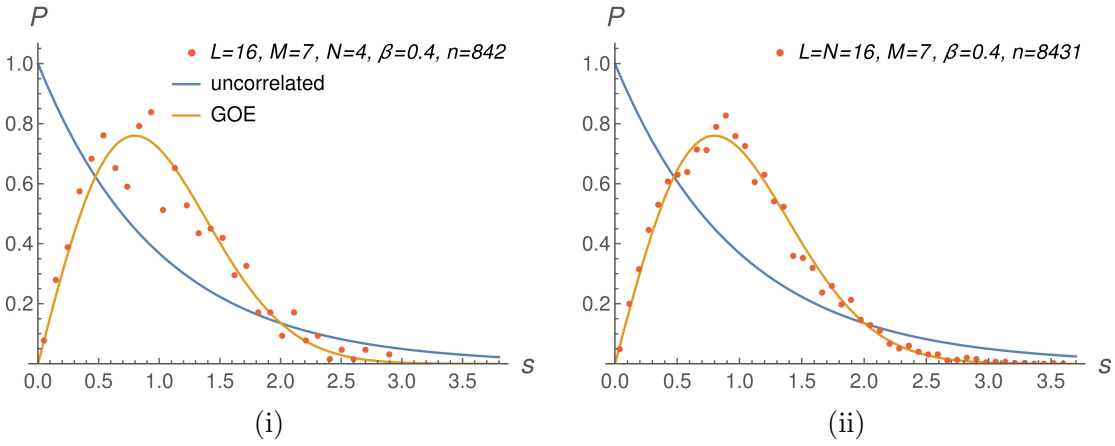


Figure 4.10: Examples of nearest-neighbour spacing distributions for non-planar spectra in the β-deformed $\mathfrak{su}(2)$ sector with (i) $N < L$ and (ii) $N = L$.

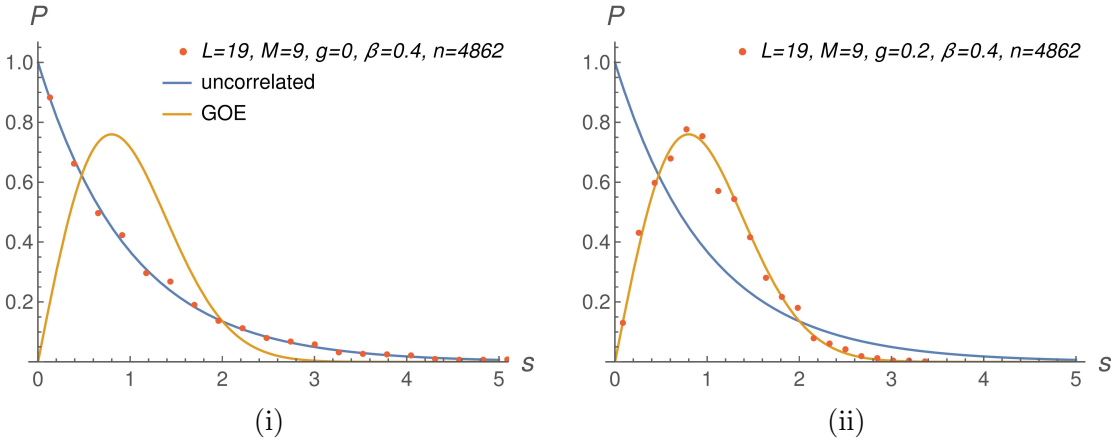


Figure 4.11: Nearest-neighbour spacing distributions for planar spectra in the β-deformed $\mathfrak{su}(2)$ sector (i) at purely one-loop order and (ii) with a small two-loop contribution.

Results to two-loop order. In the $\mathfrak{su}(2)$ sector of $\mathcal{N} = 4$ sYM theory the dilatation operator is also known at two-loop order, cf. equation (1.37). The corresponding planar Hamiltonian with next-to-nearest neighbour interactions is given by

$$H^{(0)} = \sum_{i=1}^{L} \left[\left(1 - \frac{3\lambda}{16\pi^2}\right) \mathbb{1}_{i,i+1} - 4\left(1 - \frac{\lambda}{4\pi^2}\right) \vec{S}_i \otimes \vec{S}_{i+1} - \frac{\lambda}{4\pi^2} \vec{S}_i \otimes \vec{S}_{i+2} \right] \tag{4.46}$$

with spin operators $\vec{S}_i$. The corresponding Hamiltonian in the β-deformed case can also be found and just corresponds to the undeformed Hamiltonian with appropriate β-dependent phase factors. The planar integrability of undeformed and β-deformed $\mathcal{N} = 4$ sYM theory is supposed to hold at all loop orders, but when truncating the perturbative expansion at two-loop order and using finite values of λ the resulting system cannot be expected to be integrable. Indeed, a level-statistics analysis shows that the corresponding model is chaotic and the level spacing distribution is well-described by the GOE, cf. Figure 4.11 where $g = \lambda/16\pi^2$.

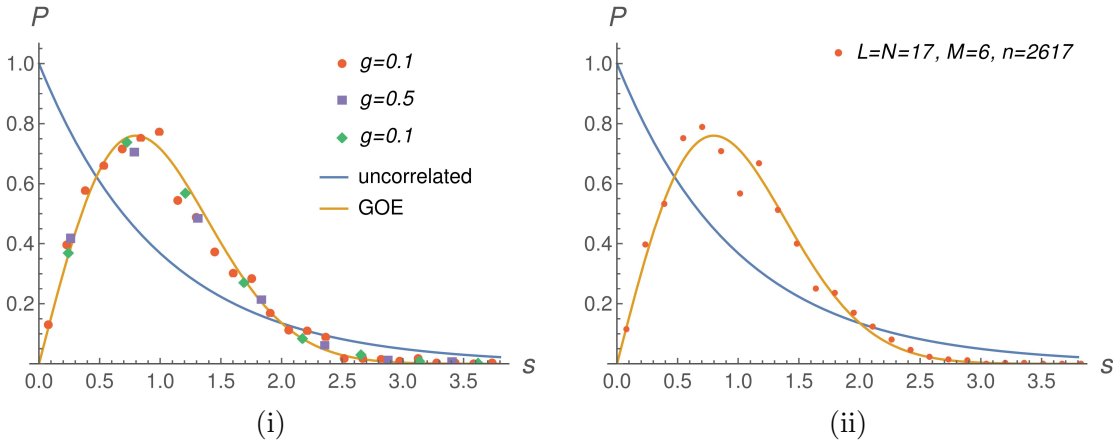


Figure 4.12: Distribution of spacings for (i) the dilatation operator through two-loops for $g = 0.1, 0.5, 1$, and for (ii) the two-loop part of the dilatation operator by itself in the $\mathfrak{su}(2)$ sector.

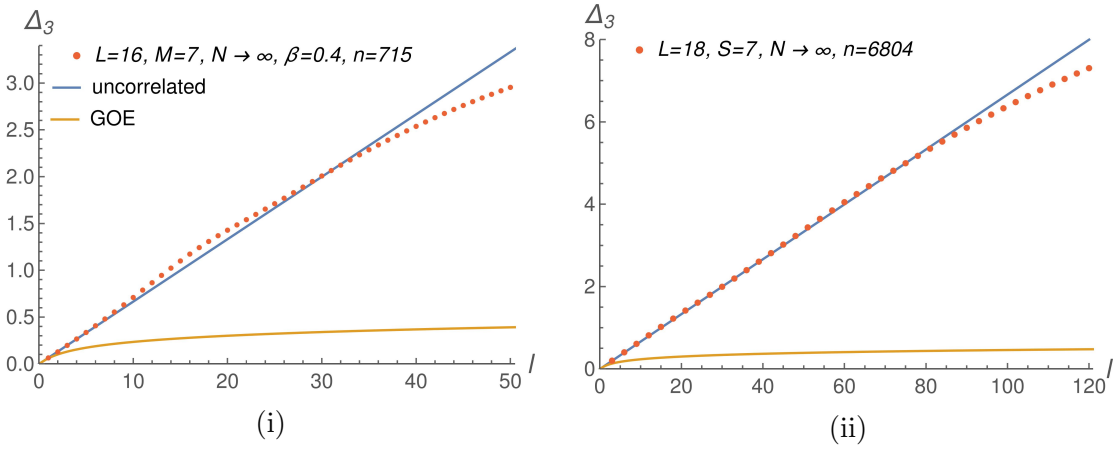


Figure 4.13: Dyson–Mehta statistic for planar spectra in the (i) β-deformed $\mathfrak{su}(2)$ and (ii) undeformed $\mathfrak{sl}(2)$ sector.

If we include the effects of the two-loop dilatation operator at finite N, the qualitative behaviour is unchanged as can be seen in Figure 4.12 (i). This is in part a consequence of the fact that the spectrum of the two-loop part of the dilatation operator, by itself, has a GOE Wigner–Dyson distribution Figure 4.12 (ii) with $\alpha = 0.89$. As the perturbative expansion is asymptotic at finite-N, we are not making any conclusions about the finite-g behaviour of the spectrum, but rather are exploring the qualitative effect of including higher-order terms in the dilatation operator.

4.2.3 Long-range correlations

Next we discuss the results from an analysis of long-range correlations in sYM theory spectra and in particular study the Dyson–Mehta statistic Δ_3.

Planar results. In Figure 4.13 we show that planar spectra in both the $\mathfrak{su}(2)$ sector of β-deformed $\mathcal{N} = 4$ sYM theory and in the $\mathfrak{sl}(2)$ sector of the undeformed

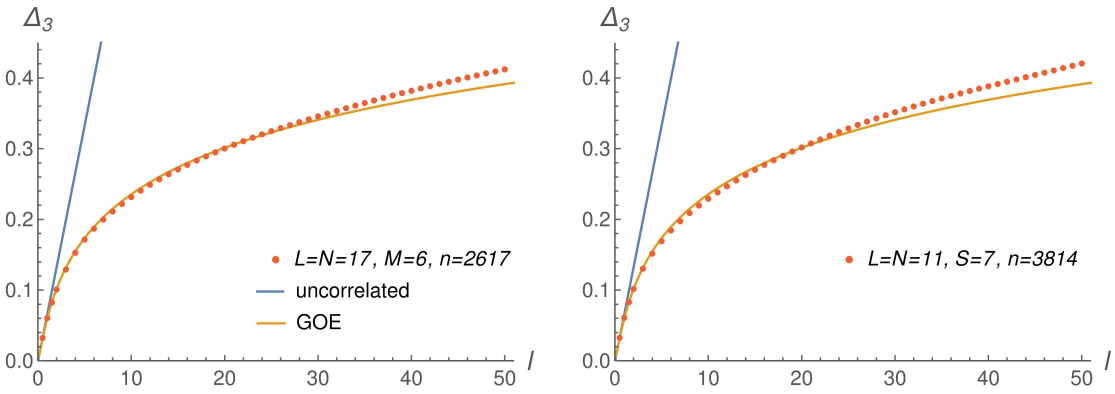


Figure 4.14: Dyson–Mehta statistic for non-planar spectra in the (i) $\mathfrak{su}(2)$ and (ii) $\mathfrak{sl}(2)$ sector of $\mathcal{N} = 4$ sYM theory.

theory follow the linear behaviour (4.20) up to some length l_{max}. This behaviour of Δ_3 reflects the independence of energy levels which is characteristic for spectra of integrable models. The maximum length l_{max} differs for different spectra and grows with the size of the Hilbert space. In the case of the $\mathfrak{su}(2)$ spectrum of Figure 4.13 (i) the linear behaviour is followed up to a length of $l_{\text{max}} \simeq 15$, whereas for the larger $\mathfrak{sl}(2)$ spectrum of Figure 4.13 (ii) it is $l_{\text{max}} \simeq 80$. For interval lengths $l > l_{\text{max}}$ the Dyson–Mehta statistic bends down from the linear behaviour, indicating that levels become correlated over large distances in the spectrum. This is in agreement with the behaviour of Δ_3 for spectra of semi-classical integrable systems [211] which saturates for large l.

Non-planar results. For non-planar Yang–Mills theories, we find that $\Delta_3(l)$ for $l \geq 1$ clearly grows more slowly than the integrable case and follows the GOE prediction of (4.22) up to some non-universal l_{max}, as demonstrated in Figure 4.14 where we find excellent agreement up to $l_{\text{max}} \simeq 25$. However, we see that Δ_3 of the β-deformed theory at $N = 4$ does not match the GOE prediction quite as well, see Figure 4.15 (ii) where the results are consistently above the predicted GOE behaviour for $l \geq 8$. This could be a result of the smaller Hilbert space than in Figure 4.15 (i), where we find good agreement up to $l_{\text{max}} \simeq 70$. Note that above the maximal interval length $l > l_{\text{max}}$ the Dyson–Mehta statistic $\Delta_3(l)$ does not saturate as in the integrable case, but deviates upwards. This is in contradiction with semiclassical chaotic theories as considered by Berry in [211] where Δ_3 saturates, but is a known feature e.g. in the XXZ spin chain [210].

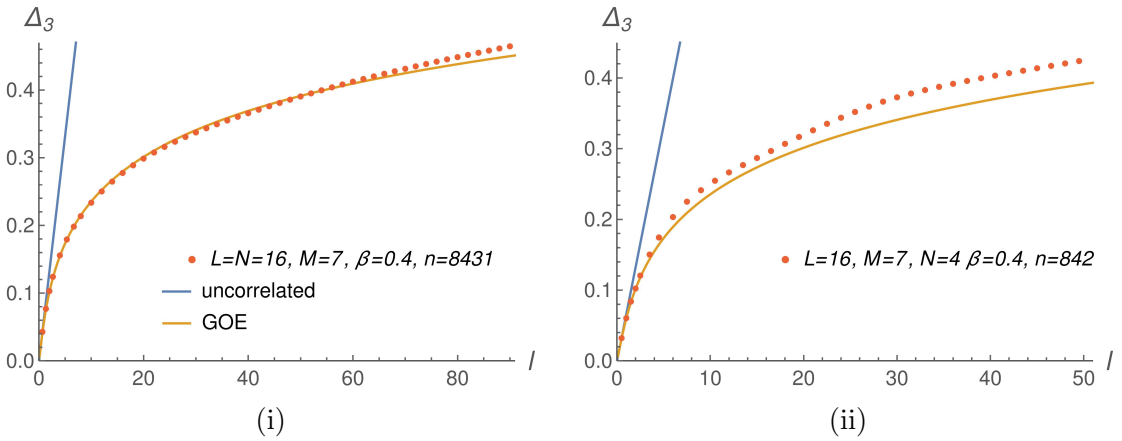


Figure 4.15: Dyson–Mehta statistic for non-planar spectra in the $\mathfrak{su}(2)$ sector of the β-deformed theory with (i) $N = 16$ and (ii) $N = 4$.

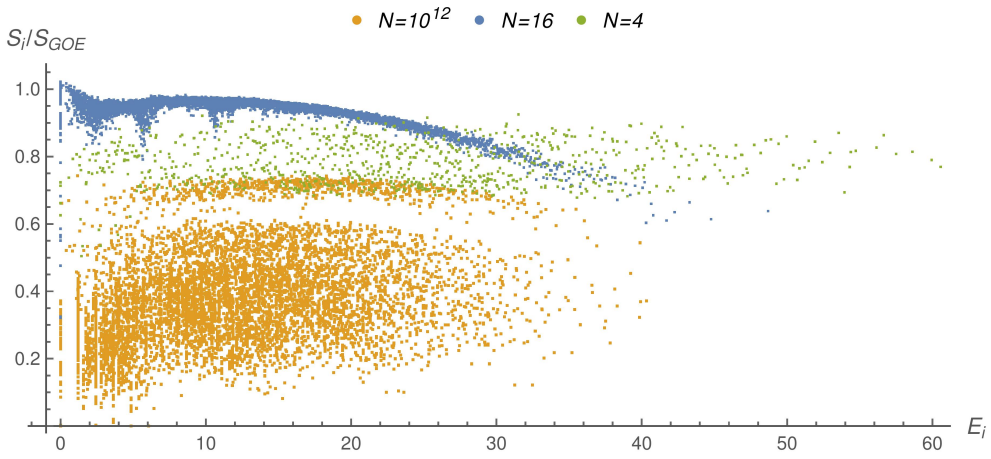


Figure 4.16: Information entropy of eigenvectors in the sector with $L = 16$, $M = 7$, $\beta = 0.4$, $n = 8512$ for different values of N.

4.2.4 Statistics of eigenstates

Going beyond the energy spectrum it is interesting to study the properties of the eigenstates at different values of N. GOE RMT makes a number of predictions for the distribution of chaotic eigenstates, in particular that they are spread out over any non-finely tuned reference basis, i.e. they are delocalised. As a measure of this spreading we use the information entropy (4.33). As our choice of reference basis we simply take the multi-trace operators with fixed numbers of excitations which was used to compute the dilatation operator matrix elements. The GOE RMT prediction is (4.37) for all eigenvectors. However in most physical systems, for example in nuclei [204] or spin-1/2 spin chains [217], the RMT result is only approached near the middle of the energy band, while the states at the edges have significantly lower entropy.

Figure 4.16 shows the information entropy S_i of each eigenvector $|E_i\rangle$, normalised to the corresponding RMT values, for the β-deformed theory for different values of N. Comparing the results for $N = 16$ and $N = 10^{12}$, it is clear that the entropy is larger

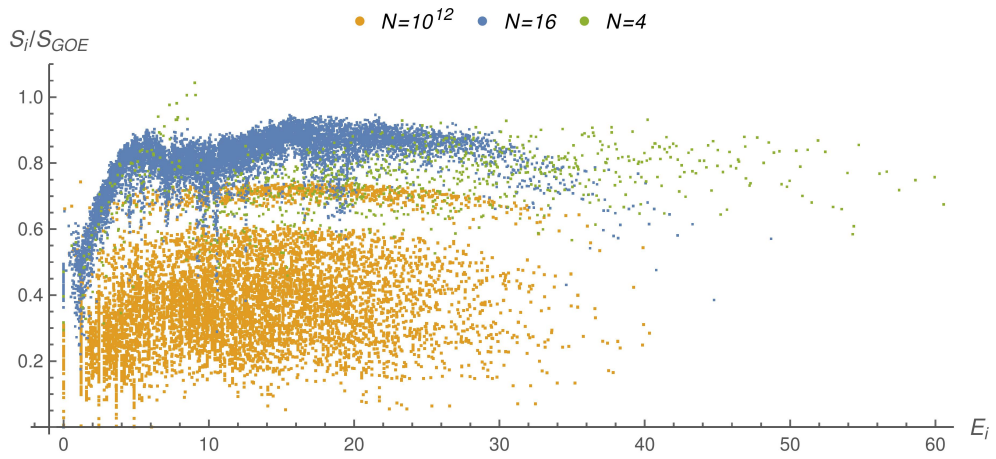


Figure 4.17: Information entropy of eigenvectors from the diagonalisation of $\mathfrak{D}_2^T$ in the sector with $L = 16,\ M = 7,\ \beta = 0.4,\ n = 8512$ for different values of N.

for the smaller value of N and the mean entropy normalised to the GOE prediction at $N = L$ is 0.94, which is significantly larger than the value at large N which is 0.38. Perhaps even more noticeably, the fluctuations in the entropy values are much smaller at $N = L$ where S_i somewhat resembles a smooth function of the energy. We also plot the case of $L = N^2$, and again one finds that the mean value is well above the integrable large-N result, and the maximum value 0.92 approaches the RMT bound. As the dimension of the $N < L$ Hilbert space is smaller, the statistics are perhaps not as reliable, but one interesting feature is the uniformity of the entropy with mixing being almost independent of the energy.

One issue in comparing RMT with the gauge theory is that while the dilatation operator has real eigenvalues and possesses discrete symmetries analogous to time-reversal, with the choice of a scalar product for which the basis of multi-trace operators is orthonormal, it is in fact not symmetric. As a consequence, its eigenvectors are complex rather than real and are not mutually orthogonal with respect to our scalar product. This can be seen in the different entropies of the eigenvectors of the transposed matrix, Figure 4.17, but the finite-N states generally still have larger entropy.

The information entropy can be similarly computed for the two-loop dilatation operator in the undeformed theory. In Figure 4.18 we consider the value $g = 0.1$, and while there is a number of differences in the structure of the states, qualitatively the results are similar. We can repeat these calculations for the $\mathfrak{sl}(2)$ sector and while we find that the mean entropy is still larger at small N than large N, it is generally quite low and significantly further from the GOE RMT bound.

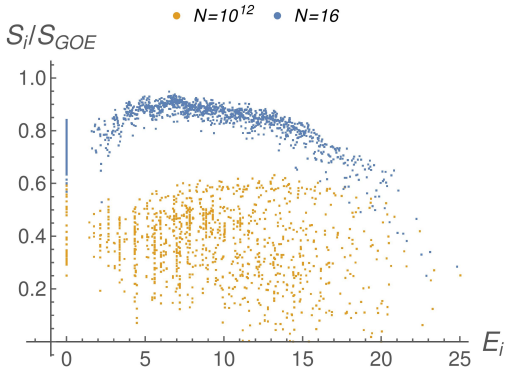


Figure 4.18: Information entropy of the two-loop dilatation operator of the undeformed theory in the positive-parity sector for different values of N in the sector with $L = 16,\ M = 7,\ n = 1201$.

4.2.5 GOE and discrete symmetries

We have provided numerical evidence that the non-planar spectrum of $\mathcal{N} = 4$ sYM shares the universal properties we expect of chaotic quantum many-body systems and, specifically is well-described by the GOE of RMT. While we have only displayed results for specific choices of operator lengths and excitation numbers, we have found comparable results for all other charges that we have considered.

That GOE describes the non-planar distribution of energy levels best is, in some respects, to be expected due to the conjecture of [35] that the level statistics of time-reversal invariant systems are characterised by GOE. This connection is manifest for RMT systems with space-rotation invariance with generator K, i.e. $[H, K] = 0$, where the Hamiltonian can be made real and symmetric, see e.g. [206]. The planar Hamiltonian (3.3) is invariant under time-reversal symmetry with operator $\mathcal{T} = KC$, where C takes the complex conjugate of an operator and $K = \prod_{j=1}^{L} i\sigma_j^y$. This symmetry can be extended to the non-planar theory where K essentially exchanges Z and X fields in the dilatation operator. While the undeformed Hamiltonian in the $\mathfrak{su}(2)$ sector is invariant under this symmetry, the introduction of the deformation parameter breaks the rotation invariance. Fortunately, there is an additional symmetry that explains why also spectra in the β-deformed theory show GOE behaviour: The deformed, as well as the undeformed, dilatation operator in the $\mathfrak{su}(2)$ sector are invariant under a time-reversal symmetry of the form $\mathcal{T} = \mathcal{P}C$, where C is again the complex conjugation operator, while $\mathcal{P}$ is the parity operator which acts as (3.30) on spin chains, and (4.39) on general multi-trace operators. One can show [206] that in a suitable basis, the operator $\mathcal{P}$ satisfies $\mathcal{P}\mathcal{P}^* = 1$ and this guarantees that any operator commuting with $\mathcal{T} = \mathcal{P}C$ can be chosen to be real and symmetric, and is thus a member of the GOE.

Conclusions

We have considered various aspects of non-planar anomalous dimensions in pure and deformed $\mathcal{N} = 4$ sYM theory, and in the following we summarise our results and discuss possible future directions of research.

We began with a review of the close connection between the dilatation operator and scattering amplitudes revealed in [31, 32]. This connection allows us to obtain anomalous dimensions of a given set of operators via an on-shell approach based on a handful of form factors of these operators, as well as the theory's scattering amplitudes. We illustrated this approach for the mixing of $SU(4)$-invariant dimension-4 operators in $\mathcal{N} = 4$ sYM theory, and as a by-product of this computation we re-derived the theory's on-shell Lagrangian. This computation may be generalised to the mixing of a larger set of marginal operators, with fewer symmetries and more general scattering amplitudes, to extract the Leigh–Strassler theories, or maybe even new conformal deformations of $\mathcal{N} = 4$ sYM theory.

While one can obtain specific anomalous dimensions directly from this on-shell approach, it also allows one to extract the general form of dilatation operators. Here we computed the finite-N one-loop dilatation operator of β-deformed $\mathcal{N} = 4$ sYM theory in the scalar sector including non-planar double-trace contributions. This operator can in turn be diagonalised in order to obtain non-planar anomalous dimensions and we did so for the spectrum of length-two scalar operators. We reproduced results for the Konishi operator previously calculated by direct field-theory methods [75]. The deformed dilatation operator is also an important ingredient into the perturbative approach to non-planar anomalous dimensions and we review these results further below. In the calculation of the dilatation operator, there occur infrared divergences. They have to be subtracted and can be fixed by exploiting their universality and calculating their explicit form from known protected operators; on more general grounds we also discussed IR divergences of scattering amplitudes, in particular in the context of coherent states and asymptotic symmetries.

While the direct diagonalisation of the one-loop dilatation operator for short local operators is relatively straightforward, this task gets more and more involved as the number of field insertions in operators grows. In the planar limit alternative approaches to the direct diagonalisation based on integrability methods exist, but there

is no systematic direct approach to non-planar anomalous dimensions. In this work we have studied the leading non-planar corrections to the planar spectrum perturbatively. The approach involves two steps: first one must obtain the mixing matrix between single- and multi-trace operators at leading non-planar order, and then find its eigenvalues with the method of quantum-mechanical perturbation theory. In this work we have mostly focused on the first half of the problem by finding general expressions for the off-diagonal matrix elements of the one-loop dilatation operator in terms of the Bethe rapidities. When expressed in terms of off-shell scalar products, they can be computed efficiently using the algebraic Bethe ansatz. While the direct application of the dilatation operator can in many cases yield the mixing matrix in a similarly efficient fashion, our formulas are given in terms of partitions of the Bethe rapidities, and therefore they are especially advantageous when the number of excitations is small. In those cases we are able to easily evaluate the overlaps even for long operators, where direct diagonalisation would be infeasible. A bottleneck in the second step of computing anomalous dimensions is the determination of the Bethe rapidities. While there are tools for efficiently computing such rapidities, most notably the Baxter Q-function method of [174], carrying out the sums over solutions is still non-trivial, and so we restricted ourselves to a few examples to illustrate the method. The problem of summing over intermediate states increases with the excitation number of the operators under consideration and will rapidly become unfeasible. It might be possible that the sum over solutions is simpler than the individual terms, but this needs a better understanding of the involved sums over Bethe states.

At a more conceptual level, we find that the off-diagonal matrix elements can be written in terms of hexagon-like objects satisfying both the Watson and decoupling conditions. While our methods are not obviously related to the hexagonalisation of the torus, this decomposition hints at the possibility of an approach similar to [218], where four-point functions are built through the OPE, with the OPE data computed within an integrability framework. Similarly, the matrix elements of the dilatation operator might have a more general description which determines their form at higher orders in the perturbative expansion. In order to study this further it would be useful to determine the overlaps at higher loops and to investigate if hexagon-like objects can be found in other sectors of the theory.

One issue in our approach to the diagonalisation of the mixing matrix is that it assumes a non-degenerate spectrum of planar Bethe states. There are however many degeneracies in the planar spectrum of $\mathcal{N} = 4$ sYM theory, and so we also considered the β-deformed theory where many degeneracies are lifted. The action of the dilatation operator in the deformed theory, obtained from the form-factor on-shell approach discussed above, yields several new structures and for the purpose of evaluating $1/N^2$ corrections to the spectrum it is necessary to include an additional

diagonal overlap. We show that degeneracies between excited states in the undeformed theory are lifted when deforming the theory and this reduced degeneracy increases the number of operators for which we can compute the non-planar corrections to the energies by using non-degenerate perturbation theory. A second advantage of the β-deformation is that it provides a useful regularisation of the singular solutions occurring in the $\mathfrak{su}(2)$ sector of the undeformed $\mathcal{N} = 4$ spin chain. Such solutions correspond to finite energies, but have singular wavefunctions and so matrix elements in the undeformed theory are not well defined. Instead they can be computed in the deformed theory, where the deformation parameter regularises singularities [165], and the limit of vanishing deformation parameter can be smoothly taken. As an application of our method we computed anomalous dimensions of two-excitation states in the BMN limit through subleading order. We compared these analytical results with numerical data at lower lengths and find that the results agree with at least 8 digits of precision. Since the degeneracies occurring in other sectors of the undeformed theory are not all lifted by the β-deformation, additional twists are needed to extend our results to other sectors. For example, to study the $\mathfrak{sl}(2)$ sector it may be useful to consider the integrable dipole deformation [219].

In addition to computing anomalous dimensions of specific operators, it is also of interest to understand their general properties. To this end we analysed the distribution of level spacings and found that at infinite N the one-loop spectra of both deformed and undeformed $\mathcal{N} = 4$ sYM theory are well described by the Poisson distribution characteristic of integrable systems. At finite N the distribution transitions to the Wigner–Dyson distribution of chaotic quantum many-body systems and the statistical properties of the finite-N spectrum are well described by the GOE random matrix model. We constrained this analysis to rank-one sectors of the theories, which considerably simplifies the analysis, but still showcased the universal chaotic behaviour at finite N. While we have only displayed results for specific choices of operator length and excitation number in these sectors, we have found comparable results for all other charges we considered. Level repulsion, which is characteristic for chaotic quantum systems, was previously found in $\mathcal{N} = 4$ sYM theory in [215], where it was shown that finite-N anomalous dimensions do not cross as the 't Hooft coupling λ varies. Studying the one-loop spectrum as a function of the deformation parameter β, we similarly find that at finite N the anomalous dimensions repel and it is only at large N that they cross.

We have thus provided numerical evidence that the non-planar spectrum of $\mathcal{N} = 4$ sYM theory, in spite of its maximal amount of supersymmetry, exact conformal invariance and planar integrability, is quite generic and shares the universal properties we expect of chaotic quantum many-body systems. The appearance of quantum chaos in the spectrum is in fact quite natural if we view the dilatation operator as

the Hamiltonian of the theory defined on $\mathbb{R} \times S^3$ and multi-trace operators as defining states somewhat analogous to large nuclei in QCD. While it is possible that by resumming the perturbative series and including non-perturbative effects the qualitative behaviour will change, it is natural to conjecture that the non-planar spectrum is described by GOE RMT at finite values of both g_{YM} and N. Such a conjecture is further motivated by the fact that, at strong coupling $\lambda \gg 1$, operators with scaling dimensions $\Delta \sim N^2$ are holographically dual to black holes and so are expected to exhibit chaotic properties [37–40]. This connection was pursued recently in [220], where the authors considered operators dual to a system of giant gravitons. In that sector the one-loop dilatation operator can be reformulated as a Hamiltonian acting on a graph [221], with a large-N counting of graphs matching the expected black hole entropy and therefore suggesting an interpretation of the operators as black hole microstates. The natural basis in this context is that of restricted Schur polynomials, which diagonalise the free-theory two-point functions at finite N. It could be particularly interesting to study the energy eigenfunctions in this basis, where the dilatation operator becomes symmetric, and find the implications for the information entropy.

While the strong-coupling regime can be difficult to access from the gauge-theory side of the AdS/CFT correspondence, an interesting connection with gravity has been made in the context of the holographic duality between the SYK model and Jackiw-Teitelboim gravity on AdS_2 [222–225]. The distribution of the level spacings in the SYK model of N Majorana fermions with random couplings was numerically computed in [226], see also [41, 227, 228], and was shown to be of Wigner–Dyson type with all three ensembles, GOE, GUE and GSE, occurring depending on the value of N. It would be naturally interesting to study this chaotic behaviour at higher loop-orders in $\mathcal{N} = 4$ sYM theory and whether, by the holographic correspondence, we can describe the properties of interacting quantum strings on AdS space by RMT.

To conclude, we have studied the computation and universal properties of non-planar anomalous dimensions in conformal sYM theories. We have applied powerful techniques, including integrability and on-shell methods, as well as quantum-mechanical perturbation theory, to extract the spectrum of anomalous dimensions. These are important steps towards a systematic framework for the calculation of the finite-N data of sYM theories, but further work is necessary to generalise the results to other sectors, and higher orders in both the coupling- and $1/N$-expansion, as well as to turn them into more compact expressions. We furthermore found universal behaviours in numerical anomalous-dimension spectra: at infinite N they are consistent with planar integrability, while at finite N they point towards the quantum-chaotic nature of the theories and are described by random matrix theory. These studies open the door for further explorations of universal features of YM theories and, by the holographic principle, their gravity duals.

Appendix A

Non-minimal form factors in the $SU(4)$-invariant marginal sector

In this appendix we compute tree-level non-minimal form factors relevant for the mixing of operators G, F and $\vec{\mathrm{S}}$ given in (2.46). Doing so, we will make use of the result of [131] for all tree-level MHV form factors of the on-shell Lagrangian (1.17) given in (2.7). For two external states the Lagrangian effectively becomes G and the only-non-vanishing contribution comes from two external negative-helicity gluons. Then (2.7) reproduces the first line in (2.47). For three external states the non-minimal MHV form factor of G can be obtained from (2.7) by removing the contributions coming from the minimal form factor of $-g_{\mathrm{YM}}\mathrm{F}$ in the on-shell Lagrangian and we find

$$\begin{aligned}
\langle g_-(p_1), g_-(p_2), g_+(p_3)|\mathrm{G}|0\rangle &= -2\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\ , \\
\langle g_-(p_1), \bar{\psi}_{ABC}(p_2), \psi_D(p_3)|\mathrm{G}|0\rangle &= +2\varepsilon_{ABCD}\frac{\langle 12\rangle^2}{\langle 23\rangle}\ , \\
\langle g_-(p_1), \phi_{AB}(p_2), \phi_{CD}(p_3)|\mathrm{G}|0\rangle &= -2\varepsilon_{ABCD}\frac{\langle 12\rangle\langle 31\rangle}{\langle 23\rangle}\ .
\end{aligned} \tag{A.1}$$

The remaining form factors that are not related to the above by cyclicity and reflection vanish at this order. These expressions can furthermore be obtained by the BCFW recursion for form factors discussed in Section 2.1. Similarly we obtain the non-minimal form factors of G and F with four external states by BCFW recursion and obtain the non-vanishing components

$$\begin{aligned}
\langle \psi_A(p_1), \psi_B(p_2), \bar{\psi}^C(p_3), \bar{\psi}^D(p_4)|\mathrm{F}|0\rangle &= -2(\delta_A^C\delta_B^D - \delta_A^D\delta_B^C)\frac{\langle 34\rangle}{\langle 12\rangle}\ , \\
\langle g_+(p_1), \phi_{AB}(p_2), \bar{\psi}^A(p_3), \bar{\psi}^B(p_4)|\mathrm{F}|0\rangle &= -2\frac{\langle 24\rangle\langle 34\rangle}{\langle 12\rangle\langle 14\rangle}\ ,
\end{aligned}$$

$$\begin{aligned}
\langle g_+(p_1),\bar\psi^A(p_2),\phi_{AB}(p_3),\bar\psi^B(p_4)|\mathrm{F}|0\rangle &= 2\frac{\langle 24\rangle^2}{\langle 12\rangle\langle 14\rangle}\ ,\\
\langle \psi_A(p_1),\phi_{BC}(p_2),\phi_{DE}(p_3),\bar\psi^E(p_4)|\mathrm{F}|0\rangle &= 2\frac{\langle 24\rangle}{\langle 12\rangle}\varepsilon_{ABCD}\ ,\\
\langle \psi_A(p_1),\phi_{BC}(p_2),\bar\psi^D(p_3),\phi_{EF}(p_4)|\mathrm{F}|0\rangle &= 2\frac{\langle 23\rangle}{\langle 12\rangle}\varepsilon_{ABCF}\delta^D_E+2\frac{\langle 34\rangle}{\langle 14\rangle}\varepsilon_{ABEF}\delta^D_C\ . \qquad \text{(A.2)}
\end{aligned}$$

and

$$\begin{aligned}
\langle g_+(p_1),g_+(p_2),g_-(p_3),g_-(p_4)|\mathrm{G}|0\rangle &= -2\frac{\langle 34\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 41\rangle}\ ,\\
\langle g_+(p_1),g_-(p_2),g_+(p_3),g_-(p_4)|\mathrm{G}|0\rangle &= -2\frac{\langle 24\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\ ,\\
\langle g_+(p_1),\psi_A(p_2),\bar\psi^A(p_3),g_-(p_4)|\mathrm{G}|0\rangle &= 2\frac{\langle 24\rangle\langle 34\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 41\rangle}\ ,\\
\langle g_+(p_1),\bar\psi^A(p_2),\psi_A(p_3),g_-(p_4)|\mathrm{G}|0\rangle &= -2\frac{\langle 24\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 41\rangle}\ ,\\
\langle g_+(p_1),\psi_A(p_2),g_-(p_3),\bar\psi^A(p_4)|\mathrm{G}|0\rangle &= -2\frac{\langle 34\rangle^2}{\langle 12\rangle\langle 41\rangle}\ ,\\
\langle g_+(p_1),\phi_{AB}(p_2),\bar\phi^{AB}(p_3),g_-(p_4)|\mathrm{G}|0\rangle &= -2\frac{\langle 34\rangle\langle 24\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 41\rangle}\ ,\\
\langle g_+(p_1),\phi_{AB}(p_2),g_-(p_3),\bar\phi^{AB}(p_4)|\mathrm{G}|0\rangle &= 2\frac{\langle 23\rangle\langle 34\rangle}{\langle 12\rangle\langle 41\rangle}\ ,\\
\langle \psi_A(p_1),\psi_B(p_2),\bar\phi^{AB}(p_3),g_-(p_4)|\mathrm{G}|0\rangle &= 2\frac{\langle 24\rangle\langle 34\rangle}{\langle 12\rangle\langle 23\rangle}\ ,\\
\langle \psi_A(p_1),\bar\phi^{AB}(p_2),\psi_B(p_3),g_-(p_4)|\mathrm{G}|0\rangle &= 2\frac{\langle 24\rangle^2}{\langle 12\rangle\langle 23\rangle}\ ,\\
\langle \psi_A(p_1),\phi_{BC}(p_2),\phi_{DE}(p_3),\bar\psi^F(p_4)|\mathrm{G}|0\rangle &= -2\frac{\langle 24\rangle\langle 34\rangle}{\langle 23\rangle\langle 41\rangle}\varepsilon_{BCDE}\delta^F_A\ ,\\
\langle \psi_A(p_1),\psi_B(p_2),\bar\psi^C(p_3),\bar\psi^D(p_4)|\mathrm{G}|0\rangle &= 2\frac{\langle 34\rangle^2}{\langle 23\rangle\langle 41\rangle}\delta^D_A\delta^C_B\ ,\\
\langle \psi_A(p_1),\bar\psi^B(p_2),\psi_C(p_3),\bar\psi^D(p_4)|\mathrm{G}|0\rangle &= 2\frac{\langle 24\rangle^2}{\langle 12\rangle\langle 34\rangle}\delta^B_A\delta^D_C-2\frac{\langle 24\rangle^2}{\langle 23\rangle\langle 41\rangle}\delta^D_A\delta^B_C\ ,\\
\langle \phi_{AB}(p_1),\phi_{CD}(p_2),\phi_{EF}(p_3),\phi_{GH}(p_4)|\mathrm{G}|0\rangle&\\
= 2\left(\left(\frac{\langle 24\rangle\langle 31\rangle}{\langle 23\rangle\langle 41\rangle}-\frac{1}{2}\right)\varepsilon_{ABGH}\varepsilon_{CDEF}\right.&\left.-\left(\frac{\langle 24\rangle\langle 31\rangle}{\langle 12\rangle\langle 34\rangle}+\frac{1}{2}\right)\varepsilon_{ABCD}\varepsilon_{EFGH}\right)\ . \qquad \text{(A.3)}
\end{aligned}$$

Note that the last expression cannot be obtained by component BCFW as it contains boundary terms. Instead it was derived from (2.7) and removing the contribution from $-g_{\mathrm{YM}}\mathrm{F}$ and $-g^2_{\mathrm{YM}}\mathrm{S}_1/2^4$.

Appendix B

Overlaps from the algebraic Bethe ansatz

The algebraic Bethe ansatz, see [139, 229, 230] for introductions, provides a powerful framework for studying integrable systems such as the spin chains arising in the one-loop planar dilatation operator. Of particular interest in this work are the computationally efficient formulas for scalar products of Bethe states [167, 231]. These scalar products have previously appeared in the context of $\mathcal{N} = 4$ sYM structure constants and we will mostly follow the conventions of [162].

Central to the algebraic Bethe ansatz approach is the monodromy matrix $\hat{T}_a(u)$, which is an operator depending on the spectral parameter $u \in \mathbb{C}$, and acting on the tensor product $(\mathbb{C}^2)^{\otimes L}$ of the L spin-chain sites, and an extra auxiliary space $V \simeq \mathbb{C}^2$ labelled by the index a. Treating $\hat{T}_a(u)$ as a 2×2 matrix in auxiliary space whose entries are operators acting on the spin chain, we can write

$$\hat{T}_a(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix} . \tag{B.1}$$

The commutation relations of these entries can be found from the so-called RTT relations

$$R_{a_1 a_2}(u-v)\hat{T}_{a_1}(u)\hat{T}_{a_2}(v) = \hat{T}_{a_2}(v)\hat{T}_{a_1}(u)R_{a_1 a_2}(u-v) \tag{B.2}$$

where the R-matrix $R_{a_1 a_2}(u-v)$ is an operator acting on two auxiliary spaces labelled by a_1 and a_2 and, for the theories we consider, it only depends on the difference of the spectral parameters u and v. Viewed as a 4×4 matrix mapping as $(\mathbb{C}^2)^{\otimes 2} \to (\mathbb{C}^2)^{\otimes 2}$,

the R-matrix is given by

$$R_{a_1a_2}(u-v) = \begin{pmatrix} f(u,v) & 0 & 0 & 0 \\ 0 & 1 & g(u,v) & 0 \\ 0 & g(u,v) & 1 & 0 \\ 0 & 0 & 0 & f(u,v) \end{pmatrix} \tag{B.3}$$

where we have introduced the functions

$$f(u,v) \equiv f(u-v) = 1 + \frac{i}{u-v} \quad \text{and} \quad g(u,v) \equiv g(u-v) = \frac{i}{u-v} \ . \tag{B.4}$$

The trace of the monodromy matrix over the auxiliary space defines the transfer matrix, $\hat{T}(u) = \text{Tr}\ \hat{T}_a(u)$, and it follows from (B.2) that transfer matrices with different spectral parameters commute. The Hamiltonian of the spin chain is given by the logarithmic derivative of the transfer matrix evaluated at $u = i/2$ while the higher conserved charges can be found by further expanding the logarithm of the transfer matrix near $u = i/2$. The eigenstates of the transfer matrix thus simultaneously diagonalise the Hamiltonian and all higher charges. One can define Bethe states as

$$|\{u\}\rangle^{\text{alg}} = \prod_{i=1}^{M} \mathcal{B}(u_i)\,|0\rangle \tag{B.5}$$

where the pseudovacuum is defined by $\mathcal{C}(u)\,|0\rangle = 0$ and satisfies

$$\mathcal{A}(u)\,|0\rangle = a(u)\,|0\rangle \quad \text{and} \quad \mathcal{D}(u)\,|0\rangle = d(u)\,|0\rangle \tag{B.6}$$

with $a(u) = (u+i/2)^L$ and $d(u) = (u-i/2)^L$. When the spectral parameters $\{u_i\}$ in (B.5) satisfy the Bethe equations (3.13), using the parametrisations (3.17) and (3.18), the Bethe states are eigenstates of the transfer matrix with eigenvalues

$$\hat{T}(v)\,|\{u\}\rangle = T(v,\{u\})\,|\{u\}\rangle \ \text{ with } T(v,\{u\}) = a(v)\prod_{i=1}^{M} f(v,u_i) + d(v)\prod_{i=1}^{M} f(u_i,v) \ . \tag{B.7}$$

The operators $\mathcal{B}(u_i)$ can thus be viewed as creating excited states whose relative normalisation is given by, see [162],

$$|\{p\}\rangle = \frac{1}{\sqrt{S_<^{\{u\}} f_<^{\{u\}} d^{\{u\}} g^{\{u+i/2\}}}}\ |\{u\}\rangle^{\text{alg}} \ , \tag{B.8}$$

where we use the product notation (3.23). The dual states in the algebraic Bethe

ansatz are defined by

$$ {}^{\text{alg}}\langle\{u\}| = (-1)^M \langle 0| \prod_{i=1}^{M} \mathcal{C}(u_i^*) \tag{B.9}$$

where the dual vacuum satisfies $\langle 0|\,\mathcal{B}(u) = 0$ and

$$\langle 0|\,\mathcal{A}(u) = \langle 0|\,a(u) \quad \text{and} \quad \langle 0|\,\mathcal{D}(u) = \langle 0|\,d(u)\ . \tag{B.10}$$

These dual states are related to Bethe states by Hermitian conjugation using the definition

$$|0\rangle = \langle 0|^\dagger\ , \quad \text{and} \quad \mathcal{C}(u^*) = -\mathcal{B}^\dagger(u) \tag{B.11}$$

and are dual eigenstates of $\hat{T}(u)$ when the spectral parameters satisfy the Bethe equations.

We will be interested in the quantity $I_M(\{v\},\{u\})$, which is related to the scalar product of Bethe states by the definition

$$I_M(\{v\},\{u\}) \equiv \langle 0| \prod_{j=1}^{M} \mathcal{C}(v_j) \prod_{j=1}^{M} \mathcal{B}(u_j)\,|0\rangle \tag{B.12}$$

$$= (-1)^M\ {}^{\text{alg}}\langle\{v^*\}|\{u\}\rangle^{\text{alg}} \tag{B.13}$$

and, following [167], can be written as a sum over partitions of the excitations. The partitions are defined by splitting each set of excitations, $\{u\}$ and $\{v\}$, into subsets, $\alpha\cup\bar{\alpha} = \{u\}$ and $\beta\cup\bar{\beta} = \{v\}$, with the cardinality of α equal to that of β. The scalar product is then given as

$$I_M(\{v\},\{u\}) = \quad g_<^{\{u\}} g_>^{\{v\}} \sum_{\substack{\alpha\cup\bar{\alpha}=\{u\}\\ \beta\cup\bar{\beta}=\{v\}}} \text{sgn}(\alpha)\text{sgn}(\beta) d^\alpha a^{\bar{\alpha}} a^\beta d^{\bar{\beta}} k^{\alpha\beta} k^{\bar{\beta}\bar{\alpha}} k^{\alpha\bar{\alpha}} k^{\bar{\beta}\beta} \det t^{\alpha\beta} \det t^{\bar{\beta}\bar{\alpha}} \tag{B.14}$$

where

$$k(u,v) = \frac{f(u,v)}{g(u,v)} = 1 - i(u-v)\ , \quad \text{and} \quad t(u,v) = \frac{g^2(u,v)}{f(u,v)} = \frac{-1}{(u-v)(u-v+i)} \tag{B.15}$$

and $\text{sgn}(\alpha)$ is the signature of the permutation required to put $\alpha\cup\bar{\alpha}$ into the canonical order $\{u\}$. This formula is valid for arbitrary Bethe states, even those whose rapidities do not satisfy the Bethe equations and which are thus said to be "off-shell". In the case where one set of rapidities satisfies the Bethe equations (they are said to be

"on-shell"), the formula can be dramatically simplified to the calculation of a single determinant [231]. There is a further simplification when both sets of rapidities are on-shell and equal. In this case, as the set of rapidities is invariant under complex conjugation, the quantity I_M is related to the norm of the Bethe state and is given by Gaudin's formula

$$I_M(\{u\},\{u\}) = d^{\{u\}} a^{\{u\}} f_{>}^{\{u\}} f_{<}^{\{u\}} \det_{j,k} \partial_{u_j} \phi_k \tag{B.16}$$

where ϕ_k is defined in (3.13).

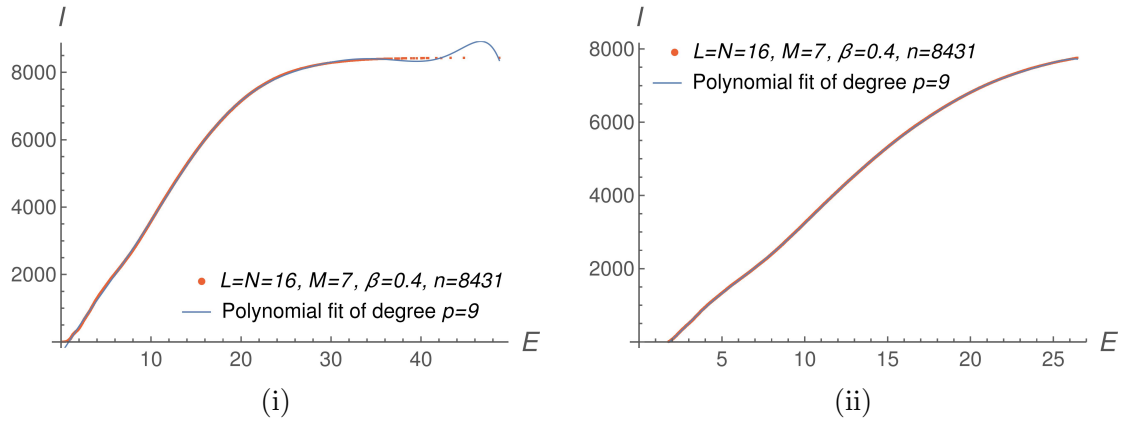
Appendix C

Data preparation

In this appendix we discuss the data preparation of numerical spectra obtained from integrability methods or direct diagonalisation of the dilatation operator, as introduced in the main text. We perform the computation of these spectra in Mathematica. The major bottleneck when trying to achieve better statistics by going to larger spectra in the direct diagonalisation method is the calculation of the mixing matrix whose size grows roughly exponentially when going to higher operator lengths L. Therefore we perform these calculations on the Londsdale cluster of the Trinity Centre for High-Performance Computing. The remaining calculations, including the data preparation of the spectra, as well as the computation of statistical observables, can be performed on a standard desktop or laptop computer.

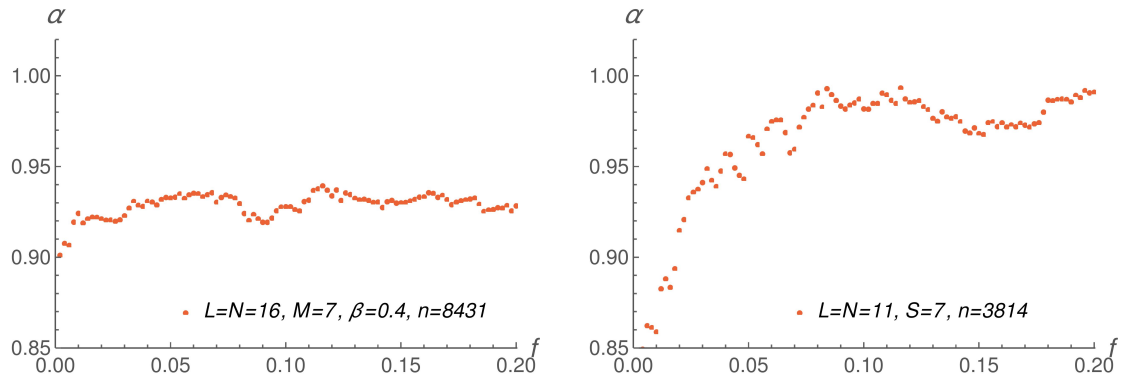
Unfolding method. As discussed in Section 4.1.2, when comparing spectra of physical systems and RMT spectra we must first separate the overall energy dependence $I_{\text{av}}(E)$ from the fluctuations. The former typically depends on the specifics of the model, whereas the latter reveal the underlying chaos or integrability. We unfold the spectrum via (4.8) by first approximating I_{av}. In order to do so, one can for example select each i-th energy state, with $i \gg 1$, and perform a piecewise linear interpolation. However we find that some of the subsequent results for the NNS distribution and Dyson–Mehta statistic considerably vary as one varies i. Therefore, we choose to approximate I_{av} with a polynomial fit to the set of $\{I(E_i)\}$. The degree p of the polynomial is a parameter that needs to be tuned, but we find that the results are virtually unchanged for a wide range of values.

Specifics for NNS distribution. Since $I(E)$ is usually quite flat at the ends of the spectrum, we find it useful to clip at least 4% of the states on each end before fitting the distribution, see Figure C.1. Fitting $I_{\text{av}}(E)$ on the whole of the spectrum yields poor behaviour for extreme values of E, while clipping states at the ends of the spectrum yields a fit that captures $I_{\text{av}}(E)$ more accurately. While this clipping is



(i) (ii)

Figure C.1: Integrated spectral density $I(E)$ for the β-deformed theory and a polynomial fit approximating its average behaviour $I_{\text{av}}(E)$ for (i) the whole spectrum and (ii) with 4% of the states at each end clipped.

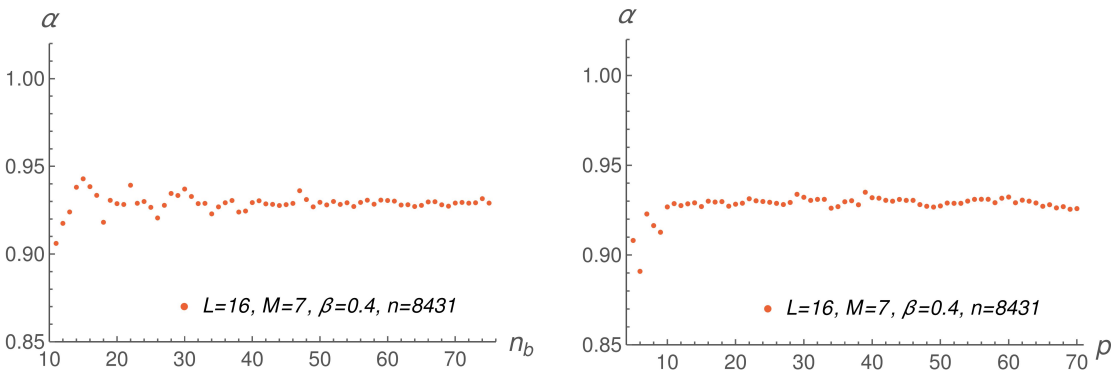


(i) (ii)

Figure C.2: Values of α for the Wigner–Dyson distribution which best fit the data under different clipping fractions f of low-energy states (with 4% of the high-energy levels clipped) for a spectrum in (i) the $\mathfrak{su}(2)$ sector of the β-deformed theory, and (ii) the $\mathfrak{sl}(2)$ sector ($n_b = 50$ in both cases).

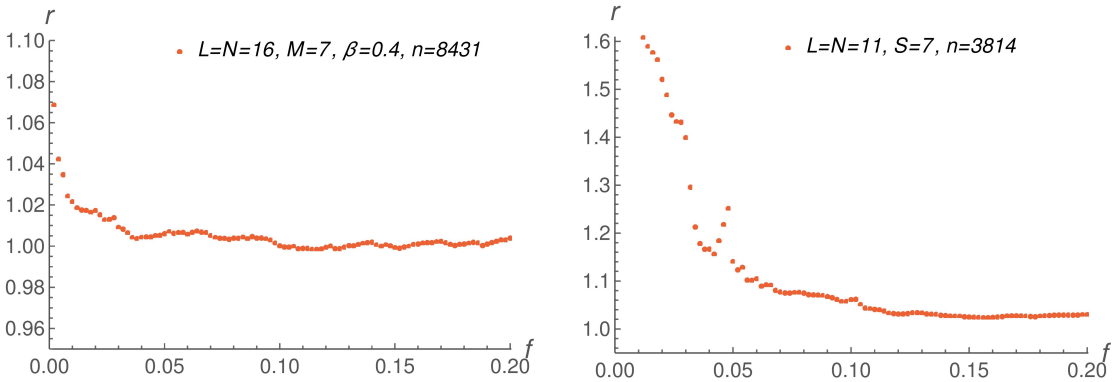
necessary in order to obtain a good unfolding, there are still states at the ends of the spectrum that do not exhibit the chaotic properties of RMT. In order to assess which minimal clippings lead to stable results for the NNS distribution, we fit the results to the Wigner–Dyson distribution (4.41). The minimal percentage of states that need to be clipped to lead to stable values of α is theory-dependent, but can be found systematically, see Figure C.2. Note that in order for the results to be stable one needs to clip a larger percentage of states in the $\mathfrak{sl}(2)$ sector. Regarding the degree p of the polynomial unfolding and the number of bins n_b, the variations are small, see Figure C.3. The results are rather insensitive to variations in these parameters, as long as $p \geq 11$ and $n_b \geq 20$. We use $p = 17$ in all examples shown in this work.

Specifics for the Dyson–Mehta statistics. For the Dyson–Mehta statistic $\Delta_3(l)$ it is similarly important to remove a sufficient number of low-energy states. In Figure



(i) (ii)

Figure C.3: Variation of the best fit parameter α in the β-deformed theory, with 8% (4%) clipping of low-(high-)energy states, as (i) we vary the number of bins n_b used for obtaining the NNS distribution with $p = 17$ fixed, and as (ii) we vary the degree p of the polynomial unfolding of the spectrum with $n_b = 50$.



(i) (ii)

Figure C.4: Plot of the ratio r between $\Delta_3(l)$ for a spectrum in (i) the β-deformed $\mathfrak{su}(2)$ sector, and (ii) the undeformed $\mathfrak{sl}(2)$ sector, and its GOE prediction at $l = 30$ as we vary the clipping fraction f.

C.4 we plot the ratio r of $\Delta_3(30)$ for a given theory with that of the GOE prediction. One can see that the ratio converges as one increases the clipping fraction. The stabilisation in the case of Δ_3 seems to occur at slightly higher values of f than those found in the analogous analysis of the spacings in Figure C.2, but is otherwise similar. Meanwhile the dependence on the degree of the polynomial unfolding is negligible for $p \geq 11$ and also consistent with the plot in Figure C.3.

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